

## Research Article

# Resonance between the Representation Function and Exponential Functions over Arithmetic Progression

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Let  $r(n)$  denote the number of representations of a positive integer  $n$  as a sum of two squares, i.e.,  $n = x_1^2 + x_2^2$ , where  $x_1$  and  $x_2$  are integers. We study the behavior of the exponential sum twisted by  $r(n)$  over the arithmetic progressions  $\sum_{\substack{n \sim X \\ n \equiv l \pmod{q}}} r(n)e(\alpha n^\beta)$ ,

where  $0 \neq \alpha \in \mathbb{R}$ ,  $0 < \beta < 1$ ,  $e(x) = e^{2\pi i x}$ , and  $n \sim X$  means  $X < n \leq 2X$ . Here,  $X > 1$  is a large parameter,  $1 \leq l \leq q$  are integers, and  $(l, q) = 1$ . We obtain the upper bounds in different situations.

## 1. Introduction

In analytic number theory, the problems concerning nonlinear exponential twisting arithmetic functions arise naturally in investigating equidistribution theory, zero-distribution of  $L$ -functions, and so on. Let  $a_n$  be some arithmetic number-theoretic function. We usually consider the general nonlinear exponential sum of the form

$$S(X, \alpha) = \sum_{n \sim X} a_n e(\alpha n^\beta), \quad 0 \neq \alpha \in \mathbb{R}, 0 \leq \beta \leq 1. \quad (1)$$

Here,  $n \sim X$  means  $X \leq n \leq 2X$ , and  $e(z) = e^{2\pi i z}$ . When  $\beta = 1/2$  and  $a_n = \Lambda(n)$  is the von Mangoldt function, the sum  $S(X, \alpha)$  was studied by Vinogradov [1]. For  $a_n = \Lambda(n)$  and  $a_n = \mu(n)$  ( $\mu$  is the Möbius function), the sums  $S(X, \alpha)$  were studied by Iwaniec, Luo, and Sarnak, and they showed these sums are intimately related to  $L$ -functions of  $GL_2$ . If  $f$  is a holomorphic cusp form of even weight on the upper half plane, they also proved that a good upper bound of  $S(X, \alpha)$  implies a quasi-Riemann hypothesis for  $L(s, f)$  [2]. In addition, studying the behavior of the Fourier coefficients of automorphic forms has great significance in modern number theory. Analytic number

theorists always estimate the mean value or the twisted sums such as  $S(X, \alpha)$  mentioned above to obtain some information about the Fourier coefficients (for examples, see [3–10]).

If  $\beta$  is a variable and  $a_n$  are the Fourier coefficients of automorphic forms, these sums were studied by Ren and Ye [7] and Sun and Wu [11]. They proved that the resonance phenomenon occurs only when  $\beta = 1/2$  and  $|\alpha|$  is close to  $2\sqrt{k}$ ,  $k \in \mathbb{Z}^+$ . Let  $r(n)$  denote the number of representations of a positive integer  $n$  as a sum of two squares, i.e.,  $n = x_1^2 + x_2^2$ , where  $x_1$  and  $x_2$  are integers. Sun and Wu [11] also studied the case that  $a_n = r(n)$  and obtained the resonance phenomenon. Yan [12] studied the nonlinear exponential sum twisting the Fourier coefficients of Maass forms over the arithmetic progress and obtained an asymptotic formula for the sum

$$\sum_{\substack{n \sim X \\ n \equiv l \pmod{q}}} \lambda_g(n) e(\alpha n^\beta), \quad 0 \neq \alpha \in \mathbb{R}, 0 \leq \beta \leq 1, \quad (2)$$

where  $g$  is a Maass cusp form for  $SL(2, \mathbb{Z})$  and  $\lambda_g(n)$  is the  $n$ -th Fourier coefficient of  $g$ . These analogues to the arithmetic progression are the main motivation of this paper.

In this paper, we study the nonlinear exponential sum

$$\sum_{\substack{n \sim X \\ n \equiv l \pmod q}} r(n)e(\alpha n^\beta), \tag{3}$$

where  $0 \neq \alpha \in \mathbb{R}$ ,  $0 < \beta < 1$ . Here,  $X > 1$  is a large parameter.  $1 \leq l \leq q$  are integers, and  $(l, q) = 1$ . We consider the case that  $q$  tends to infinity as  $X \rightarrow \infty$  and obtain analogues of the result of Sun and Wu [11].

The principal aim of this paper is to prove the following result.

