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Retraction

Retracted: Bipolar Fuzzy Implicative Ideals of BCK-Algebras

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

[1] G. Muhiuddin and D. Al-Kadi, "Bipolar Fuzzy Implicative Ideals of BCK-Algebras," *Journal of Mathematics*, vol. 2021, Article ID 6623907, 9 pages, 2021.

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Research Article

Bipolar Fuzzy Implicative Ideals of BCK-Algebras

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The notion of bipolar fuzzy implicative ideals of a BCK-algebra is introduced, and several properties are investigated. The relation between a bipolar fuzzy ideal and a bipolar fuzzy implicative ideal is studied. Characterizations of a bipolar fuzzy implicative ideal are given. Conditions for a bipolar fuzzy set to be a bipolar fuzzy implicative ideal are provided. Extension property for a bipolar fuzzy implicative ideal is stated.

1. Introduction

Fuzzy sets are characterized by a membership function which associates elements with real numbers in the interval [0, 1] that represents its membership degree to the fuzzy set. Several kinds of fuzzy set extensions have been introduced such as interval-valued, intuitionistic, and bipolar-valued fuzzy sets. The bipolar-valued fuzzy set notion [1] was introduced to treat imprecision as in traditional fuzzy sets, where the degree of membership belongs to the interval [0, 1], and we cannot tell apart unrelated elements from the opposite elements. The extension here enlarges the range of the membership degree from the interval [0, 1] to the interval [−1, 1] to solve such a problem (we refer the reader to [2–4]). The membership degrees which lie in the interval [−1, 1] represent the satisfaction degree to the corresponding property in a fuzzy set and its counter property as follows: having a membership degree in the interval [−1, 0) means that the elements are satisfying implicit counter property, having (0, 1] means that the elements are satisfying the property, and having 0 means that the elements are unrelated to the corresponding property.

The bipolar-valued fuzzification has been used to study different notions in BCK/BCI-algebras such as subalgebras and ideals of BCK/BCI-algebras [5], a-ideals of BCI-algebras [6], and more, see the references [7–10]. Other researches also added their contribution to the study in this field on different branches of algebra in various aspects (see, e.g., [11–26]). Also, some more general concepts on bipolar fuzzy have been studied in [27–31].

Recently, the bipolar fuzzy BCI-implicative ideals of BCI-algebras were studied in [32]. Moreover, new types of bipolar fuzzy ideals of BCK-algebras have been investigated in [33], typically bipolar fuzzy (closed, positive implicative, and implicative) ideals. Moreover, some related concepts on fuzzy sets and their useful generalizations were applied in various algebraic structures (see, e.g., [33–53]).

In this paper, we apply the notion of a bipolar-valued fuzzy set to implicative ideals of BCK-algebras and obtain further results in this manner. Furthermore, we consider the relation of a bipolar fuzzy ideal with a bipolar fuzzy implicative ideal. We provide characterizations of a bipolar fuzzy implicative ideal. Moreover, we display conditions for a bipolar fuzzy set to be a bipolar fuzzy implicative ideal. Finally, we discuss extension property for a bipolar fuzzy implicative ideal.

2. Preliminaries

The basic results on BCK-algebras are given in this section.

By a *BCK-algebra*, we mean an algebra $(Ł;^{*}, 0)$ of type (2, 0) satisfying the axioms:

(a1) $(\forall \varkappa, \ell, \upsilon \in Ł)(((\varkappa * \ell) * (\varkappa * \upsilon)) * (\upsilon * \ell) = 0)$
(a2) $(\forall \varkappa, \ell \in Ł)((\varkappa * (\varkappa * \ell)) * \ell = 0)$

(a3) $(\forall \varkappa \in Ł)(\varkappa * \varkappa = 0, 0 * \varkappa = 0)$

(a4) $(\forall \varkappa, \ell \in Ł)(\varkappa * \ell = 0, \ell * \varkappa = 0 \Longrightarrow \varkappa = \ell)$

We can define a partial ordering $\leq$ by $\varkappa \leq \ell$ if and only if $\varkappa * \ell = 0$.

In any BCK-algebra Ł, the following hold:

(b1) $(\forall \varkappa \in Ł)(\varkappa * 0 = \varkappa)$

(b2) $(\forall \varkappa, \ell, \upsilon \in Ł)((\varkappa * \ell) * \upsilon = (\varkappa * \upsilon) * \ell)$

(b3) $(\forall \varkappa, \ell, \upsilon \in Ł)((\varkappa * \upsilon) * (\ell * \upsilon) \leq \varkappa * \ell)$

(b4) $(\forall \varkappa, \ell, \upsilon \in Ł)(\varkappa \leq \ell \Rightarrow \varkappa * \upsilon \leq \ell * \upsilon, \upsilon * \ell \leq \upsilon * \varkappa)$

Let us consider a subset $(\varnothing \neq I)$ of a BCK-algebra Ł. We say I is an *ideal* if

(c1) $0 \in I$, (c2) $(\forall \varkappa \in Ł)(\forall \ell \in I)(\varkappa * \ell \in I \Rightarrow \varkappa \in I)$

A nonempty subset I of a BCK-algebra Ł is called an *implicative ideal* of Ł if it satisfies (c1) and

(c3) $(\forall \varkappa, \ell, \upsilon \in Ł)((\varkappa * \ell) * \upsilon \in I, \ell * \upsilon \in I \Rightarrow \varkappa * \upsilon \in I)$

3. Bipolar Fuzzy Ideals

In the following sections, Ł denotes a BCK-algebra.

For any family $\{\delta_i | i \in \Delta\}$ of real numbers, we define

$$\vee\{\delta_i | i \in \Delta\} := \begin{cases} \max\{\delta_i | i \in \Delta\}, & \text{if } \Delta \text{ is finite,} \\ \sup\{\delta_i | i \in \Delta\}, & \text{otherwise,} \end{cases}$$
$$\wedge\{\delta_i | i \in \Delta\} := \begin{cases} \min\{\delta_i | i \in \Delta\}, & \text{if } \Delta \text{ is finite,} \\ \inf\{\delta_i | i \in \Delta\} & \text{otherwise.} \end{cases} \tag{1}$$

Moreover, if $\Delta = \{1, 2, \ldots, n\}$, then $\vee\{\delta_i | it \in n\Delta\}$ and $\wedge\{\delta_i | it \in n\Delta\}$ are denoted by $\delta_1 \vee \delta_2 \vee \cdots \vee \delta_n$ and $\delta_1 \wedge \delta_2 \wedge \cdots \wedge \delta_n$, respectively.

