Hindawi
Journal of Mathematics
Volume 2021, Article ID 6624321, 12 pages
https://doi.org/10.1155/2021/6624321

*Research Article*

# Strong Convergence Results of Split Equilibrium Problems and Fixed Point Problems

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Received 24 November 2020; Revised 22 December 2020; Accepted 13 January 2021; Published 29 January 2021

Academic Editor: George Psihoyios



In this paper, we investigate the split equilibrium problem and fixed point problem in Hilbert spaces. We propose an iterative scheme for solving such problem in which the involved equilibrium bifunctions $f$ and $g$ are pseudomonotone and monotone, respectively, and the operators $S$ and $T$ are all pseudocontractive. We show that the suggested scheme converges strongly to a solution of the considered problem.

## 1. Introduction

Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two real Hilbert spaces. Let $C$ and $Q$ be two nonempty, closed, and convex subsets of $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively. Let $f\colon C\times C\longrightarrow\mathbb{R}$ be a bifunction. Recall that the equilibrium problem is to find a point $x^*\in C$ such that

$$f(x^*,x)\geq 0,\quad \forall x\in C. \tag{1}$$

Use $\mathrm{SEP}(C,f)$ to denote the solution set of equilibrium problem (1).

Equilibrium problems have been considered broadly in the literature (see e.g. [1–5]). Now, it is known that variational inequalities ([6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]) and fixed point problems ([18, 19, 20, 21, 22, 23]) can be transformed in the form of (1). For every $\sigma>0$ and $x\in H$, there exists a unique point $z\in C$ such that $f(z,y)+(1/\sigma)\langle z-x,y-x\rangle\geq 0$, $\forall y\in C$ (see [2]). Thus, for solving equilibrium problem (1), an important technique is to use the resolvent of bifunction $f$ ([2]). Another important method for solving equilibrium problem (1) is to use linear search technique [4].

Let $S\colon C\longrightarrow C$ and $T\colon Q\longrightarrow Q$ be two operators. Let $\mathrm{Fix}(S)$ and $\mathrm{Fix}(T)$ be the fixed point sets of $S$ and $T$, respectively. Let $g\colon Q\times Q\longrightarrow\mathbb{R}$ be a bifunction. Let $A\colon H_1\longrightarrow H_2$ be a bounded linear operator. In this paper, we concern the following split problem of finding a point $\tilde{u}\in C$ such that

$$\begin{aligned}&\tilde{u}\in\mathrm{SEP}(C,f)\cap\mathrm{Fix}(S),\\&A\tilde{u}\in\mathrm{SEP}(Q,g)\cap\mathrm{Fix}(T).\end{aligned}\tag{2}$$

Denote the solution set of (2) by $\Gamma$, i.e., $\Gamma=\{x^*\in\mathrm{SEP}(C,f)\cap\mathrm{Fix}(S),Ax^*\in\mathrm{SEP}(Q,g)\cap\mathrm{Fix}(T)\}$.

The split problem has received many concerns (see [13, 24–28]) due to its extensive applications in image recovery and signal processing, control theory, and so on. Note that the split problem (2) includes the following split problems as special cases:

(i) The split equilibrium problem studied in [29, 30] can be formulated to find an element $\tilde{u}\in C$ such that

$$\begin{aligned}\tilde{u} &\in \mathrm{SEP}(C, f),\\ A\tilde{u} &\in \mathrm{SEP}(Q, g).\end{aligned} \tag{3}$$

The solution set of (3) is denoted by $\Gamma_1$.

(ii) The split fixed point problem considered in [31, 32, 33, 34] reduces to find a point $\tilde{u} \in C$ such that

$$\begin{aligned}\tilde{u} &\in \mathrm{Fix}(S),\\ A\tilde{u} &\in \mathrm{Fix}(T).\end{aligned} \tag{4}$$

The solution set of (4) is denoted by $\Gamma_2$.

Numerical iterative algorithms have been proposed for finding a split problem of the set of solutions of equilibrium problems and the set of fixed points of nonexpansive operators; see, for example, [35–39] and the references therein. Recently, Yao et al. [40] proposed an iterative scheme for solving the split problem (2) and they obtained the weak convergence of the suggested scheme.

In this paper, we continuously study the split problem (2) in which the involved equilibrium bifunctions $f$ and $g$ are pseudomonotone and monotone, respectively, and the operators $S$ and $T$ are all pseudocontractive. We propose an iterative scheme for solving the split problem (2) and strong convergence results are obtained.

## 2. Preliminaries

Let $\mathscr{H}_1$ be a real Hilbert space with its inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let $C$ be a nonempty, convex, and closed subset of $\mathscr{H}_1$. Let $P_C\colon \mathscr{H}_1 \longrightarrow C$ be the metric projection defined by

$$P_C(x) = \arg\min_{y\in C} \|y - x\|. \tag{5}$$

$P_C$ satisfies: for given $x \in H_1$,

$$\langle x - P_C(x), y - P_C(x)\rangle \le 0, \quad \forall y \in C. \tag{6}$$

Let $f\colon C \times C \longrightarrow \mathbb{R}$ be a bifunction. Recall that $f$ is said to be monotone if

$$f\left(u^\dagger, v^\dagger\right) + f\left(v^\dagger, u^\dagger\right) \le 0, \quad \forall u^\dagger, v^\dagger \in C. \tag{7}$$

$f$ is said to be pseudomonotone if

$$f\left(u^\dagger, v^\dagger\right) \ge 0 \text{ implies } f\left(v^\dagger, u^\dagger\right) \le 0, \quad \forall u^\dagger, v^\dagger \in C. \tag{8}$$

Let $S$: $C \longrightarrow C$ be an operator. $S$ is called pseudocontractive if

$$\left\|Sx - Sx^\dagger\right\|^2 \le \left\|x - x^\dagger\right\|^2 + \left\|(I - S)x - (I - S)x^\dagger\right\|^2, \quad \forall x, x^\dagger \in C. \tag{9}$$

$S$ is called $L$-Lipschitz if there exists a constant $L \ge 0$ such that

$$\left\|Sx - Sx^\dagger\right\| \le L\left\|x - x^\dagger\right\|, \quad \forall x, x^\dagger \in C. \tag{10}$$

If $L = 1$, then $S$ is said to be nonexpansive. If $L < 1$, then $S$ is said to be $L$-contraction.

