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Research Article

Coupled Fixed-Point Theorems in Theta-Cone-Metric Spaces

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This paper gives further generalizations of some well-known coupled fixed-point theorems. Specifically, Theorem 3 of the paper is the generalization of the Baskar–Lackshmikantham coupled fixed-point theorem, and Theorem 5 is the generalization of the Sahar Mohamed Ali Abou Bakr fixed-point theorem, where the underlying space is complete θ-cone-metric space.

1. Introduction and Preliminaries

Since 1922, the pioneering fixed-point principle of Banach [1] showed exclusive interest of researchers because it has many applications, including variational linear inequalities and optimization, and applications in differential equations, in the field of approximation theory, and in minimum norm problems.

Since then, several types of contraction mappings have been introduced and many research papers have been written to generalize this Banach contraction principle.

In 1987, Guo and Lakshmikantham [2] introduced one of the most interesting concepts of coupled fixed point.

Definition 1. An element $(x, y) \in E \times E$ is said to be a coupled fixed point of the mapping $T\colon E \times E \longrightarrow E$ if and only if $T(x, y) = x$ and $T(y, x) = y$.

In 2006, Bhaskar and Lakshmikantham [3] introduced the concept of the mixed monotone property as follows.

Definition 2. Let $(E, \leq)$ be a partially ordered set and T be a mapping from $E \times E$ to E. Then,

(1) T is said to be monotone nondecreasing in x if and only if, for any $y \in E$,

$$\text{if } x_1, x_2 \in E \text{ and } x_1 \leq x_2, \text{ then } T(x_1, y) \leq T(x_2, y), \tag{1}$$

(2) T is said to be monotone nonincreasing in y if and only if, for any $x \in E$,

$$\text{if } y_1, y_2 \in E \text{ and } y_1 \leq y_2, \text{ then } T(x, y_1) \geq T(x, y_2), \tag{2}$$

(3) T is said to have a mixed monotone property if and only if $T(x, y)$ is both monotone nondecreasing in x and monotone nonincreasing in y

Definition 3. An element $(x_0, y_0) \in E \times E$ is said to be a lower-anti-upper coupled point of the mapping $T\colon E \times E \longrightarrow E$ if and only if

$$x_0 \leq T(x_0, y_0) \text{ and } y_0 \geq T(y_0, x_0). \tag{3}$$

A mapping $T\colon E \times E \longrightarrow E$ is said to have a lower-upper property if and only if T has at least one lower-anti-upper coupled point.

Definition 4. Let $(E, \leq, \|.\|)$ be a partially ordered normed space. Then,

(1) E is said to be a sequentially lower ordered space if it fulfills the condition: If $\{x_n\}_{n\in N}$ is a nondecreasing

sequence in E such that $\{x_n\}_{n\in N}$ converges strongly to x, then $x_n \leq x$ for all $n \in N$

(2) E is said to be a sequentially upper-ordered space if it fulfills the condition: If $\{y_n\}_{n\in N}$ is a nonincreasing sequence in E such that $\{y_n\}_{n\in N}$ converges strongly to y, then $y_n \geq y$ for all $n \in N$

(3) E is said to be a sequentially lower-upper ordered space if it is both a lower- and upper-ordered space

In 2006, Bhaskar and Lakshmikantham [3] proved the existence of coupled fixed points for mixed monotone mappings with weak contractivity assumption in a partial-ordered Banach space $(E, \|.\|, \leq)$ as follows.

Theorem 1 (see [3]). *Let E be a sequentially both lower- and upper-ordered Banach space and $T\colon E \times E \longrightarrow E$ be a mapping with mixed monotone and lower-upper properties. If there is a real number $0 \leq k < 1$ such that*

$$\|T(x,y) - T(z,w)\| \leq \frac{k}{2}[\|x-z\| + \|y-w\|], \quad \forall x,y,z,w \in E, z \leq x, \text{ and } y \leq w, \tag{4}$$

then T has coupled fixed points in E.

In 2013, Mohamed Ali [4] introduced novel contraction type of mappings and proved the following fixed-point theorem.

Theorem 2 (see [4]). *Let $(E, \|.\|)$ be a Banach space and T be a mapping from $E \times E$ into E, and we suppose there are three constants $a, b, c \in [0,1)$ and $a + b + c < 1$ such that*

$$\|T(x,y) - T(y,z)\| \leq a\|x-y\| + b\|T(x,y) - x\| + c\|T(y,z) - y\|, \quad \forall x,y,z \in E. \tag{5}$$

Then, there is a unique point $x_0 \in E$ such that $T(x_0, x_0) = x_0$.