**Theorem 1.** *Let  $X > 1$ ,  $0 < \beta < 1$ , and  $0 \neq \alpha \in \mathbb{R}$ . Let  $l, q \in \mathbb{N}$  and  $1 \leq q \leq X^{1/2}$ . Let  $\delta = (q, 4)$ .*

(i) *If  $|\alpha|\beta q X^\beta < (1/4)\sqrt{\delta X}$ , then we have*

$$\sum_{\substack{n \sim X \\ n \equiv l \pmod q}} r(n)e(\alpha n^\beta) \ll q^{(1/3)+\varepsilon} X(|\alpha|\beta X^\beta)^{-1}. \tag{4}$$

(ii) *If  $|\alpha|\beta q X^\beta \geq (1/4)\sqrt{\delta X}$  and  $\beta \neq (1/2)$ , then we have*

$$\sum_{\substack{n \sim X \\ n \equiv l \pmod q}} r(n)e(\alpha n^\beta) \ll q^{(1/2)+\varepsilon} |2\beta - 1|^{-(1/2)} (|\alpha|\beta X^\beta)^{1+\varepsilon}. \tag{5}$$

(iii) *If  $|\alpha|\beta q X^\beta \geq (1/4)\sqrt{\delta X}$  and  $\beta = 1/2$ , then for  $|\alpha| < 1/q$  or  $|\alpha| > \sqrt{X}/q$ , we have*

$$\sum_{\substack{n \sim X \\ n \equiv l \pmod q}} r(n)e(\alpha n^\beta) \ll q^{(1/2)+\varepsilon} X^{(1/4)+\varepsilon} |\alpha|^{(1/2)+\varepsilon}. \tag{6}$$

For  $(1/q) \leq |\alpha| \leq (\sqrt{X}/q)$ , we have

$$\begin{aligned} \sum_{\substack{n \sim X \\ n \equiv l \pmod q}} r(n)e(\alpha n^\beta) &= \frac{1}{q} \sum_{c|q} \varepsilon(a, n_c) c^{-3/2} X^{3/4} r(n_c, Q^*) |S(-l, -\overline{D_c} n_c; c)| n_c^{-1/4} G_Q(c, a) \\ &+ O(q^{(1/2)+\varepsilon} X^{(1/4)+\varepsilon} |\alpha|^{(1/2)+\varepsilon}), \end{aligned} \tag{7}$$

where

$$\varepsilon(a, n_c) = \frac{\delta_c \varepsilon_a a_1}{2} \int_1^2 u^{-1/4} e\left(\operatorname{sgn}(\alpha) \left(|\alpha| - \frac{2\sqrt{\delta n_c}}{2c}\right) \sqrt{Xu}\right) du, \tag{8}$$

and  $\delta_c = 1$  or  $0$  according to if there exists a positive integer  $n_c$  for  $c|q$  satisfying

$$|c|\alpha| - \sqrt{\delta n_c}| \leq X^{-1/2}, \tag{9}$$

or not.

(iv) *In particular, if  $|\alpha| = (\sqrt{\delta k}/q)$  with  $1 \leq k \leq (X/\delta)$ , then we have*

$$\begin{aligned} \sum_{\substack{n \sim X \\ n \equiv l \pmod q}} r(n)e\left(\pm \frac{\sqrt{\delta kn}}{q}\right) &= \frac{2\sqrt{2}}{3} (2^{3/4} - 1) \delta^{-1/4} (1 \pm i) |S(-l, -\overline{D_c} k; q)| q^{-5/2} X^{3/4} r(k, Q^*) k^{-1/4} G_Q(q, a) \\ &+ O(k^{-1/4} X^{1/4} (qXk)^\varepsilon). \end{aligned} \tag{10}$$

To prove Theorem 1, we shall follow the steps in [7, 11, 12] first. Then, we will use a new Voronoi-type summation formula generalized by Hu et al. [13] to get the asymptotic formula, and this is the key to success. Thus, we can get the Kloosterman sum, use Weil's bound to get the saving in the  $q$ -aspect, and then obtain a similar main term as that in [12].

## 2. Some Lemmas

To prove Theorem 1, we need to quote some lemmas. First, we consider the Kloosterman sum, which is defined as

$$S(m, n; c) = \sum_{d \pmod c}^* e\left(\frac{md + n\overline{d}}{c}\right), \tag{11}$$

where  $\bar{d}$  denotes the inverse of  $d$  modulo  $c$ . The famous Weil's bound of the Kloosterman sum is

$$|S(m, n; c)| \leq (m, n, c)^{1/2} c^{1/2} \tau(c), \tag{12}$$

where  $d(c)$  denotes the divisor function.

Let  $J_\nu$  denote the standard J-Bessel function. Let  $r(n, Q)$  denote the number of representations of  $n$  by the quadratic form  $Q$ , namely,

$$r(n, Q) = \#\{m \in \mathbb{Z}^l : Q(m) = n\}. \tag{13}$$

If  $Q(x) = x_1^2 + x_2^2 + \dots + x_l^2$ , we denote  $r(n, Q) = r_l(n)$ . Let  $A$  be a symmetric positive definite integral matrix

associated to  $Q$ , and let  $D = \det A$  denote the discriminant of  $A$ . Let  $G_Q(c, a) = \sum_{x \pmod{c}} e(aQ(x)/c)$ ,  $\delta = (c, D)$ ,  $D = \delta D_c$ , and  $Q^*$  be a positive definite integral quadratic form, which is defined in terms of the Smith normal form of  $Q^\dagger$  (see [14]), and  $Q^\dagger$  the adjoint form of  $Q$ .

Then, we have the following Voronoi summation formula [13].

