Hindawi
Journal of Mathematics
Volume 2021, Article ID 6613928, 5 pages
https://doi.org/10.1155/2021/6613928

Research Article

Composition Semigroups on Weighted Bergman Spaces Induced by Doubling Weights

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Received 31 December 2020; Accepted 20 January 2021; Published 4 February 2021

Academic Editor: Dimitri Mugnai

We prove that composition semigroups are strongly continuous on weighted Bergman spaces with doubling weights. Point spectra and compact resolvent operators of infinitesimal generators of composition semigroups are characterized.

1. Introduction

Let $\mathscr{H}(\mathbb{D})$ denote the space of analytic functions in the unit disc $\mathbb{D} = \{z \in \mathbb{C}\colon |z| < 1\}$. For a nonnegative function $\omega \in L^1([0,1))$, the extension to $\mathbb{D}$, defined by $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$, is called a radial weight. For $0 < p < \infty$ and a radial weight ω, the weighted Bergman space A^p_ω consists of $f \in \mathscr{H}(\mathbb{D})$ such that

$$\|f\|^p_{A^p_\omega} = \int_{\mathbb{D}} |f(z)|^p \omega(z) dA(z) < \infty, \tag{1}$$

where $dA(z) = dx\, dy/\pi$ is the normalized Lebesgue area measure on $\mathbb{D}$. As usual, A^p_α stands for the classical weighted Bergman space induced by the standard radial weight $\omega(z) = (1-|z|^2)^\alpha$, where $-1 < \alpha < \infty$.

For a radial weight ω, write $\widehat{\omega}(z) = \int_{|z|}^1 \omega(s) ds$ for all $z \in \mathbb{D}$. In this paper, we always assume $\widehat{\omega}(z) > 0$, for otherwise $A^p_\omega = \mathscr{H}(\mathbb{D})$ for each $0 < p < \infty$. A weight ω belongs to the class $\widehat{\mathscr{D}}$ if there exists a constant $C = C(\omega) \geq 1$ such that $\widehat{\omega}(r) \leq C\widehat{\omega}(1 + r/2)$ for all $0 \leq r < 1$. For more knowledge about those Bergman spaces, see [1–4] and the reference therein.

A family $(\varphi_t)_{t\geq 0}$ of analytic self-maps of the unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$ is said to be a semigroup if the following conditions hold:

(i) φ_0 is the identity map of $\mathbb{D}$

(i) $\varphi_t \circ \varphi_s = \varphi_{t+s}$, for $t, s \geq 0$

(ii) For each $z \in \mathbb{D}$, $\varphi_t(z) \longrightarrow z$, as $t \longrightarrow 0^+$

A semigroup $(\varphi_t)_{t\geq 0}$ is said to be trivial if each φ_t is the identity of $\mathbb{D}$. The infinitesimal generator of $(\varphi_t)_{t\geq 0}$ is defined as the function

$$G(z) = \lim_{t\mapsto 0^+} \frac{\varphi_t(z) - z}{t}, \quad z \in \mathbb{D}. \tag{2}$$

For any nontrivial semigroup $(\varphi_t)_{t\geq 0}$, there exist a point $b \in \overline{\mathbb{D}}$ and an analytic function $P\colon \mathbb{D} \mapsto \mathbb{C}$ with $\mathrm{Re} P \geq 0$ such that

$$G(z) = (\overline{b}z - 1)(z - b)P(z). \tag{3}$$

Refer [5] for the details. Representation (3) is unique, and the point b is said to be the Denjoy–Wolff point of $(\varphi_t)_{t\geq 0}$.

Notice that each semigroup $(\varphi_t)_{t\geq 0}$ gives rise to a semigroup $(C_t)_{t\geq 0}$ consisting of composition operators on $H(\mathbb{D})$, the set of analytic functions on $\mathbb{D}$, where

$$C_t(f) := C_{\varphi_t} f = f^\circ \varphi_t, \quad f \in H(\mathbb{D}). \tag{4}$$

Given a semigroup $(\varphi_t)_{t\geq 0}$ and a Banach space X of analytic functions on $\mathbb{D}$, we say that $(\varphi_t)_{t\geq 0}$ generates a strongly continuous composition operator on X if C_t is bounded on X and

$$\lim_{t\longrightarrow 0^+} \|C_t x - x\|_X = 0 \quad \text{for all } x \in X. \tag{5}$$