There are many interesting coupled fixed-point theorems concerning some other type of contraction mappings, see [5–10].

Recently, more advanced approaches for studying coupled fixed points have been presented by the authors in [11–13].

In 2007, Huang and Zhang [14] introduced the concept of cone-metric spaces as follows: First, a subset M of the real Banach space $\mathscr{E}$ is said to be a cone in $\mathscr{E}$ if and only if

(1) M is nonempty closed and $M \neq \{\Theta\}$, where Θ is the zero (neutral) element of $\mathscr{E}$

(2) $\lambda M + \mu M \subset M$ for all nonnegative real numbers λ, μ

(3) $M \cap -M = \{\Theta\}$

If int$M$ is the set of all interior points of M, then a cone M in a normed space $\mathscr{E}$ induces the following ordered relations:

$$u \prec v \Leftrightarrow v - u \in M, \quad u < v \Leftrightarrow (v - u \in M, \text{and } u \neq v), \quad u \prec_{\neq} v \Leftrightarrow v - u \in \operatorname{int} M. \tag{6}$$

If E is a nonempty set, the distance $d(x,y)$ between any two elements $x, y \in E$ is defined to be a vector in the cone M, and the space (E, d) is said to be a cone-metric space if and only if d satisfied the three axioms of metric but using the ordered relation $\prec$ induced by M for the triangle inequality instead. They studied the topological characterizations of such a defined space, and then, they applied their concept to have more generalizations of some previous fixed-point theorems for contractive type of mappings.

A mapping $T\colon E \longrightarrow E$ is said to be a contraction if and only if there is a constant $\alpha \in [0,1)$ such that

$$d(T(x), T(y)) \prec \alpha d(x,y), \quad \forall x, y \in E. \tag{7}$$

In 2019, Mohamed Ali Abou Bakr [15] proved the existence of a unique common fixed point of generalized joint cyclic Banach algebra contractions and Banach algebra Kannan type of mappings on cone quasimetric spaces.

In 2013, Khojasteh et al. [10] introduced the notion of θ-action function, $\theta\colon [0,\infty) \times [0,\infty) \longrightarrow [0,\infty)$, the concept of θ-metric, and then, they studied the topological structures of θ-metric spaces in detail. Their work led to a step-forward generalization of metric spaces.

In 2020, Mohamed Ali Abou Bakr [16] replaced $[0,\infty)$ by a cone M in a normed space and used the ordered relation induced by this cone to introduce the following analogous generalization of θ-action function.

Definition 5. Let $(\mathscr{E}, \prec)$ be an ordered normed space, where $\prec$ is an ordered relation induced by some cone $M \subset \mathscr{E}$ and $\theta\colon M \times M \longrightarrow M$ be a continuous mapping with respect to each variable, and we denote

$$\operatorname{Im}(\theta) = \{t\colon t \in M \text{ such that } \exists u_0, v_0 \in \mathrm{E}, \quad \theta(u_0, v_0) = t\}. \tag{8}$$

Then, θ is said to be an ordered action mapping on $\mathscr{E}$ if and only if it satisfies the following conditions:

(1) $\theta(\Theta, \Theta) = \Theta$ and $\theta(u,v) = \theta(v,u)$ for every $u, v \in M$

(2)

$$\theta(u,v) < \theta(w,t) \text{ if either } \begin{cases} u < w \text{ and } v \prec t; \\ \text{or} \\ u \prec w \text{ and } v < t \end{cases} \tag{9}$$

(3) For every $u \in \operatorname{Im}(\theta)$ and every $\Theta \prec v \prec u$, there is $\Theta \prec w \prec u$ such that $\theta(v,w) = u$

(4) $\theta(u, \Theta) \prec u$ for every $u \in (M/\{\Theta\})$

Because $x - \Theta \in M$ for every $x \in M$, one can write instead $\Theta \prec x$ for every $x \in M$, ($\Theta < x$ for every $x \in (M/\{\Theta\})$).

In addition, Mohamed Ali Abou Bakr [16] gave further replacement, replaced the set of nonnegative real numbers $\mathbb{R}^+$ by a cone M in a normed space, and used θ-ordered actions to introduce the concept of θ-cone-metric space as follows.