**Lemma 1.** *Let  $(c, a) = 1$  and  $F \in C_0^\infty(\mathbb{R}^+)$  be a smooth compact function. We have*

$$\sum_{n=1}^\infty r(n, Q) e\left(\frac{an}{c}\right) F(n) = \frac{(2\pi)^{m/2} c^{-m}}{\Gamma(m/2)\sqrt{D}} G_Q(c, a) \Phi\left(\frac{m}{2}\right) + \sum_{n=1}^\infty r(n, Q^*) e\left(\frac{-a\bar{D}_c n}{c}\right) (\delta n)^{(1-(m/2))/2} G(n), \tag{14}$$

where

$$G(n) = \frac{4\pi D^{(m/4)-1}}{c^{(m/2)+1}} G_Q(c, a) \int_0^\infty F(x) x^{((m/2)-1)/2} J_{(m/2)-1}\left(\frac{4\pi\sqrt{\delta n x}}{c\sqrt{D}}\right) dx, \tag{15}$$

and  $\Phi(s)$  is the Mellin transform of  $F(x)$ , which is given by

$$\Phi(s) = \int_0^\infty F(x) x^{s-1} dx. \tag{16}$$

*Remark 1.* In our situation,  $D = 4$ , but we still want to compute the dependence of  $D$  in our proof. If one can obtain

the asymptotic formula for general  $r(n, Q)$ , then our result can be applied directly to get the analogues for  $r(n, Q)$ .

For asymptotic expansions of the Bessel functions, we quote the following lemma.

**Lemma 2.** *For  $z > 0$  large, we have*

$$J_{\pm\nu}(z) = \sqrt{\frac{2}{\pi z}} \left\{ \cos\left(z \mp \frac{\pi}{2} \nu - \frac{\pi}{4}\right) - \frac{\nu^2 - (1/4)}{2z} \sin\left(z \mp \frac{\pi}{2} \nu - \frac{\pi}{4}\right) + O(z^{-2}) \right\}. \tag{17}$$

For the mean value of  $r(n)$ , we have the classical result [16].

**Lemma 3.**

$$\sum_{n \leq x} r(n) = \pi x + O(x^{1/3}). \tag{18}$$

We also need the following result [17].

**Lemma 4.** *Let  $G(x)$  and  $F(x)$  be real functions in  $[a, b]$  with  $G(x)/F'(x)$  being monotonic. Suppose that  $|G(x)| \leq M$ .*

(a) *If  $F'(x) \geq u > 0$  or  $F'(x) \leq -u < 0$ , then*

$$\int_a^b G(x) e(F(x)) dx \ll \frac{M}{u}. \tag{19}$$

(b) *If  $F''(x) \geq v > 0$  or  $F''(x) \leq -v < 0$ , then*

$$\int_a^b G(x) e(F(x)) dx \ll \frac{M}{\sqrt{v}}. \tag{20}$$

### 3. Proof of Theorem 1

In this section, we will finish the proof of Theorem 1. By the formula of the Ramanujan sum

$$\sum_{d|q} \sum_{a=1}^d e\left(\frac{an}{d}\right) = \begin{cases} 0, & \text{if } q \nmid n, \\ q, & \text{if } q|n, \end{cases} \tag{21}$$

we get

$$\sum_{\substack{n \sim X \\ n \equiv l \pmod{q}}} r(n)e(\alpha n^\beta) = \frac{1}{q} \sum_{c|q} \sum_{a \pmod{c}}^* e\left(-\frac{al}{c}\right) \sum_{n \sim X} r(n)e\left(\frac{an}{c}\right)e(\alpha n^\beta), \tag{22}$$

where  $\sum^*$  means the summation is restricted by  $(a, c) = 1$ .

Let  $\Delta > 1$ , and let  $0 \leq \phi(x) \leq 1$  be a  $C^\infty$  function supported on  $[1, 2]$ , which is identically 1 on  $[1 + \Delta^{-1}, 2 - \Delta^{-1}]$  and satisfies  $\phi^{(r)}(x) \ll \Delta^r$  for  $r \geq 0$ . Using the bound in Lemma 3, we get

$$\sum_{n \sim X} r(n)e\left(\frac{an}{c}\right)e(\alpha n^\beta) = \sum_{n=1}^\infty r(n)e\left(\frac{an}{c}\right)W(n) + R_1(N, X, \Delta), \tag{23}$$

where

$$W(x) = \phi\left(\frac{x}{X}\right)e(\alpha x^\beta), \tag{24}$$

$$\begin{aligned} R_1(N, X, \Delta) &\ll \sum_{X < n \leq X+(X/\Delta)} |r(n)| \\ &\ll \left(X + \frac{X}{\Delta}\right) + \left(X + \frac{X}{\Delta}\right)^{(1/3)} - X - X^{(1/3)} \\ &\ll \left(\frac{X}{\Delta}\right)^{1+\varepsilon}. \end{aligned} \tag{25}$$