$(C_t)_{t\geq 0}$ is called uniformly continuous on X if $\lim_{t\longrightarrow 0^+}\|C_t - I\| = 0$, where I is the identity map on X. The infinitesimal generator of a strongly continuous semigroup $(C_t)_{t\geq 0}$ on a Banach space X is the operator

$$\Gamma x := \lim_{t\longrightarrow 0^+} \frac{C_t(x) - x}{t}, \tag{6}$$

which is densely defined for every x in the domain

$$D(\Gamma) := \left\{ x \in X\colon \lim_{t\longrightarrow 0^+} \frac{C_t(x) - x}{t} \text{exists} \right\}. \tag{7}$$

Refer [6] for more information about operator semigroup.

In 1978, Berkson and Porta [5] initially studied the strong continuity of semigroups of composition operators acting on the classical Hardy space $H^p(\mathbb{D})$. They proved that for $1\leq p<\infty$, $(C_t)_{t\geq 0}$ is strongly continuous on H^p. Later, Siskakis demonstrated that $(C_t)_{t\geq 0}$ is strongly continuous on the Bergman space A^p_α $(1\leq p<\infty; -1<\alpha<\infty)$ and the Dirichlet space $\mathscr{D}$ in [7, 8]. Moreover, he showed that the infinitesimal generator Γ of a semigroup of composition operators on these spaces is of the form $\Gamma f = Gf'$ with a certain domain. Refer [9–16] for more results of composition semigroups on other various spaces.

In this paper, we consider composition semigroups on weighted Bergman spaces A^p_ω with $\omega\in\widehat{\mathscr{D}}$. We will show that every $(C_t)_{t\geq 0}$ is strongly continuous on A^p_ω if $1\leq p<\infty$ and $\omega\in\widehat{\mathscr{D}}$. The corresponding infinitesimal generator Γ of $(C_t)_{t\geq 0}$ and its point spectrum can also be identified. In addition, if the Denjoy–Wolff point of $(\varphi_t)_{t\geq 0}$ belongs to $\mathbb{D}$, then we can also characterize the compactness of resolvent operator $R(\lambda,\Gamma)$ of Γ, provided λ belongs to the resolvent set of Γ.

Throughout the paper, the symbol $A\approx B$ means that $A\lesssim B\lesssim A$. We say that $A\lesssim B$ if there exists a constant C such that $A\leq CB$.

2. Strongly Continuous Composition Semigroup

We need more information about $(\varphi_t)_{t\geq 0}$ before presenting our results.

Assume a semigroup $(\varphi_t)_{t\geq 0}$ consisting of analytic self-maps of $\mathbb{D}$ with infinitesimal generator G and Denjoy–Wolff point b. In general, $(\varphi_t)_{t\geq 0}$ can be classified into two classes: $b\in\mathbb{D}$ and $b\in\partial\mathbb{D}$, the boundary of $\mathbb{D}$. In particular, there exists a unique univalent function $h\colon \mathbb{D}\longrightarrow\mathbb{C}$, called Koenigs function, such that

(1) If $b\in\mathbb{D}$, then $h(b)=0$, $h'(b)=1$ and

$$(h^\circ\gamma_b)(\varphi_t(z)) = e^{G'(b)t}(h^\circ\gamma_b)(z) \quad \text{for } z\in\mathbb{D} \text{ and } \geq 0, \tag{8}$$

where $\gamma_b(z) = z - b/1 - \overline{b}z$. Moreover,

$$G(z)(h^\circ\gamma_b)(z) = G'(b)(h^\circ\gamma_b)(z), \quad z\in\mathbb{D}. \tag{9}$$

(2) If $b\in\partial\mathbb{D}$, then $h(0)=0$ and

$$h(\varphi_t(z)) = h(z) + G(0)t \quad \text{for } z\in\mathbb{D} \text{ and } t\geq 0. \tag{10}$$

Moreover,

$$h'(z)G(z) = G(0), \quad z\in\mathbb{D}. \tag{11}$$

Lemma 1 shows that every composition operator is bounded on A^p_ω if $0<p<\infty$ and $\omega\in\widehat{\mathscr{D}}$. The proof can be easily obtained by a simple combination of Theorem 15 and (4.7) in [17].

