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Research Article

A System of Two Diophantine Inequalities with Primes

Xue Han, Huafeng Liu, and Deyu Zhang

School of Mathematics and Statistics, Shandong Normal University, Jinan, Shandong 250358, China

Correspondence should be addressed to Huafeng Liu; hfliu_sdu@hotmail.com

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Let $1 < d < c < 128/119$, $1 < \alpha < \beta < 6^{1-d/c}$. In this paper, we prove that there exist positive real numbers $N_1^{(0)}$ and $N_2^{(0)}$ depending on c, d, α, β such that for all real numbers $N_1 > N_1^{(0)}$, $N_2 > N_2^{(0)}$ and $\alpha \le N_2/N_1^{d/c} \le \beta$, the system of two Diophantine inequalities $|p_1^c + \cdots + p_6^c - N_1| < N_1^{-(1/c)(128/119-c)}\log^{109} N_1$, $|p_1^d + \cdots + p_6^d - N_2| < N_2^{-(1/d)(128/119-d)}\log^{109} N_2$ is solvable in prime variables $p_1, \ldots, p_6$.

1. Introduction

Assume that $c > 1$ is not an integer and ε is a sufficiently small positive number. Let $H(c)$ denote the least integer s such that the Diophantine inequality

$$\left|p_1^c + p_2^c + \cdots + p_s^c - N\right| < \varepsilon \tag{1}$$

is solvable in primes $p_1, p_2, \ldots, p_s$ for sufficiently large N. In 1952, Piatetski-Shapiro [1] proved that

$$\limsup_{c\longrightarrow\infty} \frac{H(c)}{c \log c} \le 4. \tag{2}$$

Piatetski-Shapiro also proved $H(c) \le 5$ for $1 < c < 3/2$. Tolev [2] first improved this result for c close to one. More precisely, Tolev proved that if $1 < c < 15/14$‘, the Diophantine inequality

$$\left|p_1^c + p_2^c + p_3^c - N\right| < \varepsilon(N) \tag{3}$$

has prime solutions p_1, p_2, p_3 for large N, where $\varepsilon(N) = N^{-(1/c)(15/14-c)}\log^9 N$. Later, this result was improved by many authors (see [3–9]), and many analogous problems of this type were studied (for example, see [10–17]).

In 1995, Tolev [18] first considered the system of two Diophantine inequalities with primes

$$\begin{aligned} &\left|p_1^c + \cdots + p_5^c - N_1\right| < \varepsilon_1^*(N_1),\\ &\left|p_1^d + \cdots + p_5^d - N_2\right| < \varepsilon_2^*(N_2). \end{aligned} \tag{4}$$

He established that for all real numbers N_1, N_2 satisfying $N_1 > N_1^{(0)}, N_2 > N_2^{(0)}$ and $\alpha \le N_2/N_1^{d/c} \le \beta$, system (4) with

$$\begin{aligned} \varepsilon_1^*(N_1) &= N_1^{-(1/c)(35/34-c)}\log^{12} N_1, \varepsilon_2^*(N_2)\\ &= N_2^{-(1/d)(35/34-d)}\log^{12} N_2 \end{aligned} \tag{5}$$

has solutions in primes $p_1, \ldots, p_5$, where c, d, α, β are real numbers satisfying the conditions

$$1 < d < c < 35/34, 1 < \alpha < \beta < 5^{1-d/c}, \tag{6}$$

and $N_1^{(0)}, N_2^{(0)}$ depending on c, d, α, β are sufficiently large numbers. Subsequently, Tolev's result was improved by Zhai [19] and Zhai and Cao [20]. Now the best result is due to Zhai and Cao [20] who proved that system (4) with

$$\begin{aligned} \varepsilon_1^*(N_1) &= N_1^{-(1/c)(27/26-c)}\log^{100} N_1, \varepsilon_2^*(N_2)\\ &= N_2^{-(1/d)(27/26-d)}\log^{100} N_2 \end{aligned} \tag{7}$$

is solvable for $1 < d < c < 27/26$.

In this paper, we consider the following system of two Diophantine inequalities over primes $p_1, \ldots, p_6$:

$$\begin{aligned}&\left|p_1^c+\cdots+p_6^c-N_1\right|<\varepsilon_1(N_1),\\&\left|p_1^d+\cdots+p_6^d-N_2\right|<\varepsilon_2(N_2),\end{aligned} \tag{8}$$

where $c>1$, $d>1$ are different numbers but close to 1 and $\varepsilon_1(N_1), \varepsilon_2(N_2)$ satisfy

$$\begin{aligned}&\varepsilon_1(N_1)\longrightarrow 0,\ \text{as}\ N_1\longrightarrow\infty,\\&\varepsilon_2(N_2)\longrightarrow 0,\ \text{as}\ N_2\longrightarrow\infty.\end{aligned} \tag{9}$$

We have to impose a condition on the orders of N_1 and N_2 due to the inequality

$$\left(x_1^c+\cdots+x_6^c\right)^{d/c}\le x_1^d+\cdots+x_6^d\le 6^{1-d/c}\left(x_1^c+\cdots+x_6^c\right)^{d/c}, \tag{10}$$

which holds for every positive $x_1,\ldots,x_6$ provided $1<d<c$. Our result is as follows.

Theorem 1. *Suppose that c, d, α, β are real numbers and satisfy conditions*

$$1<d<c<128/119, \tag{11}$$

$$1<\alpha<\beta<6^{1-d/c}. \tag{12}$$

Then, there exist positive real numbers $N_1^{(0)}$, $N_2^{(0)}$ depending on c, d, α, β such that for all real numbers $N_1>N_1^{(0)}, N_2>N_2^{(0)}$ and

$$\alpha\le N_2/N_1^{d/c}\le\beta, \tag{13}$$

system (8) with

$$\begin{aligned}&\varepsilon_1(N_1)=N_1^{-(1/c)(128/119-c)}\log^{109}N_1,\\&\varepsilon_2(N_2)=N_2^{-(1/d)(128/119-d)}\log^{109}N_2\end{aligned} \tag{14}$$

has solutions in primes $p_1,\ldots,p_6$.