Definition 6 (see [16]). Let $(\mathscr{E}, \prec)$ be an ordered normed space, where $\prec$ is the ordered relation induced by some cone $M \subset \mathscr{E}$, and θ be an ordered action on $\mathscr{E}$. If E is a nonempty set, then the function $d_\theta\colon E \times E \longrightarrow M$ is said to be a θ-cone-metric on E if and only if d_θ satisfies the following conditions:

(1) $d_\theta(x, y) = \Theta \Leftrightarrow x = y$
(2) $d_\theta(x, y) = d_\theta(y, x), \forall x, y \in E$
(3) $d_\theta(x, y) \prec \theta(d_\theta(x, z), d_\theta(z, y)), \quad \forall x, y, z \in E$

The double (E, d_θ) is defined to be a θ-cone-metric space.

The author has further given some topological characterizations of this space and then generalized some previous fixed-point theorems in this setting.

Remark 1. If $\theta(u, v) = u + v$, then we have a cone-metric space.

In this paper, we extend and generalize the coupled fixed-point theorem of Baskar–Lackshmikantham (1.5) to a more general one (2.1), where the underlying space (E, d_θ) is a complete θ-cone-metric space. On the other side, if $T\colon E \times E \longrightarrow E$ is a continuous mapping in the second argument and there are three constants $a, b, c \in [0, 1)$ and $a + b + c < 1$ such that

$$\begin{aligned} d_\theta(T(x, y), T(y, z)) \prec{}& a d_\theta(x, y) + b d_\theta(T(x, y), x) \\ &+ c d_\theta(T(y, z), y), \quad \forall x, y, z \in E, \end{aligned} \tag{10}$$

then we proved that T has a unique fixed point in the sense that there is a unique point $x \in E$ such that $T(x, x) = x$.

We also claim that some results of [6–10, 17] can be proved in the case of θ-cone-metric spaces.

2. Main Results

Let $(E, d_\theta, \leq)$ be a partially ordered θ-cone-metric space. Then, the following relation defines a partial-ordered relation on $E \times E$:

$$(x, y) \ll (z, w) \Leftrightarrow x \leq z, \text{ and } w \leq y. \tag{11}$$

We have the following coupled fixed-point theorem.

Theorem 3. *Let $(E, d_\theta, \leq)$ be a partially ordered, sequentially lower-upper ordered complete θ-cone-metric space and $G\colon E \times E \longrightarrow E$ be a mapping having mixed monotone and lower-upper properties on E. We assume that there exists $r \in [0, 1)$ with*

$$\begin{aligned} d_\theta(G(x, y), G(z, w)) \prec \frac{r}{2}[d_\theta(x, z) + d_\theta(y, w)], \\ \forall (x, y) \ll (z, w). \end{aligned} \tag{12}$$

Then, G has coupled fixed points in E.

Proof. Since G has a lower-upper property, then there exist $x_0, y_0 \in E$ such that

$$x_0 \leq G(x_0, y_0) \text{ and } G(y_0, x_0) \leq y_0. \tag{13}$$

We denote $x_1 = G(x_0, y_0)$ and $y_1 = G(y_0, x_0)$ and then give notations for the elements of the following inductively constructed sequences:

$$\begin{aligned} x_2 &= G(x_1, y_1) \coloneqq G^2(x_0, y_0), \\ y_2 &= G(y_1, x_1) \coloneqq G^2(y_0, x_0), \\ x_3 &= G(x_2, y_2) \coloneqq G^3(x_0, y_0), \\ y_3 &= G(y_2, x_2) \coloneqq G^3(y_0, x_0), \\ &\cdots \\ x_{n+1} &= G(x_n, y_n) \coloneqq G^{n+1}(x_0, y_0), \\ y_{n+1} &= G(y_n, x_n) \coloneqq G^{n+1}(y_0, x_0), \\ &\cdots \end{aligned} \tag{14}$$

Using the mixed monotonicity property of G insures that each step leads to the next step in each of the following:

$$\begin{aligned} &x_0 \leq x_1 = G(x_0, y_0) \leq G(x_1, y_0) \leq G(x_1, y_1) = x_2, \\ &y_2 = G(y_1, x_1) \leq G(y_1, x_0) \leq G(y_0, x_0) = y_1 \leq y_0, \\ &x_1 \leq x_2 = G(x_1, y_1) \leq G(x_2, y_2) = x_3, \\ &y_3 = G(y_2, x_2) \leq G(y_1, x_1) = y_2 \leq y_1 \\ &\cdots \\ &x_{n+1} = G(x_n, y_n) \leq G(x_{n-1}, y_{n-1}) = x_n, \\ &y_{n+1} = G(y_n, x_n) \leq G(y_{n-1}, x_{n-1}) = y_n, \\ &\cdots \end{aligned} \tag{15}$$