Lemma 1 (see [17]). *Let $0<p<\infty$ and $\omega\in\widehat{\mathscr{D}}$, and let φ be an analytic self-map of $\mathbb{D}$. Then, the composition C_φ is bounded on A^p_ω. Moreover, there exist constants $\eta=\eta(\omega)>1$ and $C=C(\eta,\omega,p)$ such that*

$$\left\|C_\varphi\right\| \leq C\left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^\eta. \tag{12}$$

Now, we are ready to show our results. For $f\in\mathscr{H}(\mathbb{D})$ and $0<r<1$, set

$$M_p(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^p\right)^{1/p}, \quad 0<p<\infty,$$
$$M_\infty(r,f) = \sup_{|z=r|}|f(z)|. \tag{13}$$

Theorem 1. *Let $1\leq p<\infty$ and $\omega\in\widehat{\mathscr{D}}$. Suppose $(\varphi_t)_{t\geq 0}$ is a semigroup of analytic self-maps of $\mathbb{D}$ with infinitesimal generator G. Then, the induced composition semigroup $(C_t)_{t\geq 0}$ defined in (4) is strongly continuous on A^p_ω with infinitesimal generator Γ:*

$$\Gamma f = Gf' \tag{14}$$

on its domain

$$D(\Gamma) = \{f\in A^p_\omega\colon Gf'\in A^p_\omega\}. \tag{15}$$

Moreover, $(C_t)_{t\geq 0}$ is uniformly continuous on A^p_ω if and only if $(\varphi_t)_{t\geq 0}$ is trivial.

Proof. Since $\omega\in\widehat{\mathscr{D}}$, the polynomials are dense in A^p_ω. Therefore, for any $f\in A^p_\omega$, there exists a sequence of polynomials $\{P_n\}$ such that $\lim_{n\longrightarrow\infty}\|f-P_n\|_{A^p_\omega}=0$. It follows triangle inequality that

$$\begin{aligned}\left\|C_tf - f\right\|_{A^p_\omega} &\leq \left\|C_tf - C_tP_n\right\|_{A^p_\omega} + \left\|C_tP_n - P_n\right\|_{A^p_\omega} + \left\|f-P_n\right\|_{A^p_\omega}\\ &\leq \left(\left\|C_t\right\|+1\right)\left\|f-P_n\right\|_{A^p_\omega} + \left\|C_tP_n - P_n\right\|_{A^p_\omega}.\end{aligned} \tag{16}$$

According to Lemma 1, we know that $\sup_{t\in[0,1]}\|C_t\|<\infty$. Therefore, to prove $\lim_{t\longrightarrow 0^+}\|C_tf - f\|_{A^p_\omega}=0$, it suffices to prove $\lim_{t\longrightarrow 0^+}\|C_tP - P\|_{A^p_\omega}=0$ for each polynomial P. Equivalently, we only need to show that for each $n\geq 0$, $\lim_{t\longrightarrow 0^+}\|(\varphi_t)^n - e_n\|_{A^p_\omega}=0$, where $e_n(z)=z^n$, while it can be easily obtained by Lebesgue-dominated convergence theorem. Thus, $(C_t)_{t\geq 0}$ is strongly continuous on A^p_ω.

By definition, the domain of Γ is

$$D(\Gamma) = \left\{ f \in A^p_\omega: \lim_{t \longrightarrow 0^+} \frac{C_t(f) - f}{t} \text{ exists in } A^p_\omega \right\}. \quad (17)$$

Let, now, $D = \{ f \in A^p_\omega: Gf' \in A^p_\omega \}$. We are going to show that if $f \in D(\Gamma)$, then $Gf' \in A^p_\omega$. Indeed, if $f \in D(\Gamma)$, then $\Gamma(f) \in A^p_\omega$ and

$$\lim_{t \longrightarrow 0^+} \left\| \frac{C_t f - f}{t} - \Gamma(f) \right\|_{A^p_\omega} = 0. \quad (18)$$

Since for a fixed $0 < r < 1$, the well-known inequality $M_\infty(r, f) \lesssim M_p(1 + r/2, f)(1 - r)^{-1/p}$ yields

$$M^p_\infty(r, f) \lesssim \frac{\|f\|^p_{A^p_\omega}}{(1 - r)\widehat{\omega}(r)}. \quad (19)$$

