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Research Article

Approximate Controllability of Neutral Measure Evolution Equations with Nonlocal Conditions

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In this paper, we consider a kind of neutral measure evolution equations with nonlocal conditions. By using semigroup theory and fixed point theorem, we can obtain sufficient conditions for the controllability results of such equations. Finally, an example is given to verify the reliability of the results.

1. Introduction

In the past decades, the theory of impulsive differential equations has been fully developed. An impulsive differential equation is used to simulate the evolution process of a system under perturbation during continuous evolution [1–3]. However, this type of system allows only a limited number of discontinuities within a limited range. As a result, it cannot simulate some complex phenomena, such as Zeno's behavior. However, the dynamic system with discontinuous trajectory is modeled by a measure differential equation or measure-driven equation [4–10]. Measure differential equations (MDEs) were studied in the early days [11–18]. In 1971, Das first studied measure differential equations. For a complete introduction of measure differential systems, we can refer to [12, 13].

On the other hand, the complete controllability of several nonlinear dynamic systems, such as stochastic systems of fractional order and dynamic systems of impulsive differential equations, has been extensively studied. Recently, some authors had discussed existence, stability, and nonlocal controllability of the measure evolution equation [9, 15, 19–22]. In the past few years, the existence and controllability of fractional abstract functional differential development systems with nonlocal conditions have been fully studied [2, 23–31]. However, the controllability problem of neutral measure evolution equations with nonlocal conditions is seldom studied. Therefore, whether the appropriate solution of the system exists for any given control u and whether the system is approximately controllable are studied.

In the paper, we will study the following neutral measure evolution equations with nonlocal conditions:

$$\begin{cases} \mathrm{d}[x(t)-q(t,x_t)]=Ax(t)\mathrm{d}t+[Bu(t)+f(t,x_t)]\mathrm{d}g(t), & t\in J=[0,a],\\ x(s)=\phi(s)+p(x), \quad s\in[-r,0], \end{cases} \tag{1}$$

where f, q: $[0, a] \times G([-r, 0]; X) \longrightarrow X, x_t(\theta) = x(t+\theta)$, $\theta \in [-r, 0], r > 0$, p: $G([-r, a]; X) \longrightarrow X$ is a specified function. The state variable $x(\cdot)$ takes values in Banach space X with the norm $\|\cdot\|$. A: $D(A) \subset X \longrightarrow X$ generates a uniformly bounded analytic semigroup $\{T(t)\}_{t\geq 0}$ in Banach space X. g: $[0, a] \longrightarrow R$ is a nondecreasing left-continuous function. The control function $u(t)$ takes values in another Banach space U, where U is a control set. The set $G([-r, 0]; X)$ and $G([-r, a]; X)$ represent the regulated functions space on $[-r, 0]$ and $[-r, a]$, respectively. The rest of this paper is organized as follows. In Section 2, some notations and preparation are given about Kurzweil–Henstock–Stieltjes integrals and regulated functions. In Section 3, we obtain the existence results for measure evolution system (1) by using Schauder's fixed point theorem. In Section 4, based on Krasnoselskii's fixed point theorem, we establish a controllability result for mild solutions of system (1). An example is given to prove validity of the results we obtained in Section 5.

2. Preliminaries

In this section, we will review some concepts and main results regarding Kurzweil–Henstock–Stieltjes integrals and regulated functions.

Consider a function δ: $[a, b] \longrightarrow R^+$. A tagged partition of the interval $[a, b]$ with division points $a = t_0 \leq t_1 \leq \cdots \leq t_m = b$ and tags $\tau_i \in [t_{i-1}, t_i], i = 1, \ldots, m$ is called δ-fine if $\tau_i - \delta(\tau_i) \leq t_i \leq \tau_i + \delta(\tau_i), i \in \{1, \ldots, m\}$.

Definition 1 (See [22]). Let g: $[a, b] \longrightarrow R$. A function f: $[a, b] \longrightarrow R^n$ is said to be Kurzweil-Henstock-Stieltjes integrable with respect to (w.r.t.) g on $[a, b]$ (shortly, K–H–Stieltjes integrable) if there exists a vector $I \in X$ such that for every $\varepsilon > 0$, there is a gauge δ_ε on $[a, b]$ with

$$\left\| \sum_{i=1}^{m} f(\tau_i)\left(g(t_i) - g(t_{i-1})\right) - I \right\| < \varepsilon, \tag{2}$$

for every δ_ε-fine partition $\{([t_{i-1}, t_i], \xi_i)$: $i = 1, 2, \ldots, m\}$ of $[a, b]$. The K–H–Stieltjes integrability is preserved on all subintervals of $[a, b]$. The function $\int_a^b f(t) \mathrm{d}g(t)$ is called the K–H–Stieltjes primitive of f w.r.t. g on $[a, b]$. Let $K_g^p(J; X)(p > 1)$ be a space of all p-ordered K–H–Stieltjes integral regulated functions from J to X with respect to g, with the norm $\|\cdot\|_{K_g^p}$ defined by

$$\|f\|_{K_g^p} = \left((\mathrm{KHS}) \int_0^a |f(s)|^p \mathrm{d}g(s)\right)^{(1/p)}. \tag{3}$$

Definition 2 (See [7]). Let X be a Banach space with a norm $\|\cdot\|$ and $[a, b]$ be a closed interval of the real line. A function f: $[a, b] \longrightarrow X$ is called regulated on $[a, b]$ if the limits

$$\begin{aligned} \lim_{s \longrightarrow t^-} f(s) &= f(t^-), \quad t \in (a, b], \\ \lim_{s \longrightarrow t^+} f(s) &= f(t^+), \quad t \in [a, b), \end{aligned} \tag{4}$$

exist and are finite. The space of all regulated functions f: $[a, b] \longrightarrow X$ is denoted by $G([a, b]; X)$. It is well known that the set of discontinuities of a regulated function is at most countable and that the space $G([a, b]; X)$ is a Banach space endowed with the norm $\|f\|_\infty = \sup_{t \in [a,b]} |f(t)|$. Let $r > 0$, for any element $z \in G([-r, 0]; X)$, and we define the norm $\|z\|_* = \sup_{s \in [-r,0]} |z(s)|$.

Let Y be another separable reflexive Banach space where control function u takes values. Let $E \subset Y$ be bounded, and admissible control set $U = K_g^p(J; E)(p > 1)$.

Definition 3 (See [7]). A set $\mathscr{A} \subset G([a, b]; X)$ is called equiregulated if for every $\varepsilon > 0$ and $t_0 \in [a, b]$, there is a $\theta > 0$ such that

(i) If $x \in \mathscr{A}, t \in [a, b]$ and $t_0 - \theta < t < t_0$, then $|x(t_0^-) - x(t)| < \varepsilon$

(ii) If $x \in \mathscr{A}, t \in [a, b]$ and $t_0 < t < t_0 + \theta$, then $|x(t) - x(t_0^+)| < \varepsilon$

Lemma 1 (See [7]). *Consider the functions f: $J \longrightarrow X$ and g: $J \longrightarrow R$ such that g is regulated and $\int_a^b f dg$ exists. Then, the function $h(t) = \int_a^b f dg$, $t \in [a, b]$ is regulated and satisfies*

$$\begin{aligned} h(t^+) &= h(t) + f(t)\Delta^+ g(t), \quad t \in [a, b), \\ h(t^-) &= h(t) - f(t)\Delta^- g(t), \quad t \in (a, b], \end{aligned} \tag{5}$$

where $\Delta^+ g(t) = g(t^+) - g(t)$ and $\Delta^- g(t) = g(t) - g(t^-)$, where $g(t^-)$ and $g(t^+)$ denote the left limit and the right limit of the function g at t, respectively.

