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Research Article

On Semi-c-Periodic Functions

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The main aim of this paper is to indicate that the notion of semi-c-periodicity is equivalent with the notion of c-periodicity, provided that c is a nonzero complex number whose absolute value is not equal to 1.

1. Introduction

The notion of periodicity plays a fundamental role in mathematics. A continuous function $f\colon I \longrightarrow E$, where E is a topological space and $I = \mathbb{R}$ or $I = [0, \infty)$, is said to be *periodic* if and only if there exists a real number $\omega > 0$ such that $f(x + \omega) = f(x)$ for all $x \in I$. The notion of periodicity has recently been reconsidered by Alvarez et al. [1], who proposed the following notion: a continuous function $f\colon I \longrightarrow E$, where E is a complex Banach space, is said to be (ω, c)*-periodic* ($\omega > 0$, $c \in \mathbb{C}\setminus\{0\}$) if and only if $f(x + \omega) = cf(x)$ for all $x \in I$. Due to ([1], Proposition 2.2), we know that a continuous function $f\colon I \longrightarrow E$ is (ω, c)-periodic if and only if the function $g(\cdot) \equiv c^{(-\cdot/\omega)} f(\cdot)$ is periodic and $g(x + \omega) = g(x)$ for all $x \in I$; here, $c^{(-\cdot/\omega)}$ denotes the principal branch of the exponential function (see also the research articles [2, 3] by Alvarez et al., the conference paper [4] by Pinto, where the idea for introduction of (ω, c)-periodic functions was presented for the first time, and [5, 6] for some generalizations of the concept of (ω, c)-periodicity).

In the sequel, by E we denote a complex Banach space equipped with the norm $\|\cdot\|$; $C(I\colon E)$ denotes the vector space consisting of all continuous functions $f\colon I \longrightarrow E$. A function $f \in C(I\colon E)$ is said to be c*-periodic* ($c \in \mathbb{C}\setminus\{0\}$) if and only if there exists a real number $\omega > 0$ such that the function $f(\cdot)$ is (ω, c)-periodic. The class of c-periodic functions extends two important classes of functions:

(1) The class of antiperiodic functions, i.e., the class of (-1)-periodic functions: in this case, any positive real number $\omega > 0$ satisfying $f(x + \omega) = -f(x)$, $x \in I$, is said to be an antiperiod of $f(\cdot)$. Any antiperiodic function is periodic, since we can apply the above functional equality twice in order to see that $f(x + 2\omega) = -f(x)$ for all $x \in I$.

(2) The class of Bloch (ω, k)-periodic functions ($\omega > 0$, $k \in \mathbb{R}$), i.e., the class of continuous functions $f\colon I \longrightarrow E$ satisfying $f(x + \omega) = e^{ik\omega} f(x)$ for all $x \in I$. The number ω is usually called Bloch period of $f(\cdot)$, the number k is usually called the Bloch wave vector or Floquet exponent of $f(\cdot)$, and in the case that $k\omega = \pi$, the class of Bloch (ω, k)-periodic functions is equal to the class of antiperiodic functions having the number ω as an antiperiod. If the function $f(\cdot)$ is Bloch (ω, k)-periodic, then we inductively obtain $f(x + m\omega) = e^{imk\omega} f(x)$ for all $x \in I$ and $m \in \mathbb{N}$, so that the function $f(\cdot)$ must be periodic provided that $k\omega \in \mathbb{Q}$, but, if $k\omega \notin \mathbb{Q}$, then the function $f(\cdot)$ need not be periodic as the

following simple counterexample shows: the function

$$f(x) := e^{ix} + e^{i(\sqrt{2}-1)x}, \quad x \in \mathbb{R}, \tag{1}$$

is Bloch (ω, k)-periodic with $\omega = 2\pi + \sqrt{2}\pi$ and $k = \sqrt{2} - 1$ but not periodic. In ([7], Remark 1), we have recently observed that any Bloch (ω, k)-periodic function must be almost periodic (see also the research articles [8] by Hasler and [9] by Hasler and Guérékata, where it has been noted that the Bloch (ω, k)-periodic functions are unavoidable in condensed matter and solid state physics).