Notation. Throughout this paper, the letter p, with or without a subscript, always represents a prime, c and d are real numbers satisfying (11), and α and β are real numbers satisfying (12). η denotes a sufficiently small positive number depending on c and d. $\chi(t)$ denotes the characteristic function over the interval $[-1,1]$. $\rho=\beta'+i\gamma'$ is the non-trivial zero of the Riemann zeta function $\zeta(s)$. As usual, $\Lambda(n)$ and $\tau(n)$ are the von Mangoldt function and the divisor function, respectively. N_1 and N_2 are sufficiently large numbers. We set

$$\begin{aligned}&X=N_1^{1/c}, \tau_1=X^{3/4-c-\eta}, \tau_2=X^{3/4-d-\eta},\\&\varepsilon_1=X^{-(128/119-c)}\log^{107}X, \varepsilon_2=X^{-(128/119-d)}\log^{107}X,\\&K_1=X^{(128/119-c)}\log^{-106}X, K_2=X^{(128/119-d)}\log^{-106}X,\\&e(t)=e^{2\pi it}, \varphi(t)=e^{-\pi t^2}, \varphi_\delta(t)=\delta\varphi(\delta t).\end{aligned} \tag{15}$$

2. Outline of the Proof

Let λ denote a sufficiently small positive number, whose value depends on c, d, α, β and will be determined more precisely in Lemma 1. Let

$$B=\sum_{\lambda X<p_1,\ldots,p_6\le X}(\log p_1)\cdots(\log p_6)\chi\left(\frac{p_1^c+\cdots+p_6^c-N_1}{\varepsilon_1\log X}\right)\chi\left(\frac{p_1^d+\cdots+p_6^d-N_2}{\varepsilon_2\log X}\right), \tag{16}$$

$$S(x,y)=\sum_{\lambda X<p\le X}(\log p)e\left(xp^c+yp^d\right), \tag{17}$$

$$D=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}S^6(x,y)e(-N_1x-N_2y)\varphi_{\varepsilon_1}(x)\varphi_{\varepsilon_2}(y)\mathrm{d}x\mathrm{d}y. \tag{18}$$

We divide the plane into three regions: the neighbourhood of origin Ω_1, the intermediate region Ω_2, and the trivial region Ω_3, which are defined as

$$\begin{aligned}\Omega_1&=\{(x,y):\ \max(|x|/\tau_1,|y|/\tau_2)<1\},\\\Omega_2&=\{(x,y):\ \max(|x|/\tau_1,|y|/\tau_2)\ge1,\\&\qquad\max(|x|/K_1|y|/K_2)\le1\},\\\Omega_3&=\{(x,y):\ \max(|x|/K_1|y|/K_2)>1\}.\end{aligned} \tag{19}$$

Thus, the integral D can be represented as

$$D=D_1+D_2+D_3, \tag{20}$$

where

$$D_i=\iint_{\Omega_i}S^6(x,y)e(-N_1x-N_2y)\varphi_{\varepsilon_1}(x)\varphi_{\varepsilon_2}(y)\mathrm{d}x\mathrm{d}y\quad(i=1,2,3). \tag{21}$$

If we show that B tends to infinity as X tends to infinity, then Theorem 1 holds. From Lemma 4, it is sufficient to prove that D tends to infinity as X tends to infinity. To do this, noting (20), we shall prove the following:

$$|D_1|\gg\varepsilon_1\varepsilon_2X^{6-c-d}, \tag{22}$$

$$|D_2|\ll\frac{\varepsilon_1\varepsilon_2X^{6-c-d}}{\log X}, \tag{23}$$

$$\left|D_3\right| \ll 1. \tag{24}$$

In Section 3, we first give some auxiliary lemmas. Inequality (22) is proved in Section 4. Inequality (23), from which we can get the range of c and d, is proved in Section 5. In Section 6, we complete the proof of Theorem 1.

3. Auxiliary Lemmas

Lemma 1. *Let $\delta \in [\alpha, \beta]$. There exists $\lambda > 0$ depending on c, d, α, β such that for the volume V of the domain in six-dimensional space defined by*

$$t_1, \ldots, t_6 > \lambda, \left|t_1^c + \cdots + t_6^c - 1\right| < \mu_1, \left|t_1^d + \cdots + t_6^d - \delta\right| < \mu_2, \tag{25}$$

we have

$$V \gg \mu_1 \mu_2, \tag{26}$$

provided μ_1, μ_2 are sufficiently small.

Proof. This lemma is similar to Lemma 1 in Tolev [18]. We can write the volume of V as

$$V = \int \cdots \int_{\substack{t_1, \ldots, t_6 > \lambda \\ \left|t_1^c + \cdots + t_6^c - 1\right| < \mu_1 \\ \left|t_1^d + \cdots + t_6^d - \delta\right| < \mu_2}} 1 \mathrm{d}t_1 \cdots \mathrm{d}t_6. \tag{27}$$

Then, we can fix $t_1, \ldots, t_5$ to get the range of t_6 from last two inequalities and get

$$\left(1 - t_1^c + \cdots + t_5^c - \mu_1\right)^{1/c} < t_6 < \left(1 - t_1^c + \cdots + t_5^c + \mu_1\right)^{1/c}, \tag{28}$$

$$\left(\delta - t_1^d + \cdots + t_5^d - \mu_2\right)^{1/d} < t_6 < \left(\delta - t_1^d + \cdots + t_5^d + \mu_2\right)^{1/d}. \tag{29}$$

Since μ_1 and μ_2 are sufficiently small, we can adjust the value of λ to ensure that there are intersections between (28) and (29). We may also get this lemma by adjusting the value of λ and using circle method to estimate

$$\left|t_1^c + \cdots + t_6^c - 1\right| < \mu_1, \left|t_1^d + \cdots + t_6^d - \delta\right| < \mu_2. \tag{30}$$

Lemma 2 (see [18], Lemma 2). *The function $\varphi(t) = e^{-\pi t^2}$ has the following properties:*

$$\varphi(x) = \int_{-\infty}^{\infty} \varphi(t) e(-xt) \mathrm{d}t, \tag{31}$$

$$\chi(t/\varrho) \geq \varphi(t) - e^{-\pi\varrho^2} \quad \text{for } \varrho > 0, \tag{32}$$

$$\varphi(t) \geq e^{-\pi} \quad \text{for } |t| \leq 1. \tag{33}$$

Lemma 3. *Let $B > 1$ denote a real number and f be a smooth real function on $[B, 2B]$. Suppose that there exists a positive constant $A = A(f)$ such that*

$$\mathrm{AB}^{1-j} \ll_j \left|f^j(x)\right| \ll_j \mathrm{AB}^{1-j}, \quad \text{for } x \sim B \text{ and } j \in \mathbb{N}, \tag{34}$$

where the implied absolute constant depends only on j. Then, there exists an exponent pair (κ, ι) with

$$0 \leq \kappa \leq 1/2 \leq \iota < 1, \tag{35}$$

such that

$$\sum_{B < n \leq B+h} e(f(n)) \ll A^{\kappa} B^{\iota}, \quad \text{for } 1 < h \leq B. \tag{36}$$

Proof. We can find this lemma in Ivić ([21], pp. 72–79).