The mixed monotonicity property, the contractiveness of G, and the inductive process prove the following for every $n \in N$:

$$\begin{aligned} d_\theta\left(G^{n+1}(x_0, y_0), G^n(x_0, y_0)\right) \prec \left[\frac{r}{2}\right]^n [d_\theta(G(x_0, y_0), x_0) + d_\theta(G(y_0, x_0), y_0)], \\ d_\theta\left(G^{n+1}(y_0, x_0), G^n(y_0, x_0)\right) \prec \left[\frac{r}{2}\right]^n [d_\theta(G(y_0, x_0), y_0) + d_\theta(G(x_0, y_0), x_0)]. \end{aligned} \tag{16}$$

Consequently, we have

$$\begin{aligned}&\lim_{n\longrightarrow\infty} d_\theta\left(G^{n+1}(x_0,y_0),G^n(x_0,y_0)\right)\\&=\lim_{n\longrightarrow\infty} d_\theta\left(G^{n+1}(y_0,x_0),G^n(y_0,x_0)\right)=\Theta.\end{aligned}\tag{17}$$

Hence, we claim that both $\{G^n(x_0,y_0)\}_{n\in N}$ and $\{G^n(y_0,x_0)\}_{n\in N}$ are Cauchy sequences in E. Indeed, if one of them, say $\{G^n(x_0,y_0)\}_{n\in N}$, is not Cauchy, then there exist $v\in \operatorname{Im}(\theta)$, $\Theta < v$ and sequences of natural numbers $\{i_n\}_{n\in N}$ and $\{j_n\}_{n\in N}$ such that, for any $i_n > j_n > n$,

$$\begin{aligned}&v \prec d_\theta\left(G^{i_n}(x_0,y_0),G^{j_n}(x_0,y_0)\right),\\&d_\theta\left(G^{i_n-1}(x_0,y_0),G^{j_n}(x_0,y_0)\right)< v.\end{aligned}\tag{18}$$

Since any subsequence of $\{d_\theta(G^{n+1}(x_0,y_0),G^n(x_0,y_0))\}_{n\in N}$ is convergent to Θ, the properties of θ imply the following contradiction:

$$\begin{aligned}v &\prec d_\theta\left(G^{i_n}(x_0,y_0),G^{j_n}(x_0,y_0)\right)\\&\prec \theta\left(d_\theta\left(G^{i_n-1}(x_0,y_0),G^{j_n}(x_0,y_0)\right),d_\theta\left(G^{i_n}(x_0,y_0),G^{i_n-1}(x_0,y_0)\right)\right)\\&<\theta\left(v,d_\theta\left(G^{i_n}(x_0,y_0),G^{i_n-1}(x_0,y_0)\right)\right)\\&<\theta\left(v,\lim_{n\longrightarrow\infty} d_\theta\left(G^{i_n}(x_0,y_0),G^{i_n-1}(x_0,y_0)\right)\right)\\&<\theta(v,\Theta)<v.\end{aligned}\tag{19}$$

Similarly, the sequence $\{G^n(y_0,x_0)\}_{n\in N}$ is also Cauchy. Since E is a complete θ-cone-metric space, there exist $x,y\in E$ such that

$$\begin{aligned}&\lim_{n\longrightarrow\infty} d_\theta(G^n(x_0,y_0),x)=\Theta,\\&\lim_{n\longrightarrow\infty} d_\theta(G^n(y_0,x_0),y)=\Theta.\end{aligned}\tag{20}$$