Convergence in the norm of A^p_ω implies the pointwise convergence. Therefore, for every $z \in \mathbb{D}$,

$$\begin{aligned} \Gamma f(z) &= \lim_{t \longrightarrow 0^+} \frac{f(\varphi_t(z)) - f(z)}{t} \\ &= \lim_{t \longrightarrow 0^+} \frac{f(\varphi_t(z)) - f(\varphi_0(z))}{t} \frac{\partial (f(\varphi_t(z)))}{\partial t}\Big|_{t=0} \\ &= G(z) f'(z). \end{aligned} \quad (20)$$

That is, $G(z) f'(z) = \Gamma f(z) \in A^p_\omega$ and $D(\Gamma) \subseteq D$. On the other hand, for $\lambda \in \rho(\Gamma)$, the resolvent set of Γ, we have

$$\begin{aligned} D &= \{ f \in A^p_\omega: Gf' \in A^p_\omega \} = \{ f \in A^p_\omega: Gf' - \lambda f \in A^p_\omega \} \\ &= R(\lambda, \Gamma), \end{aligned} \quad (21)$$

where $R(\lambda, \Gamma) = (\lambda I - \Gamma)^{-1}$ is the resolvent operator of Γ. Since $R(\lambda, \Gamma)(A^p_\omega) \subseteq D(\Gamma)$, $D \subseteq D(\Gamma)$. Hence, $D = D(\Gamma)$.

If $(C_t)_{t \geq 0}$ is uniformly continuous on A^p_ω, then the infinitesimal generator Γ is bounded on A^p_ω. To show $(\varphi_t)_{t \geq 0}$ is trivial, it is equivalent to show $G \equiv 0$. To the end, we may consider polynomials $e_n(z) = z^n$. Then, $\Gamma(e_n) = nGe_{n-1}$, and taking $n = 1$, we see that $G \in A^p_\omega$. Since Γ is bounded on A^p_ω, for $n \geq 1$, we have $\|\Gamma e_n\| \lesssim \|e_n\|$, that is,

$$n^p \int_0^1 M^p_p(r, G) r^{(n-1)p+1} \omega(r) dr \lesssim \int_0^1 r^{np+1} \omega(r) dr. \quad (22)$$

It follows that $n^p M^p_p(1/2, G) \lesssim 1$. Since $G \in A^p_\omega$, we have $M_p(1/2, G) = 0$. Thus, $G \equiv 0$. The proof is complete. □

Theorem 2. *Let $1 \leq p < \infty$ and $\omega \in \widehat{\mathscr{D}}$. Suppose $(\varphi_t)_{t \geq 0}$ is a semigroup of analytic self-maps of $\mathbb{D}$ with Denjoy–Wolff point b, infinitesimal generator G, and associated Koenigs function h.*

(i) If $b \in \mathbb{D}$, then $P\sigma(\Gamma) \subseteq \{ kG'((b): k = 0, 1, 2, \ldots \}$. And, $kG'(b) \in P\sigma(\Gamma)$ if and only if $(h^\circ \gamma_b)^k \in A^p_\omega$.

(ii) If $b \in \partial\mathbb{D}$, then $P\sigma(\Gamma) = \{ \lambda G(0): e^{\lambda h(z)} \in A^p_\omega \}$.

Proof. (i) By (7),

$$G(z) = G'(b) \frac{(h^\circ \gamma_b)}{(h^\circ \gamma_b)'}. \quad (23)$$

Now, suppose $f \in \mathscr{H}(\mathbb{D})$ and $\lambda \neq 0$ such that $\Gamma f = Gf' = \lambda f$. Then,

$$\frac{f'}{f} = \lambda \frac{(h^\circ \gamma_b)'}{G'(b)(h^\circ \gamma_b)}. \quad (24)$$

Pick r such that $|b| < r < 1$ and f has no zeros on $|z| = r$. We have

$$\frac{1}{2\pi i} \int_{|z|=r} \frac{f'(z)}{f(z)} dz = \frac{\lambda}{G'(b)} \frac{1}{2\pi i} \int_{|z|=r} \frac{(h^\circ \gamma_b)'(z)}{(h^\circ \gamma_b)(z)} dz. \quad (25)$$

From this and the argument principle, it follows that $\lambda / G'(b) = k$, a nonnegative integer, which shows the first part of (i). Also, notice that the differential equation

$$f'(z) = k \frac{(h^\circ \gamma_b)'(z)}{(h^\circ \gamma_b)(z)} f(z) \quad (26)$$

has solution $f(z) = c(h^\circ \gamma_b)^k$, where $c \neq 0$. This shows the second part of (i).