In the sequel, we use the following symbols. Let $\{x^k\}$ be a sequence in $C$:

(i) $x^k \rightharpoonup x^\dagger$ means the weak convergence of $x^k$ to $x^\dagger$ as $k \longrightarrow \infty$

(ii) $x^k \longrightarrow x^\dagger$ means the strong convergence of $x^k$ to $x^\dagger$ as $k \longrightarrow \infty$

(iii) $\omega_w(x^k) = \{x^\dagger \colon \exists \{x^{k_i}\} \subset \{x^k\} \text{ such that } x^{k_i} \rightharpoonup x^\dagger \ (i \longrightarrow \infty)\}$

Recall that $f$ is said to be jointly sequently weakly continuous on $C \times C$, if for two sequences $x^k \in C$ and $y^k \in C$ satisfy $x^k \rightharpoonup u^\dagger$ and $y^k \rightharpoonup v^\dagger$, then we have $f(x^k, y^k) \longrightarrow f(u^\dagger, v^\dagger)$.

Let $\mathscr{H}_2$ be a real Hilbert space with its inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let Q be a nonempty, convex, and closed subset of $\mathscr{H}_2$. Let $\varphi$: $Q \longrightarrow (-\infty, +\infty]$ be a proper, lower semicontinuous, and convex function. Then, the subdifferential $\partial\varphi$ of $\varphi$ is defined by

$$\partial\varphi(u) := \left\{v^\dagger \in H_2 \colon \varphi(u) + \langle v^\dagger, u^\dagger - u\rangle \le \varphi\left(u^\dagger\right), \forall u^\dagger \in Q\right\}, \tag{11}$$

for each $u \in Q$.

It is well known that

$$u^\dagger = \arg\min_{u\in Q}\{\varphi(u)\} \Longleftrightarrow 0 \in \partial\varphi\left(u^\dagger\right) + N_Q\left(u^\dagger\right), \tag{12}$$

where $N_Q(u^\dagger) = \{\omega \in H_2 \colon \langle \omega, u - u^\dagger\rangle \le 0, \forall u \in Q\}$.

The following lemma can be found in [41]. For the completeness, we include the detail of proof.

**Lemma 1** (see [41]). *Let $S$: $C \longrightarrow C$ be an $L_1$-Lipschitz pseudocontractive operator. Then, for all $\tilde{u} \in C$ and $u^\dagger \in Fix(S)$, we have*

$$\left\|u^\dagger - S((1 - \eta)\tilde{u} + \eta S\tilde{u})\right\|^2 \le \left\|\tilde{u} - u^\dagger\right\|^2 + (1 - \eta) \cdot \|\tilde{u} - S((1 - \eta)\tilde{u} + \eta S\tilde{u})\|^2, \tag{13}$$

*where* $0 < \eta < (1/\sqrt{1 + L_1^2} + 1)$.

*Proof.* Since $u^\dagger \in Fi\tilde{u}(S)$, we have from (9) that

$$\left\|S((1 - \eta)I + \eta S)\tilde{u} - u^\dagger\right\|^2 \le \left\|(1 - \eta)\left(\tilde{u} - tu^\dagger\right) + \eta\left(S\tilde{u} - u^\dagger\right)\right\|^2 + \|(1 - \eta)\tilde{u} + \eta S\tilde{u} - S((1 - \eta)\tilde{u} + \eta S\tilde{u})\|^2, \tag{14}$$

$$\left\|S\tilde{u} - u^\dagger\right\|^2 \le \left\|\tilde{u} - u^\dagger\right\|^2 + \|S\tilde{u} - \tilde{u}\|^2, \tag{15}$$

for all $\tilde{u} \in C$.

Since $S$ is $L_1$-Lipschitzian and $\tilde{u} - ((1 - \eta)\tilde{u} + \eta S\tilde{u}) = \eta(\tilde{u} - tSn\tilde{u})$, we have

$$\|S\tilde{u} - S((1 - \eta)\tilde{u} + \eta S\tilde{u})\| \le \eta L_1\|\tilde{u} - S\tilde{u}\|. \tag{16}$$

According to (15), we obtain

$$\begin{aligned}
&\left\|(1-\eta)\left(\tilde{u}-tu^{\dagger}\right)+\eta\left(S\tilde{u}-u^{\dagger}\right)\right\|^{2}\\
&\quad=(1-\eta)\left\|\tilde{u}-u^{\dagger}\right\|^{2}+\eta\left\|S\tilde{u}-u^{\dagger}\right\|^{2}-\eta(1-\eta)\|\tilde{u}-S\tilde{u}\|^{2}\\
&\quad\leq(1-\eta)\left\|\tilde{u}-u^{\dagger}\right\|^{2}+\eta\left(\left\|\tilde{u}-u^{\dagger}\right\|^{2}+\|S\tilde{u}-\tilde{u}\|^{2}\right)\\
&\qquad-\eta(1-\eta)\|\tilde{u}-S\tilde{u}\|^{2}\\
&\quad=\left\|\tilde{u}-u^{\dagger}\right\|^{2}+\eta^{2}\|S\tilde{u}-\tilde{u}\|^{2}.
\end{aligned}\tag{17}$$

Based on (16), we conclude

$$\begin{aligned}
&\|(1-\eta)\tilde{u}+\eta S\tilde{u}-S((1-\eta)\tilde{u}+\eta S\tilde{u})\|^{2}\\
&\quad=\|(1-\eta)(\tilde{u}-tSn((1-\eta)\tilde{u}+\eta S\tilde{u}))+\eta(S\tilde{u}-S((1-\eta)\tilde{u}+\eta S\tilde{u}))\|^{2}\\
&\quad=(1-\eta)\|\tilde{u}-S((1-\eta)\tilde{u}+\eta S\tilde{u})\|^{2}+\eta\|S\tilde{u}-S((1-\eta)\tilde{u}+\eta S\tilde{u})\|^{2}\\
&\qquad-\eta(1-\eta)\|\tilde{u}-S\tilde{u}\|^{2}\\
&\quad\leq(1-\eta)\|\tilde{u}-S((1-\eta)\tilde{u}+\eta S\tilde{u})\|^{2}-\eta\left(1-\eta-\eta^{2}L_{1}^{2}\right)\|\tilde{u}-S\tilde{u}\|^{2}.
\end{aligned}\tag{18}$$