Applying Lemma 1 with  $m = 2$  and  $F(n) = W(n)$ , we have

$$\begin{aligned} \sum_{n=1}^\infty r(n)e\left(\frac{an}{c}\right)W(n) &= \frac{2\pi}{c^2\sqrt{D}}G_Q(c, a) \int_0^\infty W(x)dx \\ &\quad + \sum_{n=1}^\infty r(n, Q^*)e\left(\frac{-aD_c n}{c}\right)G(n), \end{aligned} \tag{26}$$

where

$$G(n) = \frac{4\pi}{c^2\sqrt{D}}G_Q(c, a) \int_0^\infty W(x)J_0\left(\frac{4\pi\sqrt{\delta n x}}{c\sqrt{D}}\right)dx. \tag{27}$$

For the first term, changing variables  $x = Xt$  and applying Lemma 4(a), we get

$$\int_0^\infty W(x)dx = X \int_1^2 \phi(t)e(\alpha(Xt)^\beta)dt \ll X(|\alpha|\beta X^\beta)^{-1}. \tag{28}$$

By [16], we have  $G_Q(c, a) \ll c$ . Thus, the contribution of the first term to (22) is

$$\begin{aligned} &\frac{1}{q} \sum_{c|q} \sum_{a \pmod{c}}^* e\left(-\frac{al}{c}\right) \frac{2\pi}{c^2\sqrt{D}}G_Q(c, a)X(|\alpha|\beta X^\beta)^{-1} \\ &\ll \frac{1}{q} \sum_{c|q} \sum_{a \pmod{c}}^* e\left(-\frac{al}{c}\right) \frac{1}{c}X(|\alpha|\beta X^\beta)^{-1} \\ &\ll q^{-1+\varepsilon}X(|\alpha|\beta X^\beta)^{-1}. \end{aligned} \tag{29}$$

Next, we turn to estimate the contribution from the term involving  $G(n)$ . Using Lemma 2, we have

$$J_0(z) = \frac{1-i}{2\sqrt{\pi z}}(e^{iz} + ie^{-iz}) - \frac{1+i}{16\sqrt{\pi z^3}}(e^{iz} - ie^{-iz}) + O(z^{-5/2}). \tag{30}$$

Taking  $z = (4\pi\sqrt{\delta n x})/c\sqrt{D}$ , we obtain

$$\begin{aligned} J_0\left(\frac{4\pi\sqrt{\delta x y}}{c\sqrt{D}}\right) &= \frac{(1-i)c^{1/2}D^{1/4}}{4\pi\delta^{1/4}}(xy)^{-1/4} \left( e\left(\frac{2\sqrt{\delta x y}}{c\sqrt{D}}\right) + ie\left(-\frac{2\sqrt{\delta x y}}{c\sqrt{D}}\right) \right) - \frac{(1+i)c^{3/2}D^{3/4}}{128\pi^2\delta^{3/4}}(xy)^{-3/4} \left( e\left(\frac{2\sqrt{\delta x y}}{c\sqrt{D}}\right) \right. \\ &\quad \left. - ie\left(-\frac{2\sqrt{\delta x y}}{c\sqrt{D}}\right) \right) + O(c^{5/2}(xy)^{-5/4}). \end{aligned} \tag{31}$$

Putting this in  $G(y)$ , we get

$$\begin{aligned}
 G(y) &= \frac{(1-i)}{c^{3/2}D^{1/4}\delta^{1/4}}G_Q(c,a)\int_0^\infty(xy)^{-1/4}\phi\left(\frac{x}{X}\right)e(\alpha x^\beta)\left(e\left(\frac{2\sqrt{\delta xy}}{c\sqrt{D}}\right)+ie\left(-\frac{2\sqrt{\delta xy}}{c\sqrt{D}}\right)\right)dx \\
 &\quad - \frac{(1+i)D^{1/4}}{32\pi c^{1/2}\delta^{3/4}}G_Q(c,a)\int_0^\infty(xy)^{-3/4}\phi\left(\frac{x}{X}\right)e(\alpha x^\beta)\left(e\left(\frac{2\sqrt{\delta xy}}{c\sqrt{D}}\right)-ie\left(-\frac{2\sqrt{\delta xy}}{c\sqrt{D}}\right)\right)dx \\
 &\quad + O\left(c^{3/2}\int_0^\infty(xy)^{-5/4}\phi\left(\frac{x}{X}\right)dx\right).
 \end{aligned}
 \tag{32}$$

Changing variable  $x = Xt^2$ , we obtain

$$\begin{aligned}
 G(y) &= \frac{2(1-i)}{c^{3/2}D^{1/4}\delta^{1/4}}G_Q(c,a)X^{3/4}y^{-1/4}\int_0^\infty t^{-1/2}\phi(t^2)e(\alpha X^\beta t^{2\beta})\left(e\left(\frac{2\sqrt{\delta Xy}}{c\sqrt{D}}t\right)+ie\left(-\frac{2\sqrt{\delta Xy}}{c\sqrt{D}}t\right)\right)dt \\
 &\quad - \frac{(1+i)D^{1/4}}{16\pi c^{1/2}\delta^{3/4}}G_Q(c,a)X^{1/4}y^{-3/4}\int_0^\infty t^{-1/2}\phi(t^2)e(\alpha X^\beta t^{2\beta})\left(e\left(\frac{2\sqrt{\delta Xy}}{c\sqrt{D}}t\right)-ie\left(-\frac{2\sqrt{\delta Xy}}{c\sqrt{D}}t\right)\right)dt \\
 &\quad + O\left(c^{3/2}\int_0^\infty(xy)^{-5/4}\phi\left(\frac{x}{X}\right)dx\right) \\
 &= G_1(y) + O\left(c^{3/2}X^{-1/4}y^{-5/4}\right),
 \end{aligned}
 \tag{33}$$

