Research Article

Generalizing Certain Analytic Functions Correlative to the n -th Coefficient of Certain Class of Bi-Univalent Functions

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In the present paper, the authors implement the two analytic functions with its positive real part in the open unit disk. New types of polynomials are introduced, and by using these polynomials with the Faber polynomial expansion, a formula is structured to solve certain coefficient problems. This formula is applied to a certain class of bi-univalent functions and solve the n -th term of its coefficient problems. In the last section of the article, several well-known classes are also extended to its n -th term.

1. Introduction

The structural properties and information about Geometric Function Theory depends on the estimation of coefficient of analytic functions. For example, the second coefficient estimation ($|a_2|$) in the set of univalent functions gives the growth, distortion bounds, and also covering theorems. The study of coefficient bounds for the classes of bi-univalent functions was first investigated by Levin [1] in 1967, but the interest sparks among the researchers when Brannan and Taha [2] conjectured the coefficient bounds for the classes of bi-univalent functions. A series of coefficients investigation have been carried out recently in [3–11] and coefficient properties by Rehman et al. [12]. In fact, the work of Srivastava et al. [13] has made a huge impact on the development of bi-univalent functions and appeared frequently in the literature ever since the publication of their pioneering work. In a recent development, Srivastava et al. [14] have made the use of Faber polynomial expansions with q -analysis to determine the bounds for the n -th coefficient in the Taylor–Maclaurin series expansions. In addition to this, Srivastava et al. [15] made the use of a linear combination of

three functions ($(f(z)/z)$, $f'(z)$, and $zf''(z)$) with the technique involving the Faber polynomials and determined the coefficient estimates for the general Taylor–Maclaurin functions belonging to the bi-univalent function.

Let $\mathcal{A}$ denotes the class of analytic functions in the unit disk, $\{\mathbb{U} = z \in \mathbb{C} : |z| < 1\}$ of the following form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1)$$

Furthermore, we represent by $\mathcal{T}$ the class of univalent functions, which is defined as a function $f \in \mathcal{A}$ is called univalent on $\mathbb{U}$ (or schlicht or one-to-one) if $f(z_1) \neq f(z_2)$ for all $z_1, z_2 \in \mathbb{U}$ with $z_1 \neq z_2$. The leading member of this class is the famous Kœbe's function of the form $K(z) = (z/(1-z^2))$. Some other examples are $v(z) = z$, $t(z) = (z/(1-z))$, and $u(z) = (z/(1-z^2))$ (see [16]). Next, we denote by Σ the class of bi-univalent functions that states the class of functions $f \in \mathcal{A}$ is said to be bi-univalent in the unit disk $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. For example, $f(z) = (z/(1-z))$, $d(z) = -\log(1-z)$, and $j(z) = (1/2)\log(1+z/(1-z))$ (for details, see [10,13]). According to

Kœbe's One Quarter Theorem (see [17], p. 31), the range of every function of class $\mathcal{T}$ contains the disk $\{w: |w| < (1/4)\}$. Kœbe's theorem ensures that the image of a unit disc $\mathbb{U}$, under every univalent function $f \in \mathcal{T}$, contains a disk of radius $(1/4)$. Therefore, every univalent function in $f \in \mathcal{A}$ has an inverse f^{-1} defined by

$$\begin{aligned} f^{-1}(f(z)) &= z = f(f^{-1}(z)), \quad (z \in \mathbb{U}), \\ f(f^{-1}(w)) &= w, \quad \left(|w| < r_o(f); r_o(f) \geq \frac{1}{4}\right). \end{aligned} \quad (2)$$

These type inverse functions can easily be verified by

$$\begin{aligned} \mathcal{Q}(w) &= w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 \\ &\quad + (14a_2^4 - 21a_2^2 a_3 + 3a_3^2 + 6a_2 a_4 - a_5)w^5 \dots, \end{aligned} \quad (3)$$

where $\mathcal{Q}(w) = f^{-1}(w)$.

Let $f(z)$, $g(z)$, and $w(z)$ be analytic functions in the open unit disc $\mathbb{U}$. The function $f(z)$ is said to be the subordinate to $g(z)$, expressed as $f(z) \prec g(z)$, if there exists a Schwartz function w , that is, $w(0) = 0$, $|w(z)| < 1$, and $f(z) = g(w(z))$. Particularly, if the function $g(z)$ is univalent in the unit disc $\mathbb{U}$, then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$ (cf. [18, 19]).

2. Discussion

Löwner [20] and Pommerenke [19] proved that the inverse of Kœbe's function concedes the best bounds for all $|a_n|$. However, new techniques have been adopted recently to determine the peculiar behavior of coefficients a_n for various subclasses of $\mathcal{T}$ (see, for example, [21–24]).

The series expansion of the inverse of $f \in \mathcal{T}$ in some disk about the origin is given by

$$f^{-1}(w) = w + a_2 w^2 + a_3 w^3 + a_4 w^4 + a_5 w^5 + \dots \quad (4)$$

Moreover, a function $f(z)$ which is univalent in a neighborhood about the origin with its inverse satisfies the condition $f(f^{-1}(w)) = w$, and then, equation (4) can be written as

$$\begin{aligned} w &= f^{-1}(w) + a_2 (f^{-1}(w))^2 + a_3 (f^{-1}(w))^3 \\ &\quad + a_4 (f^{-1}(w))^4 + a_5 (f^{-1}(w))^5 + \dots, \end{aligned} \quad (5)$$

where $f^{-1}(w) = w + \sum_{n=2}^{\infty} (1/n) K_{n-1}^{-n}(a_2, a_3, \dots, a_n) w^n$ is the Faber polynomial expansion of functions $f \in \mathcal{A}$ of the form (1), where

$$\begin{aligned} K_{n-1}^{-n} &= \frac{(-n)!}{(-2n+1)!(n-1)!} a_2^{n-1} + \frac{(-n)!}{(2(-n+1)!(n-3))!} a_2^{n-3} a_3 \\ &\quad + \frac{(-n)!}{(-2n+3)!(n-4)!} a_2^{n-4} a_4 + \frac{(-n)!}{(2(-n+2)!(n-5))!} a_2^{n-5} [a_5 + (-n+2)a_3^2] \\ &\quad + \frac{(-n)!}{(-2n+5)!(n-6)!} a_2^{n-6} [a_6 + (-2n+5)a_3 a_4] + \sum_{j \geq 7} a_2^{n-j} \mathcal{S}_j, \end{aligned} \quad (6)$$

with $(-n)! \equiv \Gamma(1-n) = (-n)(-n-1)\dots$ such that $(n \in \mathbb{N} \cup \{0\}, (\mathbb{N} = \{1, 2, 3, \dots\}))$ and $\mathcal{S}_j$ ($7 \leq j \leq n$) is a homogeneous polynomial in variables $(a_2, a_3, \dots, a_n)$. For more details, one may refer to the expansion of K_{n-1}^{-n} , and for the coefficients of its inverse functions, see Theorem 6.1 in [25] (p. 209) and [6]. Likewise, using the general term K_{n-1}^{-n} , we can compute the first six terms as follows:

$$\begin{aligned} K_1^{-2} &= -2a_2, \\ K_2^{-3} &= 3(2a_2^2 - a_3), \\ K_3^{-4} &= -4(5a_2^3 - 5a_2 a_3 + a_4), \\ K_4^{-5} &= 5(14a_2^4 - 21a_2^2 a_3 + 3a_3^2 + 6a_2 a_4 - a_5), \\ K_5^{-6} &= -6(42a_2^5 - 84a_2^3 a_3 + 28a_2^2 a_4 - 7a_2 a_5 - 28a_2 a_3^2 - 7a_3 a_4 - a_6). \end{aligned} \quad (7)$$

The Faber polynomials introduced in 1903 by Faber [26] (also see [27]) play an important role in various areas of mathematical sciences, in particular the geometric function theory (Gong [28], Chapter III, and Schiffer [29]). The

calculus of Faber polynomials gets more importance, especially when it was found useful in the study of inverse functions (f^{-1}) (see for details [25, 30–32]). Based on the implication of Faber polynomial expansions in determining the coefficient estimations of the bi-univalent functions and following the work of [1, 3, 9, 10, 13–15, 33, 34] and [35], we are motivated to derive new type of polynomials that collaborate with the Faber polynomial expansion to estimate the coefficient bounds for a certain class of bi-univalent functions beyond $|a_n| \geq 2$: $n \in \mathbb{N}_0 = \{2, 3, 4, \dots\}$.

Throughout the article, we consider $\mathcal{G}(z)$ to be analytic with its positive real part on the unit disk $\mathbb{U}$; obeying the conditions $\mathcal{G}(0) = 1$, $\mathcal{G}'(0) > 0$, and $\mathcal{G}(\mathbb{U})$ is symmetric with respect to real axis. This type of function can be expressed as series expansion of the following form:

$$\mathcal{G}(z) = 1 + B_1 z + B_2 z^2 + \dots = 1 + \sum_{n=1}^{\infty} B_n z^n, \quad (8)$$

for all $(B_n \in \mathbb{R})$ with $(B_1 > 0, n \in \mathbb{N} = \{1, 2, 3, \dots\})$.

We now define the class $\Sigma\mathcal{H}(\alpha, \tau, \gamma)(\mathcal{G})$ with the following conditions.