For a bipolar fuzzy set $q = (Ł; q_n, q_p)$, we define *negative* α-cut of $q = (Ł; q_n, q_p)$ and the *positive* β-cut of $q = (Ł; q_n, q_p)$, respectively, as follows:

$$N(q; \alpha) := \{\varkappa \in Ł | q_n(\varkappa) \leq \alpha\},$$
$$P(q; \beta) := \{\varkappa \in Ł | q_p(\varkappa) \geq \beta\}, \tag{2}$$

where $(\alpha, \beta) \in [-1, 0) \times (0, 1]$. The set

$$C(q; (\alpha, \beta)) := N(q; \alpha) \cap P(q; \beta) \tag{3}$$

is called the (α, β)-cut of $q = (Ł; q_n, q_p)$. For every $k \in (0, 1)$, if $(\alpha, \beta) = (-k, k)$, then the set

$$C(q; k) := N(q; -k) \cap P(q; k) \tag{4}$$

is called the k-cut of $q = (Ł; q_n, q_p)$.

Definition 1 (see [5]). A bipolar fuzzy set $q = (Ł; q_n, q_p)$ in a BCK-algebra Ł is called a *bipolar fuzzy ideal* of Ł if it satisfies the following assertions:

(i) $(\forall \varkappa \in Ł)(q_n(0) \leq q_n(\varkappa),\ q_p(0) \geq q_p(\varkappa))$

(ii) $(\forall \varkappa, \ell \in Ł)\begin{pmatrix} q_n(\varkappa) \leq q_n(\varkappa * \ell) \vee q_n(\ell), \\ q_p(\varkappa) \geq q_p(\varkappa * \ell) \wedge q_p(\ell). \end{pmatrix}$

For any $w \in Ł$ and any bipolar fuzzy set $q = (Ł; q_n, q_p)$ in Ł, we let

$$I(w) = \{\varkappa \in Ł | q_n(\varkappa) \leq q_n(w), q_p(\varkappa) \geq q_p(w)\}. \tag{5}$$

Obviously, $w \in I(w)$. If $q = (Ł; q_n, q_p)$ is a bipolar fuzzy ideal of Ł, then $0 \in I(w)$. The following is our question: *For a bipolar fuzzy set $q = (Ł; q_n, q_p)$ in* Ł *satisfying* Definition 1 (*i*), *is $I(w)$an ideal of* Ł? The following example provides a negative answer; that is, there exists an element $w \in Ł$ such that $I(w)$ is not an ideal of Ł.

Example 1. Let $Ł = \{\theta, \ell, \upsilon, \omega, \delta\}$ be a set with a Cayley table which is given in Table 1.

Then, $(Ł; *, \theta)$ is a BCK-algebra. Let $q = (Ł; q_n, q_p)$ be a bipolar fuzzy set in Ł defined by

	θ	l	υ	w	δ
q_n	−0.7	−0.5	−0.3	−0.1	−0.4
q_p	0.8	0.7	0.4	0.2	0.5

Then, $q = (Ł; q_n, q_p)$ satisfies Definition 1 (i), and it is not a bipolar fuzzy ideal of Ł because

$$q_n(\upsilon) = -0.3 > -0.4 = q_n(\upsilon * \delta) \vee q_n(\delta), \tag{6}$$

and/or

$$q_p(\upsilon) = 0.4 < 0.5 = q_p(\upsilon * \delta) \wedge q_p(\delta). \tag{7}$$

Then, $I(\delta) = \{\theta, \ell, \delta\}$ is not an ideal of Ł since $\upsilon * \delta = \theta \in I(\delta)$ and $\delta \in I(\delta)$, while $\upsilon \notin I(\delta)$. Note that $I(\upsilon) = \{\theta, \ell, \upsilon, \delta\}$ is an ideal of Ł.

We give conditions for the set $I(w)$ to be an ideal.

Theorem 1. *Let $w \in Ł$. If $q = (Ł; q_n, q_p)$ is a bipolar fuzzy ideal of Ł, then $I(w)$ is an ideal of Ł.*

Proof. We recall that $0 \in I(w)$. Let $\varkappa, \ell \in Ł$ such that $\varkappa * \ell \in I(w)$ and $\ell \in I(w)$. Then, $q_n(w) \geq q_n(\varkappa * \ell)$, $q_p(w) \leq q_p(\varkappa * \ell)$, $q_n(w) \geq q_n(\ell)$ and $q_p(w) \leq q_p(\ell)$. Since $q = (Ł; q_n, q_p)$ is a bipolar fuzzy ideal of Ł, we have from Definition 1 (ii) that

$$q_n(\varkappa) \leq q_n(\varkappa * \ell) \vee q_n(\ell) \leq q_n(w),$$
$$q_p(\varkappa) \geq q_p(\varkappa * \ell) \wedge q_p(\ell) \geq q_p(w), \tag{8}$$

and so, $\varkappa \in I(w)$. Therefore, $I(w)$ is an ideal of Ł. □

Theorem 2. *Let $q = (Ł; q_n, q_p)$ be a bipolar fuzzy set in Ł and $w \in Ł$.*

(1) If $I(w)$ is an ideal of Ł, then $q = (Ł; q_n, q_p)$ satisfies the following implications for all $\varkappa, \ell, \upsilon \in Ł$:

$$q_n(\varkappa) \geq q_n(\ell * \upsilon) \vee q_n(\upsilon) \Rightarrow q_n(\varkappa) \geq q_n(\ell),$$
$$q_p(\varkappa) \leq q_p(\ell * \upsilon) \wedge q_p(\upsilon) \Rightarrow q_p(\varkappa) \leq q_p(\ell). \tag{9}$$

(2) If $q = (Ł; q_n, q_p)$ satisfies Definition 1(i) and (9), then $I(w)$ is an ideal of Ł.