The notion of almost periodicity was introduced by Harald Bohr, a younger brother of Nobel Prize winner Niels Bohr, around 1925 and later generalized by many other mathematicians. In [10], we have analyzed the following generalization of the notion of almost periodicity, called c-almost periodicity ($c \in \mathbb{C} \setminus \{0\}$): let $f: I \longrightarrow E$ be a continuous function, and let a number $\epsilon > 0$ be given. We call a number $\tau > 0$ an (ϵ, c)-period for $f(\cdot)$ if and only if $\|f(x+\tau) - cf(x)\| \leq \epsilon$ for all $x \in I$; by $\vartheta_c(f, \epsilon)$ we denote the set consisting of all (ϵ, c)-periods for $f(\cdot)$. It is said that $f(\cdot)$ is *c-almost periodic* if and only if for each $\epsilon > 0$ the set $\vartheta_c(f, \epsilon)$ is *relatively dense* in $[0, \infty)$, which means that for each $\epsilon > 0$ there exists a finite real number $l > 0$ such that any subinterval I' of $[0, \infty)$ of length l meets $\vartheta_c(f, \epsilon)$. Any c-periodic function is c-almost periodic and any c-almost periodic function is almost periodic ([10]); if $c = 1$, resp. $c = -1$, then we also say that the function $f(\cdot)$ is *almost periodic*, resp. *almost antiperiodic* (for the primary source of information about almost periodic functions and their applications, we refer the reader to the research monographs by Besicovitch [11], Diagana [12], Fink [13], Guérékata [14], Kostić [15], and Zaidman [16]).

In [10], besides the class of c-almost periodic functions, we have introduced and analyzed the classes of c-uniformly recurrent functions, semi-c-periodic functions, and their Stepanov generalizations, where $c \in \mathbb{C}$ and $|c| = 1$ (the classes of semiperiodic functions and semi-antiperiodic functions, i.e., the classes of semi-1-periodic functions and semi-(-1)-periodic functions, have been previously considered by Andres and Pennequin in [17], the research article of invaluable importance for us, and Chaouchi et al. in [7]; the notion of semi-Bloch k-periodicity, where $k \in \mathbb{R}$, has been also analyzed in [7], but it differs from the notion of semi-c-periodicity analyzed in [10] and this paper). If $|c| = 1$, then we know that a function $f \in C(I: E)$ is semi-c-periodic if and only if there exists a sequence (f_n) of c-periodic functions in $C(I: E)$ such that $\lim_{n \longrightarrow \infty} f_n(x) = f(x)$ uniformly in I; in this case, a semi-c-periodic function need not be c-periodic [10]. For example, we have the following (see ([17], Example 1), ([7], Example 4 and Example 5), and ([10], Example 2.16)): let p and q be odd natural numbers such that $p - 1 \equiv 0 \pmod{q}$, and let $c = e^{(i\pi p/q)}$. The function

$$f(x) := \sum_{n=1}^{\infty} \frac{e^{(ix/(2nq+1))}}{n^2}, \quad x \in \mathbb{R}, \tag{2}$$

is semi-c-periodic because it is a uniform limit of $[\pi \cdot (1+2q) \ldots (1+2Nq)]$-periodic functions

$$f_N(x) := \sum_{n=1}^{N} \frac{e^{(ix/(2nq+1))}}{n^2}, \quad x \in \mathbb{R} \ (N \in \mathbb{N}). \tag{3}$$

Our main result, Theorem 1, states that the following phenomenon occurs in case $|c| \neq 1$: if (f_n) is a sequence of c-periodic functions and $\lim_{n \longrightarrow \infty} f_n(x) = f(x)$ uniformly in I, then $f(\cdot)$ is c-periodic. Therefore, in this case, any concept of semi-c-periodicity introduced below coincides with the concept of c-periodicity (more precisely, in this paper, we analyze the concepts of semi-c-periodicity of type i (i_+), where $i = 1, 2$ and $c \in \mathbb{C} \setminus \{0\}$; if $|c| = 1$, all these concepts are equivalent and reduced to the concept of semi-c-periodicity, while in case $|c| \neq 1$, all these concepts are equivalent and reduced to the concept of c-periodicity).