Definition 1. Let $0 \leq \gamma \leq 1$, $\tau \in \mathbb{C}, \{0\}$, and $0 \leq \alpha \leq 1$. A function $f \in \Sigma$ is in the class $\Sigma\mathcal{H}(\alpha, \tau, \gamma)(\mathcal{G})$ if the following subordinations hold:

$$\alpha + \frac{1}{\tau^\alpha} \left[(\alpha - \gamma) \frac{f(z)}{z} + \gamma f'(z) - \alpha \right] \prec \mathcal{G}(z), \quad (z \in \mathbb{U}), \quad (9)$$

$$\alpha + \frac{1}{\tau^\alpha} \left[(\alpha - \gamma) \frac{\mathcal{Q}(w)}{w} + \gamma \mathcal{H}'(w) - \alpha \right] \prec \mathcal{G}(w), \quad (w \in \mathbb{U}), \quad (10)$$

where $\mathcal{Q}(w) = f^{-1}(w)$.

Remark 1. If we set $\alpha = 1$ in $\Sigma\mathcal{H}(\alpha, \tau, \gamma)(\mathcal{G})$, the class interacts with the class introduced in [3]. Next, if we set $\alpha = \tau = 1, \gamma = 0$, and $\mathcal{G}(z) = (1 + \mathcal{L}z/1 + \mathcal{F}z)$, $(-1 \leq \mathcal{F} < \mathcal{L} \leq 1; z \in \mathbb{U})$ in Definition 1, we get the following class $\Sigma\mathcal{H}(\mathcal{L}, \mathcal{F})$.

Definition 2 (see [3]). A function $f \in \Sigma$ is in the class $\Sigma\mathcal{H}(\mathcal{L}, \mathcal{F})$, if the following subordinations hold:

$$\begin{aligned} \frac{f(z)}{z} &\prec \frac{1 + \mathcal{L}z}{1 + \mathcal{F}z}, \\ \frac{\mathcal{Q}(w)}{w} &\prec \frac{1 + \mathcal{L}w}{1 + \mathcal{F}w}, \quad (z, w \in \mathbb{U}). \end{aligned} \quad (11)$$

In fact, the class we introduced is mainly inspired by Bansal and Sokol [3], Kumar et al. [7], and [13, 33]. For example, for $\alpha = 0$ in $\Sigma\mathcal{H}(\alpha, \tau, \gamma)(\mathcal{G})$, the class interacts with [7]; for $\alpha = \tau = \gamma = 1$ in $\Sigma\mathcal{H}(\alpha, \tau, \gamma)(\mathcal{G})$, the class meets with [3]; and for $\alpha = \tau = 1$, $\mathcal{G}(z) = (1 + z/1 - z)^\eta$, $(0 \leq \eta < 1)$, the class joins to the function class given in [33].

In order to prove our main result, we need the following lemma.

Lemma 1 (see [17]). Let the function $b \in \mathcal{T}$ be given by the series

$$b(z) = 1 + h_1 z + h_2 z^2 + h_3 z^3 + \dots, \quad (z \in \mathbb{U}), \quad (12)$$

then the sharp estimate $|h_n| \leq 2$, $(n \in \mathbb{N})$, holds.

The two important functions that frequently appear in the literature of bi-univalent functions are considered by many authors for determining the initial coefficient bounds of certain class of bi-univalent functions. Functions $b_1(z)$ and $b_2(w)$, respectively, are defined by

$$b_1(z) = \frac{1 + s(z)}{1 - s(z)} = 1 + q_1 z + q_2 z^2 + q_3 z^3 + \dots + q_n z^n, \quad (13)$$

$$b_2(w) = \frac{1 + t(w)}{1 - t(w)} = 1 + v_1 w + v_2 w^2 + v_3 w^3 + \dots + v_n w^n, \quad (14)$$

where by Lemma 1, $\{|q_n| \text{ and } |v_n|\} \leq 2$.

3. Preliminary Results

In order to accomplish the intended formula, we introduce two polynomials and define them as $\mathfrak{R}$ - and $\mathfrak{X}$ - polynomials, respectively, as follows:

$$\begin{aligned} s(z) &= \mathfrak{R}_1 z + \mathfrak{R}_2 z^2 + \mathfrak{R}_3 z^3 + \dots \\ &= \sum_{n=1}^{\infty} \mathfrak{R}_n z^n, \quad (n \in \mathbb{N} = \{1, 2, 3, \dots\}), \end{aligned} \quad (15)$$

$$\begin{aligned} t(w) &= \mathfrak{X}_1 w + \mathfrak{X}_2 w^2 + \mathfrak{X}_3 w^3 + \dots \\ &= \sum_{n=1}^{\infty} \mathfrak{X}_n w^n, \quad (n \in \mathbb{N} = \{1, 2, 3, \dots\}), \end{aligned} \quad (16)$$

where in equation (15),

$$\begin{aligned} \mathfrak{R}_1 &= \frac{1}{2} q_1, \\ \mathfrak{R}_2 &= \frac{1}{2} (q_2 - \mathfrak{R}_1 q_1), \\ \mathfrak{R}_3 &= \frac{1}{2} (q_3 - \mathfrak{R}_1 q_2 - \mathfrak{R}_2 q_1), \\ \mathfrak{R}_4 &= \frac{1}{2} (q_4 - \mathfrak{R}_1 q_3 - \mathfrak{R}_2 q_2 - \mathfrak{R}_3 q_1), \\ \mathfrak{R}_5 &= \frac{1}{2} (q_5 - \mathfrak{R}_1 q_4 - \mathfrak{R}_2 q_3 - \mathfrak{R}_3 q_2 - \mathfrak{R}_4 q_1), \\ &\vdots \\ \mathfrak{R}_n &= \frac{1}{2} (q_n - \mathfrak{R}_1 q_{n-1} - \mathfrak{R}_2 q_{n-2} - \mathfrak{R}_3 q_{n-3} - \mathfrak{R}_4 q_{n-4} \\ &\quad - \dots - \mathfrak{R}_{n-1} q_1), \end{aligned} \quad (17)$$

and in equation (16),

$$\begin{aligned} \mathfrak{X}_1 &= \frac{1}{2} v_1, \\ \mathfrak{X}_2 &= \frac{1}{2} (v_2 - \mathfrak{X}_1 v_1), \\ \mathfrak{X}_3 &= \frac{1}{2} (v_3 - \mathfrak{X}_1 v_2 - \mathfrak{X}_2 v_1), \\ \mathfrak{X}_4 &= \frac{1}{2} (v_4 - \mathfrak{X}_1 v_3 - \mathfrak{X}_2 v_2 - \mathfrak{X}_3 v_1), \\ \mathfrak{X}_5 &= \frac{1}{2} (v_5 - \mathfrak{X}_1 v_4 - \mathfrak{X}_2 v_3 - \mathfrak{X}_3 v_2 - \mathfrak{X}_4 v_1), \\ &\vdots \\ \mathfrak{X}_n &= \frac{1}{2} (v_n - \mathfrak{X}_1 v_{n-1} - \mathfrak{X}_2 v_{n-2} - \mathfrak{X}_3 v_{n-3} - \mathfrak{X}_4 v_{n-4} \\ &\quad - \dots - \mathfrak{X}_{n-1} v_1). \end{aligned} \quad (18)$$

The triangular arrays in (17) and (18) lead us to form the general term of $\mathfrak{R}_n$ and $\mathfrak{X}_n$, respectively, as follows:

$$\mathfrak{R}_n = \frac{1}{2} \left(q_n - \sum_{j=1}^{n-1} \mathfrak{R}_j q_{n-j} \right), \quad (n \in \mathbb{N} = \{1, 2, 3, \dots\}), \quad (19)$$

$$\mathfrak{X}_n = \frac{1}{2} \left(v_n - \sum_{j=1}^{n-1} \mathfrak{X}_j v_{n-j} \right), \quad (n \in \mathbb{N} = \{1, 2, 3, \dots\}). \quad (20)$$

Therefore, equation (15) becomes

$$s(z) = \frac{1}{2} \left(q_n - \sum_{j=1}^{n-1} \mathfrak{R}_j q_{n-j} \right) z^n, \quad (n \in \mathbb{N} = \{1, 2, 3, \dots\}), \quad (21)$$

and equation (16) yields $t(w)$:

$$t(w) = \frac{1}{2} \sum_{n=1}^{\infty} \left(v_n - \sum_{j=1}^{n-1} \mathfrak{X}_j v_{n-j} \right) w^n, \quad (n \in \mathbb{N} = \{1, 2, 3, \dots\}). \quad (22)$$

Now, using the above series of expressions in (17) and (18), we enlist below some of the terms such as $\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3, \mathfrak{R}_4, \mathfrak{R}_5, \mathfrak{R}_6$, etc. and $\mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_3, \mathfrak{X}_4, \mathfrak{X}_5, \mathfrak{X}_6$, etc., respectively:

$$\begin{aligned} \mathfrak{R}_1 &= \frac{1}{2} q_1, \\ \mathfrak{R}_2 &= \frac{1}{2} \left[q_2 - \frac{1}{2} q_1^2 \right], \\ \mathfrak{R}_3 &= \frac{1}{2} \left[\left(q_3 - \frac{1}{2} q_1 q_2 \right) - \frac{1}{2} q_1 \left(q_2 - \frac{1}{2} q_1^2 \right) \right], \\ \mathfrak{R}_4 &= \frac{1}{2} \left[\left(q_4 - \frac{1}{2} q_1 q_3 \right) - \frac{1}{2} q_2 \left(q_2 - \frac{1}{2} q_1^2 \right) - \frac{1}{2} q_1 \left\{ \left(q_3 - \frac{1}{2} q_1 q_2 \right) - \frac{1}{2} q_1 \left(q_2 - \frac{1}{2} q_1^2 \right) \right\} \right], \\ \mathfrak{R}_5 &= \frac{1}{2} \left[\left(q_5 - \frac{1}{2} q_1 q_4 \right) - \frac{1}{2} q_3 \left(q_2 - \frac{1}{2} q_1^2 \right) - \frac{1}{2} q_2 \left\{ \left(q_3 - \frac{1}{2} q_1 q_2 \right) - \frac{1}{2} q_1 \left(q_2 - \frac{1}{2} q_1^2 \right) \right\} \right. \\ &\quad \left. - \frac{1}{2} q_1 \left\{ \left(q_4 - \frac{1}{2} q_1 q_3 \right) - \frac{1}{2} q_2 \left(q_2 - \frac{1}{2} q_1^2 \right) - \frac{1}{2} q_1 \left(\left(q_3 - \frac{1}{2} q_1 q_2 \right) - \frac{1}{2} q_1 \left(q_2 - \frac{1}{2} q_1^2 \right) \right) \right\} \right], \\ \mathfrak{R}_6 &= \frac{1}{2} \left[\left(q_6 - \frac{1}{2} q_1 q_5 \right) - \frac{1}{2} q_4 \left(q_2 - \frac{1}{2} q_1^2 \right) - \frac{1}{2} q_3 \left\{ \left(q_3 - \frac{1}{2} q_1 q_2 \right) - \frac{1}{2} q_1 \left(q_2 - \frac{1}{2} q_1^2 \right) \right\} \right. \\ &\quad \left. - \frac{1}{2} q_2 \left\{ \left(q_4 - \frac{1}{2} q_1 q_3 \right) - \frac{1}{2} q_2 \left(q_2 - \frac{1}{2} q_1^2 \right) - \frac{1}{2} q_1 \left(\left(q_3 - \frac{1}{2} q_1 q_2 \right) - \frac{1}{2} q_1 \left(q_2 - \frac{1}{2} q_1^2 \right) \right) \right\} \right. \\ &\quad \left. - \frac{1}{2} q_1 \left\{ \left(q_5 - \frac{1}{2} q_1 q_4 \right) - \frac{1}{2} q_3 \left(q_2 - \frac{1}{2} q_1^2 \right) - \frac{1}{2} q_2 \left(\left(q_3 - \frac{1}{2} q_1 q_2 \right) - \frac{1}{2} q_1 \left(q_2 - \frac{1}{2} q_1^2 \right) \right) \right\} \right]. \end{aligned} \quad (23)$$

and these expressions go on.

Upon using equation (18), we obtain the following list of

$\mathfrak{X}_n$: $n \in \mathbb{N} = \{1, 2, 3, \dots\}$:

$$\begin{aligned}
\mathfrak{X}_1 &= \frac{1}{2}v_1, \\
\mathfrak{X}_2 &= \frac{1}{2}\left[v_2 - \frac{1}{2}v_1^2\right], \\
\mathfrak{X}_3 &= \frac{1}{2}\left[\left(v_3 - \frac{1}{2}v_1v_2\right) - \frac{1}{2}v_1\left(v_2 - \frac{1}{2}v_1^2\right)\right], \\
\mathfrak{X}_4 &= \frac{1}{2}\left[\left(v_4 - \frac{1}{2}v_1v_3\right) - \frac{1}{2}v_2\left(v_2 - \frac{1}{2}v_1^2\right) - \frac{1}{2}v_1\left\{\left(v_3 - \frac{1}{2}v_1v_2\right) - \frac{1}{2}v_1\left(v_2 - \frac{1}{2}v_1^2\right)\right\}\right], \\
\mathfrak{X}_5 &= \frac{1}{2}\left[\left(v_5 - \frac{1}{2}v_1v_4\right) - \frac{1}{2}v_3\left(v_2 - \frac{1}{2}v_1^2\right) - \frac{1}{2}v_2\left\{\left(v_3 - \frac{1}{2}v_1v_2\right) - \frac{1}{2}v_1\left(v_2 - \frac{1}{2}v_1^2\right)\right\}\right. \\
&\quad \left.- \frac{1}{2}v_1\left\{\left(v_4 - \frac{1}{2}v_1v_3\right) - \frac{1}{2}v_2\left(v_2 - \frac{1}{2}v_1^2\right) - \frac{1}{2}v_1\left(\left(v_3 - \frac{1}{2}v_1v_2\right) - \frac{1}{2}v_1\left(v_2 - \frac{1}{2}v_1^2\right)\right)\right\}\right], \\
\mathfrak{X}_6 &= \frac{1}{2}\left[\left(v_6 - \frac{1}{2}v_1v_5\right) - \frac{1}{2}v_4\left(v_2 - \frac{1}{2}v_1^2\right) - \frac{1}{2}v_3\left\{\left(v_3 - \frac{1}{2}v_1v_2\right) - \frac{1}{2}v_1\left(v_2 - \frac{1}{2}v_1^2\right)\right\}\right. \\
&\quad \left.- \frac{1}{2}v_2\left\{\left(v_4 - \frac{1}{2}v_1v_3\right) - \frac{1}{2}v_2\left(v_2 - \frac{1}{2}v_1^2\right) - \frac{1}{2}v_1\left(\left(v_3 - \frac{1}{2}v_1v_2\right) - \frac{1}{2}v_1\left(v_2 - \frac{1}{2}v_1^2\right)\right)\right\}\right. \\
&\quad \left.- \frac{1}{2}v_1\left\{\left(v_5 - \frac{1}{2}v_1v_4\right) - \frac{1}{2}v_3\left(v_2 - \frac{1}{2}v_1^2\right) - \frac{1}{2}v_2\left(\left(v_3 - \frac{1}{2}v_1v_2\right) - \frac{1}{2}v_1\left(v_2 - \frac{1}{2}v_1^2\right)\right)\right\}\right],
\end{aligned} \tag{24}$$

and the expressions go on.

Note 1. The following notations will be used in coming sections:

- (1) ${}^{\mathfrak{u}}\mathfrak{R}_{t_i}$: the set of all possible $\mathfrak{R}'$ s for $t_i = \{1, 2, 3, \dots\}$
- (2) ${}^{\mathfrak{u}}\mathfrak{R}_{t_i}$: list of usable $\mathfrak{R}'$ s
- (3) ${}^nB_{t_i}$: list of all possible B' s, where B_{t_k} is chosen from (8)
- (4) ${}^pB_{t_i}$: number of previous B' s for $t_i = \{1, 2, 3, \dots\}$
- (5) $\binom{t_k}{m}$: m , shows the degree of each combination of ${}^{\mathfrak{u}}\mathfrak{R}_{t_i}$ and t_k , represents the number of ${}^{\mathfrak{u}}\mathfrak{R}_{t_i}$ in a single combination
- (6) $B_{t_k}\{{}^{\mathfrak{u}}\mathfrak{R}_{t_i}\}$: the multipliers of B_{t_k} in terms of ${}^{\mathfrak{u}}\mathfrak{R}_{t_i}$.

Remark 2. The relation to determine ${}^{\mathfrak{u}}\mathfrak{R}_{t_i}$, associated with B_{t_k} , is given by

$${}^{\mathfrak{u}}\mathfrak{R}_{t_i} = n - {}^pB_{t_i}, \tag{25}$$

where n is taken from the coefficient of z^n : $n = \{1, 2, 3, \dots\}$. For example, referring to (6) in Note 1, if we need to find $B_3\{{}^{\mathfrak{u}}\mathfrak{R}_{t_i}\}$ for the coefficient of z^4 , then ${}^pB_3 = \{B_1, B_2\} = 2B'$ s. This implies $\{{}^{\mathfrak{u}}\mathfrak{R}_{t_i}\} = 4 - 2 = \{\mathfrak{R}_1, \mathfrak{R}_2\}$. Therefore, for the coefficient of z^4 , the multipliers of $B_3 = B_3\{\mathfrak{R}_1, \mathfrak{R}_2\}$.

4. Main Results

The study of bi-univalent functions shows that most authors use the comparison method between their newly introduced

bi-univalent classes and the analytic functions under certain conditions to estimate the coefficient bounds for their function classes. The same type of studies can be found in [3, 7, 10] and many others. The main crux to estimate the n -th coefficient bounds of such bi-univalent functions lies in the generalization of the two analytic functions defined in (13) and (14) and the generalization of the class of certain bi-univalent functions. Hence, we present the generalization of two analytic functions that assist in structuring the formula for estimating the n -th coefficient bound for certain class of bi-univalent functions.

4.1. Analytic Functions Correlative to Bi-Univalent Functions

Theorem 1. If $\{t_i, t_j, t_k, (t_k - n)\} > 0$ and the function $b_1(z) = (1 + s(z)/1 - s(z))$ is analytic in the unit disk, then for a certain bi-univalent function, the comparative coefficients of $\{z^n: n \in \mathbb{N}, z \in \mathbb{C}\}$ may be represented by the following expansions:

$$\begin{aligned}
\frac{(\alpha + n\gamma)}{\tau^\alpha} a_{n+1} &= \sum_{i=1}^n B_{t_k} \left(\sum_{t_i=1}^{t_k} \prod_{t_i=1}^{t_k-1} \mathfrak{R}_{t_i}^{t_k-1} \mathfrak{R}_{n-(t_k-1)t_i} \right. \\
&\quad \left. + \sum_{t_i \neq t_j}^{t_k-n} \mathfrak{R}_{t_i}^{t_k-2} \mathfrak{R}_{t_j} \mathfrak{R}_{n-(t_k-2)t_i-t_j} \right) \binom{t_k}{m}.
\end{aligned} \tag{26}$$

Proof. The given function $b_1(z) = (1 + s(z)/1 - s(z))$ is analytic in $\mathbb{U}$ as $b_1(0) = 1$, and thus, function b_1 has the following Taylor–Maclaurin series expansion:

$$b_1(z) = \frac{1+s(z)}{1-s(z)} = 1 + q_1z + q_2z^2 + q_3z^3 + \dots \quad (27)$$