Now, we are going to show that (x,y) is a coupled fixed point of G. Since the sequence $\{G^n(x_0,y_0)=x_n\}_{n\in N}$ is nondecreasing with $\lim_{n\longrightarrow\infty}G^n(x_0,y_0)=x$, then $G^n(x_0,y_0)\le x$, and since the sequence $\{G^n(y_0,x_0)=y_n\}_{n\in N}$ is nonincreasing with $\lim_{n\longrightarrow\infty}G^n(y_0,x_0)=y$, then $y\le G^n(y_0,x_0)$ for every $n\in N$, and accordingly, we have

$$\begin{aligned}d_\theta(G(x,y),x)&\prec\theta\left(d_\theta\left(G(x,y),G^{n+1}(x_0,y_0)\right),d_\theta\left(G^{n+1}(x_0,y_0),x\right)\right),\\&=\theta\left(d_\theta(G(x,y),G(x_n,y_n)),d_\theta\left(G^{n+1}(x_0,y_0),x\right)\right)\\&\prec\theta\left(\left[\frac{r}{2}\right][d_\theta(x,x_n)+d_\theta(y,y_n)],d_\theta\left(G^{n+1}(x_0,y_0),x\right)\right)\\&\prec\theta\left(\left[\frac{r}{2}\right][d_\theta(x,G^n(x_0,y_0))+d_\theta(y,G^n(y_0,x_0))],d_\theta\left(G^{n+1}(x_0,y_0),x\right)\right).\end{aligned}\tag{21}$$

Taking the limit as $n\longrightarrow\infty$ with the help of equation (20), we find that

$$d_\theta(G(x,y),x)\prec\theta\left(\left[\frac{r}{2}\right][\Theta+\Theta],\Theta\right)=\theta(\Theta,\Theta)=\Theta.\tag{22}$$

Hence, $d_\theta(G(x,y),x)=\Theta$; therefore, $G(x,y)=x$. Similarly, $G(y,x)=y$. □

If the partial-ordered relation on $E\times E$ is defined as

$$(x,y)\ll_2(z,w)\Leftrightarrow x\ge z,\text{ and } y\le w,\tag{23}$$

then the following theorem is similarly proved.

Theorem 4. *Let $(E,d_\theta,\le)$ be a partially ordered, sequentially lower-upper ordered complete θ-cone-metric space and $G\colon E\times E\longrightarrow E$ be a mapping having mixed monotone property, and we suppose that there are $x_0,y_0\in E$ such that $T(x_0,y_0)\le x_0$ and $y_0\le T(y_0,x_0)$. If there exists $r\in[0,1)$ with*

$$\begin{aligned}&d_\theta(G(x,y),G(z,w))\prec\frac{r}{2}[d_\theta(x,z)+d_\theta(y,w)],\\&\qquad\forall(x,y)\ll_2(z,w),\end{aligned}\tag{24}$$

then G has coupled fixed points in E.

Corollary 1. *Let E be a sequentially both lower- and upper-ordered Banach space and $T\colon E\times E\longrightarrow E$ be a mapping with mixed monotone and lower-upper properties. If there is a real number $0\le k<1$ such that*

$$\|T(x,y)-T(z,w)\| \leq \frac{k}{2}[\|x-z\|+\|y-w\|],$$
$$\forall x,y,z,w \in E, z \leq x, \text{ and } y \leq w, \tag{25}$$

then T has coupled fixed point in E.

Proof. We just notice that any Banach space $(E, \|.\|)$ is a θ-cone-metric space (E, d_θ), where $(\mathscr{E}, \prec) = (\mathbb{R}, |.|)$ is the Banach space of real numbers with the absolute value metric and with the usual ordered relation of real numbers, $\theta(u,v) = u+v$, $\theta\colon [0,\infty)\times[0,\infty) \longrightarrow [0,\infty)$, and the metric d_θ is the metric induced by the norm $\|.\|$ on E, $d_\theta(x,y) = \|x-y\|$. □

Remark 2. Corollary 1 is Baskar–Lackshmikantham coupled fixed-point Theorem 1. This proves that Theorem 3 is a quite good generalization of the Baskar–Lackshmikantham coupled fixed-point theorem.

On the other side, we have the following results:

Lemma 1. *Let (E, d_θ) be a θ-cone-metric space and T be a mapping, $T\colon E\times E \longrightarrow E$. It is supposed that there are constants $a,b,c \in [0,1)$ and $a+b+c<1$ such that*

$$d_\theta(T(x,y),T(y,z)) \prec a d_\theta(x,y) + b d_\theta(T(x,y),x) + c d_\theta(T(y,z),y), \quad \forall x,y,z \in E. \tag{26}$$

If x_1 and x_2 are arbitrary elements in E, then the sequence $\{x_n\}_{n=3}^{\infty}$ defined iteratively by

$$x_n = T(x_{n-1}, x_{n-2}), \quad \forall n \in \mathscr{N}, n>2, \tag{27}$$

which satisfies the following:

$$d_\theta(x_{n+1},x_n) \prec t d_\theta(x_n,x_{n-1}), \quad \forall n>2, \tag{28}$$

$$d_\theta(x_{n+1},x_n) \prec t^n d_\theta(x_2,x_1), \quad \forall n>2, \tag{29}$$

where $t = (a+c/1-b)$. Moreover, the sequence $\{x_n\}_{n\in\mathscr{N}}$ is a Cauchy sequence.