For any function $f \in C(I: E)$, we set $\|f\|_\infty := \sup_{x \in I} \|f(x)\|$. The notion of c-uniform recurrence plays an important role in the proof of our main result [10].

Definition 1. A continuous function $f: I \longrightarrow E$ is said to be c-uniformly recurrent ($c \in \mathbb{C} \setminus \{0\}$) if and only if there exists a strictly increasing sequence (α_n) of positive real numbers such that $\lim_{n \longrightarrow +\infty} \alpha_n = +\infty$ and

$$\lim_{n \longrightarrow +\infty} \|f(\cdot + \alpha_n) - cf(\cdot)\|_\infty = 0. \tag{4}$$

The space consisting of all c-uniformly recurrent functions from the interval I into E will be denoted by $UR_c(I: E)$. If $c = 1$, resp. $c = -1$, then we also say that the function $f(\cdot)$ is *uniformly recurrent*, resp. *uniformly antirecurrent*.

Although the notion of uniform recurrence was analyzed already by Bohr in his landmark paper [18] (1924), the precise definition of a uniformly recurrent function was firstly given by Haraux and Souplet [19] in 2004, who proved that the function $f: \mathbb{R} \longrightarrow \mathbb{R}$, given by

$$f(x) := \sum_{n=1}^{\infty} \frac{1}{n} \sin^2\left(\frac{x}{2^n}\right), \quad x \in \mathbb{R}, \tag{5}$$

is unbounded, Lipschitz continuous and uniformly recurrent; moreover, we have that $f(\cdot)$ is c-uniformly recurrent if and only if $c = 1$ (see [10], Example 2.19(i)). The first example of a uniformly antirecurrent function has recently been constructed in ([10], Example 2.20), where we have proved that the function $g: \mathbb{R} \longrightarrow \mathbb{R}$, given by

$$g(x) := (\sin x) \cdot \sum_{n=1}^{\infty} \frac{1}{n} \sin^2\left(\frac{x}{3^n}\right), \quad x \in \mathbb{R}, \tag{6}$$

is unbounded, Lipschitz continuous and uniformly antirecurrent. Any c-almost periodic function is c-uniformly recurrent, while the converse statement does not hold in general.

For completeness, we will include all details of the proof of the following auxiliary lemma from [10].

Lemma 1 *(A). Suppose that* $f \in UR_c(I: E)$ *and* $c \in \mathbb{C} \setminus \{0\}$ *satisfies* $|c| \neq 1$. *Then,* $f \equiv 0$.

Proof. Without loss of generality, we may assume that $I = [0, \infty)$. Suppose to the contrary that there exists $x_0 \geq 0$ such that $f(x_0) \neq 0$. Inductively, (4) implies

$$|c|^k m - \frac{|c|^k - 1}{n(|c| - 1)} \leq \|f(x)\| \leq |c|^k M - \frac{|c|^k - 1}{n(|c| - 1)}, \tag{7}$$

provided that $k \in \mathbb{N}$ and $x \in [k\alpha_n, (k+1)\alpha_n]$. Consider now case $|c| < 1$. Let $0 < \epsilon < c\|f(x_0)\|$. Then, (7) yields that there exist integers $k_0 \in \mathbb{N}$ and $n \in \mathbb{N}$ such that for each $k \in \mathbb{N}$ with $k \geq k_0$, we have $\|f(x)\| \leq (\epsilon/2)$, $x \in [k\alpha_n, (k+1)\alpha_n]$. Then, the contradiction is obvious because for each $m \in \mathbb{N}$ with $m > n$, there exists $k \in \mathbb{N}$ such that $x_0 + \alpha_m \in [k\alpha_n, (k+1)\alpha_n]$, and therefore $\|f(x_0 + \alpha_m)\| \geq |c|\|f(x_0)\| - (1/m) \longrightarrow |c|\|f(x_0)\| > \epsilon$, $m \longrightarrow +\infty$. Consider now case $|c| > 1$; let $n \in \mathbb{N}$ be such that $\|f(x_0)\| > (1/(n(|c| - 1)))$ and $M := \max_{x \in [0, 2\alpha_n]} \|f(x)\| > 0$. Then, for each $m \in \mathbb{N}$ with $m > n$, there exists $k \in \mathbb{N}$ such that $\alpha_m \in [(k-1)\alpha_n, k\alpha_n]$, and therefore $\|f(x + \alpha_m)\| \leq 1 + |c|M$, $x \in [0, 2\alpha_n]$. On the other hand, we obtain inductively from (4) that