Since $s: \mathbb{U} \longrightarrow \mathbb{U}$, the analytic function b_1 has a positive real part on $\mathbb{U}$, and in view of Lemma 1, we have $|q_n| \leq 2: n \in \mathbb{N}$. Hence, solving (27) for $s(z)$, we get

$$s(z) = \frac{b_1(z) - 1}{b_1(z) + 1} = \frac{q_1z + q_2z^2 + \dots + q_nz^n}{2 + (q_1z + q_2z^2 + \dots + q_nz^n)} = \Re_1z + \Re_2z^2 + \dots + \Re_nz^n. \quad (28)$$

Now, utilizing equations (8) and (9) and equations (27) and (28), we obtain

$$\begin{aligned} \alpha + \frac{1}{\tau^\alpha} \left\{ (\alpha - \gamma) \frac{f(z)}{z} + \gamma f'(z) - \alpha \right\} &= \mathcal{G} \left(\frac{b_1(z) - 1}{b_1(z) + 1} \right) = 1 + B_1 \Re_1 z + (B_1 \Re_2 + B_2 \Re_1^2) z^2 + (B_1 \Re_3 + 2B_2 \Re_1 \Re_2 + B_3 \Re_1^3) z^3 \\ &+ (B_1 \Re_4 + B_2 (\Re_2^2 + 2\Re_1 \Re_3) + 3B_3 \Re_1^2 \Re_2 + B_4 \Re_1^4) z^4 + \left(\frac{B_1 \Re_5 + 2B_2 (\Re_2 \Re_3 + \Re_1 \Re_4)}{+3B_3 (\Re_1 \Re_2^2 + \Re_1^2 \Re_3) + 4B_4 \Re_1^3 \Re_2 + B_5 \Re_1^5} \right) z^5 + \dots \end{aligned} \quad (29)$$

Since a function f belonging to the class of bi-univalent functions is in the form of (1), a calculation shows that the

$f^{-1}(z) = \mathcal{Q}(w)$ has similar expansion as in expression (3). So, by using (1) in (29), we obtain

$$\begin{aligned} \alpha + \frac{2\alpha}{\tau^\alpha} + \left\{ \frac{\alpha + \gamma}{\tau^\alpha} a_2 z + \frac{\alpha + 2\gamma}{\tau^\alpha} a_3 z^2 + \dots + \frac{\alpha + n\gamma}{\tau^\alpha} a_{n+1} z^n \right\} \\ + (B_1 \Re_4 + B_2 (\Re_2^2 + 2\Re_1 \Re_3) + 3B_3 \Re_1^2 \Re_2 + B_4 \Re_1^4) z^4 + \left(\frac{B_1 \Re_5 + 2B_2 (\Re_2 \Re_3 + \Re_1 \Re_4)}{+3B_3 (\Re_1 \Re_2^2 + \Re_1^2 \Re_3) + 4B_4 \Re_1^3 \Re_2 + B_5 \Re_1^5} \right) z^5 + \dots \end{aligned} \quad (30)$$

Remark 3. In the above equation, the left-hand side is the generalized form of the class $\Sigma \mathcal{H}(\alpha, \tau, \gamma)(\mathcal{G})$, while the right-hand side is the newly inducted polynomial function associated with (8), (27), and (28).

Remark 4. The polynomial on the right-hand side of (30) is extendable to z^n and thus needs to determine the coefficient of z^n .

Now, by comparing the coefficients of $z, z^2, z^3, \dots, z^n$, we have the following expressions:

□

$$\begin{aligned}
\frac{(\alpha + \gamma)}{\tau^\alpha} a_2 &= B_1 \mathfrak{R}_1, \\
\frac{(\alpha + 2\gamma)}{\tau^\alpha} a_3 &= B_1 \mathfrak{R}_2 + B_2 \mathfrak{R}_1^2, \\
\frac{(\alpha + 3\gamma)}{\tau^\alpha} a_4 &= B_1 \mathfrak{R}_3 + 2B_2 \mathfrak{R}_1 \mathfrak{R}_2 + B_3 \mathfrak{R}_1^3, \\
\frac{(\alpha + 4\gamma)}{\tau^\alpha} a_5 &= B_1 \mathfrak{R}_4 + B_2 (\mathfrak{R}_2^2 + 2\mathfrak{R}_1 \mathfrak{R}_3) + 3B_3 \mathfrak{R}_1^2 \mathfrak{R}_2 + B_4 \mathfrak{R}_1^4, \\
\frac{(\alpha + 5\gamma)}{\tau^\alpha} a_6 &= B_1 \mathfrak{R}_5 + 2B_2 (\mathfrak{R}_2 \mathfrak{R}_3 + \mathfrak{R}_1 \mathfrak{R}_4) + 3B_3 (\mathfrak{R}_1 \mathfrak{R}_2^2 + \mathfrak{R}_1^2 \mathfrak{R}_3) + 4B_4 \mathfrak{R}_1^3 \mathfrak{R}_2 + B_5 \mathfrak{R}_1^5, \\
\frac{(\alpha + 6\gamma)}{\tau^\alpha} a_7 &= B_1 \mathfrak{R}_6 + B_2 (\mathfrak{R}_3^2 + 2\mathfrak{R}_2 \mathfrak{R}_4 + 2\mathfrak{R}_1 \mathfrak{R}_5) + B_3 (\mathfrak{R}_2^3 + 3\mathfrak{R}_1 \mathfrak{R}_2 \mathfrak{R}_3 + 3\mathfrak{R}_1^2 \mathfrak{R}_4) \\
&\quad + B_4 (6\mathfrak{R}_1^2 \mathfrak{R}_2^2 + 4\mathfrak{R}_1^3 \mathfrak{R}_3) + 5B_5 \mathfrak{R}_1^4 \mathfrak{R}_2 + B_6 \mathfrak{R}_1^6 \\
&\quad \vdots = \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \\
\frac{(\alpha + n\gamma)}{\tau^\alpha} a_{n+1} &= B_1 \mathfrak{R}_n + \sum_{i=2}^{n-1} B_{t_k} \left(\sum_{t_i=1}^{t_k} \mathfrak{R}_{t_i}^{t_k-1} \mathfrak{R}_{n-(t_k-1)t_i} + \sum_{t_i \neq t_j}^{t_k-n} \mathfrak{R}_{t_i}^{t_k-2} \mathfrak{R}_{t_j} \mathfrak{R}_{n-(t_k-2)t_i-t_j} \right) \binom{t_k}{m} + B_n \mathfrak{R}_1^n.
\end{aligned} \tag{31}$$

Thus, by iteration, we deduce a general formula that relates (8) and (9) with $\mathfrak{R}$ -polynomial defined in (15). So the

acquired formula that finds the series of expressions asserted in (31) is given as follows:

$$\frac{(\alpha + n\gamma)}{\tau^\alpha} a_{n+1} = \sum_{i=1}^n B_{t_k} \left(\sum_{t_i=1}^{t_k} \mathfrak{R}_{t_i}^{t_k-1} \mathfrak{R}_{n-(t_k-1)t_i} + \sum_{t_i \neq t_j}^{t_k-n} \mathfrak{R}_{t_i}^{t_k-2} \mathfrak{R}_{t_j} \mathfrak{R}_{n-(t_k-2)t_i-t_j} \right) \binom{t_k}{m}. \tag{32}$$

This proves our assertion in (26).

Note 2. Observe that the above equation consists of two parts in which the first part of t_i takes single values, while the second part takes simultaneously two different values such that $t_i \neq t_j$ for $i \neq j$. In the case of $\{t_i, t_j, t_k, (t_k - n)\} < 0$, the whole term is omitted. Also, $m \in \mathbb{N}$ shows the degree of each combination of $(\mathfrak{R}_{t_i}: t_i = \{1, 2, 3, \dots\})$ such that the values

of $\mathfrak{R}_n$ are chosen from expression (17) or from (19). For numerical explanation of (32), see Example 1 in Section 4.

Theorem 2. If $\{t_i, t_j, t_k, (t_k - n)\} > 0$ and the function $b_2(w) = (1 + t(w))/1 - t(w)$ is analytic in the unit disk, then for a certain bi-univalent function, the comparative coefficients of $\{w^n: n \in \mathbb{N}, w \in \mathbb{C}\}$ may be represented by the following expansions:

$$\frac{(\alpha + n\gamma)}{\tau^\alpha} \frac{K_n^{-(n+1)}}{n+1} = \sum_{i=1}^n B_{t_k} \left(\sum_{t_i=1}^{t_k} \mathfrak{X}_{t_i}^{t_k-1} \mathfrak{X}_{n-(t_k-1)t_i} + \sum_{t_i \neq t_j}^{t_k-n} \mathfrak{X}_{t_i}^{t_k-2} \mathfrak{X}_{t_j} \mathfrak{X}_{n-(t_k-2)t_i-t_j} \right) \binom{t_k}{m}. \tag{33}$$

Proof. The given function $b_2(w) = (1 + t(w))/1 - t(w)$ is analytic in $\mathbb{U}$ as $b_2(0) = 1$, and thus, function $b_2(w)$ has the following Taylor–Maclaurin series expansion:

$$b_2(w) = \frac{1 + t(w)}{1 - t(w)} = 1 + v_1 w + v_2 w^2 + v_3 w^3 + \dots \tag{34}$$