By (14), (17), and (18), we obtain

$$\begin{aligned}
\left\|S((1-\eta)I+\eta S)\tilde{u}-u^{\dagger}\right\|^{2}&\leq\left\|\tilde{u}-u^{\dagger}\right\|^{2}+\eta^{2}\|\tilde{u}-S\tilde{u}\|^{2}\\
&\quad+(1-\eta)\|\tilde{u}-S((1-\eta)\tilde{u}+\eta S\tilde{u})\|^{2}\\
&\quad-\eta\left(1-\eta-\eta^{2}L_{1}^{2}\right)\|\tilde{u}-S\tilde{u}\|^{2}\\
&=\left\|\tilde{u}-u^{\dagger}\right\|^{2}+(1-\eta)\|\tilde{u}-S((1-\eta)I+\eta S)\tilde{u}\|^{2}\\
&\quad-\eta\left(1-2\eta-\eta^{2}L_{1}^{2}\right)\|\tilde{u}-S\tilde{u}\|^{2}.
\end{aligned}\tag{19}$$

Since $\eta<(1/\sqrt{1+L_1^2}+1)$, $1-2\eta-\eta^2L_1^2>0$. Hence, we can deduce the desired result from (19). □

**Lemma 2** (see [42]). *Let $S\colon C\longrightarrow C$ be a continuous pseudocontractive operator. Then,*

*(i) $\mathrm{Fix}(S)\subset C$ is closed and convex*

*(ii) $S$ is demiclosedness, i.e., if $x^k\rightharpoonup\tilde{u}$ and $Sx^k\longrightarrow z^{\dagger}$ as $k\longrightarrow\infty$, then $S\tilde{u}=z^{\dagger}$.*

Here, we state some conditions on $f$ and $g$ which will be used in the sequel.

Let $C$ and $Q$ be two nonempty, closed, and convex subsets of real Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively. Let $f\colon C\times C\longrightarrow\mathbb{R}$ and $g\colon Q\times Q\longrightarrow\mathbb{R}$ be two bifunctions. Assume that

(i) (A1): $f(z^{\dagger},z^{\dagger})=0$ for all $z^{\dagger}\in C$

(ii) (A2): $f$ is pseudomonotone on $\mathrm{SEP}(C,f)$

(iii) (A3): $f$ is jointly sequently weakly continuous on $C\times C$

(iv) (A4): $f(z^{\dagger},\cdot)$ is convex and subdifferentiable on $C$ for all $z^{\dagger}\in C$

(v) (B1): $g(z^{\dagger},z^{\dagger})=0$ for all $z^{\dagger}\in Q$

(vi) (B2): $g$ is monotone on $Q$

(vii) (B3): $g(u,\cdot)$ is convex and lower semicontinuous on $Q$ for each $u\in Q$

(viii) (B4): for all $u,v,w\in Q$, $\limsup_{\lambda\downarrow 0}g(\lambda w+(1-\lambda)u,v)\leq g(u,v)$

**Lemma 3** (see [1, 2]). *Assume that $g$ satisfies conditions (B1)–(B4). For $\varsigma>0$ and $u\in H_2$, there exists $w\in Q$ such that*

$$g(w,v)+\frac{1}{\varsigma}\langle v-w,w-u\rangle\geq 0,\quad\forall v\in Q.\tag{20}$$

Let the operator $J_{\varsigma}^{g}$ be defined by

$$J_{\varsigma}^{g}(u)=\left\{w\in Q\colon g(w,v)+\frac{1}{\varsigma}\langle v-w,w-u\rangle\geq 0,\forall v\in Q\right\}.\tag{21}$$

We have the following conclusions:

(i) $J_{\varsigma}^{g}$ is single-valued and firmly nonexpansive, that is, for any $u,v\in H_2$,

$$\left\|J_{\varsigma}^{g}(u)-J_{\varsigma}^{g}(v)\right\|^{2}\leq\langle J_{\varsigma}^{g}(u)-J_{\varsigma}^{g}(v),u-v\rangle. \tag{22}$$

(ii) $\mathrm{SEP}(Q,g)$ is closed and convex and $\mathrm{SEP}(Q,g)=\mathrm{Fix}(J_{\varsigma}^{g})$.

(iii) For $\varsigma_1,\varsigma_2>0$ and $u,v\in H_2$, we have

$$\left\|J_{\varsigma_1}^{g}(u)-J_{\varsigma_2}^{g}(v)\right\|\leq\|u-v\|+\frac{|\varsigma_2-\varsigma_1|}{\varsigma_2}\left\|J_{\varsigma_2}^{g}(v)-v\right\|. \tag{23}$$

**Lemma 4** (see [4]). *Assume that $f$ satisfies conditions (A1)-(A4). Let $\{\beta_k\}$ be a sequence satisfying $\beta_k\in[\underline{\beta},\overline{\beta}]\subset(0,1]$. For given $v^k\in C$, let the sequence $\{y^k\}$ be generated by*

$$y^k=\arg\min_{u^{\dagger}\in C}\left\{f\left(v^k,u^{\dagger}\right)+\frac{1}{2\beta_k}\left\|v^k-u^{\dagger}\right\|^2\right\}. \tag{24}$$

Then the boundedness of $\{v^k\}$ implies that $\{y^k\}$ is bounded.

**Lemma 5** (see [5]). *Assume that $f$ satisfies conditions (A1)–(A4). For given two points $\overline{u},\overline{v}\in C$ and two sequences $\{a^k\}\subset C$ and $\{b^k\}\subset C$, if $a^k\rightharpoonup\overline{u}$ and $b^k\rightharpoonup\overline{v}$, respectively, then, for any $\varepsilon>0$, there exist $\vartheta>0$ and $N_\varepsilon\in\mathbb{N}$ such that*

$$\partial_2 f\left(b^k,a^k\right)\subset\partial_2 f(\overline{v},t\overline{u})+\frac{\varepsilon}{\vartheta}B, \tag{25}$$

*for every $k\geq N_\varepsilon$, where $B:=\{b\in H_1:\|b\|\leq 1\}$.*

**Lemma 6** (see [43]). *Let $\{a_n\}\subset(0,\infty)$, $\{b_n\}\subset(0,1)$, and $\{c_n\}$ be three real number sequences. If*

$$a_{n+1}\leq(1-b_n)a_n+c_n, \tag{26}$$

*for all $n\geq 0$ with $\sum_{n=1}^{\infty}b_n=\infty$ and $\limsup_{n\longrightarrow\infty}(c_n/b_n)\leq 0$ or $\sum_{n=1}^{\infty}|c_n|<\infty$, then $\lim_{n\longrightarrow\infty}a_n=0$.*

## 3. Main Results

In this section, in order to solve problem (2), we first present an iterative algorithm and consequently prove its strong convergence.

Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two real Hilbert spaces. Let $C$ and $Q$ be two nonempty, closed, and convex subsets of $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively. Assume that

(i) $h: C\longrightarrow C$ is a $\kappa$-contractive operator

(ii) $S: C\longrightarrow C$ is an $L_1$-Lipschitz pseudocontractive operator and $T: Q\longrightarrow Q$ is an $L_2$-Lipschitz pseudocontractive operator with $L_1>1$ and $L_2>1$

(iii) $f$ and $g$ are two bifunctions satisfying conditions (A1)–(A4) and conditions (B1)–(B4), respectively

(iv) $A:\mathscr{H}_1\longrightarrow\mathscr{H}_2$ is a bounded linear operator and $A^*$ is its adjoint

Let $\{\delta_k\}$, $\{\eta_k\}$, $\{\beta_k\}$, $\{\tau_k\}$, $\{\varsigma_k\}$, $\{\zeta_k\}$, $\{\lambda_k\}$, and $\{\mu_k\}$ be real number sequences and $\alpha$, $\vartheta$, and $\gamma$ be constants. Next, we introduce our iterative algorithm.

*Algorithm 1.* Fix an initial point $x^0\in C$. Set $k=0$.

Step 1: assume that $x^k$ is known and compute

$$v^k=(1-\delta_k)x^k+\delta_k S\left[(1-\eta_k)x^k+\eta_k Sx^k\right]. \tag{27}$$

Step 2: compute

$$y^k=\arg\min_{y^{\dagger}\in C}\left\{f\left(v^k,y^{\dagger}\right)+\frac{1}{2\beta_k}\left\|v^k-y^{\dagger}\right\|^2\right\}. \tag{28}$$

If $y^k=v^k$, then set $u^k=v^k$ and go to Step 5. Otherwise, go to Step 3.

Step 3: let $m_k=\min\{1,2,\ldots,k,\ldots\}$ such that

$$f\left(z^{k,m_k},v^k\right)-f\left(z^{k,m_k},y^k\right)\geq\frac{\alpha}{2\beta_k}\left\|v^k-y^k\right\|^2, \tag{29}$$

where

$$z^{k,m_k}=(1-\vartheta^{m_k})v^k+\vartheta^{m_k}y^k. \tag{30}$$

Write $\vartheta_k=\vartheta^{m_k}$ and $z^k=z^{k,m_k}$.

Step 4: compute

$$u^k=P_C\left(v^k-\tau_k\iota_k\nu^k\right), \tag{31}$$

where $\nu^k\in\partial_2 f(z^k,v^k)$ and $\iota_k=(f(z^k,v^k)/\|\nu^k\|^2)$.

Step 5:

For any $v\in Q$, find $w^k$ such that

$$g\left(w^k,v\right)+\frac{1}{\varsigma_k}\langle v-w^k,w^k-Au^k\rangle\geq 0. \tag{32}$$

Compute

$$q^k=(1-\zeta_k)w^k+\zeta_k T\left[(1-\lambda_k)w^k+\lambda_k Tw^k\right]. \tag{33}$$

Step 6: compute

$$x^{k+1}=\mu_k h\left(x^k\right)+(1-\mu_k)P_C\left[u^k+\gamma A^*\left(q^k-Au^k\right)\right]. \tag{34}$$

Step 7: set $k:=k+1$ and return to Step 1.

In order to demonstrate the convergence of Algorithm 1, we need some additional assumptions on the iterative parameters. Suppose that the following conditions are satisfied:

(C1): $0<\underline{\delta}<\delta_k<\overline{\delta}<\eta_k<\overline{\eta}<(1/\sqrt{1+L_1^2}+1)(\forall k\geq 0)$ and $\alpha,\vartheta\in(0,1)$

(C2): $\beta_k\in[\gamma_1,\gamma_2]\subset(0,1]$; $\tau_k\in[\tau_1,\tau_2]\subset(0,2)$ and $0<\varsigma\leq\varsigma_k<+\infty$

(C3): $0<\underline{\zeta}<\zeta_k<\overline{\zeta}<\lambda_k<\overline{\lambda}<(1/\sqrt{1+L_2^2}+1)(\forall k\geq 0)$ and $\gamma\in(0,1/\|A\|^2)$

(C4): $\lim_{k\longrightarrow+\infty}\mu_k=0$ and $\sum_{k=0}^{+\infty}\mu_k=+\infty$

We have the following remark which can be found in [4].

*Remark 1*

(1) If $y^k=v^k$, then $y_n\in\mathrm{SEP}(C,f)$

(2) The linesearch rule (29) is well defined

(3) $0 \notin \partial_2 f(z^k, v^k)$

(4) $f(z^k, v^k) > 0$

(5) $\|u^k - p\|^2 \le \|v^k - p\|^2 - \tau_k(2-\tau_k)(\iota_k\|v^k\|)^2$ for all $p \in \text{SEP}(C, f)$

Next, we prove our main result.