(ii) By (10), we have $G(z) = G(0)/h'(z)$. If $f(z) = e^{\lambda h(z)} \in A^p_\omega$, then $\Gamma(f) = \lambda G(0) f$, so $\lambda G(0) \in P\sigma(\Gamma)$. Conversely, if $\Gamma f = G' f = \lambda G(0) f$, then $f'(z) = \lambda h'(z) f(z)$, which follows that $f(z) = c e^{\lambda h(z)}$, where $c \neq 0$. The proof is complete. □

3. Resolvent Operator

In [18], Siskakis characterized the compactness of $R(\lambda, \Gamma)$ on the Hardy space H^p and the weighted Bergman space A^p_α if the Denjoy–Wolff point of $(\varphi_t)_{t \geq 0}$ is in the interior of $\mathbb{D}$. In this section, we will consider the compactness of $R(\lambda, \Gamma)$ on A^p_ω with $\omega \in \widehat{\mathscr{D}}$.

For $\alpha \geq 1$ and $\omega \in \widehat{\mathscr{D}}$, $\mathscr{C}^\alpha(\omega^*)$ consists of all $f \in \mathscr{H}(\mathbb{D})$ such that

$$\|f\|^2_{\mathscr{C}^\alpha(\omega^*)} = |g(0)|^2 + \sup_{I \in \partial\mathbb{D}} \frac{\int_{S(I)} |f(z)|^2 \omega^*(z) dA(z)}{(\omega(S(I)))^\alpha} < \infty. \quad (27)$$

$\mathscr{C}^\alpha_0(\omega^*)$ consists of all $f \in \mathscr{H}(\mathbb{D})$ such that

$$\lim_{|I| \longrightarrow 0} \frac{\int_{S(I)} |f'(z)|^2 \omega^*(z) dA(z)}{(\omega(S(I)))^\alpha} = 0, \quad (28)$$

where $S(I) = \{ re^{it} \in \mathbb{D}: e^{it} \in I, 1 - |I| \leq r < 1 \}$ is Carleson square associated with I and $\omega^*(z) = \int_{|z|}^1 \log s/|z| s \, d\omega(s)$ for all $z \in \mathbb{D}/\{0\}$. Refer [1, 2] for more information about these spaces.

To prove the main result in this section, we need the following lemma, which characterizes the boundedness and compactness of integral operator V_g on A^p_ω. Here, for a $g \in \mathscr{H}(\mathbb{D})$, V_g is defined as

$$V_g(f)(z) = \frac{1}{z}\int_0^z f(\zeta)g'(\zeta)f\zeta, \quad z \in \mathbb{D}, f \in \mathscr{H}(\mathbb{D}). \tag{29}$$

Lemma 2 (see [1, 2]). *Let $0 < p < \infty$, $\omega \in \widehat{\mathscr{D}}$, and $g \in \mathscr{H}(\mathbb{D})$. Then,*

(i) V_g is bounded on A^p_ω if and only if $g \in \mathscr{C}^1(\omega^*)$

(ii) V_g is compact on A^p_ω if and only if $g \in \mathscr{C}^1_0(\omega^*)$

Lemma 3 is critical to our result.

Lemma 3. *Let $1 \le p < \infty$ and $\omega \in \widehat{\mathscr{D}}$, and let $(\varphi_t)_{t\ge 0}$ be a nontrivial semigroup of self-maps on $\mathbb{D}$ with Denjoy–Wolff point 0, infinitesimal generator G, and Koenigs function h. Suppose $(C_t)_{t\ge 0}$ is the corresponding composition semigroup on A^p_ω, with the generator Γ. Then, for all $\lambda \in \rho(\Gamma)$, the resolvent operator of Γ has the following representation:*

$$R(\lambda,\Gamma)f(z) = -\frac{1}{G'(0)}\frac{1}{(h(z))^{\lambda/-G'(0)}} \cdot \int_0^z f(\zeta)(h(\zeta))^{\lambda/-G'(0)-1}h'(\zeta)\mathrm{d}\zeta. \tag{30}$$