Since $t: \mathbb{U} \longrightarrow \mathbb{U}$, the analytic function b_2 has a positive real part on $\mathbb{U}$, and in the view of Lemma 1, we have $|v_n| \leq 2: n \in \mathbb{N}$. Thus, solving (34) for $t(w)$, we obtain

$$\begin{aligned}
t(w) &= \frac{b_2(w) - 1}{b_2(w) + 1} = \frac{v_1 w + v_2 w^2 + \dots + v_n w^n}{2 + (v_1 w + v_2 w^2 + \dots + v_n w^n)} \\
&= \mathfrak{X}_1 w + \mathfrak{X}_2 w^2 + \dots + \mathfrak{X}_n w^n,
\end{aligned} \tag{35}$$

so, from equations (8) and (10) and equations (34) and (35), we get

$$\begin{aligned} \alpha + \frac{1}{\tau^\alpha} \left[(\alpha - \gamma) \frac{\mathcal{Q}(w)}{w} + \gamma \mathcal{H}'(w) - \alpha \right] &= \mathcal{G} \left(\frac{b_2(w) - 1}{b_2(w) + 1} \right) = 1 + B_1 \mathfrak{x}_1 w + (B_1 \mathfrak{x}_2 + B_2 \mathfrak{x}_1^2) w^2 + (B_1 \mathfrak{x}_3 + 2B_2 \mathfrak{x}_1 \mathfrak{x}_2 + B_3 \mathfrak{x}_1^3) w^3 \\ &\quad + (B_1 \mathfrak{x}_4 + B_2 (\mathfrak{x}_2^2 + 2\mathfrak{x}_1 \mathfrak{x}_3) + 3B_3 \mathfrak{x}_1^2 \mathfrak{x}_2 + B_4 \mathfrak{x}_1^4) w^4 \\ &\quad + \left(\begin{array}{c} B_1 \mathfrak{x}_5 + 2B_2 (\mathfrak{x}_2 \mathfrak{x}_3 + \mathfrak{x}_1 \mathfrak{x}_4) \\ + 3B_3 (\mathfrak{x}_1 \mathfrak{x}_2^2 + \mathfrak{x}_1^2 \mathfrak{x}_3 + 4B_4 \mathfrak{x}_1^3 \mathfrak{x}_2 + B_5 \mathfrak{x}_1^5) \end{array} \right) w^5 + \dots \end{aligned} \quad (36)$$

Since a function f belonging to the class of bi-univalent functions has the Maclaurin series given by (1), a calculation

shows that the inverse of $\mathcal{Q}(w) = f^{-1}(w)$ has similar expansion as in (3).

Now, by using (2), (5), and (7) in (36), we get

$$\begin{aligned} \alpha + \left\{ \frac{\alpha + \gamma}{\tau^\alpha} \frac{K_1^{-2}}{2} w + \frac{\alpha + 2\gamma}{\tau^\alpha} \frac{K_2^{-3}}{3} w^2 + \dots + \frac{\alpha + n\gamma}{\tau^\alpha} \frac{K_n^{-(n+1)}}{n+1} w^n \right\} &= 1 + B_1 \mathfrak{x}_1 w \\ &\quad + (B_1 \mathfrak{x}_2 + B_2 \mathfrak{x}_1^2) w^2 + (B_1 \mathfrak{x}_3 + 2B_2 \mathfrak{x}_1 \mathfrak{x}_2 + B_3 \mathfrak{x}_1^3) w^3 + (B_1 \mathfrak{x}_4 + B_2 (\mathfrak{x}_2^2 + 2\mathfrak{x}_1 \mathfrak{x}_3) + 3B_3 \mathfrak{x}_1^2 \mathfrak{x}_2 + B_4 \mathfrak{x}_1^4) w^4 \\ &\quad + (B_1 \mathfrak{x}_5 + 2B_2 (\mathfrak{x}_2 \mathfrak{x}_3 + \mathfrak{x}_1 \mathfrak{x}_4) + 3B_3 (\mathfrak{x}_1 \mathfrak{x}_2^2 + \mathfrak{x}_1^2 \mathfrak{x}_3) + 4B_4 \mathfrak{x}_1^3 \mathfrak{x}_2 + B_5 \mathfrak{x}_1^5) w^5 + \dots \end{aligned} \quad (37)$$

Thus, by comparing the coefficients of $w, w^2, w^3, \dots, w^n$, we have

$$\begin{aligned} \frac{(\alpha + \gamma)}{\tau^\alpha} \frac{K_1^{-2}}{2} &= B_1 \mathfrak{x}_1, \quad \frac{(\alpha + 2\gamma)}{\tau^\alpha} \frac{K_2^{-3}}{3} = B_1 \mathfrak{x}_2 + B_2 \mathfrak{x}_1^2, \\ \frac{(\alpha + 3\gamma)}{\tau^\alpha} \frac{K_3^{-4}}{4} &= B_1 \mathfrak{x}_3 + 2B_2 \mathfrak{x}_1 \mathfrak{x}_2 + B_3 \mathfrak{x}_1^3, \\ \frac{(\alpha + 4\gamma)}{\tau^\alpha} \frac{K_4^{-5}}{5} &= B_1 \mathfrak{x}_4 + B_2 (\mathfrak{x}_2^2 + 2\mathfrak{x}_1 \mathfrak{x}_3) + 3B_3 \mathfrak{x}_1^2 \mathfrak{x}_2 + B_4 \mathfrak{x}_1^4, \\ \frac{(\alpha + 5\gamma)}{\tau^\alpha} \frac{K_5^{-6}}{6} &= B_1 \mathfrak{x}_5 + 2B_2 (\mathfrak{x}_2 \mathfrak{x}_3 + \mathfrak{x}_1 \mathfrak{x}_4) + 3B_3 (\mathfrak{x}_1 \mathfrak{x}_2^2 + \mathfrak{x}_1^2 \mathfrak{x}_3) + 4B_4 \mathfrak{x}_1^3 \mathfrak{x}_2 + B_5 \mathfrak{x}_1^5, \\ \frac{(\alpha + 6\gamma)}{\tau^\alpha} \frac{K_6^{-7}}{7} &= B_1 \mathfrak{x}_6 + B_2 (\mathfrak{x}_3^2 + 2\mathfrak{x}_2 \mathfrak{x}_4 + 2\mathfrak{x}_1 \mathfrak{x}_5) + B_3 (\mathfrak{x}_2^3 + 6\mathfrak{x}_1 \mathfrak{x}_2 \mathfrak{x}_3 + 3\mathfrak{x}_1^2 \mathfrak{x}_4) \\ &\quad + B_4 (6\mathfrak{x}_1^2 \mathfrak{x}_2^2 + 4\mathfrak{x}_1^3 \mathfrak{x}_3) + 5B_5 \mathfrak{x}_1^4 \mathfrak{x}_2 + B_6 \mathfrak{x}_1^6 \\ &\quad \vdots \quad = \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \\ \frac{(\alpha + n\gamma)}{\tau^\alpha} \frac{K_n^{-(n+1)}}{n+1} &= B_1 \mathfrak{x}_n + \sum_{i=2}^{n-1} B_{t_k} \left(\sum_{t_i=1}^{t_k} \prod_{t_i=1}^{t_k-1} \mathfrak{x}_{t_i}^{t_k-1} \mathfrak{x}_{n-(t_k-1)t_i} + \sum_{t_i \neq t_j} \prod_{t_i=1}^{t_k-n} \mathfrak{x}_{t_i}^{t_k-2} \mathfrak{x}_{t_j} \mathfrak{x}_{n-(t_k-2)t_i-t_j} \right) \binom{t_k}{m} + B_n \mathfrak{x}_1^n. \end{aligned} \quad (38)$$

Similar to (32), we obtain the following formula that relates (8) and (10) with $\mathfrak{X}$ -polynomial defined in (16):

$$\frac{(\alpha + n\gamma)}{\tau^\alpha} \frac{K_n^{-(n+1)}}{n+1} = \sum_{i=1}^n B_{t_k} \left(\sum_{t_i=1}^{t_k} \mathfrak{X}_{t_i}^{t_k-1} \mathfrak{X}_{n-(t_k-1)t_i} + \sum_{t_i \neq t_j}^{t_k-n} \mathfrak{X}_{t_i}^{t_k-2} \mathfrak{X}_{t_j} \mathfrak{X}_{n-(t_k-2)t_i-t_j} \right) \binom{t_k}{m}, \quad (39)$$

where m represents the degree of each combination of ${}^{\mathfrak{U}}\mathfrak{X}_{t_i}$: $t_i = \{1, 2, 3, \dots\}$ such that the values of $(\mathfrak{X}_n: n \in \mathbb{N} = \{1, 2, 3, \dots\})$ are chosen from expression (18) and values of $((K_n^{-(n+1)}/n+1): n \in \mathbb{N} = \{1, 2, 3, \dots\})$ are obtained from (7) or in general from (6). This statement completes the proof. $\square$

4.2. Applicability of $\mathfrak{R}$ - and $\mathfrak{X}$ -Polynomials. In this section, we employ $\mathfrak{R}$ - and $\mathfrak{X}$ -polynomials on our class stated in Definition 2. Here, the aim is to test our newly constructed relation defined in (30) on our class $\Sigma\mathcal{H}(\alpha, \tau, \gamma)(\mathcal{G})$ in order to obtain the initial coefficient estimation.