**Theorem 1.** *Suppose that $\Gamma \neq \emptyset$. Then, the sequence $\{x^k\}$ generated by (34) converges strongly to $q^\dagger = P_\Gamma h(q^\dagger)$.*

*Proof.* Let $x^* \in \Gamma$. We have $x^* \in \text{SEP}(C, f) \cap \text{Fix}(S)$ and $Ax^* \in \text{SEP}(Q, g) \cap \text{Fix}(T)$. By (27) and Lemma 1, we get

$$\begin{aligned}
\left\|v^k - x^*\right\|^2 &= \left\|(1-\delta_k)\left(x^k - x^*\right) + \delta_k\left(S\left[(1-\eta_k)x^k + \eta_k Sx^k\right] - x^*\right)\right\|^2 \\
&= (1-\delta_k)\left\|x^k - x^*\right\|^2 + \delta_k\left\|S\left[(1-\eta_k)x^k + \eta_k Sx^k\right] - x^*\right\|^2 \\
&\quad - (1-\delta_k)\delta_k\left\|S\left[(1-\eta_k)x^k + \eta_k Sx^k\right] - x^k\right\|^2 \\
&\le (1-\delta_k)\left\|x^k - x^*\right\|^2 + \delta_k(1-\eta_k)\left\|S\left[(1-\eta_k)x^k + \eta_k Sx^k\right] - x^k\right\|^2 \\
&\quad + \delta_k\left\|x^k - x^*\right\|^2 - (1-\delta_k)\delta_k\left\|S\left[(1-\eta_k)x^k + \eta_k Sx^k\right] - x^k\right\|^2 \\
&= \left\|x^k - x^*\right\|^2 - \delta_k(\eta_k - \delta_k)\left\|S\left[(1-\eta_k)x^k + \eta_k Sx^k\right] - x^k\right\|^2 \\
&\le \left\|x^k - x^*\right\|^2.
\end{aligned} \tag{35}$$

From (31) and Remark 1, we have

$$\begin{aligned}
\left\|u^k - x^*\right\|^2 &\le \left\|v^k - x^*\right\|^2 - \tau_k(2-\tau_k)\left(\iota_k\left\|v^k\right\|\right)^2 \\
&\le \left\|v^k - x^*\right\|^2.
\end{aligned} \tag{36}$$

According to (32) and Lemma 3, we have $w^k = J^g_{\varsigma_k} Au^k$ and $Ax^* \in \text{Fix}(J^g_{\varsigma_k})$. Since $J^g_{\varsigma_k}$ is firmly nonexpansive, we deduce

$$\begin{aligned}
\left\|w^k - Ax^*\right\|^2 &= \left\|J^g_{\varsigma_k} Au^k - J^g_{\varsigma_k} Ax^*\right\|^2 \\
&\le \langle J^g_{\varsigma_k} Au^k - J^g_{\varsigma_k} Ax^*, Au^k - Ax^*\rangle \\
&= \langle w^k - Ax^*, Au^k - Ax^*\rangle \\
&= \frac{1}{2}\left(\left\|w^k - Ax^*\right\|^2 + \left\|Au^k - Ax^*\right\|^2 - \left\|w^k - Au^k\right\|^2\right).
\end{aligned} \tag{37}$$

It follows that

$$\left\|w^k - Ax^*\right\|^2 \le \left\|Au^k - Ax^*\right\|^2 - \left\|w^k - Au^k\right\|^2. \tag{38}$$

By virtue of (33) and Lemma 1, we obtain

$$\begin{aligned}
\left\|q^k - Ax^*\right\|^2 &= \left\|(1-\zeta_k)\left(w^k - Ax^*\right) + \zeta_k\left(T\left[(1-\lambda_k)w^k + \lambda_k Tw^k\right] - Ax^*\right)\right\|^2 \\
&= (1-\zeta_k)\left\|w^k - Ax^*\right\|^2 + \zeta_k\left\|T\left[(1-\lambda_k)w^k + \lambda_k Tw^k\right] - Ax^*\right\|^2 \\
&\quad - (1-\zeta_k)\zeta_k\left\|T\left[(1-\lambda_k)w^k + \lambda_k Tw^k\right] - w^k\right\|^2 \\
&\le \left\|w^k - Ax^*\right\|^2 + \zeta_k(1-\lambda_k)\left\|T\left[(1-\lambda_k)w^k + \lambda_k Tw^k\right] - w^k\right\|^2 \\
&\quad - (1-\zeta_k)\zeta_k\left\|T\left[(1-\lambda_k)w^k + \lambda_k Tw^k\right] - w^k\right\|^2 \\
&= \left\|w^k - Ax^*\right\|^2 - (\lambda_k - \zeta_k)\zeta_k\left\|T\left[(1-\lambda_k)w^k + \lambda_k Tw^k\right] - w^k\right\|^2 \\
&\le \left\|w^k - Ax^*\right\|^2.
\end{aligned} \tag{39}$$

Thanks to (38) and (39), we get

$$\left\|q^k - Ax^*\right\|^2 \le \left\|Au^k - Ax^*\right\|^2 - \left\|w^k - Au^k\right\|^2. \quad (40)$$

Consequently,

$$\begin{aligned}\langle u^k - x^*, A^*(q^k - Au^k)\rangle &= \langle Au^k - Ax^*, q^k - Au^k\rangle \\ &= \langle q^k - Ax^*, q^k - Au^k\rangle - \left\|q^k - Au^k\right\|^2 \\ &= \frac{1}{2}\left[\left\|q^k - Ax^*\right\|^2 + \left\|q^k - Au^k\right\|^2 - \left\|Au^k - Ax^*\right\|^2\right] \\ &\quad - \left\|q^k - Au^k\right\|^2 \\ &= \frac{1}{2}\left[\left\|q^k - Ax^*\right\|^2 - \left\|Au^k - Ax^*\right\|^2\right] - \frac{1}{2}\left\|q^k - Au^k\right\|^2 \\ &\le -\frac{1}{2}\left\|w^k - Au^k\right\|^2 - \frac{1}{2}\left\|q^k - Au^k\right\|^2.\end{aligned} \quad (41)$$

Set $t^k = P_C[u^k + \gamma A^*(q^k - Au^k)]$ for all $k \ge 0$. In view of (35), (36), and (41), using the nonexpansivity of $P_C$, we have

$$\begin{aligned}\left\|t^k - x^*\right\|^2 &= \left\|P_C\left[u^k + \gamma A^*(q^k - Au^k)\right] - P_C[x^*]\right\|^2 \\ &\le \left\|u^k - x^* + \gamma A^*(q^k - Au^k)\right\|^2 \\ &= \left\|u^k - x^*\right\|^2 + \left\|\gamma A^*(q^k - Au^k)\right\|^2 + 2\gamma\langle A^*(q^k - Au^k), u^k - x^*\rangle \\ &\le \left\|u^k - x^*\right\|^2 + \gamma^2\|A\|^2\left\|q^k - Au^k\right\|^2 - \gamma\left\|w^k - Au^k\right\|^2 \\ &\quad - \gamma\left\|q^k - Au^k\right\|^2 \\ &= \left\|u^k - x^*\right\|^2 - \gamma(1 - \gamma\|A\|^2)\left\|q^k - Au^k\right\|^2 - \gamma\left\|w^k - Au^k\right\|^2 \\ &\le \left\|x^k - x^*\right\|^2 - \tau_k(2 - \tau_k)\left(\iota_k\left\|\nu^k\right\|\right)^2 - \delta_k(\eta_k - \delta_k)\left\|S[(1 - \eta_k)x^k + \eta_k Sx^k] - x^k\right\|^2 \\ &\quad - \gamma(1 - \gamma\|A\|^2)\left\|q^k - Au^k\right\|^2 - \gamma\left\|w^k - Au^k\right\|^2 \\ &\le \left\|x^k - x^*\right\|^2.\end{aligned} \quad (42)$$