Lemma 2 (See [7]). *Let X be a Banach space. Assume that $\mathscr{A} \subset G([a, b]; X)$ is equiregulated, and for every $t \in [a, b]$, the set $\{x(t)$: $t \in \mathscr{A}\}$ is relatively compact in X. Then, the set $\mathscr{A}$ is relatively compact in $G([a, b]; X)$.*

Lemma 3 (See [7]). *Let X be a Banach space and $E \subseteq X$ be a bounded, closed, and convex set. If the operator N: $E \longrightarrow E$ is completely continuous, then N has a fixed point on E.*

Lemma 4 (See [7]). *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of functions from $[a, b]$ to X. If x_n converges pointwisely to x_0 as $n \longrightarrow \infty$ and the sequence $\{x_n\}_{n=1}^{\infty}$ is equiregulated, then x_n converges uniformly to x_0.*

For $0 < \alpha < 1$, $(-A)^\alpha$ can be defined as a closed linear invertible operator with its domain $D(-A)^\alpha$ being dense in X. We denote by X_α the Banach space $D(-A)^\alpha$ endowed with norm $\|x\|_\alpha = \|(-A)^\alpha x\|$ which is equivalent to the graph norm of $(-A)^{\alpha}$. We have $X_{\beta} \hookrightarrow X_{\alpha}$ for $0<\alpha<\beta$, and the embedding is continuous [32].

Lemma 5 (See [32]). *ie following properties hold:*

(1) If $0<\alpha<\beta$, then $X_{\alpha} \subset X_{\beta}$ and the embedding is compact whenever the resolvent operator of A is compact

(2) For every $0<\alpha<1$, there exists $C_{\alpha}>0$ such that $\|(-A)^{\alpha} T(t)\| \leq (C_{\alpha}/t^{\alpha})$

Lemma 6 (See [31]). *Let M be a closed convex nonempty subset of a Banach space $(S, |\cdot|)$. We suppose that P and Q map M into S such that*

(i) $Px+Qy \in M (\forall x, y \in M)$

(ii) P is continuous and PM is contained in a compact set

(iii) Q is a contraction with constant $k<1$

Then, there is a $y \in M$ with $Py+Qy=y$.

3. Existence Results

In this section, we will study the existence of the mild solution of system (1). We first give the definition of solutions for measure system (1). Using the methods same as in [33], we can obtain the following definition.

Definition 4. The function $x \in G(J;X)$ is called a mild solution of system (1) on J if it satisfies the following measure integral equation:

$$\begin{aligned} x(t) = {} & T(t)\left[\phi(0)+p(x)-q\left(0, x_0\right)\right]+q\left(t, x_t\right) \\ & +\int_0^t A T(t-s) q\left(s, x_s\right) \mathrm{d} s \\ & +\int_0^t T(t-s)\left[B u(s)+f\left(s, x_s\right)\right] \mathrm{d} g(s). \end{aligned} \tag{6}$$

We introduce the following assumptions:

(H1): let $M=\sup_{t \in[0, a]}|T(t)|$, and $T(t)$ is a compact operator for every $t>0$

(H2): for each $t \in[0, a]$, the function q: $[0, a] \times G([-r, 0]; X) \longrightarrow X$ is continuous and there exists a constant $\alpha \in(0,1)$, and $H, H_1>0$ such that $q \in D\left((-A)^{\alpha}\right)$, and for any $\phi_1, \phi_2, \phi \in G([-r, 0]; X), t \in[0, a]$, the function $(-A)^{\alpha} q(\cdot, \phi)$ is strongly measurable and $(-A)^{\alpha} q(t, \cdot)$ satisfies the Lipschitz condition,

$$\left|(-A)^{\alpha} q\left(t, \phi_1\right)-(-A)^{\alpha} q\left(t, \phi_2\right)\right| \leq H\left\|\phi_1-\phi_2\right\|_*, \tag{7}$$

and the inequality

$$\left|(-A)^{\alpha} q(t, \phi)\right| \leq H_1\left(\|\phi\|_*+1\right) \tag{8}$$

(H3): the function f: $[0, a] \times G([-r, 0]; X) \longrightarrow X$ satisfies the following:

(i) $f(\cdot, \phi)$ is measurable for all $\phi \in G([-r, 0]; X)$, and $f(t, \cdot)$: $G([-r, 0]; X) \longrightarrow K_g^p(J; X)$ is continuous for a.e.$t \in J$

(ii) There is a function $m \in K_g^p(J; X)$ and a nondecreasing continuous function Φ: $R^{+} \longrightarrow R^{+}$ such that

$$\left|f\left(t, x_t\right)\right| \leq m(t) \Phi(\|x\|), \tag{9}$$

for all $x_t \in G([-r, 0]; X)$, almost all $t \in J$, and $\lim_{l \longrightarrow+\infty} \inf (\Phi(l)/l)=\varphi<+\infty$

(H4): p: $G([-r, a]; X) \longrightarrow X$ is continuous and compact, and there exist positive constants e and f such that

$$\|p(x)\| \leq e\|x\|_{\infty}+f, \tag{10}$$

(H5): B: $E \longrightarrow X$ is a linear and bounded operator, so there is a positive constant M_B such that $\|B\| \leq M_B$

Theorem 1. *If assumptions (H1)–(H5) hold, then problem (1) has a mild solution provided that*

$$\begin{aligned} & M e+(M+1)\left\|(-A)^{-\alpha}\right\| H_1+\frac{C_{1-\alpha} H_1 a^{\alpha}}{\alpha} \\ & \quad+M \varphi[g(a)-g(0)]^{(1/q)}\|m\|_{K_g^p}<1. \end{aligned} \tag{11}$$

Proof. We define the operator Q: $G(J; X) \longrightarrow G(J; X)$ by

$$\begin{aligned} (Q x)(t) = {} & T(t)\left[\phi(0)+p(x)-q\left(0, x_0\right)\right]+q\left(t, x_t\right) \\ & +\int_0^t A T(t-s) q\left(s, x_s\right) \mathrm{d} s \\ & +\int_0^t T(t-s)\left[B u(s)+f\left(s, x_s\right)\right] \mathrm{d} g(s). \end{aligned} \tag{12}$$

Let $l>0$ be a constant and $B_l=\left\{G(J; X):\|x\|_{\infty} \leq l\right\}$, where B_l is a bounded, closed, and convex set, $Q\left(B_l\right)=\left\{Q(x): x \in B_l\right\}$.