Theorem 3. Let $f(z) \in \Sigma\mathcal{H}(\alpha, \tau, \gamma)(\mathcal{G})$ be of the form (2), then

$$|a_2| \leq \frac{|\tau^\alpha| B_1^{(3/2)}}{\sqrt{|\tau^\alpha B_1^2(\alpha + 2\gamma) + (\alpha + \gamma)^2(B_1 - B_2)|}}, \quad (40)$$

$$|a_3| \leq B_1 |\tau^\alpha| \left\{ \frac{|\tau^\alpha| B_1}{(\alpha + \gamma)^2} + \frac{1}{(\alpha + 2\gamma)} \right\}. \quad (41)$$

Proof. Let $f \in \Sigma\mathcal{H}(\alpha, \tau, \gamma)(\mathcal{G})$ and $\mathcal{Q}(w) = f^{-1}(z)$, then there exist analytic functions $s, t: \mathbb{U} \longrightarrow \mathbb{U}$ with $s(0) = t(0) = 0$, satisfying

$$\alpha + \frac{1}{\tau^\alpha} \left[(\alpha - \gamma) \frac{f(z)}{z} + \gamma f'(z) - \alpha \right] < \mathcal{G}(s(z)), \quad (42)$$

$$\alpha + \frac{1}{\tau^\alpha} \left[(\alpha - \gamma) \frac{\mathcal{Q}(w)}{w} + \gamma \mathcal{H}'(w) - \alpha \right] < \mathcal{G}(t(w)).$$

Approaching towards the estimation of a_2 , we compare the coefficients of z . For this purpose, utilizing the formula in (32) by putting $n = 1$, we get

$$\begin{aligned} {}^n B_{t_i} &= \{B_1\}, \\ {}^{\mathfrak{U}}\mathfrak{R}_{t_i} &= \{\mathfrak{R}_1\}, \\ {}^{\mathfrak{U}}\mathfrak{R}_{t_i} = n - {}^P B_{t_i} &\Rightarrow 1 - 0 = 1 = \{\mathfrak{R}_1\}. \end{aligned} \quad (43)$$

$\square$

Note 3. In the case of singleton $\{\mathfrak{R}_1\}$, the second part of the equation (32) is no more usable as it takes simultaneously two different values (see Note 1).

Therefore, by putting $n = 1$ in (32), we obtain

$$\frac{(\alpha + 1\gamma)}{\tau^\alpha} a_{1+1} = \sum_{i=1}^1 B_1 \left(\sum_{t_i=1}^1 \mathfrak{R}_{t_i}^{1-1} \mathfrak{R}_{1-(1-1)t_i} \right) \binom{1}{m} = B_1(\mathfrak{R}_1) \binom{1}{1}, \quad (44)$$

$$\frac{(\alpha + \gamma)}{\tau^\alpha} a_2 = B_1 \mathfrak{R}_1.$$

For the next corresponding expression, utilizing the formula in (39) by putting $n = 1$, we get

$$\begin{aligned} {}^n B_{t_i} &= \{B_1\}, \\ {}^{\mathfrak{U}}\mathfrak{X}_{t_i} &= \{\mathfrak{X}_1\}, \\ {}^{\mathfrak{U}}\mathfrak{X}_{t_i} = n - {}^P B_{t_i} &\Rightarrow 1 - 0 = 1 = \{\mathfrak{X}_1\}. \end{aligned} \quad (45)$$

Once again in view of Note 3, the second part of the formula in (39) is dropped as it needs simultaneously two values (see Note 1). Thus, putting $n = 1$ in (39), we obtain

$$\frac{(\alpha + 1\gamma)}{\tau^\alpha} \frac{K_1^{-(1+1)}}{1+1} = \sum_{i=1}^1 B_1 \left(\sum_{t_i=1}^1 \mathfrak{X}_{t_i}^{1-1} \mathfrak{X}_{1-(1-1)t_i} \right) \binom{1}{m} = B_1(\mathfrak{X}_1) \binom{1}{1}, \quad (46)$$

$$\frac{(\alpha + \gamma)}{\tau^\alpha} \frac{K_1^{-2}}{2} = B_1 \mathfrak{X}_1,$$

such that the value of $(K_1^{-2}/2)$ is provided by the Faber polynomial expansions given in (7) or could be calculated from (6).

Remark 5. Note that, by comparing (44), (46), and (7) together with (23) and (24), we get $\mathfrak{R}_1 = -\mathfrak{X}_1$ that leads to $q_1 = -v_1$.

Now, by squaring (44), we obtain the following initial value of a_2^2 :

$$\begin{aligned} a_2^2 &= \frac{B_1^2 \tau^{2\alpha}}{(\alpha + \gamma)^2} \mathfrak{R}_1^2 = \frac{B_1^2 \tau^{2\alpha}}{(\alpha + \gamma)^2} \mathfrak{X}_1^2, \\ a_2^2 &= \frac{1}{4} \frac{B_1^2 \tau^{2\alpha}}{(\alpha + \gamma)^2} v_1^2. \end{aligned} \quad (47)$$

Again, for the conclusive value of a_2^2 , we compare the coefficients of z^2 and using the formula in (32) by putting $n = 2$, we get

$$\begin{aligned} {}^{11}B_{t_k} &= \{B_1, B_2\}, \\ {}^{11}\mathfrak{R}_{t_i} &= \{\mathfrak{R}_1, \mathfrak{R}_2\}, \\ {}^{11}\mathfrak{R}_{t_i} &= n - {}^pB_{t_i} \Rightarrow 2 - 1 = 1 = \{\mathfrak{R}_1\}. \end{aligned} \quad (48)$$

Again, for a singleton $\{\mathfrak{R}_1\}$, the second part of the formula is no more valid (see Note 3). Thus, equation (32) reduces to

$$\begin{aligned} \frac{(\alpha + 2\gamma)}{\tau^\alpha} a_{2+1} &= \sum_{i=1}^2 B_{t_k} \left(\sum_{t_i=1}^{t_k} \mathfrak{R}_{t_i}^{t_k-1} \mathfrak{R}_{2-(t_k-1)t_i} \right) \binom{t_k}{m} = B_1(\mathfrak{R}_2) \binom{1}{m} + B_2(\mathfrak{R}_1^1 \mathfrak{R}_1) \binom{2}{m} \\ &= B_1 \mathfrak{R}_2 \binom{1}{1} + B_2 \mathfrak{R}_1^2 \binom{2}{2}, \\ \frac{(\alpha + 2\gamma)}{\tau^\alpha} a_3 &= B_1 \mathfrak{R}_2 + B_2 \mathfrak{R}_1^2. \end{aligned} \quad (49)$$

For the next expression, utilizing the formula in (39) by putting $n = 2$, we get

$$\frac{(\alpha + 2\gamma)}{\tau^\alpha} \frac{K_2^{-(2+1)}}{2+1} = B_1 \mathfrak{X}_2 \binom{1}{1} + B_2 \mathfrak{X}_1^2 \binom{2}{2}, \quad (50)$$

$$\frac{(\alpha + 2\gamma)}{\tau^\alpha} \frac{K_2^{-3}}{3} = B_1 \mathfrak{X}_2 + B_2 \mathfrak{X}_1^2,$$

where the value of $(K_2^{-3}/3)$ is given by the Faber polynomial expansion defined in (7) or could be calculated from (6).

Now, adding (49) and (50) and then using (17), (18), (23), and (24) together with the value of v_1^2 from (47), we conclude the following value of a_2^2 :

$$\begin{aligned} \frac{2(\alpha + 2\gamma)}{\tau^\alpha} a_2^2 &= B_1 \mathfrak{R}_2 + B_2 \mathfrak{R}_1^2 + B_1 \mathfrak{X}_2 + B_2 \mathfrak{X}_1^2 = B_1 \mathfrak{R}_2 + B_1 \mathfrak{X}_2 + 2B_2 \mathfrak{X}_1^2 = \frac{1}{2} B_1 (q_2 + v_2) - \frac{1}{2} v_1^2 (B_1 - B_2), \\ a_2^2 &= \frac{B_1^3 \tau^{2\alpha} [q_2 + v_2]}{4 \{B_1^2 \tau^\alpha (\alpha + 2\gamma) + (\alpha + \gamma)^2 (B_1 - B_2)\}}. \end{aligned} \quad (51)$$

Next, subtracting (49) from (50) and then using (17), (18), (23), and (24) together with the value of v_1^2 from (47), we obtain the following value of a_3 :

$$\begin{aligned} \frac{2(\alpha + 2\gamma)}{\tau^\alpha} (a_3 - a_2^2) &= B_1 \mathfrak{R}_2 + B_2 \mathfrak{R}_1^2 - B_1 \mathfrak{X}_2 - B_2 \mathfrak{X}_1^2 = B_1 (\mathfrak{R}_2 - \mathfrak{X}_2) = \frac{1}{2} B_1 (q_2 - v_2), \\ a_3 &= \frac{1}{4} B_1 \tau^\alpha \left\{ \frac{\tau^\alpha B_1}{(\alpha + \gamma)^2} v_1^2 + \frac{q_2 - v_2}{(\alpha + 2\gamma)} \right\}. \end{aligned} \quad (52)$$

So, applying Lemma 1 on (51) and (52), we obtain the proof asserted in (40) and (41).

Remark 6. If we set out the value of $\alpha = 1$, we receive Theorem 1 of [3].

Remark 7. If we put the value of $\alpha = 1$, $\gamma = 1$, and $\tau = 1$, we receive Theorem 1 of [33].

4.3. Certain Coefficient of z^n . In this subsection, we offer an example that demonstrates the calculation of the certain coefficient of z^n , and here, we take $n = 4$.

Example 1. How to determine the expression by comparing the coefficient of z^4 , with the corresponding class of bi-univalent function?