where

$$\begin{aligned}
 G_1(y) &= a_1c^{-3/2}X^{3/4}y^{-1/4}G_Q(c,a)\left(P_+\left(\frac{2\sqrt{\delta Xy}}{c\sqrt{D}}\right)+iP_+\left(-\frac{2\sqrt{\delta Xy}}{c\sqrt{D}}\right)\right) \\
 &\quad + a_2c^{-1/2}X^{1/4}y^{-3/4}G_Q(c,a)\left(P_-\left(\frac{2\sqrt{\delta Xy}}{c\sqrt{D}}\right)-iP_-\left(-\frac{2\sqrt{\delta Xy}}{c\sqrt{D}}\right)\right),
 \end{aligned}
 \tag{34}$$

with

$$a_1 = \frac{2(1-i)}{D^{1/4}\delta^{1/4}}, \tag{35}$$

$$a_2 = -\frac{(1+i)D^{1/4}}{16\pi\delta^{3/4}},$$

$$P_\pm(w) = \int_0^\infty t^{\pm 1/2}\phi(t^2)e(\alpha X^\beta t^{2\beta} + wt)dt. \tag{36}$$

The  $O$ -term contributes

$$\begin{aligned}
 &\frac{1}{q}\sum_{c|q}\sum_{n=1}^\infty r(n,Q^*)|S(-l,-\overline{D}_c n;c)|c^{3/2}X^{-1/4}n^{-5/4} \\
 &\ll q^{1+\varepsilon}X^{-(1/4)+\varepsilon}.
 \end{aligned}
 \tag{37}$$

The integral  $P_\pm(w)$  defined in (36) was studied by Ren and Ye [7], Sun and Wu [11], and Yan [12]. Here, we follow their steps and choose the parameters with a few differences to get the  $q$ -aspect saving.

Apply the method given in [7], we obtain

$$P_\pm(w) \ll \frac{\Delta^{h-1}}{(\max\{|\alpha|\beta X^\beta, |w|\})^h}, \quad h = 1, 2. \tag{38}$$

We will take  $w = \pm(2\sqrt{\delta yX})/(c\sqrt{D})$  here. Then,  $|w| \in [(1/2)|\alpha|\beta X^\beta, 4|\alpha|\beta X^\beta]$  implies  $y \in I$ , where

$$I = \left[\frac{1}{16}\delta^{-1}c^2D(|\alpha|\beta)^2X^{2\beta-1}, 4\delta^{-1}c^2D(|\alpha|\beta)^2X^{2\beta-1}\right]. \tag{39}$$

Thus, for  $y = n \notin I$ , one obtains by (34) and (38) that

$$G_1(y) \ll (c^{-3/2}X^{3/4}y^{-1/4} + c^{-1/2}X^{1/4}y^{-3/4})G_Q(c,a)R_2(X,y), \tag{40}$$

where

$$R_2(X,y) \ll \frac{\Delta^{h-1}}{(\sqrt{Xy}/c)^h}, \quad h = 1, 2. \tag{41}$$

Thus, the contribution from  $G_1(n)$  with  $n \notin I$  is

$$\begin{aligned} & \frac{1}{q} \sum_{clq} \sum_{a \bmod c}^* e\left(-\frac{al}{c}\right) \sum_{n \notin I} r(n, Q^*) e\left(\frac{-a\overline{D_c}n}{c}\right) G_1(n) \\ & \ll \frac{1}{q} \sum_{clq} \sum_{n=1}^{\infty} r(n, Q^*) |S(-l, -\overline{D_c}n; c)| (c^{-1/2} X^{3/4} n^{-1/4} + c^{1/2} X^{1/4} n^{-3/4}) R_2(X, n). \end{aligned} \tag{42}$$

Let

$$Y = \Delta^2 c^2 X^{-1}. \tag{43}$$

$$h = \begin{cases} 1, & \text{if } n \leq Y, \\ 2, & \text{if } n > Y, \end{cases} \tag{44}$$

we get

Applying (41) with

$$\begin{aligned} & \ll \frac{1}{q} \sum_{clq} \sum_{n \leq Y} r(n, Q^*) |S(-l, -\overline{D_c}n; c)| (c^{-1/2} X^{3/4} n^{-1/4} + c^{1/2} X^{1/4} n^{-3/4}) (cX^{-1/2} n^{-1/2}) \\ & + \frac{1}{q} \sum_{clq} \sum_{n > Y} r(n, Q^*) |S(-l, -\overline{D_c}n; c)| (c^{-1/2} X^{3/4} n^{-1/4} + c^{1/2} X^{1/4} n^{-3/4}) (c^2 \Delta X^{-1} n^{-1}) \\ & =: \sum_1 + \sum_2. \end{aligned} \tag{45}$$