From (34), we get

$$\begin{aligned}\left\|x^{k+1} - x^*\right\| &= \left\|\mu_k(h(x^k) - x^*) + (1 - \mu_k)(t^k - x^*)\right\| \\ &\le \mu_k\left\|h(x^k) - h(x^*)\right\| + \mu_k\left\|h(x^*) - x^*\right\| + (1 - \mu_k)\left\|t^k - x^*\right\| \\ &\le \mu_k\kappa\left\|x^k - x^*\right\| + \mu_k\left\|h(x^*) - x^*\right\| + (1 - \mu_k)\left\|x^k - x^*\right\| \\ &= [1 - (1 - \kappa)\mu_k]\left\|x^k - x^*\right\| + \mu_k\left\|h(x^*) - x^*\right\| \\ &\le \max\left\{\left\|x^k - x^*\right\|, \frac{\left\|h(x^*) - x^*\right\|}{(1 - \kappa)}\right\}.\end{aligned} \quad (43)$$

By induction, we can obtain that $\|x^k - x^*\| \le \max\{\|x^0 - x^*\|, (\|h(x^*) - x^*\|/(1-\kappa))\}$. Thus, the sequences $\{x^k\}$, $\{u^k\}$, and $\{v^k\}$ are all bounded.

Based on (34), we have

$$\begin{aligned}\left\|x^{k+1} - x^*\right\|^2 &= \left\|\mu_k\left(h\left(x^k\right) - x^*\right) + (1-\mu_k)\left(t^k - x^*\right)\right\|^2 \\ &\le (1-\mu_k)^2\left\|t^k - x^*\right\|^2 + 2\mu_k\langle h\left(x^k\right) - x^*, x^{k+1} - x^*\rangle \\ &\le (1-\mu_k)^2\left\|t^k - x^*\right\|^2 + 2\mu_k\kappa\left\|x^k - x^*\right\|\left\|x^{k+1} - x^*\right\| + 2\mu_k\langle h(x^*) - x^*, x^{k+1} - x^*\rangle \\ &\le (1-\mu_k)^2\left\|t^k - x^*\right\|^2 + \mu_k\kappa\left\|x^k - x^*\right\|^2 + \mu_k\kappa\left\|x^{k+1} - x^*\right\|^2 \\ &\quad + 2\mu_k\langle h(x^*) - x^*, x^{k+1} - x^*\rangle.\end{aligned} \tag{44}$$

It follows that

$$\begin{aligned}\left\|x^{k+1} - x^*\right\|^2 &\le \frac{(1-\mu_k)^2}{1-\kappa\mu_k}\left\|t^k - x^*\right\|^2 + \frac{\kappa\mu_k}{1-\mu_k\kappa}\left\|x^k - x^*\right\|^2 + \frac{2\mu_k}{1-\kappa\mu_k}\langle h(x^*) - x^*, x^{k+1} - x^*\rangle \\ &\le \frac{(1-\mu_k)^2}{1-\kappa\mu_k}\left[\left\|x^k - x^*\right\|^2 - \tau_k(2-\tau_k)\left(\iota_k\left\|v^k\right\|\right)^2 - \gamma\left(1-\gamma\|A\|^2\right)\left\|q^k - Au^k\right\|^2\right. \\ &\quad \left. -\delta_k(\eta_k - \delta_k)\left\|S\left[(1-\eta_k)x^k + \eta_k Sx^k\right] - x^k\right\|^2 - \gamma\left\|w^k - Au^k\right\|^2\right] \\ &\quad + \frac{\kappa\mu_k}{1-\mu_k\kappa}\left\|x^k - x^*\right\|^2 + \frac{2\mu_k}{1-\kappa\mu_k}\langle h(x^*) - x^*, x^{k+1} - x^*\rangle \\ &= \left(1 - \frac{2(1-\kappa) - \mu_k}{1-\kappa\mu_k}\mu_k\right)\left\|x^k - x^*\right\|^2 + \frac{(1-\mu_k)^2\mu_k}{1-\kappa\mu_k}\left\{-\tau_k(2-\tau_k)\frac{\left(\iota_k\left\|v^k\right\|\right)^2}{\mu_k}\right. \\ &\quad - \delta_k(\eta_k - \delta_k)\frac{\left\|S\left[(1-\eta_k)x^k + \eta_k Sx^k\right] - x^k\right\|^2}{\mu_k} - \gamma\left(1-\gamma\|A\|^2\right)\frac{\left\|q^k - Au^k\right\|^2}{\mu_k} \\ &\quad \left. - \frac{\gamma\left\|w^k - Au^k\right\|^2}{\mu_k} + \frac{2}{(1-\mu_k)^2}\langle h(x^*) - x^*, x^{k+1} - x^*\rangle\right\}.\end{aligned} \tag{45}$$

Set $a_k = \|x^k - x^*\|^2$, $b_k = (2(1-\kappa) - \mu_k/1 - \kappa\mu_k)\mu_k$, and

$$\begin{aligned}c_k = \frac{(1-\mu_k)^2}{2(1-\kappa) - \mu_k}&\left\{-\tau_k(2-\tau_k)\frac{\left(\iota_k\left\|v^k\right\|\right)^2}{\mu_k} - \gamma\left(1-\gamma\|A\|^2\right)\frac{\left\|q^k - Au^k\right\|^2}{\mu_k}\right. \\ &- \frac{\gamma\left\|w^k - Au^k\right\|^2}{\mu_k} + \frac{2}{(1-\mu_k)^2}\langle h(x^*) - x^*, x^{k+1} - x^*\rangle \\ &\left. -\delta_k(\eta_k - \delta_k)\frac{\left\|S\left\|(1-\eta_k)x^k + \eta_k Sx^k\right\| - x^k\right\|^2}{\mu_k}\right\}.\end{aligned} \tag{46}$$

for all $k \geq 0$.