Proof. Using the contractiveness property of the given mapping gives

$$\begin{aligned} d_\theta(x_{n+1},x_n) &= (d_\theta(T(x_n,x_{n+1})), T(x_{n-1},x_{n-2})) \\ &\prec a d_\theta(x_n,x_{n-1}) + b d_\theta(x_{n+1},x_n) \\ &\quad + c d_\theta(x_n,x_{n-1}). \end{aligned} \tag{30}$$

Hence,

$$d_\theta(x_{n+1},x_n) \prec \left(\frac{a+c}{1-b}\right) d_\theta(x_n,x_{n-1}), \quad \forall n>2, \tag{31}$$

and repeating the last step $n-2$ times with the term $d_\theta(x_n,x_{n-1})$ proves the inequalities given in (29). To prove that the sequence (27) is Cauchy, we take the limit of both sides of (29) as $n \longrightarrow \infty$ gives $\lim_{n\longrightarrow\infty} d_\theta(x_{n+1},x_n) = \Theta$ and suppose that $\{x_n\}_{n\in N}$ is not Cauchy; then, there exist $v \in \operatorname{Im}(\theta)$, $\Theta < v$ and sequences of natural numbers $\{i_n\}_{n\in N}$ and $\{j_n\}_{n\in N}$ such that, for any $i_n > j_n > n$,

$$\begin{aligned} v &\prec d_\theta\left(x_{i_n}, x_{j_n}\right), \\ d_\theta\left(x_{i_n-1}, x_{j_n}\right) &< v. \end{aligned} \tag{32}$$

Since any subsequence of $\{d_\theta(x_{n+1},x_n)\}_{n\in N}$ is convergent to Θ, the continuity and the properties of θ imply the following contradiction:

$$\begin{aligned} v &\prec d_\theta\left(x_{i_n}, x_{j_n}\right) \prec \theta\left(d_\theta\left(x_{i_n-1}, x_{j_n}\right), d_\theta\left(x_{i_n}, x_{i_n-1}\right)\right) \\ &< \theta\left(v, d_\theta\left(x_{i_n}, x_{i_n-1}\right)\right) \longrightarrow_{n\longrightarrow\infty} \theta(v,\Theta) < v. \end{aligned} \tag{33}$$

□

Theorem 5. *Let (E, d_θ) be a complete θ-cone-metric space and $T\colon E\times E \longrightarrow E$ be a continuous mapping in the second argument, and we suppose there are three constants $a,b,c \in [0,1)$ and $a+b+c<1$ such that*

$$d_\theta(T(x,y),T(y,z)) \prec a d_\theta(x,y) + b d_\theta(T(x,y),x) + c d_\theta(T(y,z),y), \quad \forall x,y,z \in E, \tag{34}$$

and then, T has a unique fixed point in the sense that there is a unique point $x_0 \in E$ such that $T(x_0,x_0) = x_0$.

Proof. Since (E, d_θ) is complete, the Cauchy sequence $\{x_n\}_{n=3}^{\infty}$ given in Lemma 1 is converging to some element x_0 in E. We show that x_0 is fixed point of T. Using the properties of θ and the continuity of T, we see that