Table 1: Cayley table.

$*$	θ	ℓ	v	ω	δ
θ	θ	θ	θ	θ	θ
ℓ	ℓ	θ	ℓ	θ	θ
v	v	v	θ	v	θ
ω	ω	ω	ω	θ	ω
δ	δ	δ	δ	δ	θ

Proof

(1) We assume that $I(w)$ is an ideal of Ł for each $w \in$ Ł. We suppose that $q_n(\varkappa) \geq q_n(\ell * v) \vee q_n(v)$ and $q_p(\varkappa) \leq q_p(\ell * v) \wedge q_p(v)$ for all $\varkappa, \ell, v \in$ Ł. Then, $\ell * v \in I(\varkappa)$ and $v \in I(\varkappa)$. Since $I(\varkappa)$ is an ideal of Ł, it follows that $\ell \in I(\varkappa)$, that is, $q_n(\varkappa) \geq q_n(\ell)$ and $q_p(\varkappa) \leq q_p(\ell)$.

(2) We suppose that $q = (\text{Ł}; q_n, q_p)$ satisfies Definition 1 (i) and (9). For each $w \in$ Ł, let $\varkappa, \ell \in$ Ł such that $\varkappa * \ell \in I(w)$ and $\ell \in I(w)$. Then, $q_n(\varkappa * \ell) \leq q_n(w)$, $q_p(\varkappa * \ell) \geq q_p(w)$, $q_n(\ell) \leq q_n(w)$, and $q_p(\ell) \geq q_p(w)$, which imply that $q_n(w) \geq q_n(\varkappa * \ell) \vee q_n(\ell)$ and $q_p(w) \leq q_p(\varkappa * \ell) \wedge q_p(\ell)$. Using (9), we have $q_n(w) \geq q_n(\varkappa)$ and $q_p(w) \leq q_p(\varkappa)$, and so, $\varkappa \in I(w)$. Since $q = (\text{Ł}; q_n, q_p)$ satisfies Definition 1 (i), it follows that $0 \in I(w)$. Therefore, $I(w)$ is an ideal of Ł. □

Lemma 1 (see [5]). *Every bipolar fuzzy ideal $q = (\text{Ł}; q_n, q_p)$ of Ł satisfies the following implication:*

$$(\forall \varkappa, \ell \in \text{Ł})\left(\varkappa \leq \ell \Rightarrow q_n(\varkappa) \leq q_n(\ell), q_p(\varkappa) \geq q_p(\ell)\right). \tag{10}$$

Proposition 1. *For any bipolar fuzzy ideal $q = (\text{Ł}; q_n, q_p)$ of Ł, the following are equivalent:*

(1) $(\forall \varkappa, \ell \in \text{Ł})\begin{pmatrix} q_n(\varkappa * \ell) \leq q_n((\varkappa * \ell) * \ell), \\ q_p(\varkappa * \ell) \geq q_p((\varkappa * \ell) * \ell). \end{pmatrix}$

(2) $(\forall \varkappa, \ell, v \in \text{Ł})\begin{pmatrix} q_n((\varkappa * v) * (\ell * v)) \leq q_n((\varkappa * \ell) * v), \\ q_p((\varkappa * v) * (\ell * v)) \geq q_p((\varkappa * \ell) * v). \end{pmatrix}$

Proof We assume that condition (2) is valid. Note that

$$((\varkappa * (\ell * v)) * v) * v = ((\varkappa * v) * (\ell * v)) * v \leq (\varkappa * \ell) * v, \tag{11}$$

for all $\varkappa, \ell, v \in$ Ł by using (b2), (b3), and (b4). It follows from Lemma 1 that

$$\begin{aligned} q_n((\varkappa * \ell) * v) &\geq q_n(((\varkappa * (\ell * v)) * v) * v), \\ q_p((\varkappa * \ell) * v) &\leq q_p(((\varkappa * (\ell * v)) * v) * v). \end{aligned} \tag{12}$$

So, from (b2) and (2), it follows that

$$\begin{aligned} q_n((\varkappa * v) * (\ell * v)) &= q_n((\varkappa * (\ell * v)) * v) \\ &\leq q_n(((\varkappa * (\ell * v)) * v) * v) \\ &\leq q_n((\varkappa * \ell) * v), \\ q_p((\varkappa * v) * (\ell * v)) &= q_p((\varkappa * (\ell * v)) * v) \\ &\geq q_p(((\varkappa * (\ell * v)) * v) * v) \\ &\geq q_p((\varkappa * \ell) * v). \end{aligned} \tag{13}$$

Thus, (9) holds. Now, we suppose that (9) is valid. Using (b1), (a3), and (9) with replacing v by ℓ, we have

$$\begin{aligned} q_n(\varkappa * \ell) &= q_n((\varkappa * \ell) * 0) = q_n((\varkappa * \ell) * (\ell * \ell)) \\ &\leq q_n((\varkappa * \ell) * \ell), \\ q_p(\varkappa * \ell) &= q_p((\varkappa * \ell) * 0) = q_p((\varkappa * \ell) * (\ell * \ell)) \\ &\geq q_p((\varkappa * \ell) * \ell), \end{aligned} \tag{14}$$

which proves (2). □

Proposition 2 (see [5]). *A bipolar fuzzy set $q = (\text{Ł}; q_n, q_p)$ in Ł is a bipolar fuzzy ideal of Ł if and only if for all $\varkappa, \ell, v \in$ Ł, $(\varkappa * \ell) * v = 0$ implies $q_n(\varkappa) \leq q_n(\ell) \vee q_n(v)$ and $q_p(\varkappa) \geq q_p(\ell) \wedge q_p(v)$.*

As a generalization of Proposition 2, we have the following results.