Lemma 4. *The quantities B and D satisfy*

$$B \geq D + O(1). \tag{37}$$

Proof. From (31) and (32) in Lemma 2, we have

$$\begin{aligned}
&\chi\left(\frac{p_1^c + \cdots + p_6^c - N_1}{\varepsilon_1 \log X}\right) \geq \varphi\left(\frac{p_1^c + \cdots + p_6^c - N_1}{\varepsilon_1}\right) - e^{-\pi(\log X)^2} \\
&\varphi\left(\frac{p_1^c + \cdots + p_6^c - N_1}{\varepsilon_1}\right) = \varphi\left(\frac{N_1 - \left(p_1^c + \cdots + p_6^c\right)}{\varepsilon_1}\right) \\
&= \int_{-\infty}^{\infty} \varphi(x) e\left(-x \frac{N_1 - \left(p_1^c + \cdots + p_6^c\right)}{\varepsilon_1}\right) \mathrm{d}x \\
&= \int_{-\infty}^{\infty} e\left(x\left(p_1^c + \cdots + p_6^c\right)\right) e\left(-x N_1\right) \varphi_{\varepsilon_1}(x) \mathrm{d}x,
\end{aligned} \tag{38}$$

where we substitute $\varepsilon_1 x$ for x. Similarly, we have

$$\begin{aligned}
&\chi\left(\frac{p_1^d + \cdots + p_6^d - N_2}{\varepsilon_2 \log X}\right) \geq \varphi\left(\frac{p_1^d + \cdots + p_6^d - N_2}{\varepsilon_2}\right) - e^{-\pi(\log X)^2} \\
&\varphi\left(\frac{p_1^d + \cdots + p_6^d - N_2}{\varepsilon_2}\right) = \int_{-\infty}^{\infty} e\left(y\left(p_1^d + \cdots + p_6^d\right)\right) e\left(-y N_2\right) \varphi_{\varepsilon_2}(y) \mathrm{d}y.
\end{aligned} \tag{39}$$

Then,

$$B \geq \sum_{\lambda X < p_1, \ldots, p_6 \leq X} (\log p_1) \cdots (\log p_6) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e\left(x\left(p_1^c + \cdots + p_6^c\right) + y\left(p_1^d + \cdots + p_6^d\right)\right) e\left(-xN_1 - yN_2\right) \varphi_{\varepsilon_1}(x)\varphi_{\varepsilon_2}(y)\mathrm{d}x\mathrm{d}y + O(1) \geq D + O(1). \tag{40}$$

Lemma 5. *There are two real functions $G(x), F(x)$ defined in $[a,b]$, $|G(x)| \leq H$ for $a \leq x \leq b$ and $G(x)/F(x)$ is a monotonous function. Write*

$$I = \int_a^b G(x) e(F(x)) \mathrm{d}x. \tag{41}$$

If $F'(x) \geq h > 0$ or $F'(x) \leq -h < 0$ for all $x \in [a,b]$, then

$$|I| \ll H/h. \tag{42}$$

If $F''(x) \geq h > 0$ for all $x \in [a,b]$, then

$$|I| \ll H/\sqrt{h}. \tag{43}$$

Proof. This lemma can be found in Ivić ([21], pp. 56-57).

Lemma 6. *Let $\Psi(u) = \sum_{n \leq u} \Lambda(n)$ and $2 \leq t \leq u$; then,*

$$\Psi(u) = u - \sum_{|\gamma'| \leq t} \frac{u^\rho}{\rho} + O\left(\frac{u \log^2 u}{t}\right), \tag{44}$$

where $\rho = \beta' + i\gamma'$ is the nontrivial zero of $\zeta(s)$.

Proof. This is a well-known explicit formula, which can be found in Karatsuba ([22], p. 80).

4. The Estimate of the Integral D_1

In this section, we give the estimate of the integral D_1 and set $T = X^{3/4 + 119c/512 - \eta}$.

Lemma 7. *If $1 \leq T_1 \leq T$, then*

$$\frac{1}{\sqrt{T_1}} \sum_{0 < \gamma' \leq T_1} X^{\beta'} \ll X e^{-(\log X)^{1/4}}, \tag{45}$$

and

$$\frac{1}{T_1} \sum_{0 < \gamma' \leq T_1} X^{\beta'} \ll X e^{-(\log X)^{1/4}}. \tag{46}$$

Proof. We can get this lemma from Lemma 11 and Lemma 12 of Tolev [2].

Lemma 8. *If $\max(|x|/\tau_1, |y|/\tau_2) < 1$, then*

$$|J(x,y)| \ll X e^{-(\log X)^{1/5}}, \tag{47}$$

where

$$J(x,y) = \sum_{|\gamma'| \leq T} I_\rho(x,y), I_\rho(x,y) = \int_{\lambda X}^{X} e\left(t^c x + t^d y\right) t^{\rho - 1} \mathrm{d}t. \tag{48}$$

Proof. Without loss of generality, we consider the case $0 \leq x < \tau_1, 0 \leq y < \tau_2$. From $\rho = \beta' + i\gamma'$, we have

$$I_\rho(x,y) = \int_{\lambda X}^{X} t^{\beta' - 1} e\left(t^c x + t^d y + \frac{\gamma'}{2\pi} \log t\right) \mathrm{d}t. \tag{49}$$

Let

$$F(t) = t^c x + t^d y + \frac{\gamma'}{2\pi} \log t. \tag{50}$$

We define three sets of nontrivial zeroes of $\zeta(s)$ as

$$\begin{aligned} M_1 &= \left\{\rho: |\gamma'| \leq T, -\gamma'/2\pi > 3\left(cX^c x + dX^d y\right)/2\right\}, \\ M_2 &= \left\{\rho: |\gamma'| \leq T, \left(c(\lambda X)^c x + d(\lambda X)^d y\right)/2 \right. \\ &\quad \left. \leq -\gamma'/2\pi \leq 3\left(cX^c x + dX^d y\right)/2\right\}, \\ M_3 &= \left\{\rho: |\gamma'| \leq T, -\gamma'/2\pi < \left(c(\lambda X)^c x + d(\lambda X)^d y\right)/2\right\}, \end{aligned} \tag{51}$$

and the set M_2 may be empty.