Step I: we prove that there exists a constant $l>0$ such that $Q\left(B_l\right) \subset B_l$. Assuming that this conclusion is not true, for each $l>0$, there will exist $x^l \in B_l, t_l \in J$ such that $\left\|Q x^l\left(t_l\right)\right\|>l$.

According to (H1)–(H5),

$$
\begin{aligned}
&\left\|Qx^l(t_l)\right\| \\
&= \left\|T(t_l)\left[\phi(0) + p\left(x^l\right) - q(0, x_0)\right] + q\left(t_l, x^l_{t_l}\right) + \int_0^{t_l} AT(t_l - s)q\left(s, x^l_s\right)\mathrm{d}s + \int_0^{t_l} T(t_l - s)\left[Bu(s) + f\left(s, x^l_s\right)\right]\mathrm{d}g(s)\right\| \\
&\leq M\left\|\phi(0) + p\left(x^l\right)\right\| + M\left\|q(0, x_0)\right\| + \left\|q\left(t_l, x^l_{t_l}\right)\right\| + \left\|\int_0^{t_l} (-A)^{1-\alpha}T(t_l - s)(-A)^{\alpha}q\left(s, x^l_s\right)\mathrm{d}s\right\| \\
&+ \left\|\int_0^{t_l} T(t_l - s)Bu(s)\mathrm{d}g(s)\right\| + \left\|\int_0^{t_l} T(t_l - s)f\left(s, x^l_s\right)\mathrm{d}g(s)\right\| \\
&\leq M\left\|\phi(0) + e\left\|x^l\right\| + f\right\| + M\left\|(-A)^{-\alpha}\right\|H_1\left(\left\|x_0\right\| + 1\right) + \left\|(-A)^{-\alpha}\right\|H_1\left(\left\|x^l\right\| + 1\right) + C_{1-\alpha}H_1 \\
&\times\left(\left\|x^l\right\| + 1\right)\left\|\int_0^{t_l} (t_l - s)^{\alpha-1}\mathrm{d}s\right\| + MM_B\left\|\int_0^{t_l} u(s)\mathrm{d}g(s)\right\| + M\left\|\int_0^{t_l} f\left(s, x^l_s\right)\mathrm{d}g(s)\right\| \\
&\leq M\left\|\phi(0) + el + f\right\| + (M + 1)\left\|(-A)^{-\alpha}\right\|H_1(l + 1) + \frac{C_{1-\alpha}H_1(l + 1)a^{\alpha}}{\alpha} \\
&+ MM_B[g(a) - g(0)]^{(1/q)}\|u\|_{K^p_g} + M[g(a) - g(0)]^{(1/q)}\Phi\left(\left\|x^l\right\|\right)\left(\int_0^{t_l} |m(s)|^p\mathrm{d}g(s)\right)^{(1/p)} \\
&\leq M\left\|\phi(0) + el + f\right\| + (M + 1)\left\|(-A)^{-\alpha}\right\|H_1(l + 1) + \frac{C_{1-\alpha}H_1(l + 1)a^{\alpha}}{\alpha} \\
&+ MM_B[g(a) - g(0)]^{(1/q)}\|u\|_{K^p_g} + M[g(a) - g(0)]^{(1/q)}\Phi(l)\|m\|_{K^p_g}.
\end{aligned}
\tag{13}
$$

In the formula (13), $q > 0$, $(1/q) + (1/p) = 1$, combining (13) with the fact $l < \|Qx^l(t_l)\|$, we can obtain

$$
\begin{aligned}
l < \left\|Qx^l(t_l)\right\| &\leq M\left\|\phi(0) + el + f\right\| + (M + 1)\left\|(-A)^{-\alpha}\right\|H_1(l + 1) + \frac{C_{1-\alpha}H_1(l + 1)a^{\alpha}}{\alpha} \\
&+ MM_B[g(a) - g(0)]^{(1/q)}\|u\|_{K^p_g} + M[g(a) - g(0)]^{(1/q)}\Phi(l)\|m\|_{K^p_g}.
\end{aligned}
\tag{14}
$$

Dividing both sides of (14) by l and taking limit as $l \longrightarrow \infty$, we have

$$
Me + (M + 1)\left\|(-A)^{-\alpha}\right\|H_1 + \frac{C_{1-\alpha}H_1a^{\alpha}}{\alpha} + M\varphi[g(a) - g(0)]^{(1/q)}\|m\|_{K^p_g} > 1,
\tag{15}
$$

which contradicts.

Step II: $Q(B_l)$ is equiregulated.

$$\begin{aligned}\left\|Qx(t)-Qx(t_0^+)\right\| &\le \left\|T(t)-T(t_0^+)\right\|\left\|\phi(0)+p(x)-q(0,x_0)\right\|+\left\|q(t,x_t)-q(t_0^+,x_{t_0^+})\right\|\\ &+\left\|\int_0^{t_0^+}(-A)[T(t-s)-T(t_0^+-s)]q(s,x_s)\mathrm{d}s\right\|+\left\|\int_{t_0^+}^{t}(-A)T(t-s)q(s,x_s)\mathrm{d}s\right\|\\ &+\left\|\int_0^{t_0^+}[T(t-s)-T(t_0^+-s)]Bu(s)\mathrm{d}g(s)\right\|+\left\|\int_{t_0^+}^{t}T(t-s)Bu(s)\mathrm{d}g(s)\right\|\\ &+\left\|\int_0^{t_0^+}[T(t-s)-T(t_0^+-s)]f(s,x_s)\mathrm{d}g(s)\right\|+\left\|\int_{t_0^+}^{t}T(t-s)f(s,x_s)\mathrm{d}g(s)\right\|\\ &=I_1+I_2+I_3+I_4+I_5+I_6+I_7+I_8,\end{aligned} \tag{16}$$

where

$$\begin{aligned}I_1&=\left\|T(t)-T(t_0^+)\right\|\left\|\phi(0)+p(x)-q(0,x_0)\right\|,\\ I_2&=\left\|q(t,x_t)-q(t_0^+,x_{t_0^+})\right\|,\\ I_3&=\left\|\int_0^{t_0^+}(-A)[T(t-s)-T(t_0^+-s)]q(s,x_s)\mathrm{d}s\right\|,\\ I_4&=\left\|\int_{t_0^+}^{t}(-A)T(t-s)q(s,x_s)\mathrm{d}s\right\|,\\ I_5&=\left\|\int_0^{t_0^+}[T(t-s)-T(t_0^+-s)]Bu(s)\mathrm{d}g(s)\right\|,\\ I_6&=\left\|\int_{t_0^+}^{t}T(t-s)Bu(s)\mathrm{d}g(s)\right\|,\\ I_7&=\left\|\int_0^{t_0^+}[T(t-s)-T(t_0^+-s)]f(s,x_s)\mathrm{d}g(s)\right\|,\\ I_8&=\left\|\int_{t_0^+}^{t}T(t-s)f(s,x_s)\mathrm{d}g(s)\right\|.\end{aligned} \tag{17}$$