Theorem 3. *If a bipolar fuzzy set $q = (\text{Ł}; q_n, q_p)$ in Ł is a bipolar fuzzy ideal of Ł, then for all $\varkappa, w_1, w_2, \ldots, w_n \in$ Ł,*

$$\prod_{i=1}^{n} \varkappa * w_i = 0 \Rightarrow \begin{pmatrix} q_n(\varkappa) \leq q_n(w_1) \vee q_n(w_2) \vee \cdots \vee q_n(w_n), \\ q_p(\varkappa) \geq q_p(w_1) \wedge q_p(w_2) \wedge \cdots \wedge q_p(w_n), \end{pmatrix} \tag{15}$$

where $\prod_{i=1}^{n} \varkappa * w_i = (\cdots((\varkappa * w_1) * w_2) * \cdots) * w_n$.

Proof. The proof is by induction on n. Let $q = (\text{Ł}; q_n, q_p)$ be a bipolar fuzzy ideal of Ł. Lemma 1 and Proposition 2 show that condition (15) is valid for $n = 1, 2$. We assume that $q = (\text{Ł}; q_n, q_p)$ satisfies condition (15) for $n = k$, that is, for all $\varkappa, w_1, w_2, \ldots, w_k \in$ Ł, $\prod_{i=1}^{k} \varkappa * w_i = 0$ implies

$$\begin{aligned} q_n(\varkappa) &\leq q_n(w_1) \vee q_n(w_2) \vee \cdots \vee q_n(w_k), \\ q_p(\varkappa) &\geq q_p(w_1) \wedge q_p(w_2) \wedge \cdots \wedge q_p(w_k). \end{aligned} \tag{16}$$

Let $\varkappa, w_1, w_2, \ldots, w_k, w_{k+1} \in$ Ł such that $\prod_{i=1}^{k+1} \varkappa * w_i = 0$. Then,

$$\begin{aligned} q_n(\varkappa * w_1) &\leq q_n(w_2) \vee q_n(w_3) \vee \cdots \vee q_n(w_{k+1}), \\ q_p(\varkappa * w_1) &\geq q_p(w_2) \wedge q_p(w_3) \wedge \cdots \wedge q_p(w_{k+1}). \end{aligned} \tag{17}$$

Since $q = (\text{Ł}; q_n, q_p)$ is a bipolar fuzzy ideal of Ł, it follows from Definition 1 (ii) that

$$\begin{aligned} q_n(\varkappa) &\le q_n(\varkappa * w_1)\vee q_n(w_1) \\ &\le q_n(w_1)\vee q_n(w_2)\vee\cdots\vee q_n(w_{k+1}), \\ q_p(\varkappa) &\ge q_p(\varkappa * w_1)\wedge q_p(w_1) \\ &\ge q_p(w_1)\wedge q_p(w_2)\wedge\cdots\wedge q_p(w_{k+1}). \end{aligned} \tag{18}$$

This completes the proof. □

Now, we consider the converse of Theorem 3.

Theorem 4. *Let $q = (Ł; q_n, q_p)$ be a bipolar fuzzy set in Ł satisfying condition (15). Then, $q = (Ł; q_n, q_p)$ is a bipolar fuzzy ideal of Ł.*

Proof. Note that $(\cdots((0 * \underbrace{\varkappa) * \varkappa) * \cdots) * \varkappa}_{n\,\text{times}} = 0$ for all $\varkappa \in Ł$. It follows from (15) that $q_n(0) \le q_n(\varkappa)$ and $q_p(0) \ge q_p(\varkappa)$ for all $\varkappa \in Ł$. Let $\varkappa, \ell, v \in Ł$ such that $\varkappa * \ell \le v$. Then,

$$0 = (\varkappa * \ell) * v = (\cdots(((\varkappa * \ell) * v) * \underbrace{0) * \cdots) * 0}_{n-2\,\text{times}}, \tag{19}$$

and so,

$$\begin{aligned} q_n(\varkappa) &\le q_n(\ell)\vee q_n(v)\vee q_n(0) = q_n(\ell)\vee q_n(v), \\ q_p(\varkappa) &\ge q_p(\ell)\wedge q_p(v)\wedge q_p(0) = q_p(\ell)\wedge q_p(v). \end{aligned} \tag{20}$$

Hence, by Proposition 2, we conclude that $q = (Ł; q_n, q_p)$ is a bipolar fuzzy ideal of Ł. □

4. Bipolar Fuzzy Implicative Ideals

Definition 2. A bipolar fuzzy set $q = (Ł; q_n, q_p)$ in Ł is called a *bipolar fuzzy implicative ideal* of Ł if both the nonempty negative α-cut and the nonempty positive β-cut of $q = (Ł; q_n, q_p)$ are implicative ideals of Ł for all $(\alpha, \beta) \in [-1, 0] \times [0, 1]$.

Example 2. Let $Ł = \{\theta, \ell, v\}$ be a set in which the operation $*$ is defined by Table 2.

Table 2: Cayley table.

$*$	θ	ℓ	v
θ	θ	θ	θ
ℓ	ℓ	θ	θ
v	v	v	θ

Then, $(Ł; *, \theta)$ is a BCK-algebra. Let $(t_0, s_0), (t_1, s_1) \in [-1, 0] \times [0, 1]$ satisfy $(t_0, s_0) > (t_1, s_1)$, that is, $t_0 < t_1$ and $s_0 > s_1$. Let $q = (Ł; q_n, q_p)$ be a bipolar fuzzy set in Ł given by

	θ	l	v
q_n	t_0	t_0	t_1
q_p	s_0	s_0	s_1

By routine calculations, we know that $q = (Ł; q_n, q_p)$ is a bipolar fuzzy implicative ideal of Ł.