By the bound  $\sum_{n \leq x} r(n) = \pi x + O(x^{1/3})$ , we obtain

$$\begin{aligned} \sum_1 & = \frac{1}{q} \sum_{clq} \sum_{n \leq Y} r(n, Q^*) |S(-l, -\overline{D_c}n; c)| (c^{1/2} X^{1/4} n^{-3/4} + c^{3/2} X^{-1/4} n^{-5/4}) \\ & \ll \frac{1}{q} \sum_{clq} d(c) c X^{1/4} \sum_{n \leq Y} r(n, Q^*) (l, \overline{D_c}n; c)^{1/2} n^{-3/4} \\ & + \frac{1}{q} \sum_{clq} d(c) c^2 X^{-1/4} \sum_{n \leq Y} r(n, Q^*) (l, \overline{D_c}n; c)^{1/2} n^{-5/4} \\ & \ll q^{(1/2)+\varepsilon} \Delta^{(1/2)+\varepsilon}. \end{aligned} \tag{46}$$

Similarly, using the bound  $\sum_{n=1}^{\infty} (|r(n, Q^*)|^2/n^{1+\varepsilon}) \ll 1$ , we get

$$\begin{aligned}
 \sum_2 &= \frac{1}{q} \sum_{c|q} \sum_{n>Y} r(n, Q^*) |S(-l, -\overline{D}_c n; c)| \Delta (c^{3/2} X^{-1/4} n^{-5/4} + c^{5/2} X^{-3/4} n^{-7/4}) \\
 &\ll \frac{1}{q} \Delta \sum_{c|q} d(c) c^{2+\varepsilon} X^{-1/4} \sum_{n>Y} r(n, Q^*) (l, \overline{D}_c n; c)^{1/2} n^{-5/4} \\
 &\quad + \frac{1}{q} \Delta \sum_{c|q} d(c) c^{3+\varepsilon} X^{-3/4} \sum_{n>Y} r(n, Q^*) (l, \overline{D}_c n; c)^{1/2} n^{-7/4} \\
 &\ll \frac{1}{q} \Delta \sum_{c|q} c^{2+\varepsilon} X^{-1/4} \left( \sum_{n>Y} \frac{|r(n, Q^*)|^2}{n^{1+\varepsilon}} \right)^{1/2} \left( \sum_{n>Y} n^{-3/2} \right)^{1/2} \\
 &\quad + \frac{1}{q} \Delta \sum_{c|q} c^{3+\varepsilon} X^{-3/4} \left( \sum_{n>Y} \frac{|r(n, Q^*)|^2}{n^{1+\varepsilon}} \right)^{1/2} \left( \sum_{n>Y} n^{-5/2} \right)^{1/2} \\
 &\ll \frac{1}{q} \Delta \sum_{c|q} c^{2+\varepsilon} X^{-1/4} Y^{-(1/4)+\varepsilon} + \frac{1}{q} \Delta \sum_{c|q} c^{3+\varepsilon} X^{-3/4} Y^{-(3/4)+\varepsilon} \\
 &\ll q^{(1/2)+\varepsilon} \Delta^{(1/2)+\varepsilon}.
 \end{aligned} \tag{47}$$

Finally, we obtain the contribution of (45) as

$$\ll q^{(1/2)+\varepsilon} \Delta^{(1/2)+\varepsilon}. \tag{48}$$

If  $|\alpha|\beta q X^\beta < (1/2)\sqrt{\delta X/D}$ , then  $I \cap Z^+ = \emptyset$ , by (25), (29), (37), and (48), and choosing

$$\Delta = X^{2/3} q^{-1/3}, \tag{49}$$

we get

$$\sum_{n-Xn \equiv l \pmod{q}} r(n) e(\alpha n^\beta) \ll q^{(1/3)+\varepsilon} X (|\alpha|\beta X^\beta)^{-1}. \tag{50}$$

Next, we will handle the situation that  $y = n \in I$ . By the trivial estimate  $\int_1^{\sqrt{2}} t^{\pm 1/2} \phi(t^2) e(f(t)) dt \ll 1$  and putting it into (22), we get the second term in the right-hand side of (34) which contributes

$$\begin{aligned}
 &\ll \frac{1}{q} \sum_{c|q} \sum_{n \in I} r(n, Q^*) |S(-l, -\overline{D}_c n; c)| c^{-1/2} X^{1/4} n^{-3/4} G_Q(c, a) \\
 &\ll q^{(1/2)+\varepsilon} (|\alpha|\beta X^\beta)^{(1/2)+\varepsilon}.
 \end{aligned} \tag{51}$$

Applying (38) to bound the terms with  $\alpha w > 0$ , we have

$$P_+ \left( \frac{2\sqrt{\delta X y}}{c\sqrt{D}} \right) + iP_+ \left( -\frac{2\sqrt{\delta X y}}{c\sqrt{D}} \right) = \varepsilon_\alpha \int_0^\infty t^{1/2} \phi(t^2) e(f_1(t)) dt + O(R_2(X, y, c)), \tag{52}$$

where

$$\varepsilon_\alpha = \begin{cases} i, & \alpha > 0, \\ 1, & \alpha < 0, \end{cases} \tag{53}$$

$$f_1(t) = f_1(t, y) = \operatorname{sgn}(\alpha) \left( |\alpha| X^\beta t^{2\beta} - \frac{2\sqrt{\delta X y}}{c\sqrt{D}} t \right). \tag{54}$$