The combination of the compactness of semigroup $T(t)$, $(t>0)$ and its strong continuity shows the continuity of $T(t)$ in the sense of uniform operator topology and $p(x), q(t,x_t), f(t,x_t)$ are bounded, and applying dominated convergence theorem, I_1, I_3, $I_5, I_7 \longrightarrow 0$, as $t \longrightarrow t_0^+$, independently on particular choices of $x(\cdot)$. Combining $1=(-A)^{-\alpha}(-A)^{\alpha}$ and $(-A)=(-A)^{\alpha}(-A)^{1-\alpha}$ with (H2), $I_2, I_4 \longrightarrow 0$.

$$\begin{aligned}I_6&=\left\|\int_{t_0^+}^{t}T(t-s)Bu(s)\mathrm{d}g(s)\right\|\\ &\le MM_B\|u\|_{K_g^p}\left[g(t)-g(t_0^+)\right]^{(1/q)},\end{aligned} \tag{18}$$

as $t \longrightarrow t_0^+, I_6 \longrightarrow 0$. Moreover, let $H(t)=\int_0^t m(s)\mathrm{d}g(s)$, according to Lemma 1, and $H(t)$ is a regulated function; then,

$$I_8 \le \left\|\int_{t_0^+}^{t}T(t-s)f(s,x_s)\mathrm{d}g(s)\right\| \le M\Phi(l)\left\|\int_{t_0^+}^{t}m(s)\mathrm{d}g(s)\right\| \le M\Phi(l)\left\|H(t)-H(t_0^+)\right\| \longrightarrow 0, \tag{19}$$

as $t \longrightarrow t_0^+$. Also, we can follow the similar procedure to show $\|(Qx)(t_0^-)-(Qx)(t)\| \longrightarrow 0$ as $t \longrightarrow t_0^-$ for each $t_0 \in (0,a]$. Therefore, $Q(B_l)$ is equiregulated on J.

Step III: Q: $B_l \longrightarrow B_l$ is a continuous operator. Let x^n be a convergent sequence in B_l with $x^n \longrightarrow x \in B_l$ as $n \longrightarrow \infty$. According to hypothesis (H2)–(H5) and the boundedness of $T(t)$, we have, for each $s \in J$,

$$\begin{aligned}f(s,x_s^n) \longrightarrow f(s,x_s), q(s,x_s^n) \longrightarrow q(s,x_s),\\ \cdot\, p(x^n) \longrightarrow p(x),\end{aligned} \tag{20}$$

as $n \longrightarrow \infty$, and then, by the dominated convergence theorem, we get

$$\begin{aligned}\|Q(x^n)-Q(x)\| &\leq M\|p(x^n)-p(x)\|+\|q(t,x_t^n)-q(t,x_t)\| \\ &\quad+\left\|\int_0^t(-A)^{1-\alpha}T(t-s)(-A)^{\alpha}[q(s,x_s^n)-q(s,x_s)]\mathrm{d}s\right\| \\ &\quad+\int_0^t T(t-s)\|f(s,x_s^n)-f(s,x_s)\|\mathrm{d}g(s) \\ &\leq M\|p(x^n)-p(x)\|+\|q(t,x_t^n)-q(t,x_t)\| \\ &\quad+C_{1-\alpha}\int_0^t(t-s)^{\alpha-1}(-A)^{\alpha}\|q(s,x_s^n)-q(s,x_s)\|\mathrm{d}s \\ &\quad+M\int_0^t\|f(s,x_s^n)-f(s,x_s)\|\mathrm{d}g(s)\longrightarrow 0.\end{aligned} \tag{21}$$

In addition, the analysis same as in Step II demonstrates that $\{Qx^n\}_{n=1}^{\infty}$ is equiregulated on J. This property and the abovementioned verification together with Lemma 4 imply that Qx^n converges uniformly to $Qx(t)$ as $n \longrightarrow \infty$, namely,

$$\|Qx^n-Qx\|_{\infty}=\sup_{t\in J}|Qx^n(t)-Qx(t)|\longrightarrow 0, \quad \text{as } n\longrightarrow\infty. \tag{22}$$

Therefore, QB_l is a continuous operator on J.

Step IV: for every $t \in J$, the set $V(t)=\{(Qx)(t)\colon x(\cdot)\in B_l\}$ is relatively compact in X.

The case where $t=0$ is trivial, since $V(0)=\{\phi(0)+p(x), x\in B_l\}$. Let $t\,(0\leq t\leq a)$ be fixed, and let η be a given real number satisfying $0<\eta<t$. For every $x(\cdot)\in B_l$, we define

$$\begin{aligned}(Q_\eta x)(t) &= T(t)[\phi(0)+p(x)-q(0,x_0)]+q(t,x_t)+T(\eta)\int_0^{t-\eta}AT(t-s-\eta)q(s,x_s)\mathrm{d}s \\ &\quad+T(\eta)\int_0^{t-\eta}T(t-s-\eta)[Bu(s)+f(s,x_s)]\mathrm{d}g(s).\end{aligned} \tag{23}$$

Since $T(\eta)$ is compact, the set $V_\eta(t)=\{(Q_\eta x)(t)\colon x(\cdot)\in B_l\}$ is relatively compact in X for every η, $0<\eta<t$. On the other hand, for every $x(\cdot)\in B_l$ in view of assumption (H2)-(H3), we have

$$\begin{aligned}\|(Qx)-(Q_\eta x)\| &= \left\|\int_{t-\eta}^t(-A)^{1-\alpha}T(t-s)(-A)^{\alpha}q(s,x_s)\mathrm{d}s\right\|+\left\|\int_{t-\eta}^t T(t-s)Bu(s)\mathrm{d}g(s)\right\| \\ &\quad+\left\|\int_{t-\eta}^t T(t-s)f(s,x_s)\mathrm{d}g(s)\right\| \\ &\leq C_{1-\alpha}H_1(l+1)\left\|\int_{t-\eta}^t(t-s)^{\alpha-1}\mathrm{d}s\right\|+MM_B\left\|\int_{t-\eta}^t u(s)\mathrm{d}g(s)\right\| \\ &\quad+M\int_{t-\eta}^t\|f(s,x_s)\|\mathrm{d}g(s) \\ &\leq\frac{C_{1-\alpha}H_1(l+1)\eta^{\alpha}}{\alpha}+MM_B[g(a)-g(a-\eta)]^{(1/q)}\|u\|_{K_g^p} \\ &\quad+M\Phi(l)\left\|\int_0^t m(s)\mathrm{d}g(s)-\int_0^{t-\eta}m(s)\mathrm{d}g(s)\right\|\longrightarrow 0.\end{aligned} \tag{24}$$

By the left continuity of g and Lemma 1, as $\eta\longrightarrow 0^+$, $\|(Qx)-(Q_\eta x)\|\longrightarrow 0$. Therefore there are relatively compact sets arbitrarily close to the set $V(t)$. Hence, for each $t\in J$, $V(t)$ is a relatively compact set in X. Step II and Step IV together with Lemma 2 imply that the set $Q(B_l)$ is relatively compact in $G(J;X)$. Hence, Q is a completely continuous operator on B_l. By Schauder's fixed point theorem (Lemma 3), Q has a fixed point in B_l, which is a mild solution of measure control system (1). The proof is completed.

4. Controllability Result

In this section, to investigate the controllability of system (1), we first give the definition of controllability.