We first consider $X^{-c} \leq x \leq \tau_1$ or $X^{-d} \leq y \leq \tau_2$. In this situation, we consider the following three cases.

Case 1. When $\rho \in M_1$. In this case, we have

$$\begin{aligned} F'(t) &= ct^{c-1} x + dt^{d-1} y + \frac{\gamma'}{2\pi t} \\ &\leq \frac{1}{\lambda X}\left(c(\lambda X)^c x + d(\lambda X)^d y - \frac{-\gamma'}{2\pi}\right) < 0. \end{aligned} \tag{52}$$

Applying Lemma 5, we have

$$\left|I_\rho(x,y)\right| \ll \frac{X^{\beta'}}{-cX^c x - dX^d y - \gamma'/2\pi} \ll \frac{X^{\beta'}}{|\gamma'|}. \tag{53}$$

Therefore, by (46), we can obtain

$$\sum_{\rho\in M_1}\left|I_\rho(x,y)\right| \ll \sum_{0<\gamma'\le T}\frac{X^{\beta'}}{\gamma'}$$
$$\ll (\log X)\max_{1\le T_1\le T}\left(\frac{1}{T_1}\sum_{0<\gamma'\le T_1}X^{\beta'}\right) \tag{54}$$
$$\ll Xe^{-(\log X)^{1/5}}.$$

Case 2. When $\rho\in M_2$. In this case, we have

$$F''(t) = c(c-1)t^{c-2}x + d(d-1)t^{d-2}y - \frac{\gamma'}{2\pi t^2}$$
$$\asymp \left(X^c x + X^d y\right)X^{-2}, \tag{55}$$
$$\left|I_\rho(x,y)\right| \ll \frac{X^{\beta'}}{\sqrt{X^c x + X^d y}}.$$

Hence, we use (45) and get

$$\sum_{\rho\in M_2}\left|I_\rho(x,y)\right| \ll \frac{1}{\sqrt{X^c x + X^d y}}\sum_{0<\gamma'\le X^c x + X^d y}X^{\beta'}$$
$$\ll Xe^{-(\log X)^{1/4}}. \tag{56}$$

If the set M_2 is empty, then the upper bound is trivial.

Case 3. When $\rho\in M_3$. In this case, we have

$$F'(t) = ct^{c-1}x + dt^{d-1}y + \frac{\gamma'}{2\pi t} \gg \frac{1}{X}\left(c(\lambda X)^c x + d(\lambda X)^d y - \frac{-\gamma'}{2\pi}\right) > 0,$$
$$\left|I_\rho(x,y)\right| \ll \frac{X^{\beta'}}{c(\lambda X)^c x + d(\lambda X)^d y + \gamma'/2\pi}. \tag{57}$$

From (46), we have

$$\sum_{\rho\in M_3}\left|I_\rho(x,y)\right| \ll \sum_{-\pi c(\lambda X)^c x - \pi d(\lambda X)^d y < \gamma' \le T}\frac{X^{\beta'}}{c(\lambda X)^c x + d(\lambda X)^d y + \gamma'/2\pi}$$
$$\ll \sum_{-\pi c(\lambda X)^c x - \pi d(\lambda X)^d y < \gamma' \le X^c x + X^d y}\frac{X^{\beta'}}{X^c x + X^d y} + \sum_{X^c x + X^d y < \gamma' \le T}\frac{X^{\beta'}}{\gamma'}$$
$$\ll (\log X)\max_{1\le T_1\le T}\left(\frac{1}{T_1}\sum_{0<\gamma'\le T_1}X^{\beta'}\right) \tag{58}$$
$$\ll Xe^{-(\log X)^{1/5}}.$$

Therefore, from (54)–(58), we obtain

$$|J(x,y)| \ll Xe^{-(\log X)^{1/5}}. \tag{59}$$

Now we consider the remaining case $0\le x\le X^{-c}$ and $0\le y\le X^{-d}$. We use the trivial estimate of $|I_\rho(x,y)|$ and get

$$\sum_{\rho\in M_2}\left|I_\rho(x,y)\right| \ll \sum_{\rho\in M_2}X^{\beta'} \ll X^{\beta_0}, \tag{60}$$

where $\beta_0 = \max_{\rho\in M_2}\beta' < 1$. $\sum_{\rho\in M_1}|I_\rho(x,y)|$ and $\sum_{\rho\in M_3}|I_\rho(x,y)|$ are estimated analogically as in the previous case. Then, the estimate for $|J(x,y)|$ is established.

Lemma 9. *If* $\max(|x|/\tau_1, |y|/\tau_2) < 1$, *then*

$$S(x, y) = I(x, y) + O\left(Xe^{-(\log X)^{1/5}}\right), \tag{61}$$

where

$$I(x, y) = \int_{\lambda X}^{X} e\left(xt^c + yt^d\right)\mathrm{d}t. \tag{62}$$

Proof. Noting

$$\sum_{\substack{\lambda X < p^v \le X \\ v > 1}} (\log p) e\left(p^{cv} x + p^{dv} y\right) \ll \sum_{p^2 \le X} \log p \sum_{\frac{\log(\lambda X)}{\log p} < v < \frac{\log X}{\log p}} 1 \ll X^{1/2}, \tag{63}$$

we have

$$\begin{aligned} S(x, y) &= \sum_{\lambda X < n \le X} \Lambda(n) e\left(n^c x + n^d y\right) - \sum_{\substack{\lambda X < p^v \le X \\ v > 1}} (\log p) e\left(p^{cv} x + p^{dv} y\right) \\ &= \sum_{\lambda X < n \le X} \Lambda(n) e\left(n^c x + n^d y\right) + O\left(X^{1/2}\right) := U(x, y) + O\left(X^{1/2}\right), \end{aligned} \tag{64}$$

where v denotes an integer number. Applying Abel's transformation, we get

$$\begin{aligned} U(x, y) = &-\int_{\lambda X}^{X} (\Psi(t) - \Psi(\lambda X)) \frac{\mathrm{d}}{\mathrm{d}t}\left(e\left(t^c x + t^d y\right)\right)\mathrm{d}t \\ &+ (\Psi(X) - \Psi(\lambda X)) e\left(X^c x + X^d y\right). \end{aligned} \tag{65}$$