Since the contribution from the  $O$ -term is absorbed by (51), we only need to estimate

$$\varepsilon_\alpha \frac{a_1}{q} \sum_{c|q} c^{-3/2} X^{3/4} \sum_{n \in I} r(n, Q^*) |S(-l, -\overline{D}_c n; c)| n^{-1/4} G_Q(c, a) \int_1^{\sqrt{2}} t^{1/2} \phi(t^2) e(f_1(t, n)) dt. \tag{55}$$

We distinguish two cases according to  $\beta = 1/2$  or not.

If  $\beta \neq 1/2$ , we have

$$f_1''(t) = \text{sgn}(\alpha)|\alpha|(2\beta)(2\beta - 1)X^\beta t^{2\beta-2} \gg |\alpha|\beta|2\beta - 1|X^\beta, \quad \text{for } t \in [1, \sqrt{2}]. \tag{56}$$

By integration by parts and [17], we have

$$\int_1^{\sqrt{2}} t^{1/2} \phi(t^2) e(f_1(t, n)) dt \ll \max_{1 \leq t \leq \sqrt{2}} \left| \int_1^t e(f_1(u)) du \right| \ll (|\alpha|\beta|2\beta - 1|X^\beta)^{-1/2}. \tag{57}$$

Hence, (55) is

$$\begin{aligned} &\ll \frac{1}{q} \sum_{c|q} c^{-3/2} X^{3/4} \sum_{n \in I} r(n, Q^*) |S(-l, -\overline{D}_c n; c)| n^{-1/4} G_Q(c, a) |\alpha|\beta|2\beta - 1|X^\beta)^{-1/2} \\ &\ll q^{(1/2)+\varepsilon} |2\beta - 1|^{-(1/2)} (|\alpha|\beta X^\beta)^{1+\varepsilon}. \end{aligned} \tag{58}$$

If  $\beta = 1/2$ , let

$$H_r = 2^{-r} c |\alpha| X^{1/2}, \quad 1 \leq r \leq r_0 = \lceil \log_2(c|\alpha|X^{1/2}) \rceil + 1, \tag{59}$$

and write

$$A(H_r) = \left\{ n: H_r X^{-1/2} < \left| c|\alpha| - 2\sqrt{\frac{\delta n}{D}} \right| \leq 2H_r X^{-1/2} \right\}. \tag{60}$$

Then,

$$|A(H_r)| \leq H_r c |\alpha| X^{-1/2}. \tag{61}$$

Moreover, for  $n \in A(H_r)$ , we have

$$|f_1'(t)| = \left| |\alpha| - \frac{2\sqrt{\delta n}}{c\sqrt{D}} \right| X^{1/2} > H_r c^{-1}. \tag{62}$$

Then, we obtain

$$\int_1^{\sqrt{2}} t^{1/2} \phi(t^2) e(f_1(t)) dt \ll H_r^{-1} c. \tag{63}$$

Hence, in this situation, (55) contributes

$$\begin{aligned} &\ll \frac{1}{q} \sum_{c|q} c^{-1/2} X^{(3/4)+\varepsilon} \sum_{r=1}^{r_0} H_r^{-1} \sum_{n \in A(H_r)} r(n, Q^*) |S(-l, -\overline{D}_c n; c)| n^{-1/4} G_Q(c, a) \\ &\ll \frac{1}{q} \sum_{c|q} c^{-1/2} X^{(3/4)+\varepsilon} \sum_{r=1}^{r_0} H_r^{-1} (|\alpha|^2 c^2)^{-(1/4)+\varepsilon} c^{(3/2)+\varepsilon} |A(H_r)| \\ &\ll q^{(1/2)+\varepsilon} X^{(1/4)+\varepsilon} |\alpha|^{(1/2)+\varepsilon}. \end{aligned} \tag{64}$$

It remains to estimate

$$\varepsilon_\alpha \frac{a_1}{q} \sum_{c|q} c^{-3/2} X^{3/4} \sum_{n \in I \cap I_0} r(n, Q^*) |S(-l, -\overline{D}_c n; c)| n^{-1/4} G_Q(c, a) \int_1^{\sqrt{2}} t^{1/2} \phi(t^2) e(f_1(t, n)) dt, \tag{65}$$

where



$$I_0 = \left\{ n: \left| c|\alpha| - 2\sqrt{\frac{\delta n}{D}} \right| \leq X^{-1/2} \right\}. \quad (66)$$

If  $|\alpha| \leq (1/q)$ , then  $I_0 = \emptyset$ , and we have (65) which vanishes. And if  $|\alpha| \geq (\sqrt{X}/q)$ , then by trivial estimate, (65) contributes