Definition 5. The system (1) is said to be controllable on the interval J if for every initial function $x_0 \in G([-r,0];X)$ and $x_1 \in X$, there exists a control $u \in K_g^p(J;E)\,(p>1)$ such that the mild solution $x(t)$ of (1) satisfies $x(a)=x_1$.

We assume the following conditions:

(H6): for all $\phi_1,\phi_2 \in G([-r,0];X)$, we assume that function f satisfies

$$\left|f(s,\phi_1)-f(s,\phi_2)\right| \le K_1\left\|\phi_1-\phi_2\right\|_{*}, \tag{25}$$

and for all $\varphi_1,\varphi_2 \in G([-r,a];X)$, function p satisfies

$$\left|p(\varphi_1)-p(\varphi_2)\right| \le L_p\left\|\varphi_1-\varphi_2\right\|. \tag{26}$$

(H7): the linear operator $W\colon K_g^p(J;U) \longrightarrow X$, defined by $Wu=\int_0^a T(a-s)Bu(s)\mathrm{d}g(s)$, has an invertible operator W^{-1} taking values in $K_g^p(J;U)/\mathrm{Ker}\,W$, and there exists a positive constant M' such that $\|BW^{-1}\| \le M'$

(H8): $N = M\Phi(l')[g(a)-g(0)]^{(1/q)}\|m\|_{K_g^p} + MM'[g(a)-g(0)]\{\|x_1\|+M[\|\phi(0)\|+el'+f+H_1\|(-A)^{-\alpha}\|\times(\|x_0\|+1)]+\|(-A)^{-\alpha}\|H_1(l'+1)+((C_{1-\alpha}a^{\alpha}H_1(l'+1))/\alpha)+M\Phi(l')[g(a)-g(0)]^{(1/q)}\|m\|_{K_g^p}\} + M[el'+f+H_1\|(-A)^{-\alpha}\|\quad(\|x_0\|+1)]+H_1\|(-A)^{-\alpha}\|(l'+1)+((C_{1-\alpha}a^{\alpha}H_1(l'+1))/\alpha)<l$,

where $l' = l + M|\phi(0)|$

(H9): $k\colon = ML_p + H(\|(-A)^{-\alpha}\| + (a^{\alpha}C_{1-\alpha}/\alpha)) < 1$

Theorem 2. *If hypotheses (H1)–(H9) are satisfied, system (1) is controllable on J.*

Proof. Using assumption (H7) for an arbitrary function $x(\cdot)$, we define the control

$$\begin{aligned} u_x(t) = W^{-1}\Big[&x_1 - T(a)[\phi(0)+p(x)-q(0,x_0)] - q(a,x_a) - \int_0^a AT(a-s)q(s,x_s)\mathrm{d}s \\ &- \int_0^a T(a-s)f(s,x_s)\mathrm{d}g(s)\Big](t). \end{aligned} \tag{27}$$

In what follows, it suffices to show that when using this control, the operator F defined by

$$\begin{aligned} Fx(t) &= \phi(t)+p(x), t\in[-r,0], \\ Fx(t) &= T(t)[\phi(0)+p(x)-q(0,x_0)] + q(t,x_t) + \int_0^t AT(t-s)q(s,x_s)\mathrm{d}s \\ &\quad + \int_0^t T(t-s)[Bu_x(s)+f(s,x_s)]\mathrm{d}g(s), \quad t\in J, \end{aligned} \tag{28}$$

has a fixed point $x(\cdot)$ from which it follows that this fixed point is a mild solution of system (1). Clearly, $x(a)=(Fx)(a)=x_1$, from which we conclude that the system is controllable. Let $x(t)=y(t)+\widehat{\phi}(t), t\in[-r,a]$, where $\widehat{\phi}(t)$ is taken as $\phi(t)+p(x)$ for $t\in[-r,0]$ while for $t\in J$, it is defined as $T(t)\phi(0)$.

We define the operators F_1 and F_2 by

$$F_1y(t) = \begin{cases} 0, \quad t\in[-r,0], \\ \int_0^t T(t-s)f(s,y_s+\widehat{\phi}_s)\mathrm{d}g(s) + \int_0^t T(t-s)BW^{-1}[x_1 - T(a)[\phi(0)+p(y+\widehat{\phi}) \\ -q(0,x_0)] - q(a,y_a+\widehat{\phi}_a) - \int_0^a AT(a-\tau)q(\tau,y_\tau+\widehat{\phi}_\tau)\mathrm{d}\tau \\ -\int_0^a T(a-s)f(\tau,y_\tau+\widehat{\phi}_\tau)\mathrm{d}g(\tau)](s)\mathrm{d}g(s), \quad t\in J, \end{cases}$$

$$F_2y(t) = \begin{cases} 0, \quad t\in[-r,0], \\ T(t)[p(y+\widehat{\phi})-q(0,x_0)] + q(t,y_t+\widehat{\phi}_t) + \int_0^t AT(t-s)q(s,y_s+\widehat{\phi}_s)\mathrm{d}s, \quad t\in J. \end{cases} \tag{29}$$

Obviously, the operator F has a fixed point if only if operator F_1+F_2 has a fixed point. Thus, we shall employ Krasnoselskii's fixed point theorem.

Step I: we have to show $(F_1+F_2)B_l\subset B_l$, i.e., any $\phi_1,\phi_2\in B_l$ implies that $F_1\phi_1+F_2\phi_2\in B_l$. It is easy from hypotheses (H1–H9) and Lemma 1 to see that

$$\begin{aligned}
\left\|F_1\phi_1(t)+F_2\phi_2(t)\right\| &= \left\|\int_0^t T(t-s)f(s,\phi_{1s}+\widehat{\phi}_s)\mathrm{d}g(s)\right\| \\
&+\int_0^t T(t-s)BW^{-1}\left[\left\|x_1\right\|+\left\|T(a)\left[\phi(0)+p(\phi_1+\widehat{\phi})+q(0,x_0)\right]\right\|\right. \\
&+\left\|q(a,\phi_{1a}+\widehat{\phi}_a)\right\|+\left\|\int_0^a AT(a-\tau)q(\tau,\phi_{1\tau}+\widehat{\phi}_\tau)\mathrm{d}\tau\right\| \\
&+\left.\left\|\int_0^a T(a-s)f(\tau,\phi_{1\tau}+\widehat{\phi}_\tau)\mathrm{d}g(\tau)\right\|\right](s)\mathrm{d}g(s) \\
&+\left\|T(t)[p(\phi_2+\widehat{\phi})+q(0,x_0)]\right\|+\left\|q(t,\phi_{2t}+\widehat{\phi}_t)\right\| \\
&+\left\|\int_0^t AT(t-s)q(s,\phi_{2s}+\widehat{\phi}_s)\mathrm{d}s\right\| \\
&\leq M\Phi(l')[g(a)t{-}ngq(0)]^{(1/q)}\|m\|_{K_g^p}+MM'[g(a)-g(0)]\Big\{\left\|x_1\right\|+M[\|\phi(0)\|+el' \\
&+f+H_1\left\|(-A)^{-\alpha}\right\|(\left\|x_0\right\|+1)]+\left\|(-A)^{-\alpha}\right\|H_1(l'+1)+\frac{C_{1-\alpha}a^\alpha H_1(l'+1)}{\alpha} \\
&+M\Phi(l')[g(a)-g(0)]^{(1/q)}\|m\|_{K_g^p}\Big\}+M\left[el'+f+H_1\left\|(-A)^{-\alpha}\right\|(\left\|x_0\right\|+1)\right] \\
&+H_1\left\|(-A)^{-\alpha}\right\|(l'+1)+\frac{C_{1-\alpha}a^\alpha H_1(l'+1)}{\alpha}=N<l,
\end{aligned} \tag{30}$$

and thus, condition (i) in Lemma 6 is verified.