Then, Lemma 6 implies

$$\begin{aligned} U(x, y) &= -\int_{\lambda X}^{X} \left(t - \lambda X - \sum_{|\gamma'| \le T} \frac{t^\rho - (\lambda X)^\rho}{\rho} + O\left(\frac{X \log^2 X}{T}\right)\right) \frac{\mathrm{d}}{\mathrm{d}t}\left(e\left(t^c x + t^d y\right)\right)\mathrm{d}t \\ &\quad + \left(X - \lambda X - \sum_{|\gamma'| \le T} \frac{X^\rho - (\lambda X)^\rho}{\rho} + O\left(\frac{X \log^2 X}{T}\right)\right) e\left(X^c x + X^d y\right) \\ &= -\int_{\lambda X}^{X} \left(t - \lambda X - \sum_{|\gamma'| \le T} \frac{t^\rho - (\lambda X)^\rho}{\rho}\right) \frac{\mathrm{d}}{\mathrm{d}t}\left(e\left(t^c x + t^d y\right)\right)\mathrm{d}t \\ &\quad + \left(X - \lambda X - \sum_{|\gamma'| \le T} \frac{X^\rho - (\lambda X)^\rho}{\rho}\right) e\left(X^c x + X^d y\right) + O\left(\frac{X \log^2 X\left(X^c \tau_1 + X^d \tau_2\right)}{T}\right). \end{aligned} \tag{66}$$

Using integration by parts, we have

$$U(x,y)=\int_{\lambda X}^{X} e\left(t^c x+t^d y\right)\left(1-\sum_{|\gamma'|\le T} t^{\rho-1}\right)\mathrm{d}t+O\left(\frac{X\log^2 X\left(X^c\tau_1+X^d\tau_2\right)}{T}\right)$$
$$=I(x,y)-J(x,y)+O\left(\frac{X\log^2 X\left(X^c\tau_1+X^d\tau_2\right)}{T}\right). \tag{67}$$

From Lemma 8, this lemma follows.

Lemma 10 (see [18], Lemma 8). *For $S(x,y)$ defined by (17), we have*

$$\iint_{\Omega_1} |S(x,y)|^4\varphi_{\varepsilon_1}(x)\varphi_{\varepsilon_2}(y)\mathrm{d}x\mathrm{d}y\ll \varepsilon_1\varepsilon_2 X^{4-c-d}\log^8 X. \tag{68}$$

Lemma 11 (see [18], Lemma 9). *For $I(x,y)$ defined in Lemma 9, we have*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |I(x,y)|^4\varphi_{\varepsilon_1}(x)\varphi_{\varepsilon_2}(y)\mathrm{d}x\mathrm{d}y\ll \varepsilon_1\varepsilon_2 X^{4-c-d}\log^4 X. \tag{69}$$

Write

$$H=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I^6(x,y)e\left(-N_1x-N_2y\right)\varphi_{\varepsilon_1}(x)\varphi_{\varepsilon_2}(y)\mathrm{d}x\mathrm{d}y, \tag{70}$$

and

$$H_1=\iint_{\Omega_1} I^6(x,y)e\left(-N_1x-N_2y\right)\varphi_{\varepsilon_1}(x)\varphi_{\varepsilon_2}(y)\mathrm{d}x\mathrm{d}y. \tag{71}$$

Lemma 12. *For integrals H and H_1 defined by (70) and (71), respectively, we have*

$$\left|H-H_1\right|\ll\frac{\varepsilon_1\varepsilon_2X^{6-c-d}}{\log X}. \tag{72}$$

Proof. We have

$$\left|H-H_1\right|\ll\iint_{\mathbb{R}^2/\Omega_1}|I(x,y)|^6\varphi_{\varepsilon_1}(x)\varphi_{\varepsilon_2}(y)\mathrm{d}x\mathrm{d}y$$
$$\ll \max_{\mathbb{R}^2\setminus\Omega_1}|I(x,y)|^2\iint_{\mathbb{R}^2}|I(x,y)|^4\varphi_{\varepsilon_1}(x)\varphi_{\varepsilon_2}(y)\mathrm{d}x\mathrm{d}y. \tag{73}$$

Applying Lemma 5 with $G(t)=1$ and $F(t)=xt^c+yt^d$, from $F''(t)=c(c-1)xt^{c-2}+d(d-1)yt^{d-2}$, we can get

$$\max_{\mathbb{R}^2\setminus\Omega_1}|I(x,y)|\ll X^{5/8+\eta/2}. \tag{74}$$

Then, by Lemma 11, we can get this lemma.

Lemma 13. *For the integral H defined by (70), we have*

$$H\gg\varepsilon_1\varepsilon_2X^{6-c-d}. \tag{75}$$

Proof. We have

$$H=\int\cdots\int_{\lambda X<t_1,\ldots,t_6\le X}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e\left(x\left(t_1^c+\cdots+t_6^c-N_1\right)\right)e\left(y\left(t_1^d+\cdots+t_6^d-N_2\right)\right)\times\varphi_{\varepsilon_1}(x)\varphi_{\varepsilon_2}(y)\mathrm{d}x\mathrm{d}y\mathrm{d}t_1\cdots\mathrm{d}t_6$$
$$=\int\cdots\int_{C_1}\Delta\mathrm{d}t_1\cdots\mathrm{d}t_6+\int\cdots\int_{C_2}\Delta\mathrm{d}t_1\cdots\mathrm{d}t_6, \tag{76}$$

where

$$C_1=\left\{\lambda X<t_1,\ldots,t_6\le X\colon \left|\frac{t_1^c+\cdots+t_6^c-N_1}{\varepsilon_1}\right|<1 \text{ and } \left|\frac{t_1^d+\cdots+t_6^d-N_2}{\varepsilon_2}\right|<1\right\},$$
$$C_2=\left\{\lambda X<t_1,\ldots,t_6\le X\colon \left|\frac{t_1^c+\cdots+t_6^c-N_1}{\varepsilon_1}\right|\ge 1 \text{ or } \left|\frac{t_1^d+\cdots+t_6^d-N_2}{\varepsilon_2}\right|\ge 1\right\}, \tag{77}$$
$$\Delta=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e\left(x\left(t_1^c+\cdots+t_6^c-N_1\right)\right)e\left(y\left(t_1^d+\cdots+t_6^d-N_2\right)\right)\varphi_{\varepsilon_1}(x)\varphi_{\varepsilon_2}(y)\mathrm{d}x\mathrm{d}y.$$