Theorem 5. *A bipolar fuzzy set $q = (Ł; q_n, q_p)$ in Ł is a bipolar fuzzy implicative ideal of Ł if and only if it satisfies Definition 1 (i) and the following assertions:*

$$(\forall \varkappa, \ell, v \in Ł)\begin{pmatrix} q_n(\varkappa * v) \le q_n((\varkappa * \ell) * v)\vee q_n(\ell * v), \\ q_p(\varkappa * v) \ge q_p((\varkappa * \ell) * v)\wedge q_p(\ell * v). \end{pmatrix} \tag{21}$$

Proof. We suppose that $q = (Ł; q_n, q_p)$ is a bipolar fuzzy implicative ideal of Ł. If $q_n(0) > q_n(b)$ or $q_p(0) < q_p(d)$ for some $b, d \in Ł$, then $0 \notin N(q; q_n(b))$ or $0 \notin P(q; q_p(d))$, which contradicts the fact. Hence, $q_n(0) \le q_n(\varkappa)$ and $q_p(0) \ge q_p(\varkappa)\ \forall \varkappa \in Ł$. For some $b, d, c \in Ł$, we assume that we have the following relation:

$$q_n(b * c) > q_n((b * d) * c)\vee q_n(d * c) = s. \tag{22}$$

Then, $(b * d) * c \in N(q; s)$ and $d * c \in N(q; s)$, but $b * c \notin N(q; s)$. This is not possible; therefore, we have

$$q_n(\varkappa * v) \le q_n((\varkappa * \ell) * v)\vee q_n(\ell * v), \tag{23}$$

for all $\varkappa, \ell, v \in Ł$. If $q_p(b * c) < q_p((b * d) * c)\wedge q_p(d * c) = t$ for some $b, d, c \in Ł$, then $(b * d) * c \in P(q; t)$ and $d * c \in P(q; t)$, but $b * c \notin P(q; t)$. We reach a contradiction because $P(q; t)$ is an implicative ideal of Ł. Henceforth,

$$q_p(\varkappa * v) \ge q_p((\varkappa * \ell) * v)\wedge q_p(\ell * v), \tag{24}$$

for all $\varkappa, \ell, v \in Ł$. Consequently, a bipolar fuzzy implicative ideal $q = (Ł; q_n, q_p)$ satisfies Definition 1 (i) and (21).

Conversely, we suppose that $q = (Ł; q_n, q_p)$ satisfies Definition 1 (i) and (21) and let $(\alpha, \beta) \in [-1, 0] \times [0, 1]$ s.th. $N(q; \alpha) \ne \varnothing$ and $P(q; \beta) \ne \varnothing$. It is clear that $0 \in N(q; \alpha) \cap P(q; \beta)$. Let $\varkappa, \ell, v \in Ł$ be such that $(\varkappa * \ell) * v \in N(q; \alpha)$ and $\ell * v \in N(q; \alpha)$. Then, $q_n((\varkappa * \ell) * v) \le \alpha$ and $q_n(\ell * v) \le \alpha$. It follows from (21) that

$$q_n(\varkappa * v) \le q_n((\varkappa * \ell) * v)\vee q_n(\ell * v) \le \alpha, \tag{25}$$

and so, $\varkappa * v \in N(q; \alpha)$. Hence, $N(q; \alpha)$ is an implicative ideal of Ł. Similarly, we can show that

$$q_p(\varkappa * v) \ge q_p((\varkappa * \ell) * v)\wedge q_p(\ell * v) \ge \beta, \tag{26}$$

for all $\varkappa, \ell, v \in Ł$, and so, $\varkappa * v \in P(q; \beta)$. Therefore, $P(q; \beta)$ is an implicative ideal of Ł. Consequently, $q = (Ł; q_n, q_p)$ is a bipolar fuzzy implicative ideal of Ł. □

Next, we have the following theorems.

Theorem 6. *A bipolar fuzzy ideal $q = (Ł; q_n, q_p)$ of Ł is a bipolar fuzzy implicative ideal of Ł if and only if it satisfies Proposition 1 (1).*

Proof. Let $q = (Ł; q_n, q_p)$ be a bipolar fuzzy implicative ideal of Ł. If v is replaced by ℓ in (21), then

$$\begin{aligned} q_n(\varkappa * \ell) &\le q_n((\varkappa * \ell) * \ell)\vee q_n(\ell * \ell), \\ &= q_n((\varkappa * \ell) * \ell)\vee q_n(0), \\ &= q_n((\varkappa * \ell) * \ell), \end{aligned}$$

$$\begin{aligned} q_p(\varkappa * \ell) &\geq q_p((\varkappa * \ell) * \ell) \wedge q_p(\ell * \ell), \\ &= q_p((\varkappa * \ell) * \ell) \wedge q_p(0), \\ &= q_p((\varkappa * \ell) * \ell), \end{aligned} \tag{27}$$

which is Proposition 1 (1). Conversely, let $q = (Ł; q_n, q_p)$ be a bipolar fuzzy ideal of Ł satisfying Proposition 1 (1). Note that

$$((\varkappa * v) * v) * (\ell * v) \leq (\varkappa * v) * \ell = (\varkappa * \ell) * v, \tag{28}$$

for all $\varkappa, \ell, v \in Ł$. Using Lemma 1, we have

$$\begin{aligned} q_n((\varkappa * \ell) * v) &\geq q_n(((\varkappa * v) * v) * (\ell * v)), \\ q_p((\varkappa * \ell) * v) &\leq q_p(((\varkappa * v) * v) * (\ell * v)). \end{aligned} \tag{29}$$

It follows from Definition 1 (ii) and Proposition 1 (1) that

$$\begin{aligned} q_n(\varkappa * v) &\leq q_n((\varkappa * v) * v) \\ &\leq q_n(((\varkappa * v) * v) * (\ell * v)) \vee q_n(\ell * v) \\ &\leq q_n((\varkappa * \ell) * v) \vee q_n(\ell * v), \\ q_p(\varkappa * v) &\geq q_p((\varkappa * v) * v) \\ &\geq q_p(((\varkappa * v) * v) * (\ell * v)) \wedge q_p(\ell * v) \\ &\geq q_p((\varkappa * \ell) * v) \wedge q_p(\ell * v). \end{aligned} \tag{30}$$

Thus, $q = (Ł; q_n, q_p)$ is a bipolar fuzzy implicative ideal of Ł. □

Combining Proposition 1 and Theorem 6, we have the following characterization of a bipolar fuzzy implicative ideal.