$$\begin{aligned}
d_\theta(T(x_0,x_0),x_0) &\prec \theta(d_\theta(T(x_0,x_0),x_n),d_\theta(x_n,x_0))\\
&\prec \theta(\theta(d_\theta(T(x_0,x_0),T(x_0,x_{n-1})),d_\theta(T(x_0,x_{n-1}),x_n)),d_\theta(x_n,x_0)),\\
&= \theta(\theta(d_\theta(T(x_0,x_{n-1}),T(x_{n-1},x_{n-2})),d_\theta(T(x_0,x_0),T(x_0,x_{n-1}))),\\
&= \theta(\theta d_\theta T(x_0,x_{n-1}),T(x_{n-1},x_{n-2})),d_\theta(T(x_0,x_0),T(x_0,x_{n-1}))d_\theta(x_n,x_0))\\
&\prec \theta(\theta(ad_\theta(x_0,x_{n-1})+bd_\theta(T(x_0,x_{n-1}),x_0)+cd_\theta(T(x_{n-1},x_{n-2}),x_{n-1}),\\
&\quad d_\theta(T(x_0,x_0),T(x_0,x_{n-1}))),d_\theta(x_n,x_0))\\
&\prec \theta(\theta(ad_\theta(x_0,x_{n-1})+bd_\theta(T(x_0,x_{n-1}),x_0)+cd_\theta(x_n,x_{n-1}),\\
&\quad d_\theta(T(x_0,x_0),T(x_0,x_{n-1}))),d_\theta(x_n,x_0)) \longrightarrow_{n\longrightarrow\infty}\\
&\prec \theta(\theta(a\Theta+bd_\theta(T(x_0,x_0),x_0)+c\Theta,d_\theta(T(x_0,x_0),T(x_0,x_0))),\Theta)\\
&\prec \theta(\theta(a\Theta+bd_\theta(T(x_0,x_0),x_0)+c\Theta,\Theta),\Theta),\\
&= \theta(\theta(bd_\theta(T(x_0,x_0),x_0),\Theta),\Theta)\\
&\prec \theta(bd_\theta(T(x_0,x_0),x_0),\Theta)\\
&\prec bd_\theta(T(x_0,x_0),x_0).
\end{aligned} \tag{35}$$

Since $b<1$, we get $d_\theta(T(x_0,x_0),x_0)=\Theta$; consequently, $T(x_0,x_0)=x_0$. Now, let x and y be two arbitrarily distinct elements in E with $T(x,x)=x$ and $T(y,y)=y$, and we have

$$\begin{aligned}
&d_\theta(T(x,y),x) \prec \theta(d_\theta T(x,x),T(x,y)),d_\theta(T(x,x),x)\\
&\prec \theta(bd_\theta(T(x,x),x)+cd_\theta(T(x,y),x),d_\theta(T(x,x),x))\\
&\prec \theta(b\Theta+cd_\theta(T(x,y),x),\Theta)\\
&\prec cd_\theta(T(x,y),x).
\end{aligned} \tag{36}$$

Thus, $d_\theta(T(x,y),x)=\Theta$, that is, $T(x,y)=x$. Similarly, we get $T(y,x)=y$; therefore, (x,y) is a coupled fixed point of T. On the other hand, we have the following contradiction:

$$\begin{aligned}
d_\theta(x,y) &= d_\theta(T(x,x),T(y,y))\\
&\prec \theta(d_\theta(T(x,x),T(x,y)),d_\theta(T(x,y),T(y,y)))\\
&\prec \theta(cd_\theta(T(x,y),x),ad_\theta(x,y)+bd_\theta(T(x,y),x))\\
&\prec \theta(cd_\theta(x,x),ad_\theta(x,y)+bd_\theta(x,x))\\
&\prec \theta(\Theta,ad_\theta(x,y)+\Theta) \prec ad_\theta(x,y).
\end{aligned} \tag{37}$$

Since $a<1$, we have $d_\theta(x,y)$; consequently, $x=y$. ☐

We conclude the following.

Corollary 2. *Let $(E,\|.\|)$ be a Banach space and T be a mapping from $E\times E$ into E, and we suppose that there are three constants $a,b,c\in[0,1)$ and $a+b+c<1$ such that*

$$\begin{aligned}
\|T(x,y)-T(y,z)\| \le a\|x-y\| + b\|T(x,y)-x\|\\
+ c\|T(y,z)-y\|, \quad \forall x,y,z\in E.
\end{aligned} \tag{38}$$

Then, there is a unique point $x_0\in E$ such that $T(x_0,x_0)=x_0$.

Proof. It can be proved in a similar way of Corollary 1 with the same notice. ☐

Remark 3. Corollary 2 is the fixed-point theorem of Mohamed Ali Abou Bakr; accordingly, Theorem 5 is a generalization of fixed-point Theorem 2 in the setting of a complete θ-cone-metric space.

3. Conclusions

This paper gives further generalizations of some well-known coupled fixed-point theorems. Specifically, Theorem 3 generalizes the Baskar–Lackshmikantham coupled fixed-point theorem [3], and Theorem 5 generalizes the Sahar Mohamed Ali Abou Bakr fixed-point theorem [4]; the underlying space (E,d_θ) is a complete θ-cone-metric space, and we claim that some results of [6–10] can be proved in the case of θ-cone-metric spaces.