From (31), we get

$$\int_{-\infty}^{\infty} e\left(x\left(t_1^c+\cdots+t_6^c-N_1\right)\right)\varphi_{\varepsilon_1}(x)\mathrm{d}x=\varphi\left(\frac{t_1^c+\cdots+t_6^c-N_1}{\varepsilon_1}\right),$$
$$\int_{-\infty}^{\infty} e\left(x\left(t_1^d+\cdots+t_6^d-N_2\right)\right)\varphi_{\varepsilon_2}(x)\mathrm{d}x=\varphi\left(\frac{t_1^d+\cdots+t_6^d-N_2}{\varepsilon_2}\right). \tag{78}$$

In (76), the integral over C_2 is convergent, and hence $\int\cdots\int_{C_2}\Delta \mathrm{d}t_1\cdots\mathrm{d}t_6=O(1)$. Applying Lemma 2, we have

$$H\gg\int\cdots\int_{C_1} 1\mathrm{d}t_1\cdots\mathrm{d}t_6\gg X^6\int\cdots\int_{C_1'} 1\,\mathrm{d}\left(t_1/X\right)\cdots\mathrm{d}\left(t_6/X\right), \tag{79}$$

where

$$C_1'=\left\{\lambda<t_1/X,\ldots,t_6/X\colon \left|\left(t_1/X\right)^c+\cdots+\left(t_6/X\right)^c-1\right| <\varepsilon_1X^{-c}\text{and}\left|\left(t_1/X\right)^d+\cdots+\left(t_6/X\right)^d-N_2/X^d\right| <\varepsilon_2X^{-d}\right\}. \tag{80}$$

From (13) and Lemma 1, we have

$$H\gg\varepsilon_1\varepsilon_2X^{6-c-d}. \tag{81}$$

Lemma 14. *For the integral D_1 in (20), we have*

$$\left|D_1\right|\gg\varepsilon_1\varepsilon_2X^{6-c-d}. \tag{82}$$

Proof. We have

$$\left|D_1-H_1\right|\ll\int\int_{\Omega_1}\left|S^6(x,y)-I^6(x,y)\right|\varphi_{\varepsilon_1}(x)\varphi_{\varepsilon_2}(y)\mathrm{d}x\mathrm{d}y$$
$$\ll\max_{\Omega_1}|S(x,y)-I(x,y)|^2\int\int_{\Omega_1}\left(|S(x,y)|^4+|I(x,y)|^4\right)\varphi_{\varepsilon_1}(x)\varphi_{\varepsilon_2}(y)\mathrm{d}x\mathrm{d}y. \tag{83}$$

From Lemmas 9–11, we can obtain

$$\left|D_1-H_1\right|\ll\frac{\varepsilon_1\varepsilon_2X^{6-c-d}}{\log X}. \tag{84}$$

By Lemma 12, we have

$$D_1=H+O\left(\frac{\varepsilon_1\varepsilon_2X^{6-c-d}}{\log X}\right). \tag{85}$$

Then, from Lemma 13, we can get this lemma.

5. The Estimate for the Integral D_2

Lemma 15. *Let a_m, b_n be arbitrary complex numbers and*

$$\max\left(|x|/\tau_1,|y|/\tau_2\right)\ge 1,\ \max\left(|x|/K_1,|y|/K_2\right)\le 1,$$
$$X^{1/4}<R\le X^{1/2}, X^{1/4}<L<L_1\le 2L, \mathrm{LR}\le X. \tag{86}$$

Write

$$W=\sum_{L<n\le L_1}\sum_{X^{1/4}<m\le R} a_mb_ne\left((mn)^cx+(mn)^dy\right). \tag{87}$$

Then,

$$|W| \ll (\mathscr{B})^{1/2} X^{101/238} \log^{-13} X, \tag{88}$$

where

$$\mathscr{A} = \sum_{X^{1/4} < m \le R} |a_m|^2, \mathscr{B} = \sum_{L<n\le L_1} |b_n|^2. \tag{89}$$

Proof. Without loss of generality, we can consider the case $\tau_1 \le x \le K_1, 0 < y \le K_2$. We define $R_i\,(0 \le i \le Q)$:

$$R_0 = X^{1/4}, R_{i+1} = \min(R_i + s, R), R_Q = R, \tag{90}$$

where $s \in [1, R]$ will be determined later and $Q \ll R/s$. Thus, W can be rewritten as

$$W = \sum_{L<n\le L_1} \sum_{1\le i\le Q} \sum_{R_{i-1} < m \le R_i} a_m b_n e\left((mn)^c x + (mn)^d y\right). \tag{91}$$

By Cauchy's inequality, we have

$$|W|^2 \le \mathscr{B}Q \sum_{L<n\le L_1} \sum_{1\le i\le Q} \sum_{R_{i-1} < m_1, m_2 \le R_i} a_{m_1} \overline{a}_{m_2} e\left((m_1^c - m_2^c) n^c x + \left(m_1^d - m_2^d\right) n^d y\right). \tag{92}$$

We rearrange the sums as follows:

$$\begin{aligned} |W|^2 &\ll \mathscr{B}Q\left(\sum_{L<n\le L_1} \sum_{1\le i\le Q} \sum_{R_{i-1} < m \le R_i} |a_m|^2 \right) \\ &\quad + \mathscr{B}Q \sum_{1\le i\le Q} \sum_{\substack{R_{i-1} < m_1, m_2 \le R_i \\ m_1 \ne m_2}} \left(|a_{m_1}||a_{m_2}| \left| \sum_{L<n\le L_1} e\left((m_1^c - m_2^c) n^c x + \left(m_1^d - m_2^d\right) n^d y\right) \right| \right) \\ &\ll \mathscr{B}Q\left(\mathscr{A}L + \sum_{1\le h\le s} \sum_{1\le i\le Q} \sum_{R_{i-1} < m \le R_i - h} |a_m||a_{m+h}| \left| \sum_{L<n\le L_1} e(f(n)) \right| \right), \end{aligned} \tag{93}$$

where

$$f(n) = \left((m+h)^c - m^c\right) n^c x + \left((m+h)^d - m^d\right) n^d y. \tag{94}$$