Manipulating (31) for $n = 4$, two extreme terms are $B_1\mathfrak{R}_4$ and $B_4\mathfrak{R}_1^4$, so we only work for finding the mean terms that is $B_2\{\mathfrak{R}_i\}$ and $B_3\{\mathfrak{R}_i\}$ (also refer to Remark 1):

$$\frac{(\alpha + 4\gamma)}{\tau^\alpha} a_5 = B_1\mathfrak{R}_4 + B_2\{\mathfrak{R}_{t_i}\} + B_3\{\mathfrak{R}_{t_j}\} + B_4\mathfrak{R}_1^4,$$

$$\frac{(\alpha + 4\gamma)}{\tau^\alpha} a_5 = B_1\mathfrak{R}_4 + B_2\{\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3\} + B_3\{\mathfrak{R}_1, \mathfrak{R}_2\} + B_4\mathfrak{R}_1^4. \quad (53)$$

The expression in (53) means that $|a_5|$ involves only $\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3$, and $\mathfrak{R}_4$, in its final estimation. So, on further exploration, we obtain the following expressions:

$$\begin{aligned} \frac{(\alpha + 4\gamma)}{\tau^\alpha} a_5 &= B_1\mathfrak{R}_4 + \sum_{i=2}^2 B_{t_k} \left(\sum_{t_i=1}^{t_k} \mathfrak{R}_{t_i}^{t_k-1} \mathfrak{R}_{4-(t_k-1)t_i} + \sum_{t_i \neq t_j}^{t_k-4} \mathfrak{R}_{t_i}^{t_k-2} \mathfrak{R}_{t_j} \mathfrak{R}_{4-(t_k-2)t_i-t_j} \right) \binom{t_k}{m} + B_4\mathfrak{R}_1^4 \\ &= B_1\mathfrak{R}_4 + B_2 \left(\sum_{t_i=1}^2 \mathfrak{R}_{t_i}^1 \mathfrak{R}_{4-t_i} + \sum_{t_i \neq t_j}^{-2} \mathfrak{R}_{t_i}^0 \mathfrak{R}_{t_j} \mathfrak{R}_{4-t_i-t_j} \right) \binom{2}{m} \\ &\quad + B_3 \left(\sum_{t_i=1}^3 \mathfrak{R}_{t_i}^2 \mathfrak{R}_{4-2t_i} + \sum_{t_i \neq t_j}^{-1} \mathfrak{R}_{t_i}^1 \mathfrak{R}_{t_j} \mathfrak{R}_{4-t_i-t_j} \right) \binom{3}{m} + B_4\mathfrak{R}_1^4. \end{aligned} \quad (54)$$

Note that some of the terms in above equation violates the conditions ($t_k < 0$) stated in Theorem 1. Hence, by neglecting those terms, we get

$$\begin{aligned} \frac{(\alpha + 4\gamma)}{\tau^\alpha} a_5 &= B_1\mathfrak{R}_4 + B_2(\mathfrak{R}_1^1 \mathfrak{R}_{4-1} + \mathfrak{R}_2^1 \mathfrak{R}_{4-2} + \text{neglect}) \binom{2}{m} + B_3(\mathfrak{R}_1^2 \mathfrak{R}_{4-2} + \mathfrak{R}_2^2 \mathfrak{R}_{4-2*2} + \mathfrak{R}_3^2 \mathfrak{R}_{4-2*3} + \text{neglect}) \binom{3}{m} \\ &\quad + B_4\mathfrak{R}_1^4 = B_1\mathfrak{R}_4 + B_2(\mathfrak{R}_1^1 \mathfrak{R}_3 + \mathfrak{R}_2^1 \mathfrak{R}_2) \binom{2}{m} + B_3(\mathfrak{R}_1^2 \mathfrak{R}_2) \binom{3}{m} + B_4\mathfrak{R}_1^4, \\ \frac{(\alpha + 4\gamma)}{\tau^\alpha} a_5 &= B_1\mathfrak{R}_4 + B_2(\mathfrak{R}_2^2 + 2\mathfrak{R}_1 \mathfrak{R}_3) + 3B_3\mathfrak{R}_1^2 \mathfrak{R}_2 + B_4\mathfrak{R}_1^4. \end{aligned} \quad (55)$$

Note 4. It is worth mentioning that n preserves its value in each product combination of $\mathfrak{R}_{t_i}$. For example, in equation (55), the multipliers of B_3 are given by $\mathfrak{R}_1^2 \mathfrak{R}_2$, where $n = 2 \times 1 + 1 \times 2 = 4$. Also, observe that some of the terms are omitted in Example 1 and see the conditions stated in Note 2.

5. Corollaries and Consequences

This section discusses some interesting corollaries and other results.

Firstly, if we set $\mathcal{G}(z) = (1 + \mathcal{L}z/1 + \mathcal{F}z)$, ($-1 \leq \mathcal{F} < \mathcal{L} \leq 1; z \in \mathbb{U}$), then equation (8) becomes

$$\begin{aligned}\mathcal{G}(z) &= \frac{1 + \mathcal{L}z}{1 + \mathcal{F}z} = 1 + (\mathcal{L} - \mathcal{F})\{z - \mathcal{F}z^2 + \mathcal{F}^2z^3 - \dots\}, \\ &= 1 + (\mathcal{L} - \mathcal{F}) \sum_{n=1}^{\infty} (-\mathcal{F})^{n-1} z^n.\end{aligned}\quad (56)$$

Then, we can write $B_n = (-\mathcal{F})^{n-1}(\mathcal{L} - \mathcal{F})z^n$, so by putting the comparing values of $B_1 = (\mathcal{L} - \mathcal{F})$ and $B_2 = -\mathcal{F}(\mathcal{L} - \mathcal{F})$ in Theorem 3, we obtain the following corollary.

Corollary 1. Let $f(z) \in \Sigma\mathcal{H}(\alpha, \tau, \gamma)(\mathcal{G})(1 + \mathcal{L}z/1 + \mathcal{F}z)$ be of the form (1), then

$$\begin{aligned}|a_2| &\leq \frac{|\tau^\alpha|(\mathcal{L} - \mathcal{F})}{\sqrt{|\tau^\alpha|(\mathcal{L} - \mathcal{F})(\alpha + 2\gamma) + (\alpha + \gamma)^2(1 + \mathcal{F})|}}, \\ |a_3| &\leq (\mathcal{L} - \mathcal{F})|\tau^\alpha| \left\{ \frac{|\tau^\alpha|(\mathcal{L} - \mathcal{F})}{(\alpha + \gamma)^2} + \frac{1}{(\alpha + 2\gamma)} \right\}.\end{aligned}\quad (57)$$

Remark 8. If we put the value of $\alpha = 1$, $\gamma = 1$, $\tau = 1$, and $\mathcal{L} = 1 - 2\mu$, ($0 \leq \mu < 1$) and $\mathcal{F} = -1$ in Corollary 1, we receive Theorem 2 of [13].

Remark 9. Setting $\alpha = 1$, $\tau = 1$, and $\gamma = 0$ in Corollary 1, we obtain Corollary 2.3 of [3].

Secondly, if we set $\mathcal{G}(z) = (1 + z/1 - z)^\eta$, ($0 \leq \eta < 1$, $z \in \mathbb{U}$), then equation (8) becomes

$$\begin{aligned}\mathcal{G}(z) &= \left(\frac{1+z}{1-z}\right)^\eta = \mathfrak{G}_0 + \mathfrak{G}_1 z + \mathfrak{G}_2 z^2 + \mathfrak{G}_3 z^3 \\ &\quad + \mathfrak{G}_4 z^4 + \dots + \mathfrak{G}_n z^n \\ &= 1 + 2\eta z + 2\eta^2 z^2 + \frac{2}{3}(2\eta^3 + \eta)z^3 + \frac{2}{3}\eta^2(2 + \eta^2)z^4 \\ &\quad + \mathcal{O}[z^n] = 1 + \sum_{n=1}^{\infty} \mathfrak{G}_n z^n,\end{aligned}\quad (58)$$

where $\mathcal{O}[z^n]$ represents the sign of “Big-O Notation” (for details, see [36]) and $\mathfrak{G}_n$ is the n -th coefficient for the above series that may be generalized in the following manner:

$$\begin{aligned}\mathfrak{G}_0 &= 1, \\ \mathfrak{G}_1 &= 2\eta, \\ \mathfrak{G}_2 &= 2\eta^2, \\ \mathfrak{G}_3 &= \frac{2}{3}\eta(1 + 2\eta^2), \\ \mathfrak{G}_4 &= \frac{2}{3}\eta^2(2 + \eta^2), \\ \mathfrak{G}_5 &= \frac{2}{15}\eta(3 + 10\eta^2 + 2\eta^4), \\ &\vdots \\ \mathfrak{G}_n &= \frac{1}{n}(2\eta\mathfrak{G}_{n-1} + (n-2)\mathfrak{G}_{n-2}).\end{aligned}\quad (59)$$

We then write (59), in a general term, $B_n = (1/n)(2\eta\mathfrak{G}_{n-1} + (n-2)\mathfrak{G}_{n-2})$, so by putting the comparing values of $B_1 = \mathfrak{G}_1 = 2\eta$ and $B_2 = \mathfrak{G}_2 = 2\eta^2$ in Theorem 3, we obtain the following corollary.

Corollary 2. Let $f(z) \in \Sigma\mathcal{H}(\alpha, \tau, \gamma)(\mathcal{G})(1 + z/1 - z)^\eta$ be of the form (1), then

$$\begin{aligned}|a_2| &\leq \frac{2|\tau^\alpha|\eta}{\sqrt{|2\eta\tau^\alpha(\alpha + 2\gamma) + (\alpha + \gamma)^2(1 - \eta)|}}, \\ |a_3| &\leq 2\eta|\tau^\alpha| \left\{ \frac{2|\tau^\alpha|\eta}{(\alpha + \gamma)^2} + \frac{1}{(\alpha + 2\gamma)} \right\}.\end{aligned}\quad (60)$$

Remark 10. If we put the value of $\alpha = 1$, $\gamma = 1$, and $\tau = 1$ in Corollary 2, we receive Theorem 1 of [13].

Finally, if we set $\mathcal{G}(z) = (1 + (1 - 2\mu)z/1 - z)$, ($0 \leq \mu < 1$, $z \in \mathbb{U}$), then equation (8) becomes

$$\begin{aligned}\mathcal{G}(z) &= \frac{1 + (1 - 2\mu)z}{1 - z} = 1 + 2(1 - \mu)z + 2(1 - \mu)z^2 \\ &\quad + \dots + 2(1 - \mu)z^n = 1 + 2(1 - \mu) \sum_{n=1}^{\infty} z^n.\end{aligned}\quad (61)$$

Hence, the general term for $B_n = 2(1 - \mu)$, and by putting the comparing values of $B_1 = B_2 = B_3 = 2(1 - \mu)$ in Theorem 3, we obtain the following corollary.