Theorem 7. *Let $q = (Ł; q_n, q_p)$ be a bipolar fuzzy ideal of Ł. Then, it is a bipolar fuzzy implicative ideal of Ł if and only if it satisfies Proposition 1 (2).*

Theorem 8 (see [33]). *Every bipolar fuzzy implicative ideal is a bipolar fuzzy ideal.*

Theorem 9. *Let $q = (Ł; q_n, q_p)$ be a bipolar fuzzy set in Ł. Then, $q = (Ł; q_n, q_p)$ is a bipolar fuzzy implicative ideal of Ł if and only if it satisfies Definition 1 (i) and*

$$(\forall \varkappa, \ell, v \in Ł) \begin{pmatrix} q_n(\varkappa * \ell) \leq q_n(((\varkappa * \ell) * \ell) * v) \vee q_n(v), \\ q_p(\varkappa * \ell) \geq q_p(((\varkappa * \ell) * \ell) * v) \wedge q_p(v). \end{pmatrix} \tag{31}$$

Proof. We suppose that $q = (Ł; q_n, q_p)$ is a bipolar fuzzy implicative ideal of Ł. Then, $q = (Ł; q_n, q_p)$ is a bipolar fuzzy ideal of Ł by Theorem 8, and so, Definition 1 (i) is true. From Theorem 7, it follows that $q = (Ł; q_n, q_p)$ satisfies Proposition 1 (2). Thus,

$$\begin{aligned} q_n(\varkappa * \ell) &\leq q_n((\varkappa * \ell) * v) \vee q_n(v), \\ &= q_n(((\varkappa * v) * \ell) * (\ell * \ell)) \vee q_n(v) \\ &\leq q_n(((\varkappa * v) * \ell) * \ell) \vee q_n(v), \\ &= q_n(((\varkappa * \ell) * \ell) * v) \vee q_n(v) \\ q_p(\varkappa * \ell) &\geq q_p((\varkappa * \ell) * v) \wedge q_p(v), \\ &= q_p(((\varkappa * v) * \ell) * (\ell * \ell)) \wedge q_p(v) \\ &\geq q_p(((\varkappa * v) * \ell) * \ell) \wedge q_p(v), \\ &= q_p(((\varkappa * \ell) * \ell) * v) \wedge q_p(v), \end{aligned} \tag{32}$$

which proves (31). Conversely, let $q = (Ł; q_n, q_p)$ be a bipolar fuzzy set in Ł satisfying Definition 1 (i) and (31). Then,

$$\begin{aligned} q_n(\varkappa) &= q_n(\varkappa * 0) \leq q_n(((\varkappa * 0) * 0) * v) \vee q_n(v) \\ &= q_n(\varkappa * v) \vee q_n(v), \\ q_p(\varkappa) &= q_p(\varkappa * 0) \geq q_p(((\varkappa * 0) * 0) * v) \wedge q_p(v) \\ &= q_p(\varkappa * v) \wedge q_p(v). \end{aligned} \tag{33}$$

Thus, $q = (Ł; q_n, q_p)$ is a bipolar fuzzy ideal of Ł. Now, we take $v = 0$ in (31) and use (b1) and Definition 1 (i) to get

$$\begin{aligned} q_n(\varkappa * \ell) &\leq q_n(((\varkappa * \ell) * \ell) * 0) \vee q_n(0), \\ &= q_n((\varkappa * \ell) * \ell) \vee q_n(0), \\ &= q_n((\varkappa * \ell) * \ell), \\ q_p(\varkappa * \ell) &\geq q_p(((\varkappa * \ell) * \ell) * 0) \wedge q_p(0), \\ &= q_p((\varkappa * \ell) * \ell) \wedge q_p(0), \\ &= q_p((\varkappa * \ell) * \ell). \end{aligned} \tag{34}$$

It follows from Theorem 6 that $q = (Ł; q_n, q_p)$ is a bipolar fuzzy implicative ideal of Ł. □

Summarizing the abovementioned results, we have a characterization of a bipolar fuzzy implicative ideal of Ł.

Theorem 10. *Let $q = (Ł; q_n, q_p)$ be a bipolar fuzzy set in Ł. Then, the following assertions are equivalent:*

(1) $q = (Ł; q_n, q_p)$ is a bipolar fuzzy implicative ideal of Ł
(2) $q = (Ł; q_n, q_p)$ satisfies Definition 1 (i) and (21)
(3) $q = (Ł; q_n, q_p)$ is a bipolar fuzzy ideal of Ł satisfying Proposition 1 (1)
(4) $q = (Ł; q_n, q_p)$ is a bipolar fuzzy ideal of Ł satisfying Proposition 1 (2)
(5) $q = (Ł; q_n, q_p)$ satisfies Definition 1 (i) and (31)

Theorem 11. *Let $w \in Ł$. If $q = (Ł; q_n, q_p)$ is a bipolar fuzzy implicative ideal of Ł, then $I(w)$ is an implicative ideal of Ł.*

Proof. We recall that $0 \in I(w)$. Let $\varkappa, \ell, v \in Ł$ such that $(\varkappa * \ell) * v \in I(w)$ and $\ell * v \in I(w)$. Then, $q_n(w) \geq q_n((\varkappa * \ell) * v)$, $q_p(w) \leq q_p((\varkappa * \ell) * v)$, $q_n(w) \geq q_n(\ell * v)$, and $q_p(w) \leq q_p(\ell * v)$. Since $q = (Ł; q_n, q_p)$ is a bipolar fuzzy implicative ideal of Ł, it follows from (21) that

$$\begin{aligned} q_n(\varkappa * v) &\leq q_n((\varkappa * \ell) * v) \vee q_n(\ell * v) \leq q_n(w), \\ q_p(\varkappa * v) &\geq q_p((\varkappa * \ell) * v) \wedge q_p(\ell * v) \geq q_p(w), \end{aligned} \tag{35}$$

so that $\varkappa * v \in I(w)$. Therefore, $I(w)$ is an implicative ideal of Ł. □