Step II: next, we need to show that operator F_1 is continuous.

$$\begin{aligned}
\left|F_1\phi_1(t)-F_1\phi_2(t)\right| &= \left|\int_0^t T(t-s)[f(s,\phi_{1s}+\widehat{\phi}_s)-f(s,\phi_{2s}+\widehat{\phi}_s)]\mathrm{d}g(s)\right|+\int_0^t T(t-s)BW^{-1} \\
&\cdot[T(t)\left|p(\phi_{1s}+\widehat{\phi}_s)-p(\phi_{2s}+\widehat{\phi}_s)\right|+\left|q(a,\phi_{1a}+\widehat{\phi}_a)-q(a,\phi_{2a}+\widehat{\phi}_a)\right| \\
&+\left|\int_0^a AT(a-\tau)\left[q(\tau,\phi_{1\tau}+\widehat{\phi}_\tau-q(\tau,\phi_{2\tau}+\widehat{\phi}_\tau)\right]\mathrm{d}\tau\right| \\
&+\left|\int_0^a T(a-s)[f(\tau,\phi_{1\tau}+\widehat{\phi}_\tau)-f(\tau,\phi_{2\tau}+\widehat{\phi}_\tau)]\mathrm{d}g(\tau)\right|](s)\mathrm{d}g(s) \\
&\leq M[g(a)-g(0)]\Big[K_1(1+MM'[g(a)-g(0)])+M'(ML_p+H\left\|(-A)^{-\alpha}\right\| \\
&+\frac{HC_{1-\alpha}a^\alpha}{\alpha})\Big]\|\phi_1-\phi_2\|.
\end{aligned} \tag{31}$$

Therefore, given $\varepsilon>0$, there exists $\delta(\varepsilon)>0$ such that $\|\phi_1-\phi_2\|<\delta$ implies $|F_1\phi_1(t)-F_1\phi_2(t)|<\varepsilon$, which proves that operator F_1 is continuous.

Step III: we shall show that F_1 maps B_l into an equicontinuous family. For $y\in B_l,\theta_1,\theta_2\in J, 0<\theta_2<\theta_1\leq a$, we have

$$\begin{aligned}
\|F_1 y(\theta_1) - F_1 y(\theta_2)\| &= \left\| \int_0^{\theta_1} T(\theta_1 - s) f(s, y_s + \widehat{\phi}_s) \mathrm{d}g(s) - \int_0^{\theta_2} T(\theta_2 - s) f(s, y_s + \widehat{\phi}_s) \mathrm{d}g(s) \right\| \\
&\quad + \left\| \int_0^{\theta_1} T(\theta_1 - s) B u_x(s) \mathrm{d}g(s) - \int_0^{\theta_2} T(\theta_2 - s) B u_x(s) \mathrm{d}g(s) \right\| \\
&\leq \left\| \int_0^{\theta_2} [T(\theta_1 - s) - T(\theta_2 - s)] f(s, y_s + \widehat{\phi}_s) \mathrm{d}g(s) \right\| \\
&\quad + \left\| \int_{\theta_2}^{\theta_1} T(\theta_1 - s) f(s, y_s + \widehat{\phi}_s) \mathrm{d}g(s) \right\| \\
&\quad + \left\| \int_0^{\theta_2} [T(\theta_1 - s) - T(\theta_2 - s)] B u_x(s) \mathrm{d}g(s) \right\| + \left\| \int_{\theta_2}^{\theta_1} T(\theta_1 - s) B u_x(s) \mathrm{d}g(s) \right\| \\
&\leq \sup_{s \in [0,a]} \|T(\theta_1 - s) - T(\theta_2 - s)\| \left\| \int_0^{\theta_2} f(s, y_s + \widehat{\phi}_s) \mathrm{d}g(s) \right\| + M\Phi(l')[H(\theta_1) \\
&\quad - H(\theta_2)] + L\sup_{s \in [0,a]} \|T(\theta_1 - s) - T(\theta_2 - s)\| [g(\theta_2) - g(0)] + LM[g(\theta_1) - g(\theta_2)],
\end{aligned} \tag{32}$$

where $H(t) = \int_0^t m(s)\mathrm{d}g(s)$, $L = M'[\|x_1\| + M[\|\phi(0)\| + (el' + f) + H_1\|(-A)^{-\alpha}\|(\|x_0\| + 1)] + H_1\|(-A)^{-\alpha}\|(l' + 1) + (H_1 C_{1-\alpha} a^{\alpha}(l' + 1)/\alpha) + M\Phi(l')[g(a) - g(0)]^{(1/q)}\|m\|_{K_g^p}]$. $\|F_1 y(\theta_1) - F_1 y(\theta_2)\| \longrightarrow 0$, as $\theta_2 \longrightarrow \theta_1$. Hence, $F_1 B_l$ is equicontinuous. Since the case $\theta_2 < \theta_1 < 0$ or $\theta_2 < 0 < \theta_1$ is very simple, the proof of the equicontinuities for the two cases is omitted.

Subsequentially, we shall show that $F_1 B_l$ is precompact. Let $0 < t \leq a$ be fixed and ε be a real number satisfying $0 < \varepsilon < t$. For $y \in B_l$, we define

$$\begin{aligned}
F_1^{\varepsilon} y(t) &= T(\varepsilon) \int_0^{t-\varepsilon} T(t - \varepsilon - s) f(s, y_s + \widehat{\phi}_s) \mathrm{d}g(s) + T(\varepsilon) \int_0^{t-\varepsilon} T(t - \varepsilon - s) B W^{-1} [x_1 - T(a) \\
&\quad \times [\phi(0) + p(y + \widehat{\phi}) - q(0, x_0)] - q(a, y_a + \widehat{\phi}_a) - \int_0^a A T(a - \tau) q(\tau, y_\tau + \widehat{\phi}_\tau) \mathrm{d}\tau \\
&\quad - \int_0^a T(a - \tau) f(\tau, y_\tau + \widehat{\phi}_\tau) \mathrm{d}g(\tau)] \mathrm{d}g(s).
\end{aligned} \tag{33}$$

$T(t)$ is a compact operator, and $Y_{\varepsilon}(t) = \{(F_1^{\varepsilon} y)(t): y \in B_l\}$ is relatively compact in X. For every ε, $0 < \varepsilon < t$. Moreover, for every $y \in B_l$, we have

$$\begin{aligned}
|F_1 y(t) - F_1^{\varepsilon} y(t)| &= \int_{t-\varepsilon}^t T(t - s) f(s, y_s + \widehat{\phi}_s) \mathrm{d}g(s) + \int_{t-\varepsilon}^t T(t - s) B W^{-1} [x_1 - T(a) \\
&\quad - q(0, x_0)] - q(a, y_a + \widehat{\phi}_a) - \int_0^a A T(a - \tau) q(\tau, y_\tau + \widehat{\phi}_\tau) \mathrm{d}\tau \\
&\quad - \int_0^a T(a - \tau) f(\tau, y_\tau + \widehat{\phi}_\tau) \mathrm{d}g(\tau)] \mathrm{d}g(s).
\end{aligned} \tag{34}$$

$\varepsilon \longrightarrow 0^+$, $\{(F_1 y)(t): Y \in B_l\}$ is precompact in X. $F_1 B_l$ is uniformly bounded. By the Arzela–Ascoli theorem, it is concluded from the uniform boundedness, equicontinuity, and precompactness of the set $F_1 B_l$ that $\overline{F_1 B_l}$ is compact.