Next, we handle the exponential sum $|\sum_{L<n\le L_1} e(f(n))|$ of (93). From the derivatives of f, we get

$$\begin{aligned} f^j(n) \asymp &\left[\left((m+h)^c - m^c\right) L^{c-1} x \right. \\ &\left. + \left((m+h)^d - m^d\right) L^{d-1} y \right] L^{1-j}. \end{aligned} \tag{95}$$

Then, by Lemma 3 with the exponent pair (see also [21], p. 77)

$$BA^2BA^2B(0,1) = (13/40, 22/40), \tag{96}$$

we have

$$\begin{aligned} \left| \sum_{L<n\le L_1} e(f(n)) \right| &\ll \left[\left((m+h)^c - m^c\right) L^{c-1} x + \left((m+h)^d - m^d\right) L^{d-1} y\right]^{13/40} L^{22/40} \\ &\ll \left[\left((m+h)^c - m^c\right) L^{c-1} K_1 + \left((m+h)^d - m^d\right) L^{d-1} K_2\right]^{13/40} L^{22/40} \\ &\ll s^{13/40} \left(R^{13(c-1)/40} L^{(13c+9)/40} K_1^{13/40} + R^{13(d-1)/40} L^{(13d+9)/40} K_2^{13/40} \right). \end{aligned} \tag{97}$$

Inserting this upper bound into (93), we have

$$|W|^2 \ll \mathscr{B}Q\Big[\mathscr{A}L + s^{13/40}\Big(R^{13(c-1)/40}L^{(13c+9)/40}K_1^{13/40} + R^{13(d-1)/40}L^{(13d+9)/40}K_2^{13/40}\Big)\sum_0\Big], \tag{98}$$

where

$$\sum_0 \sum_{1\le h\le s}\sum_{1\le i\le Q}\sum_{R_{i-1}<m\le R_i-h}|a_m||a_{m+h}|. \tag{99}$$

By Cauchy's inequality, we obtain

$$\begin{aligned}\sum_0 &\le \sum_{1\le h\le s}\sum_{X^{1/4}<m\le R-h}|a_m||a_{m+h}|\\ &\le \sum_{1\le h\le s}\left(\sum_{X^{1/4}<m\le R-h}|a_m|^2\right)^{1/2}\left(\sum_{X^{1/4}<m\le R-h}|a_{m+h}|^2\right)^{1/2}\\ &\ll s\mathscr{A}.\end{aligned} \tag{100}$$

From (98) and $Q \ll R/s$, we have

$$|W|^2 \ll \mathscr{A}\mathscr{B}LR\Big[1/s + s^{13/40}\Big(R^{13(c-1)/40}L^{(13c-31)/40}K_1^{13/40} + R^{13(d-1)/40}L^{(13d-31)/40}K_2^{13/40}\Big)\Big]. \tag{101}$$

We take $1/s = s^{13/40}(R^{13(c-1)/40}L^{(13c-31)/40}K_1^{13/40} + R^{13(d-1)/40}L^{(13d-31)/40}K_2^{13/40})$, i.e.,

$$s = R^{13(1-c)/53}L^{(31-13c)/53}K_1^{-13/53} + R^{13(1-d)/53}L^{(31-13d)/53}K_2^{-13/53}, \tag{102}$$

and then $s \in [1, R]$. Inserting this value of s into (101), we get

$$|W|^2 \ll \mathscr{A}\mathscr{B}X^{101/119}\log^{-26} X, \tag{103}$$

which yields this lemma.

Lemma 16 (see [2], Lemma 9). *Let a_m, b_n be arbitrary complex numbers and $L < L_1 \le 2L, L \le X$. Write*

$$V = \sum_{L<n\le L_1}\sum_{X^{1/4}<m\le X/n} a_m b_n e\big((mn)^c x + (mn)^d y\big). \tag{104}$$

Then, there exist a_m', b_n' satisfying $|a_m'| \le |a_m|, |b_n'| \le |b_n|$ such that

$$|V| \ll (\log X)\left|\sum_{L<n\le L_1}\sum_{X^{1/4}<m\le X/L} a_m' b_n' e\big((mn)^c x + (mn)^d y\big)\right|. \tag{105}$$

Lemma 17. *Assume that $\max(|x|/\tau_1, |y|/\tau_2) \ge 1$, $\max(|x|/K_1, |y|/K_2) \le 1$; then,*

$$|S(x, y)| \ll X^{110/119}\log^{-9} X. \tag{106}$$

Proof. Without loss of generality, we can consider the case $\tau_1 \le x \le K_1, 0 < y \le K_2$. Clearly,

$$S(x, y) = V_0(x, y) - V_1(x, y) + O\big(X^{1/2}\big), \tag{107}$$

where

$$\begin{aligned}V_0(x, y) &= \sum_{X^{1/4}<n\le X}\Lambda(n)e\big(n^c x + n^d y\big),\\ V_1(x, y) &= \sum_{X^{1/4}<n\le \lambda X}\Lambda(n)e\big(n^c x + n^d y\big).\end{aligned} \tag{108}$$

Hence, it is sufficient to prove that

$$|V_0(x, y)|, |V_1(x, y)| \ll X^{110/119}\log^{-9} X. \tag{109}$$

The estimates of $V_0(x, y)$ and $V_1(x, y)$ are similar, and thus we focus on the estimate of $V_0(x, y)$. Using Vaughan's identity (see [23]), we have

$$V_0(x, y) = S_1 - S_2 - S_3, \tag{110}$$

where

$$\begin{aligned}S_1 &= \sum_{g\le X^{1/4}}\mu(g)\sum_{l\le X/g}(\log l)e\big((lg)^c x + (lg)^d y\big),\\ S_2 &= \sum_{k\le X^{1/2}}\sum_{r\le X/k} c_k e\big((kr)^c x + (kr)^d y\big),\\ S_3 &= \sum_{X^{1/4}<m\le X^{3/4}}\sum_{X^{1/4}<n\le X/m} a_m\Lambda(n)e\big((mn)^c x + (mn)^d y\big),\end{aligned} \tag{111}$$

where $|c_k| \le \log k$ and $|a_m| \le \tau(m)$.