Theorem 12. *If $q = (Ł; q_n, q_p)$ is a bipolar fuzzy implicative ideal of Ł, then for all $\varkappa, \ell, v, a, b \in Ł$,*

(1) $((\varkappa * \ell) * \ell) * a \leq b \Rightarrow \begin{pmatrix} q_n(\varkappa * \ell) \leq q_n(a) \vee q_n(b), \\ q_p(\varkappa * \ell) \geq q_p(a) \wedge q_p(b). \end{pmatrix}$

(2) $((\varkappa * \ell) * v) * a \leq b \Rightarrow \begin{pmatrix} q_n((\varkappa * v) * (\ell * v)) \leq q_n(a) \vee q_n(b), \\ q_p((\varkappa * v) * (\ell * v)) \geq q_p(a) \wedge q_p(b). \end{pmatrix}$

Proof. Let $\varkappa, \ell, a, b \in Ł$ such that $((\varkappa * \ell) * \ell) * a \leq b$. Using Proposition 2, we have $q_n((\varkappa * \ell) * \ell) \leq q_n(a) \vee q_n(b)$ and $q_p((\varkappa * \ell) * \ell) \geq q_p(a) \wedge q_p(b)$. It follows that

$$\begin{aligned} q_n(\varkappa * \ell) &\leq q_n((\varkappa * \ell) * \ell) \vee q_n(\ell * \ell), \\ &= q_n((\varkappa * \ell) * \ell) \vee q_n(0), \\ &= q_n((\varkappa * \ell) * \ell) \\ &\leq q_n(a) \vee q_n(b), \\ q_p(\varkappa * \ell) &\geq q_p((\varkappa * \ell) * \ell) \wedge q_p(\ell * \ell), \\ &= q_p((\varkappa * \ell) * \ell) \wedge q_p(0), \\ &= q_p((\varkappa * \ell) * \ell) \\ &\geq q_p(a) \wedge q_p(b). \end{aligned} \tag{36}$$

Now, let $\varkappa, \ell, v, a, b \in Ł$ such that $((\varkappa * \ell) * v) * a \leq b$, that is,

$$(((\varkappa * \ell) * v) * a) * b = 0. \tag{37}$$

Since $q = (Ł; q_n, q_p)$ is a bipolar fuzzy implicative ideal of Ł, it follows from Theorem 7 and Proposition 2 that

$$\begin{aligned} q_n((\varkappa * v) * (\ell * v)) &\leq q_n((\varkappa * \ell) * v) \leq q_n(a) \vee q_n(b), \\ q_p((\varkappa * v) * (\ell * v)) &\geq q_p((\varkappa * \ell) * v) \geq q_p(a) \wedge q_p(b). \end{aligned} \tag{38}$$

This completes the proof. □

Theorem 13. *Let $q = (Ł; q_n, q_p)$ be a bipolar fuzzy set in Ł satisfying Theorem 12 (1). Then, $q = (Ł; q_n, q_p)$ is a bipolar fuzzy implicative ideal of Ł.*

Proof. We first prove that $q = (Ł; q_n, q_p)$ is a bipolar fuzzy ideal of Ł. Let $\varkappa, \ell, v \in Ł$ such that $\varkappa * \ell \leq v$. Then,

$$\begin{aligned} (((\varkappa * 0) * 0) * \ell) * v = (\varkappa * \ell) * v = 0, \\ \text{that is, } ((\varkappa * 0) * 0) * \ell \leq v, \end{aligned} \tag{39}$$

which implies from (b1) and Theorem 12 (1) that $q_n(\varkappa) = q_n(\varkappa * 0) \leq q_n(\ell) \vee q_n(v)$ and $q_p(\varkappa) = q_p(\varkappa * 0) \geq q_p(\ell) \wedge q_p(v)$. Therefore, by Proposition 2, we know that $q = (Ł; q_n, q_p)$ is a bipolar fuzzy ideal of Ł. Note that $(((\varkappa * \ell) * \ell) * ((\varkappa * \ell) * \ell)) * 0 = 0$ for all $\varkappa, \ell \in Ł$. Using Theorem 12 (1) and Definition 1 (i), we have

$$\begin{aligned} q_n(\varkappa * \ell) &\leq q_n((\varkappa * \ell) * \ell) \vee q_n(0) = q_n((\varkappa * \ell) * \ell), \\ q_p(\varkappa * \ell) &\geq q_p((\varkappa * \ell) * \ell) \wedge q_p(0) = q_p((\varkappa * \ell) * \ell), \end{aligned} \tag{40}$$

and so, $q = (Ł; q_n, q_p)$ is a bipolar fuzzy implicative ideal of Ł by Theorem 6. □

Theorem 14. *Let $q = (Ł; q_n, q_p)$ be a bipolar fuzzy set in Ł satisfying Theorem 12 (2). Then, $q = (Ł; q_n, q_p)$ is a bipolar fuzzy implicative ideal of Ł.*

Proof. Let $\varkappa, \ell, a, b \in Ł$ such that $((\varkappa * \ell) * \ell) * a \leq b$, that is,

$$(((\varkappa * \ell) * \ell) * a) * b = 0. \tag{41}$$

Then,

$$\begin{aligned} q_n(\varkappa * \ell) &= q_n((\varkappa * \ell) * 0) = q_n((\varkappa * \ell) * (\ell * \ell)) \\ &\leq q_n(a) \vee q_n(b), \\ q_p(\varkappa * \ell) &= q_p((\varkappa * \ell) * 0) = q_p((\varkappa * \ell) * (\ell * \ell)) \\ &\geq q_p(a) \wedge q_p(b), \end{aligned} \tag{42}$$

and so, $q = (Ł; q_n, q_p)$ is a bipolar fuzzy implicative ideal of Ł by Theorem 13. □

Corollary 1. *If $q = (Ł; q_n, q_p)$ is a bipolar fuzzy implicative ideal of Ł, then*

$$\begin{aligned} q_n((\varkappa * v) * (\ell * v)) &\leq \vee\{q_n(w_i) \mid i = 1, 2, \cdots, n\}, \\ q_p((\varkappa * v) * (\ell * v)) &\geq \wedge\{q_p(w_i) \mid i = 1, 2, \cdots, n\}, \end{aligned} \tag{43}$$

whenever $\prod_{i=1}^{n} ((\varkappa * \ell) * v) * w_i = 0$ for all $\varkappa, \ell, v, w_1, \ldots, w_n \in Ł$.