Step IV: $F(B_l)$ is equiregulated on J.

$$\begin{aligned}\left\|Fx(t)-Fx(t_0^+)\right\| \le{}& \left\|T(t)-T(t_0^+)\right\|\left\|\phi(0)+p(x)-q(0,x_0)\right\| + \left\|q(t,x_t)-q\left(t_0^+,x_{t_0^+}\right)\right\| \\ &+\left\|\int_0^{t_0^+}(-A)[T(t-s)-T(t_0^+-s)]q(s,x_s)\mathrm{d}s\right\| + \left\|\int_{t_0^+}^{t}(-A)T(t-s)q(s,x_s)\mathrm{d}s\right\| \\ &+\left\|\int_0^{t_0^+}[T(t-s)-T(t_0^+-s)]Bu(s)\mathrm{d}g(s)\right\| + \left\|\int_{t_0^+}^{t}T(t-s)Bu(s)\mathrm{d}g(s)\right\| \\ &+\left\|\int_0^{t_0^+}[T(t-s)-T(t_0^+-s)]f(s,x_s)\mathrm{d}g(s)\right\| + \left\|\int_{t_0^+}^{t}T(t-s)f(s,x_s)\mathrm{d}g(s)\right\| \\ ={}& I_1+I_2+I_3+I_4+I_5+I_6+I_7+I_8,\end{aligned} \tag{35}$$

where

$$\begin{aligned} I_1 &= \left\|T(t)-T(t_0^+)\right\|\left\|\phi(0)+p(x)-q(0,x_0)\right\|, \\ I_2 &= \left\|q(t,x_t)-q(t_0^+,x_{t_0^+})\right\|, \\ I_3 &= \left\|\int_0^{t_0^+}(-A)[T(t-s)-T(t_0^+-s)]q(s,x_s)\mathrm{d}s\right\|, \\ I_4 &= \left\|\int_{t_0^+}^{t}(-A)T(t-s)q(s,x_s)\mathrm{d}s\right\|, \\ I_5 &= \left\|\int_0^{t_0^+}[T(t-s)-T(t_0^+-s)]Bu(s)\mathrm{d}g(s)\right\|, \\ I_6 &= \left\|\int_{t_0^+}^{t}T(t-s)Bu(s)\mathrm{d}g(s)\right\|, \\ I_7 &= \left\|\int_0^{t_0^+}[T(t-s)-T(t_0^+-s)]f(s,x_s)\mathrm{d}g(s)\right\|, \\ I_8 &= \left\|\int_{t_0^+}^{t}T(t-s)f(s,x_s)\mathrm{d}g(s)\right\|. \end{aligned} \tag{36}$$

According to Step II of Theroem 1, we can know that $F(B_l)$ is equiregulated on J.

Step V: we show that operator F_2 is a contraction with constant k, and we have

$$\begin{aligned}\left|F_2\phi_1(t)-F_2\phi_2(t)\right| \le{}& M\left|p\left(\phi_1+\widehat{\phi}\right)-p\left(\phi_2+\widehat{\phi}\right)\right| + \left|q\left(t,\phi_{1t}+\widehat{\phi}_t\right)-q\left(t,\phi_{2t}+\widehat{\phi}_t\right)\right| \\ &+\int_0^t AT(t-s)\left|q\left(s,\phi_{1s}+\widehat{\phi}_s\right)-q\left(s,\phi_{2s}+\widehat{\phi}_s\right)\right|\mathrm{d}s \\ \le{}& ML_p\left\|\phi_1-\phi_2\right\| + H\left\|(-A)^{-\alpha}\right\|\left\|\phi_1-\phi_2\right\| + \frac{a^\alpha HC_{1-\alpha}}{\alpha}\left\|\phi_1-\phi_2\right\| \\ \le{}& \left[ML_p + H\left(\left\|(-A)^{-\alpha}\right\| + \frac{a^\alpha C_{1-\alpha}}{\alpha}\right)\right]\left\|\phi_1-\phi_2\right\| := k\left\|\phi_1-\phi_2\right\|.\end{aligned} \tag{37}$$

Therefore, all the conditions of Krasnoselskii's fixed point theorem are satisfied, and thus, operator F_1+F_2 has a fixed point in B_l. From this, it follows that operator F has a fixed point and, hence, system (1) is controllable on J. This completes the proof.

5. Example

Consider the following equation:

$$\begin{cases} d_t\left[x(t,\theta)-\int_0^{\pi}U(\theta,y)x_t(s,y)\mathrm{d}y\right]=\dfrac{\partial^2}{\partial\theta^2}x(t,\theta)\mathrm{d}t+\left[u(t)+f(t,x_t(s,\theta))\right]\mathrm{d}g(t), \quad t\in J=[0,1],\ \theta\in[0,\pi],\\ x(t,0)=x(t,\pi)=0, \quad t\in[0,1],\\ x(s,\theta)=\phi(s,\theta)+\int_{-(1/2)}^{0}h(s)\ln(1+|x(s,\theta)|)\mathrm{d}s, \quad \theta\in[0,\pi],\ s\in\left[-\dfrac{1}{2},0\right]. \end{cases} \tag{38}$$

Let $X=L^2([0,\pi];R)$, we define $A\colon X\longrightarrow X$ by $Ax=(\partial^2/\partial\theta^2)x(t,\theta)$ and $D(A)=\{x(\cdot)\in X\colon x, (\partial x/\partial\theta)$ are absolutely continuous, and $(\partial^2x/\partial\theta^2)\in X, x(t,0)=x(t,\pi)=0\}$. Then, A generates a strongly continuous semigroup $\{T(t), t\geq 0\}$. Furthermore, A has a discrete spectrum, and the eigenvalues are $-n^2, n\in N$, with corresponding normalized eigenvectors $x_n(z)=(2/\pi)^{1/2}\sin(nz)$. We also use the following properties:

(i) For each $x\in X, T(t)x\colon=\sum_{n=1}^{\infty}e^{-n^2t}\langle x,x^n\rangle x^n$. In particular, $T(\cdot)$ is a uniformly stable semigroup and $\|T(t)\|_{L^2[0,\pi]}\leq e^{-t}\leq 1$.