Note that

$$\left| \left\{ n: \left| c|\alpha| - 2\sqrt{\frac{\delta n}{D}} \right| \leq X^{-1/2} \right\} \right| \leq c|\alpha|X^{-1/2}. \quad (67)$$

$$\begin{aligned} & \frac{1}{q} \sum_{c|q} c^{-3/2} X^{3/4} \sum_{|c|\alpha - 2\sqrt{(\delta n/D)}| \leq X^{-1/2}} r(n, Q^*) |S(-l, -\overline{D}_c n; c)|^{n-1/4} G_Q(c, a) \\ & \ll \frac{1}{q} \sum_{c|q} c^{1/2} X^{3/4} |I_0| (c^2 |\alpha|^2)^{-(1/4)+\varepsilon} \\ & \ll q^\varepsilon X^{(1/4)+\varepsilon} |\alpha|^{(1/2)+\varepsilon}. \end{aligned} \quad (68)$$

If  $(1/q) \leq |\alpha| \leq (\sqrt{X}/q)$ , then there is at most one integer satisfying

$$\left| c|\alpha| - 2\sqrt{\frac{\delta n}{D}} \right| \leq X^{-1/2}, \quad (69)$$

for every  $c|q$ , say  $n = n_c$ . Hence, (55) becomes

$$\begin{aligned} & \varepsilon_\alpha \frac{a_1}{q} \sum_{c|q} \delta_c c^{-3/2} X^{3/4} r(n_c, Q^*) |S(-l, -\overline{D}_c n_c; c)| n_c^{-1/4} G_Q(c, a) \int_1^{\sqrt{2}} t^{1/2} \phi(t^2) e(f_1(t, n_c)) dt \\ & = \frac{\varepsilon_\alpha a_1}{2q} \sum_{c|q} \delta_c c^{-3/2} X^{3/4} r(n_c, Q^*) |S(-l, -\overline{D}_c n_c; c)| n_c^{-1/4} G_Q(c, a) \int_1^2 u^{-1/4} e(f_1(u^{1/2}, n_c)) du + O(c^{-(1/2)+\varepsilon} \Delta^{-1} X^{3/4} r(n_c, Q^*) n_c^{-1/4}) \\ & = \frac{1}{q} \sum_{c|q} \varepsilon(a, n_c) c^{-3/2} X^{3/4} r(n_c, Q^*) |S(-l, -\overline{D}_c n_c; c)| n_c^{-1/4} G_Q(c, a) + O(1), \end{aligned} \quad (70)$$

where

$$\varepsilon(a, n_c) = \frac{\delta_c \varepsilon_\alpha a_1}{2} \int_1^2 u^{-1/4} e\left(\operatorname{sgn}(\alpha) \left( |\alpha| - \frac{2\sqrt{\delta n_c}}{c\sqrt{D}} \right) \sqrt{Xu} \right) du, \quad (71)$$

and  $\delta_c = 1$  or  $0$  according to if there exists such  $n_c$  or not. Recall

$$\begin{aligned} a_1 &= \frac{2(1-i)}{D^{1/4} \delta^{1/4}}, \\ \varepsilon_\alpha &= \begin{cases} i, & \alpha > 0, \\ 1, & \alpha < 0. \end{cases} \end{aligned} \quad (72)$$

When  $\beta = 1/2$  and  $(1/q) \leq |\alpha| \leq (\sqrt{X}/q)$  and applying (64) and (68), we derive that (55) equals to

$$\begin{aligned} & \frac{1}{q} \sum_{c|q} \varepsilon(a, n_c) c^{-3/2} X^{3/4} r(n_c, Q^*) |S(-l, -\overline{D}_c n_c; c)| n_c^{-1/4} G_Q(c, a) \\ & + O(q^{(1/2)+\varepsilon} X^{(1/4)+\varepsilon} |\alpha|^{(1/2)+\varepsilon}). \end{aligned} \quad (73)$$

This proves Theorem 1(iii).

In particular, if  $|\alpha| = (2/q)\sqrt{\delta k/D}$  with  $1 \leq k \leq (DX/4\delta)$ , then for  $c|q$ ,  $c \neq q$ , we have

$$\left| c|\alpha| - 2\sqrt{\frac{\delta k}{D}} \right| = 2\sqrt{\frac{\delta k}{D}} \left| \frac{c}{q} - 1 \right| \geq \frac{2}{q} \sqrt{\frac{\delta k}{D}} \geq 2\sqrt{\frac{\delta k}{DX}} \geq 1. \quad (74)$$

Thus,  $\delta_c = 0$  for  $c \neq q$  in this situation. Hence, (73) becomes

$$\sum_{n \sim X, n \equiv l \pmod{q}} r(n) e\left(\pm \frac{2}{q} \sqrt{\frac{\delta kn}{D}}\right) = \frac{4}{3} (2^{3/4} - 1) D^{-1/4} \delta^{-1/4} (1 \pm i) |S(-l, -\overline{D}_\epsilon k; q)| q^{-5/2} X^{3/4} r(k, Q^*) k^{-1/4} G_Q(q, a) + O(k^{-1/4} X^{1/4} (qXk)^\epsilon). \quad (75)$$

This proves Theorem 1(iv).

## Data Availability

The data used to support the findings of this study are included within the article.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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