Proof. Let $\varkappa, \ell, v, w_1, \ldots, w_n \in Ł$ such that $\prod_{i=1}^{n} ((\varkappa * \ell) * v) * w_i = 0$. Then,

$$\begin{aligned} q_n((\varkappa * v) * (\ell * v)) &\leq q_n((\varkappa * \ell) * v) \\ &\leq \vee\{q_n(w_i) \mid i = 1, 2, \cdots, n\}, \\ q_p((\varkappa * v) * (\ell * v)) &\geq q_p((\varkappa * \ell) * v) \\ &\geq \wedge\{q_p(w_i) \mid i = 1, 2, \cdots, n\}. \end{aligned} \tag{44}$$

This completes the proof. □

Theorem 15 (Extension Property). *Let $q = (Ł; q_n, q_p)$ and $g = (Ł; g_n, g_p)$ be bipolar fuzzy ideals of Ł such that $q_n(0) = g_n(0)$ and $q_p(0) = g_p(0)$ and $q_n(\varkappa) \geq g_n(\varkappa)$ and $q_p(\varkappa) \leq g_p(\varkappa)$ for all $\varkappa \in Ł$. If $q = (Ł; q_n, q_p)$ is a bipolar fuzzy implicative ideal of Ł, then so is $g = (Ł; g_n, g_p)$.*

Proof. We assume that $q = (Ł; q_n, q_p)$ is a bipolar fuzzy implicative ideal of Ł. For any $\varkappa, \ell, v \in Ł$, we have

$$
\begin{aligned}
&g_n(((\varkappa * \upsilon) * (\ell * \upsilon)) * ((\varkappa * \ell) * \upsilon)),\\
&\quad = g_n(((\varkappa * \upsilon) * ((\varkappa * \ell) * \upsilon)) * (\ell * \upsilon)),\\
&\quad = g_n(((\varkappa * ((\varkappa * \ell) * \upsilon)) * \upsilon) * (\ell * \upsilon))\\
&\quad \leq q_n(((\varkappa * ((\varkappa * \ell) * \upsilon)) * \upsilon) * (\ell * \upsilon))\\
&\quad \leq q_n(((\varkappa * ((\varkappa * \ell) * \upsilon)) * \ell) * \upsilon),\\
&\quad = q_n(((\varkappa * \ell) * ((\varkappa * \ell) * \upsilon)) * \upsilon),\\
&\quad = q_n(((\varkappa * \ell) * \upsilon) * ((\varkappa * \ell) * \upsilon)),\\
&\quad = q_n(0) = g_n(0),\\
&g_p(((\varkappa * \upsilon) * (\ell * \upsilon)) * ((\varkappa * \ell) * \upsilon)),\\
&\quad = g_p(((\varkappa * \upsilon) * ((\varkappa * \ell) * \upsilon)) * (\ell * \upsilon)),\\
&\quad = g_p(((\varkappa * ((\varkappa * \ell) * \upsilon)) * \upsilon) * (\ell * \upsilon))\\
&\quad \geq q_p(((\varkappa * ((\varkappa * \ell) * \upsilon)) * \upsilon) * (\ell * \upsilon))\\
&\quad \geq q_p(((\varkappa * ((\varkappa * \ell) * \upsilon)) * \ell) * \upsilon),\\
&\quad = q_p(((\varkappa * \ell) * ((\varkappa * \ell) * \upsilon)) * \upsilon),\\
&\quad = q_p(((\varkappa * \ell) * \upsilon) * ((\varkappa * \ell) * \upsilon)),\\
&\quad = q_p(0) = g_p(0).
\end{aligned}
\tag{45}
$$

It follows from Definition 1 (i) and (ii) that

$$
\begin{aligned}
&g_n((\varkappa * \upsilon) * (\ell * \upsilon))\\
&\quad \leq g_n(((\varkappa * \upsilon) * (\ell * \upsilon)) * ((\varkappa * \ell) * \upsilon)) \vee g_n((\varkappa * \ell) * \upsilon)\\
&\quad \leq g_n(0) \vee g_n((\varkappa * \ell) * \upsilon),\\
&\quad = g_n((\varkappa * \ell) * \upsilon),\\
&g_p((\varkappa * \upsilon) * (\ell * \upsilon))\\
&\quad \geq g_p(((\varkappa * \upsilon) * (\ell * \upsilon)) * ((\varkappa * \ell) * \upsilon)) \wedge g_p((\varkappa * \ell) * \upsilon)\\
&\quad \geq g_p(0) \wedge g_p((\varkappa * \ell) * \upsilon),\\
&\quad = g_p((\varkappa * \ell) * \upsilon),
\end{aligned}
\tag{46}
$$

for all $\varkappa, \ell, \upsilon \in \text{Ł}$. Hence, by Theorem 7, $g = (\text{Ł}; g_n, g_p)$ is a bipolar fuzzy implicative ideal of Ł. □

5. Conclusions

In the present paper, we apply the notion of a bipolar-valued fuzzy set to implicative ideals of BCK-algebras and obtain more related results. We considered the relation of a bipolar fuzzy ideal with a bipolar fuzzy implicative ideal and provided characterizations of a bipolar fuzzy implicative ideal. Also, we studied conditions for a bipolar fuzzy set to be a bipolar fuzzy implicative ideal. Furthermore, an extension property for a bipolar fuzzy implicative ideal is discussed.

We hope that this work will give a deep impact on the upcoming research in this field and other fuzzy algebraic study to open up new horizons of interest and innovations. One may apply this concept to study some application fields such as decision making, knowledge base system, and data analysis. In our opinion, these definitions and main results can be similarly extended to some other algebraic systems such as subtraction algebras, B-algebras, MV-algebras, d-algebras, and Q-algebras.

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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