$$\begin{aligned} \|f(x_0 + k\alpha_n)\| &\geq |c|^k \left[\|f(x_0)\| - \frac{1}{n(|c| - 1)} \right] \\ &\quad + \frac{1}{n(|c| - 1)} \longrightarrow +\infty \text{ as } k \in \mathbb{N}, \end{aligned} \tag{8}$$

which immediately yields a contradiction. □

2. Semi-c-Periodic Functions

Set $\mathbb{S} := \mathbb{N}$ if $I = [0, \infty)$, and $\mathbb{S} := \mathbb{Z}$ if $I = \mathbb{R}$. In this paper, we introduce and analyze the following notion with $c \in \mathbb{C} \setminus \{0\}$.

Definition 2. Let $f \in C(I: E)$.

(i) It is said that $f(\cdot)$ is semi-c-periodic of type 1 if and only if

$$\forall \varepsilon > 0\, \exists \omega > 0\, \forall m \in \mathbb{S}\, \forall x \in I \quad \|f(x + m\omega) - c^m f(x)\| \leq \varepsilon. \tag{9}$$

(ii) It is said that $f(\cdot)$ is semi-c-periodic of type 2 if and only if

$$\forall \varepsilon > 0\, \exists \omega > 0\, \forall m \in \mathbb{S}\, \forall x \in I \quad \|c^{-m} f(x + m\omega) - f(x)\| \leq \varepsilon. \tag{10}$$

The space of all semi-c-periodic functions of type i will be denoted by $\mathcal{SP}_{c,i}(I: E)$, $i = 1, 2$.

Definition 3. Let $f \in C(I: E)$.

(i) It is said that $f(\cdot)$ is semi-c-periodic of type 1_+ if and only if

$$\forall \varepsilon > 0\, \exists \omega > 0\, \forall m \in \mathbb{N}\, \forall x \in I \quad \|f(x + m\omega) - c^m f(x)\| \leq \varepsilon. \tag{11}$$

(ii) It is said that $f(\cdot)$ is semi-c-periodic of type 2_+ if and only if

$$\forall \varepsilon > 0\, \exists \omega > 0\, \forall m \in \mathbb{N}\, \forall x \in I \quad \|c^{-m} f(x + m\omega) - f(x)\| \leq \varepsilon. \tag{12}$$

The space of all semi-c-periodic functions of type i_+ will be denoted by $\mathcal{SP}_{c,i,+}(I: E)$, $i = 1, 2$.

The notion of semi-c-periodicity of type 1 has been introduced in ([10], Definition 2.4), where it has been simply called semi-c-periodicity. Due to ([10], Proposition 2.5), we have that the notion of a semi-c-periodicity of type i (i_+), where $i = 1, 2$, is equivalent with the notion of semi-c-periodicity introduced there, provided that $|c| = 1$.

Now we will focus our attention to the general case $c \in \mathbb{C} \setminus \{0\}$. We will first state the following.

Lemma 2 *(B).*

(i) If $|c| \geq 1$ *and* $f: I \longrightarrow E$ *is semi-c-periodic of type* 1_+, *then* $f(\cdot)$ *is semi-c-periodic of type* 2_+.

(ii) If $|c| \leq 1$ *and* $f: I \longrightarrow E$ *is semi-c-periodic of type* 2_+, *then* $f(\cdot)$ *is semi-c-periodic of type* 1_+.