In particular, $-G'(0)$ belongs to $\rho(\Gamma)$, and hence,

$$R(-G(0),\Gamma)f(z) = -\frac{1}{G'(0)h(z)}\int_0^z f(\zeta)h'(\zeta)\mathrm{d}\zeta. \tag{31}$$

Proof. Let

$$R \coloneqq -\frac{1}{G'(0)}\frac{1}{(h(z))^{\lambda/-G'(0)}}\int_0^z f(\zeta)(h(\zeta))^{\lambda/-G'(0)-1}h'(\zeta)\mathrm{d}\zeta. \tag{32}$$

It is elementary to compute that $(\lambda I - \Gamma)R = R(\lambda I - \Gamma) = I$, which shows that R is the resolvent operator of Γ. Since the Denjoy–Wolff point of $(\varphi_t)_{t\ge 0}$ is 0, it is easy to see that $\mathrm{Re}(-G'(0)) \ge 0$. If $\mathrm{Re}(-G'(0)) > 0$, by Lemma 1, we have

$$\omega_0 \coloneqq \lim_{t\longrightarrow\infty}\frac{\log\|C_t\|}{t} = 0. \tag{33}$$

So, $-G'(0) \in \rho(\Gamma)$ by Theorem 1.10 in ([6], p42). If $-G'(0)$ is a pure imaginary number, write $G(z) = -i\alpha z$, where $\alpha \in \mathbb{R}\backslash\{0\}$ and $\Gamma f = -i\alpha z f'(z)$. In this case, $(i\alpha - \Gamma)(f) = g$ has the unique analytic solution

$$f(z) = \frac{1}{i\alpha z}\int_0^z g(\zeta)\mathrm{d}\zeta. \tag{34}$$

It is not difficult to see that the operator

$$g \longrightarrow \frac{1}{i\alpha z}\int_0^z g(\zeta)\mathrm{d}\zeta \tag{35}$$

is bounded on A^p_ω. Hence, $-G'(0)$ belongs to $\rho(\Gamma)$. The proof is complete. □

Now, we are ready to prove the main result in this section.

Theorem 3. *Let $1 \le p < \infty$ and $\omega \in \widehat{\mathscr{D}}$. Suppose $(\varphi_t)_{t\ge 0}$ is a semigroup of analytic self-maps of $\mathbb{D}$ with Denjoy–Wolff point 0, infinitesimal generator G, and associated Koenigs function h. Denote by Γ the infinitesimal generator if the corresponding composition semigroup $(C_t)_{t\ge 0}$ on A^p_ω and denote by $R(\lambda,\Gamma)$ the resolvent operator for $\lambda \in \rho(\Gamma)$. Then, the following statements are equivalent:*

(i) $R(\lambda,\Gamma)$ is compact on A^p_ω

(ii) $\log h(z)/z \in \mathscr{C}^1_0(\omega^*)$

Proof. The well-known resolvent equation

$$R(\lambda,\Gamma) - R(\mu,\Gamma) = (\mu-\lambda)R(\lambda,\Gamma)R(\mu,\Gamma) \quad \lambda,\mu \in \rho(\Gamma), \tag{36}$$

shows that $R(\lambda,\Gamma)$ is compact on A^p_ω for all $\lambda \in \rho(\Gamma)$ if and only if it is compact for a certain $\lambda_0 \in \rho(\Gamma)$. So, by Lemma 3, to prove the compactness of $R(\lambda,\Gamma)$, it is suffices to identify the compactness of $R(-G'(0),\Gamma)$, or equivalently, the compactness of R_h:

$$R_h(f)(z) = \frac{1}{h(z)}\int_0^z f(\zeta)h'(\zeta)\mathrm{d}\zeta. \tag{37}$$