(ii) For each $x\in X$, $(-A)^{-(1/2)}x=\sum_{n=1}^{\infty}(1/n)\langle x,x^n\rangle x^n$. In particular, $\|(-A)^{1/2}\|_{L^2[0,\pi]}=1$.

(iii) The operator $(-A)^{(1/2)}$ is given by $(-A)^{(1/2)}x=\sum_{n=1}^{\infty}n\langle x,x^n\rangle x^n$, on the space $D((-A)^{-(1/2)})=\{x(\cdot)\in X, \sum_{n=1}^{\infty}n\langle x,x^n\rangle x^n\in X\}$.

Obviously, Lemma 5 and (H1) are satisfied. Let $(U_hx)(\theta)=\int_0^{\pi}U(\theta,y)x(y)\mathrm{d}y$, for $x\in X,\theta\in[0,\pi]$. Then, function $f\colon(0,1]\times G([-(1/2),0];X)\longrightarrow X$ is continuous. Let $F(t,x_t(\theta))=f(t,x_t)(\theta)=a\sin(x_t(\theta)), t\in[0,1],\theta\in[0,\pi]$, where a is a constant. We define

$$g(t)=\begin{cases} 1-\dfrac{1}{2}, & 0\leq t\leq 1-\dfrac{1}{2},\\ \ldots, & \\ 1-\dfrac{1}{n}, & 1-\dfrac{1}{n-1}\leq t\leq 1-\dfrac{1}{n},\ n>2,\ n\in N,\\ \ldots, & \\ 1, & t=1. \end{cases} \tag{39}$$

Let $m(t)=a, \Phi(l)=l$, and we can see that $\varphi=1$.

$$\begin{aligned} \|f(t,x_t)(\theta)\| &= \left(\int_0^{\pi}a^2\sin^2(x_t(\theta))d\theta\right)^{(1/2)}\\ &\leq \left(\int_0^{\pi}a^2(x_t(\theta))^2d\theta\right)^{(1/2)}\\ &= a\|x_t\|,\\ \|f(t,x_t^n)-f(t,x_t)\| &= \left(\int_0^{\pi}a^2\left|\sin(x_t^n(\theta))-\sin(x_t(\theta))\right|^2\mathrm{d}\theta\right)^{(1/2)}\\ &\leq a\left(\int_0^{\pi}\left|(x_t^n(\theta))-(x_t(\theta))\right|^2\mathrm{d}\theta\right)^{(1/2)}\\ &\leq a\left|x_t^n-x_t\right|. \end{aligned} \tag{40}$$

Therefore, (H3) is satisfied.

$p{:}G([-(1/2),1];X)\longrightarrow X$, and $p(\phi)(\theta)=\int_{-(1/2)}^{0}h(s)\ln(1+|\phi(s)(\theta)|)\mathrm{d}s$, $\phi\in G([-(1/2),1];X)$, $\phi(s)(\theta)=x(s,\theta)$. Supposing that $h\in L([-(1/2),0];R)$ and letting $\{\phi_n,n\in N\}$ be a convergence sequence on $G([-(1/2),1];X)$,

$$\begin{aligned}\|p(\phi_n) - p(\phi)\|_X^2 &= \int_0^{\pi} \left|p(\phi_n)(\theta) - p(\phi)(\theta)\right|^2 \mathrm{d}\theta,\\ &= \int_0^{\pi}\left(\int_{-(1/2)}^{0} h(s)\left(\ln\left(1+\phi_n(s)(\theta)\right) - \ln\left(1+\phi(s)(\theta)\right)\right)\mathrm{d}s\right)^2 \mathrm{d}\theta \\ &\leq \int_{-(1/2)}^{0} |h(s)|^2 \mathrm{d}s \int_{-(1/2)}^{0}\int_0^{\pi}\left|\phi_n(s)(\theta) - \phi(s)(\theta)\right|^2 \mathrm{d}\theta \mathrm{d}s \longrightarrow 0,\end{aligned} \tag{41}$$

as $n \longrightarrow \infty$. Thus, p is a continuous operator. Using the same method, we can obtain that, for any $\phi \in G([-(1/2),1];X)$, $\|p(\phi)(\theta+\eta) - p(\phi)(\theta)\|_X \longrightarrow 0$, as $\eta \longrightarrow 0$. Moreover, $\|p(\phi)\|_X \leq (\int_{-(1/2)}^{0} |h(s)|^2 \mathrm{d}s)^{(1/2)} \|\phi\|_\infty = e\|\phi\|_\infty$. Therefore, (H4) is satisfied.

(iv) The function $U(\theta, y), \theta, y \in [0,\pi]$ is measurable and $\int_0^{\pi}\int_0^{\pi} U^2(\theta, y)\mathrm{d}y\mathrm{d}\theta < \infty$.

(v) The function $\partial_\theta U(\theta, y)$ is measurable, and $U(0, y) = U(\pi, y) = 0$; let

$$\overline{H} = \left(\int_0^{\pi}\int_0^{\pi} \left(\partial_\theta U(\theta, y)\right)^2 \mathrm{d}y\mathrm{d}\theta\right)^{(1/2)} < \infty. \tag{42}$$

From (iv) it is clear that U_h is a bounded linear operator on X. Furthermore, $U_h(x) \in D((-A)^{(1/2)})$, and $\|((-A)^{-(1/2)})U_h\|_{L^2[0,\pi]} < \infty$.

In fact, from the definition of U_h and (v), it follows that

$$\begin{aligned}\langle U_h(x), x^n\rangle &= \int_0^{\pi} x^n(\theta)\left(\int_0^{\pi} U(\theta, y)x(y)\mathrm{d}y\right)\mathrm{d}\theta\\ &= \frac{1}{n}\left(\frac{2}{\pi}\right)^{(1/2)} \langle \overline{U}(x), \cos(n\theta)\rangle\end{aligned} \tag{43}$$

where $\overline{U}$ is defined by $\overline{U}(x)(\theta) = \int_0^{\pi} \partial_\theta U(\theta, y)x(y)\mathrm{d}y$.

From (v) we know that $\overline{U}: X \longrightarrow X$ is a bounded linear operator with $\|\overline{U}\|_{L^2[0,\pi]} \leq \overline{H}$. Hence, we can write $\|(-A)^{(1/2)}U_h(x)\|_{L^2[0,\pi]} = \|\overline{U}(x)\|_{L^2[0,\pi]}$, which implies that (H2) holds. Hence, according to Theorem 1, system (38) has a mild solution provided that the condition of Theorem 1 holds. Moreover, we can impose suitable hypotheses to verify the assumptions stated in Theorem 2. Thus, system (38) is controllable on $[0,1]$.

6. Conclusions

In this paper, the issue on approximate controllability of neutral measure evolution equations has been addressed, which can model a large class of hybrid systems without any restriction on their Zeno behavior. Firstly, by adopting Schauder's fixed point theorem, the existence results of mild solutions for this type of measure control system corresponding to some control function are obtained. Then, the approximate controllability results are provided. Finally, we also use an example to illustrate the main result. Furthermore, we will investigate measure functional evolution equations of Sobolev type in the next work.

Data Availability

There are no underlying data in the results.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors read and approved the final manuscript.

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