Proof. If $x \in I$, $\omega > 0$, $m \in \mathbb{N}$ and $|c| \geq 1$, then we have

$$\|f(x + m\omega) - c^m f(x)\| \leq \varepsilon \Rightarrow \|c^{-m} f(x + m\omega) - f(x)\| \leq \varepsilon, \tag{13}$$

which implies (i); the proof of (ii) is similar. □

The argumentation contained in the proofs of ([17], Lemma 1 and Theorem 1) can be repeated verbatim in order to see that the following important lemma holds true.

Lemma 3 *(C). Suppose that* $|c| \leq 1$, *resp.* $|c| \geq 1$, *and* $f: [0, \infty) \longrightarrow E$ *is semi-c-periodic of type* 1_+, *resp.* 2_+. *Then, there exists a sequence* $(f_n: [0, \infty) \longrightarrow E)_{n \in \mathbb{N}}$ *of c-periodic functions which converges uniformly to* $f(\cdot)$.

Now we are able to state and prove our main result.

Theorem 1. *Let* $|c| \neq 1$, $i \in \{1, 2\}$ *and* $f: I \longrightarrow E$. *Then,* $f(\cdot)$ *is c-periodic if and only if* $f(\cdot)$ *is semi-c-periodic of type i* (i_+).

Proof. Suppose that the function $f(\cdot)$ is (ω, c)-periodic. Then, we have $f(x + m\omega) = c^m f(x)$, $x \in I$, $m \in \mathbb{S}$, so that $f(\cdot)$ is automatically semi-c-periodic of type i (i_+). To prove the converse statement, let us observe that any semi-c-periodic of type i is clearly semi-c-periodic of type i_+. Suppose first that $|c| > 1$. Due to Lemma 2 B(i), it suffices to show that if $f(\cdot)$ is semi-c-periodic of type 2_+, then $f(\cdot)$ is c-periodic. Assume first $I = [0, \infty)$. Using Lemma C, we get the existence of a sequence $(f_n: (0, \infty) \longrightarrow E)_{n \in \mathbb{N}}$ of c-periodic functions which converges uniformly to $f(\cdot)$. Let $f_n(x+\omega_n) = cf_n(x)$, $x \geq 0$ for some sequence (ω_n) of positive real numbers. Consider first case that (ω_n) is bounded. Then, there exists a strictly increasing sequence (n_k) of positive integers and a number $\omega \geq 0$ such that $\lim_{k\longrightarrow+\infty}\omega_{n_k} = \omega$. Let $\epsilon > 0$ be given. Then, there exists an integer $k_0 \in \mathbb{N}$ such that $\|f(x) - f_{n_k}(x)\| \leq \epsilon/(2 + 2|c|^{-1})$ for all real numbers $x \geq 0$ and all integers $k \geq k_0$. Furthermore, we have

$$\begin{aligned}\left\|c^{-1}f\left(x+\omega_{n_k}\right) - f(x)\right\| &\leq \left\|c^{-1}f\left(x+\omega_{n_k}\right) - c^{-1}f_{n_k}\left(x+\omega_{n_k}\right)\right\| \\ &\quad + \left\|c^{-1}f_{n_k}\left(x+\omega_{n_k}\right) - f_{n_k}(x)\right\| \\ &\quad + \left\|f_{n_k}(x) - f(x)\right\| \\ &= \left\|c^{-1}f\left(x+\omega_{n_k}\right) - c^{-1}f_{n_k}\left(x+\omega_{n_k}\right)\right\| \\ &\quad + \left\|f_{n_k}(x) - f(x)\right\| \leq 2\left(1+|c|^{-1}\right) \\ &\quad \cdot \frac{\epsilon}{\left(2+2|c|^{-1}\right)} = \epsilon,\end{aligned} \tag{14}$$