For S_2, noting

$$\sum_{X^{1/4}<k\le X^{1/2}}\sum_{r\le X^{1/4}} c_k e\big((kr)^c x + (kr)^d y\big) \ll X^{3/4}\log X, \tag{112}$$

we have

$$S_2 = S_2^{(1)} + S_2^{(2)} + O\big(X^{3/4}\log X\big), \tag{113}$$

where

$$\begin{aligned}S_2^{(1)} &= \sum_{k\le X^{1/4}}\sum_{r\le X/k} c_k e\big((kr)^c x + (kr)^d y\big),\\ S_2^{(2)} &= \sum_{X^{1/4}<k\le X^{1/2}}\sum_{X^{1/4}<r\le X/k} c_k e\big((kr)^c x + (kr)^d y\big).\end{aligned} \tag{114}$$

For S_1, we have

$$|S_1| \le \sum_{g\le X^{1/4}}\left|\sum_{l\le X/g}(\log l)e\big((lg)^c x + (lg)^d y\big)\right|. \tag{115}$$

The second summation over l in (115) is

$$\ll \big(\log^2 X\big)\max_{L_2\in[L,L_1]}\left|\sum_{L<l\le L_2} e(h(l))\right|, \tag{116}$$

where $L < L_1 \le 2L$, $L_1 \le X/g$, and $h(l) = (lg)^c x + (lg)^d y$. Noting $h^j(l) \asymp (L^{c-1}g^c x + L^{d-1}g^d y)L^{1-j}$, Lemma 3 with exponent pair $(13/40, 22/40)$ gives

$$\left|\sum_{L<l\leq L_1} e(h(l))\right| \ll \left(L^{c-1}g^cK_1 + L^{d-1}g^dK_2\right)^{13/40}L^{22/40}$$
$$\ll X^{13c/40}L^{9/40}K_1^{13/40} + X^{13\,d/40}L^{9/40}K_2^{13/40}. \tag{117}$$

Recalling (115) and (116), we have

$$|S_1| \ll X^{110/119}\log^{-9} X. \tag{118}$$

Arguing similarly, we can get the following same bound of $S_2^{(1)}$:

$$\left|S_2^{(1)}\right| \ll X^{110/119}\log^{-9} X. \tag{119}$$

For S_3, we divide it into three parts:

$$S_3 = W_1 + W_2 + W_3, \tag{120}$$

where

$$W_1 = \sum_{X^{1/2}<n\leq X^{3/4}}\ \sum_{X^{1/4}<m\leq X/n} a_m\Lambda(n)e\left((mn)^c x + (mn)^d y\right),$$
$$W_2 = \sum_{X^{1/2}<n\leq X^{3/4}}\ \sum_{X^{1/4}<m\leq X/n} a_n\Lambda(m)e\left((mn)^c x + (mn)^d y\right),$$
$$W_3 = \sum_{X^{1/4}<n\leq X^{1/2}}\ \sum_{X^{1/4}<m\leq X^{1/2}} a_m\Lambda(n)e\left((mn)^c x + (mn)^d y\right). \tag{121}$$

For W_1, using dyadic subdivision and Lemma 16, we have

$$|W_1| \ll \left(\log^2 X\right)\left|W_1'(L)\right|, \tag{122}$$

where

$$W_1'(L) = \sum_{L<n\leq L_1}\ \sum_{X^{1/4}<m\leq X/L} a_m'b_n'e\left((mn)^c x + (mn)^d y\right), \tag{123}$$

where $X^{1/2} \leq L < L_1 \leq 2L \leq X^{3/4}$, $|a_m'| \leq |a_m| \leq \tau(m)$, and $|b_n'| \leq \Lambda(n)$. Lemma 15 implies

$$\left|W_1'(L)\right| \ll \left(\frac{X}{L}\log^3 X \cdot L\log X\right)^{1/2} X^{101/238}\log^{-13} X$$
$$= X^{110/119}\log^{-11} X, \tag{124}$$

where we used the mean value estimates

$$\sum_{m\leq y}\tau^2(m) \ll y\log^3 y, \sum_{n\leq y}\Lambda^2(n) \ll y\log y. \tag{125}$$

Therefore, we have

$$|W_1| \ll X^{110/119}\log^{-9} X. \tag{126}$$

We estimate W_2 and W_3 similar to W_1 and get

$$|W_2|, |W_3| \ll X^{110/119}\log^{-9} X. \tag{127}$$

For $S_2^{(2)}$, we follow a similar argument to S_3 and get

$$S_2^{(2)} = U_1 + U_2, \tag{128}$$

where

$$U_1 = \sum_{X^{1/4}<r\leq X^{1/2}}\ \sum_{X^{1/4}<k\leq X^{1/2}} c_k e\left((kr)^c x + (kr)^d y\right),$$
$$U_2 = \sum_{X^{1/2}<r\leq X^{3/4}}\ \sum_{X^{1/4}<k\leq X/r} c_k e\left((kr)^c x + (kr)^d y\right). \tag{129}$$

Using Lemmas 15 and 16 and the mean value estimate

$$\sum_{k\leq y}\log^2 k \ll y\log^2 y, \tag{130}$$

we get

$$|U_1|, |U_2| \ll X^{110/119}\log^{-9} X. \tag{131}$$

From (110) and (118)–(131), we can obtain (109).

Lemma 18 (see [18], Lemma 14). *For $S(x,y)$ defined by (17), we have*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |S(x,y)|^4\varphi_{\varepsilon_1}(x)\varphi_{\varepsilon_2}(y)\mathrm{d}x\mathrm{d}y \ll X^2\log^6 X. \tag{132}$$

Lemma 19. *For the integral D_2 in (20), we have*

$$|D_2| \ll \frac{\varepsilon_1\varepsilon_2 X^{6-c-d}}{\log X}. \tag{133}$$

Proof. By Lemmas 17 and 18, we have

$$|D_2| \ll \max_{\Omega_2}|S(x,y)|^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|S(x,y)|^4\varphi_{\varepsilon_1}(x)\varphi_{\varepsilon_2}(y)\mathrm{d}x\mathrm{d}y$$
$$\ll \left(X^{110/119}\log^{-9} X\right)^2 X^2\log^6 X$$
$$\ll X^{458/119}\log^{-12} X, \tag{134}$$

which yields this lemma.

6. Proof of Theorem 1

In this section, we first give the estimate of D_3 in (20) and then complete the proof of Theorem 1.

For the integral D_3 in (20), by Lemma 2, we can easily get

$$|D_3| \ll 1. \tag{135}$$

From Lemmas 14 and 19 and (135), we know that (22)–(24) follows, respectively. Therefore, recalling Lemma 4 and (20), we can get

$$B \gg \varepsilon_1\varepsilon_2 X^{6-c-d}, \tag{136}$$

from which we complete the proof of Theorem 1.

Data Availability

The data supporting the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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