Since $\mu_k \longrightarrow 0$ as $k \longrightarrow +\infty$, without loss of generality, we assume that $\mu_k \leq 1-\kappa$ for all $k \geq 0$. From (46), we have

$$\begin{aligned} c_k &\leq \frac{2}{2(1-\kappa)-\mu_k}\langle h(x^*)-x^*, x^{k+1}-x^*\rangle \\ &\leq \frac{2}{1-\kappa}\|h(x^*)-x^*\|\|x^{k+1}-x^*\|. \end{aligned} \tag{47}$$

So, $\limsup_{k\longrightarrow+\infty} c_k < +\infty$. Next, we show that $\limsup_{k\longrightarrow+\infty} c_k \geq -1$. Assume that $\limsup_{k\longrightarrow+\infty} c_k < -1$. Then, there exists a positive integer number $K_0$ such that $c_k < -1$ when $k \geq K_0$. We can rewrite (45) as $a_{k+1} \leq (1-b_k)a_k + b_k c_k$. Thus, for all $k \geq K_0$, from (45), we deduce

$$a_{k+1} \leq (1-b_k)a_k + b_k c_k \leq a_k - b_k, \tag{48}$$

which leads to $a_{k+1} \leq a_{K_0} - \sum_{i=K_0}^{k} b_k$. Therefore,

$$\limsup_{k\longrightarrow+\infty} a_{k+1} \leq a_{K_0} - \limsup_{k\longrightarrow+\infty}\sum_{i=K_0}^{k} b_k. \tag{49}$$

Note that $b_k = (2(1-\kappa)-\mu_k/1-\kappa\mu_k)\mu_k \geq (1-\kappa)\mu_k$. This together with the last inequality implies that $\limsup_{k\longrightarrow+\infty} a_{k+1} \leq -\infty$. It is impossible. Hence, $-1 \leq \limsup_{k\longrightarrow+\infty} c_k < +\infty$. As a result, we can select a subsequence $\{k_i\}$ of $\{k\}$ such that $x^{k_i} \rightharpoonup p^\dagger$ and

$$\begin{aligned} \limsup_{k\longrightarrow+\infty} c_k &= \lim_{i\longrightarrow+\infty} c_{k_i} \\ &= \lim_{i\longrightarrow+\infty}\frac{(1-\mu_{k_i})^2}{2(1-\kappa)-\mu_{k_i}}\Bigg\{-\tau_{k_i}(2-\tau_{k_i})\frac{(\iota_{k_i}\|\nu^{k_i}\|)^2}{\mu_{k_i}} - \frac{\gamma\|w^{k_i}-Au^{k_i}\|^2}{\mu_{k_i}} \\ &\quad - \gamma(1-\gamma\|A\|^2)\frac{\|q^{k_i}-Au^{k_i}\|^2}{\mu_{k_i}} + \frac{2}{(1-\mu_{k_i})^2}\langle h(x^*)-x^*, x^{k_i+1}-x^*\rangle \\ &\quad - \delta_{k_i}(\eta_{k_i}-\delta_{k_i})\frac{\|S[(1-\eta_{k_i})x^{k_i}+\eta_{k_i}Sx^{k_i}]-x^{k_i}\|^2}{\mu_{k_i}}\Bigg\}. \end{aligned} \tag{50}$$

Since the sequence $\{x^{k_i+1}\}$ is bounded, without loss of generality, we assume that $\lim_{i\longrightarrow+\infty}\langle h(x^*)-x^*, x^{k_i+1}-x^*\rangle$ exists. Consequently, from (50), we obtain

$$\begin{aligned} &\lim_{i\longrightarrow+\infty}\Bigg\{\tau_{k_i}(2-\tau_{k_i})\frac{(\iota_{k_i}\|\nu^{k_i}\|)^2}{\mu_{k_i}} + \delta_{k_i}(\eta_{k_i}-\delta_{k_i})\frac{\|S[(1-\eta_{k_i})x^{k_i}+\eta_{k_i}Sx^{k_i}]-x^{k_i}\|^2}{\mu_{k_i}} \\ &\quad + \gamma(1-\gamma\|A\|^2)\frac{\|q^{k_i}-Au^{k_i}\|^2}{\mu_{k_i}} + \frac{\gamma\|w^{k_i}-Au^{k_i}\|^2}{\mu_{k_i}}\Bigg\}, \end{aligned} \tag{51}$$

exists.

By the assumptions, we have $\liminf_{i\longrightarrow+\infty}\tau_{k_i}(2-\tau_{k_i}) > 0$ and $\liminf_{i\longrightarrow+\infty}\delta_{k_i}(\eta_{k_i}-\delta_{k_i}) > 0$, Therefore, we deduce

$$\lim_{i\longrightarrow+\infty}\iota_{k_i}\|\nu^{k_i}\| = 0, \tag{52}$$

$$\lim_{i\longrightarrow+\infty}\|S[(1-\eta_{k_i})x^{k_i}+\eta_{k_i}Sx^{k_i}]-x^{k_i}\| = 0, \tag{53}$$

$$\lim_{i\longrightarrow+\infty}\|q^{k_i}-Au^{k_i}\| = 0, \tag{54}$$

$$\lim_{i\longrightarrow+\infty}\|w^{k_i}-Au^{k_i}\| = 0. \tag{55}$$

By (54) and (55), we get

$$\lim_{i\longrightarrow+\infty}\|q^{k_i}-w^{k_i}\| = 0. \tag{56}$$

In addition, from (31), we have

$$\left\|u^k - v^k\right\| = \left\|P_C\left(v^k - \tau_k \iota_k v^k\right) - P_C\left(v^k\right)\right\| \leq \tau_k \iota_k \left\|v^k\right\|. \tag{57}$$

So, we get from (52) that

$$\lim_{i \longrightarrow +\infty} \left\|u^{k_i} - v^{k_i}\right\| = 0. \tag{58}$$