Data Availability

No data were used to support this study.

Disclosure

This research was performed as part of the employment of Dr. Sahar Mohamed Ali Abou Bakr at Ain Shams University, Faculty of Science, Department of Mathematics, Cairo, Egypt.

Conflicts of Interest

The author has no conflicts of interest.

Authors' Contributions

The sole author contributed 100% to the article. The author read and approved the final manuscript.

References

[1] S. Banach, "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales," *Fundamenta Mathematicae*, vol. 3, pp. 133–181, 1922.

[2] D. Guo and V. Lakshmikantham, "Coupled fixed points of nonlinear operators with applications," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 11, no. 5, pp. 623–632, 1987.

[3] T. G. Bhaskar and V. Lakshmikantham, "Fixed point theorems in partially ordered metric spaces and applications," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 65, no. 7, pp. 1379–1393, 2006.

[4] S. Mohamed Ali, "Fixed point theorems of some type contraction mappings," *Journal of Nonlinear and Convex Analysis*, vol. 14, no. 2, pp. 331–342, 2013.

[5] M. Kir, E. Yolacan, and H. Kiziltunc, "Coupled fixed point theorems in complete metric spaces endowed with a directed graph and application," *Open Math by De Gruyter Open*, vol. 15, Article ID 734744, 2017.

[6] A. Abdullah, M. Saud, and Alsulami, "Coupled coincidence points for monotone operators in partially ordered metric spaces," *Fixed Point Theory and Applications*, vol. 2011, p. 44, 2011.

[7] H. K. Nashine, Z. Kadelburg, and S. Radenović, "Coupled common fixed point theorems for w-compatible mappings in ordered cone metric spaces," *Applied Mathematics and Computation*, vol. 218, no. 9, pp. 5422–5432, 2012.

[8] M. Jain, K. Tas, and N. Gupta, "Coupled common fixed point results involving (ϕ, ψ)-contractions in ordered generalized metric spaces with application to integral equations," *Journal of Inequalities and Applications*, vol. 2013, no. 1, p. 372, 2013.

[9] Z. Kadelburg, P. Kumam, S. Radenovi, and W. Sintunavarat, "Common coupled fixed point theorems for Geraghty-type contraction mappings using monotone property," *Fixed Point Theory and Applications*, vol. 2015, p. 27, 2015.

[10] F. Khojasteh, E. Karapinar, and S. Radenovic, "Theta-metric space: a generalization," *Mathematical Problems in Engineering*, vol. 2013, Article ID 504609, , 2013.

[11] F. Gu and W. Shatanawi, "Some new results on common coupled fixed points of two hybrid pairs of mappings in partial metric spaces," *Journal of Nonlinear Function and Analysis*, vol. 2019, p. 13, 2019.

[12] A. Radouane and D. Mentagui, "A solution for the minimax problem via fixed point theory in complete ordered locally convex spaces," *Communications in Optimization Theory*, vol. 2020, p. 11, 2020.

[13] A. Petrusel and G. Petrusel, "Fixed point results for multivalued locally contractive operators," *Applications of Set-Valued Analysis to Optimization*, vol. 2, pp. 175–181, 2020.

[14] L.-G. Huang and X. Zhang, "Cone metric spaces and fixed point theorems of contractive mappings," *Journal of Mathematical Analysis and Applications*, vol. 332, no. 2, pp. 1468–1476, 2007.

[15] S. Mohamed Ali Abou Bakr, "Common fixed point of generalized cyclic Banach algebra contractions and Banach algebra Kannan types of mappings on cone quasi metric spaces," *Journal of Nonlinear Science and Appllications*, vol. 12, no. 10, pp. 644–655, 2019.

[16] S. Mohamed Ali Abou Bakr, "Theta cone metric spaces and some fixed point theorems," *Journal of Mathematics*, vol. 2020, p. 7, Article ID 8895568, 2020.

[17] S. Mohamed Ali Abou Bakr, "A study on common fixed point of joint $(A;B)$generalized cyclic $\phi - abc$ weak nonexpansive mappings and generalized cyclic $abc;r$ contractions in quasi metric spaces," *Abstract and Applied Analysis*, vol. 2020, Article ID 9427267, 2020.