Corollary

3. Let $f(z) \in \Sigma\mathcal{H}(\alpha, \tau, \gamma)(\mathcal{G})(1 + (1 - 2\mu)z/1 - z)$ be of the form (1), then

$$|a_2| \leq \sqrt{\frac{2\tau^\alpha(1-\mu)}{(\alpha+2\mu)}}, \quad (62)$$

$$|a_3| \leq 2(1-\mu)|\tau^\alpha| \left\{ \frac{2|\tau^\alpha|(1-\mu) + (\alpha+\mu)^2}{(\alpha+\mu)^2(\alpha+2\mu)} \right\}.$$

Lastly, by putting $\alpha = 0$, $\tau = 1$, and $\gamma = 1$ in Remark 1, we obtain the following new result.

Corollary 4. Let $f(z) \in \Sigma\mathcal{H}(\mathcal{L}, \mathcal{F})$ be of the form (1), then

$$|a_2| \leq \frac{\mathcal{L} - \mathcal{F}}{\sqrt{2\mathcal{L} - \mathcal{F} + 1}}, \quad (63)$$

$$|a_3| \leq (\mathcal{L} - \mathcal{F}) \left(\mathcal{L} - \mathcal{F} + \frac{1}{2} \right).$$

6. Conclusions

We presented an idea to generalize the analytic functions that are correlative to certain bi-univalent functions. By utilizing the $\mathfrak{R}$ - and $\mathfrak{X}$ -polynomials, we derived the initial estimates for our newly defined class of a bi-univalent function $\Sigma\mathcal{H}(\alpha, \tau, \gamma)(\mathcal{G})$ (see Definition 1 and Theorem 3). Furthermore, we provided an example to calculate the coefficients of z^n in terms of $\mathfrak{R}'$'s (see Example 1). In addition to this, we have given the n -th term of several classes (see (56), (59), and (61)). The $\mathfrak{R}$ - and $\mathfrak{X}$ -coefficients defined in (32) and (39) allow researchers to compare the n -th coefficients of (z) or (w) with Faber polynomial expansions to explore the n -th coefficient estimate for a certain bi-univalent function.

Data Availability

All data generated or analysed during the study are included within the submitted article.

Disclosure

All the authors agreed with the content of the manuscript.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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References

- [1] M. Lewin, "On a coefficient problem for bi-univalent functions," *Proceedings of the American Mathematical Society*, vol. 18, no. 1, p. 63, 1967.
- [2] D. A. Brannan and T. S. Taha, "On some classes of Bi-univalent functions," *Mathematical Analysis and Its Applications*, pp. 53–60, 1988.
- [3] D. Bansal and J. Sokol, "Coefficient bound for a new class of analytic and bi-univalent functions," *Journal of Fractional Calculus and Applications*, vol. 5, no. 1, pp. 122–128, 2014.
- [4] M. Darus and S. Singh, "On some new classes of Bi-univalent functions," *Journal of Applied Mathematics, Statistics and Informatics*, vol. 14, no. 2, pp. 19–26, 2018.
- [5] S. Hussain, S. Khan, M. A. Zaighum, and M. Darus, "On certain classes of bi-univalent functions related to m -fold symmetry," *Journal of Nonlinear Sciences and Applications*, vol. 11, pp. 425–434, 2018.
- [6] J. M. Jahangiri, S. G. Hamidi, and S. Halim, "Coefficients of bi-univalent functions with positive real part derivatives," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 37, no. 3, pp. 633–640, 2014.
- [7] S. S. Kumar, V. Kumar, and V. Ravichandran, "Estimates for the initial coefficients of bi-univalent functions," 2012, <http://arxiv.org/abs/1203.5480>.
- [8] S. Porwal and M. Darus, "On a new subclass of bi-univalent functions," *Journal of the Egyptian Mathematical Society*, vol. 21, no. 3, pp. 190–193, 2013.
- [9] H. M. Srivastava, S. S. Eker, S. G. Hamidi, and J. M. Jahangiri, "Faber polynomial coefficient estimates for Bi-univalent functions defined by the tremblay fractional derivative operator," *Bulletin of the Iranian Mathematical Society*, vol. 44, no. 1, pp. 149–157, 2018.
- [10] H. M. Srivastava, S. Eker, and R. Alic, "Coefficient bounds for a certain class of analytic and bi-univalent functions," *Filomat*, vol. 29, no. 8, pp. 1839–1845, 2015.
- [11] A. Zireh and E. Analoui Audegani, "Coefficient estimates for a subclass of analytic and bi-univalent functions," *Bulletin of the Iranian Mathematical Society*, vol. 42, no. 4, pp. 881–889, 2016.
- [12] H. Rehman, M. Darus, and J. Salah, "Coefficient properties involving the generalized k-mittag-leffler functions," *Journal of Mathematics and Mechanics*, vol. 9, pp. 155–164, 2017.
- [13] H. M. Srivastava, A. K. Mishra, and P. Gochhayat, "Certain subclasses of analytic and bi-univalent functions," *Applied Mathematics Letters*, vol. 23, no. 10, pp. 1188–1192, 2010.
- [14] H. M. Srivastava, S. Khan, S. Khan, Q. Z. Ahmad, N. Khan, and S. Hussain, "The Faber polynomial expansion method and its application to the general coefficient problem for some subclasses of bi-univalent functions associated with a certain q -integral operator," *Studia Universitatis Babes-Bolyai Matematica*, vol. 63, no. 4, pp. 419–436, 2018.
- [15] H. M. Srivastava, M. Sakar, and G. Özlem, "Some general coefficient estimates for a new class of analytic and bi-univalent functions defined by a linear combination," *Filomat*, vol. 32, no. 4, pp. 1313–1322, 2018.
- [16] M. J. Kozdron, "The basic theory of univalent functions," pp. 1–31, university of regina, Regina, Canada, 2007, <http://stat.math.uregina.ca/kozdron/LectureNotes/univalent.pdf>.
- [17] P. Duren, *Grundlehren Der Mathematischen Wissenschaften*, Springer, Berlin, Germany, 1983.
- [18] S. Owa and C. Y. Shen, "Generalized classes of starlike and convex functions of order α ," *International Journal of*

- Mathematics and Mathematical Sciences*, vol. 8, no. 3, pp. 455–467, 1985.
- [19] C. Pommerenke, “Univalent functions. (1975), (with a Chapter on Quadratic Differentials by Gerd Jensen),” *Studia Mathematica/Mathematische Lehrbücher*, vol. 25, 1975.
 - [20] K. Löwner, “Untersuchungen über schlichte konforme abbildungen des einheitskreises,” I, *Mathematische Annalen*, vol. 89, no. 1-2, pp. 103–121, 1923.
 - [21] W. E. Kirwan and G. Schober, “Inverse coefficients four functions of bounded boundary rotation,” *Journal d’Analyse Mathématique*, vol. 36, no. 1, pp. 167–178, 1979.
 - [22] J. G. Krzyż, R. J. Libera, and E. Zlotkiewicz, *Coefficients of Inverse of Regular Starlike Functions. Annales Universitatis Mariae Curie-Skłodowska*, vol. 74, pp. 103–109, 1979.
 - [23] R. J. Libera and E. J. Złotkiewicz, “Early coefficients of the inverse of a regular convex function,” *Proceedings of the American Mathematical Society*, vol. 85, no. 2, p. 225, 1982.
 - [24] G. Schober, *Coefficient Estimates for Inverses of Schlicht Functions, Aspects of Contemporary Complex Analysis*, University Durham, Durham, England, 1979.
 - [25] H. Airault and A. Bouali, “Differential calculus on the Faber polynomials,” *Bulletin des Sciences Mathématiques*, vol. 130, no. 3, pp. 179–222, 2006.
 - [26] G. Faber, “Über polynomische Entwicklungen,” *Mathematische Annalen*, vol. 57, no. 3, pp. 389–408, 1903.
 - [27] M. Jahangiri, “On the coefficients of powers of a class of bazilevic functions,” *Indian Journal of Pure and Applied Mathematics*, vol. 17, no. 9, pp. 1140–1144, 1986.
 - [28] S. Gong, “The bieberbach conjecture,” *American Mathematical Soc.* vol. 12, 1999.
 - [29] M. Schiffer, “Sur un problème d’extrémum de la représentation conforme,” *Bulletin de la Société mathématique de France*, vol. 2, pp. 48–55, 1938.
 - [30] H. Airault, “Symmetric sums associated to the factorization of grunsky coefficients,” in *Groups and Symmetries* Montreal, Canada, 2007.
 - [31] H. Airault, “Remarks on faber polynomials,” *International Mathematical Forum*, vol. 3, pp. 449–456, 2008.
 - [32] H. Airault and J. Ren, “An algebra of differential operators and generating functions on the set of univalent functions,” *Bulletin des Sciences Mathématiques*, vol. 126, no. 5, pp. 343–367, 2002.
 - [33] R. M. Ali, S. K. Lee, V. Ravichandran, and S. Supramaniam, “Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions,” *Applied Mathematics Letters*, vol. 25, no. 3, pp. 344–351, 2012.
 - [34] D. Brannan and J. Clunie, *Aspects of Contemporary Complex Analysis*, pp. 1–20, University of Durham, Durham, England, 1979.
 - [35] E. Netanyahu, “The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z|_r$,” *Archive for Rational Mechanics and Analysis*, vol. 32, no. 2, pp. 100–112, 1969.
 - [36] C. Stover, “big-o notation, mathworld—a wolfram web resource, Holyoke, MA, USA, 2020, <https://mathworld.wolfram.com/Big-ONotation.html>.