for all real numbers $x \geq 0$ and all integers $k \geq k_0$. Letting $k \longrightarrow +\infty$, we get $f(x+\omega) = cf(x)$ for all $x \geq 0$. If $\omega > 0$, the above yields that $f(\cdot)$ is (ω, c)-periodic while the assumption $\omega = 0$ yields $f \equiv 0$ or $c = 1$, i.e., $f(\cdot) \equiv 0$; in any case, $f(\cdot)$ is (ω, c)-periodic. Suppose now that (ω_n) is unbounded. Then, with the same notation as above, we may assume that $\lim_{k\longrightarrow+\infty}\omega_{n_k} = +\infty$. Using the same computation, it follows that $\lim_{k\longrightarrow+\infty}\|c^{-1}f(\cdot+\omega_{n_k}) - f(\cdot)\|_\infty = 0$, so that $f \in UR_c([0,\infty)\colon E)$. Due to Lemma 1 A, we get $f(\cdot) \equiv 0$. Assume now $I = \mathbb{R}$. By the foregoing arguments, we know that there exists $\omega > 0$ such that $f(x+\omega) = cf(x)$ for all $x \geq 0$. Let $x < 0$ and $\epsilon > 0$ be fixed. Since $f(\cdot)$ is semi-c-periodic, there exists $\omega_\epsilon > 0$ such that $\|c^{-m}f(x+\omega+m\omega_\epsilon) - f(x+\omega)\| \leq \epsilon$ and $\|c^{1-m}f(x+m\omega_\epsilon) - cf(x)\| \leq \epsilon$ for all $m \in \mathbb{N}$. For all sufficiently large integers $m \in \mathbb{N}$, we have $x + m\omega_\epsilon > 0$ so that $c^{-m}f(x+\omega+m\omega_\epsilon) = c^{1-m}f(x+m\omega_\epsilon)$, and therefore $\|f(x+\omega) - cf(x)\| \leq 2\epsilon$. Since $\epsilon > 0$ was arbitrary, we get $f(x+\omega) = cf(x)$, which completes the proof in case $|c| > 1$. Suppose now that $|c| < 1$. Due to Lemma 2(ii), it suffices to show that if $f(\cdot)$ is semi-c-periodic of type 1_+, then $f(\cdot)$ is c-periodic. But, then we can apply Lemma 3 again and the similar arguments as above to complete the whole proof. □

Corollary 1. *Let $c \in \mathbb{C}\setminus\{0\}$, let $i \in \{1,2\}$, and let $f(\cdot)$ be semi-c-periodic of type i (i_+). Then, there exist two finite real constants $M > 0$ and $\omega > 0$ such that $\|f(x)\| \leq M|c|^{(x/\omega)}$, $t \in I$.*

Using ([10], Theorem 2.14) and the proof of Theorem 1, we may deduce the following corollaries.

Corollary 2. *Let $f \in C(I\colon E)$ and $c \in \mathbb{C}\setminus\{0\}$. Then, $f(\cdot)$ is semi-c-periodic if and only if there exists a sequence (f_n) of c-periodic functions in $C(I\colon E)$ such that $\lim_{n\longrightarrow\infty}f_n(x) = f(x)$ uniformly in I.*

Corollary 3. *Let $f \in C(I\colon E)$ and $|c| \neq 1$. If (f_n) is a sequence of c-periodic functions and $\lim_{n\longrightarrow\infty}f_n(x) = f(x)$ uniformly in I, then $f(\cdot)$ is c-periodic.*

3. Conclusions

In this paper, the authors have studied the class of semi-c-periodic functions with values in Banach spaces. In the case that c is a nonzero complex number whose absolute value is not equal to 1, the authors have proved that the notion of semi-c-periodicity is equivalent with the notion of c-periodicity. For further information concerning Stepanov semi-c-periodic functions, composition principles for (Stepanov) semi-c-periodic functions, and related applications to the abstract semilinear Volterra integrodifferential equations in Banach spaces, the reader may consult the forthcoming research monograph [20].

Data Availability

The data that support the findings of this study are available at https://www.researchgate.net/publication/342068071_SEMI-c-PERIODIC_FUNCTIONS_AND_APPLICATIONS (an extended version of the paper).

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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