To this end, we will use the technology mentioned in [18], which points out that R_h can be decomposed as follows:

$$M_z P_h = R_h M_z, Q_h = P_h + Q_h P_h, \tag{38}$$

where

$$M_z f(z) = z f(z),$$
$$P_h f(z) = \frac{1}{z h(z)}\int_0^z f(\zeta)\zeta h'(\zeta)\mathrm{d}\zeta, \tag{39}$$
$$Q_h f(z) = \frac{1}{z}\int_0^z f(\zeta)\frac{\zeta h'(\zeta)}{h(\zeta)}\mathrm{d}\zeta.$$

Moreover, it is elementary to see that

$$Q_h f(z) = J(f)(z) + L_h M_z(f)(z), \tag{40}$$

where $J(f)(z) = 1/z\int_0^z f(\zeta)\mathrm{d}\zeta$ and

$$L_h(f)(z) = \frac{1}{z}\int_0^z f(\zeta)\left(\log\frac{h(\zeta)}{\zeta}\right)'\mathrm{d}\zeta. \tag{41}$$

Since R_h is bounded on A^p_ω, P_h is bounded on A^p_ω by (38). Furthermore, since h is univalent and $h(0) = 0$, it is well known that $\log(h(z)/z) \in \mathrm{BMOA}$. Hence, $\log(h(z)/z) \in \mathscr{C}^1(\omega^*)$. So by Lemma 2, L_h is bounded on A^p_ω. Therefore, the compactness of J and L_h and (40) yield the boundedness of Q_h on A^p_ω. Consequently, (38) and (40) imply that R_h is compact if and only if L_h is compact if and only if $\log h(z)/z \in \mathscr{C}_0^1(\omega^*)$ by Lemma 2. The proof is finished. □

Remark 1. By [1], we immediately know that $\mathrm{BMOA} \subsetneq \mathscr{C}^1(\omega^*) \subsetneq \mathscr{B}$ and $\mathrm{VMOA} \subsetneq \mathscr{C}_0^1(\omega^*) \subsetneq \mathscr{B}_0$. Therefore, in the above theorem, $\log h(z)/z \in \mathscr{C}_0^1(\omega^*)$ if and only if $\log h(z)/z \in \mathrm{VMOA}$ if and only if $\log h(z)/z \in \mathscr{B}_0$, which indicates that $R(\lambda, \Gamma)$ is compact on A_ω^p if and only if it is compact on the Hardy space H^p if and only if it is compact on the classical Bergman space A^p by Theorem 6.1 in [17]. This makes sense since A_ω^p induced by $\omega \in \widehat{\mathscr{D}}$ lies between H^p and A^p.

It is well known that if $R(\lambda, \Gamma)$ is compact on A_ω^p, then the spectrum $\sigma(\Gamma)$ of Γ is only the point spectrum $P\sigma(\Gamma)$. By a simple combination of Theorems 2 and 3, we are in a position to depict the spectra of infinitesimal generators of some special composition semigroups. We end up the paper by showing some examples.

Example 1. For a $\mathrm{Re} c \geq 0$,consider the semigroup

$$\varphi_t(z) = e^{-ct}z, \quad t \geq 0, z \in \mathbb{D}, \tag{42}$$

with infinitesimal generator $G(z) = -cz$ and Koenigs function $h(z) = z$. Apparently, its Denjoy–Wolff point is 0 and $h \in \mathscr{C}_0^1(\omega^*)$. Therefore, by Theorems 2 and 3, the spectrum $\sigma(\Gamma)$ of infinitesimal generator Γ of the corresponding composition semigroup $(C_t)_{t \geq 0}$ is

$$\sigma(\Gamma) = \{-ck\colon k = k = 0, 1, 2, \cdots\}. \tag{43}$$

Example 2. Consider the semigroup

$$\varphi_t(z) = 1 - (1 - z)^{e^{-t}}, \quad t \geq 0, z \in \mathbb{D}, \tag{44}$$

with Denjoy–Wolff point 0, infinitesimal generator $G(z) = -(1 - z)\log 1/1 - z$, and Koenigs function $h(z) = \log 1/1 - z$. Similarly, the spectrum $\sigma(\Gamma)$ of infinitesimal generator Γ of the corresponding composition semigroup $(C_t)_{t \geq 0}$ is

$$\sigma(\Gamma) = \{-k\colon k = k = 0, 1, 2, \cdots\}. \tag{45}$$

Data Availability

No datasets were generated or analysed during the current study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

Acknowledgments

This research was supported in part by the NNSF of China (Grant no. 11720101003).

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