Observe that

$$\begin{aligned}\left\|Sx^k - x^k\right\| &\leq \left\|Sx^k - S\left[(1-\eta_k)x^k + \eta_k Sx^k\right]\right\| \\ &\quad + \left\|S\left[(1-\eta_k)x^k + \eta_k Sx^k\right] - x^k\right\| \\ &\leq L\eta_k \left\|Sx^k - x^k\right\| + \left\|S\left[(1-\eta_k)x^k + \eta_k Sx^k\right] - x^k\right\|.\end{aligned} \tag{59}$$

It follows that

$$\left\|Sx^k - x^k\right\| \leq \frac{1}{1 - L\eta_k}\left\|S\left[(1-\eta_k)x^k + \eta_k Sx^k\right] - x^k\right\|. \tag{60}$$

This together with (53) implies that

$$\lim_{i \longrightarrow +\infty} \left\|Sx^{k_i} - x^{k_i}\right\| = 0. \tag{61}$$

In addition, by (27) and (53), we have

$$\left\|v^{k_i} - x^{k_i}\right\| \leq \delta_{k_i}\left\|S\left[\left(1-\eta_{k_i}\right)x^{k_i} + \eta_{k_i} Sx^{k_i}\right] - x^{k_i}\right\| \longrightarrow 0. \tag{62}$$

Since $\{v^k\}$ is bounded, by Lemma 4, $\{y^k\}$ is bounded. Consequently, the sequence $\{z^k\}$ is bounded. Applying Lemma 5, we deduce that $\{v^k\}$ is bounded. According to (52), we derive

$$\lim_{i \longrightarrow +\infty} f\left(z^{k_i}, v^{k_i}\right) = \lim_{i \longrightarrow +\infty}\left(\iota_{k_i}\left\|v^{k_i}\right\|\right)\left\|v^{k_i}\right\| = 0. \tag{63}$$

Since $f(z^{k_i}, \cdot)$ is convex, we have

$$\begin{aligned}0 = f\left(z^{k_i}, z^{k_i}\right) &= f\left(z^{k_i}, \left(1-\vartheta_{k_i}\right)v^{k_i} + \vartheta_{k_i} y^{k_i}\right) \\ &\leq \left(1-\vartheta_{k_i}\right) f\left(z^{k_i}, v^{k_i}\right) + \vartheta_{k_i} f\left(z^{k_i}, y^{k_i}\right).\end{aligned} \tag{64}$$

So, we get from (29) that

$$\begin{aligned}f\left(z^{k_i}, v^{k_i}\right) &\geq \vartheta_{k_i}\left[f\left(z^{k_i}, v^{k_i}\right) - f\left(z^{k_i}, y^{k_i}\right)\right] \\ &\geq \frac{\alpha}{2\beta_{k_i}}\vartheta_{k_i}\left\|v^{k_i} - y^{k_i}\right\|^2.\end{aligned} \tag{65}$$

Combining the above inequality with (63), we have

$$\lim_{i \longrightarrow +\infty} \vartheta_{k_i}\left\|v^{k_i} - y^{k_i}\right\|^2 = 0. \tag{66}$$

Note that $x^{k_i} \rightharpoonup p^\dagger \in C$. Then, it follows from (55), (58), and (62) that $u^{k_i} \rightharpoonup p^\dagger$, $v^{k_i} \rightharpoonup p^\dagger$, $Au^{k_i} \rightharpoonup Ap^\dagger$, $Av^{k_i} \rightharpoonup Ap^\dagger$, and $w^{k_i} \rightharpoonup Ap^\dagger \in Q$.

There are two possible cases. □

*Case 1.* $\limsup_{k \longrightarrow +\infty} \vartheta_{k_i} > 0$. Then, there exist $\overline{\vartheta} > 0$ and a subsequence of $\{\vartheta_{k_i}\}$, still denoted by $\{\vartheta_{k_i}\}$ such that for some $I_0 > 0$, $\vartheta_{k_i} > \overline{\vartheta}$ for all $i \geq I_0$. Consequently, by (66), we deduce

$$\lim_{i \longrightarrow +\infty} \left\|v^{k_i} - y^{k_i}\right\| = 0. \tag{67}$$

Noting that $v^{k_i} \rightharpoonup p^\dagger$, thus $y^{k_i} \rightharpoonup p^\dagger$. According to (28), we obtain

$$0 \in \partial_2 f\left(v^{k_i}, y^{k_i}\right) + \frac{1}{\beta_{k_i}}\left(y^{k_i} - v^{k_i}\right) + N_C\left(y^{k_i}\right), \tag{68}$$

so, there exists $\widehat{v}^{k_i} \in \partial_2 f(v^{k_i}, y^{k_i})$ such that

$$\langle \widehat{v}^{k_i}, y - y^{k_i}\rangle + \frac{1}{\beta_{k_i}}\langle y^{k_i} - v^{k_i}, y - y^{k_i}\rangle \geq 0, \quad \forall y \in C. \tag{69}$$

By the subdifferential inequality, we have

$$f\left(v^{k_i}, y\right) - f\left(v^{k_i}, y^{k_i}\right) \geq \langle \widehat{v}^{k_i}, y - y^{k_i}\rangle, \quad \forall y \in C. \tag{70}$$

Therefore,

$$f\left(v^{k_i}, y\right) - f\left(v^{k_i}, y^{k_i}\right) + \frac{1}{\beta_{k_i}}\langle y^{k_i} - v^{k_i}, y - y^{k_i}\rangle \geq 0, \quad \forall y \in C. \tag{71}$$

Since

$$\langle y^{k_i} - v^{k_i}, y - y^{k_i}\rangle \leq \left\|y^{k_i} - v^{k_i}\right\|\left\|y - y^{k_i}\right\|, \tag{72}$$

from (71), we get

$$f\left(v^{k_i}, y\right) - f\left(v^{k_i}, y^{k_i}\right) + \frac{1}{\beta_{k_i}}\left\|y^{k_i} - v^{k_i}\right\|\left\|y - y^{k_i}\right\| \geq 0. \tag{73}$$

Letting $i \longrightarrow +\infty$ in (73), from (A1), (A3), and (67), we obtain

$$f\left(p^\dagger, y\right) \geq f\left(p^\dagger, p^\dagger\right) = 0, \quad \forall y \in C, \tag{74}$$

hence $p^\dagger \in \mathrm{SEP}(C, f)$.

*Case 2.* $\lim_{i \longrightarrow +\infty} \vartheta_{k_i} = 0$. Since the sequence $\{y^{k_i}\}$ is bounded, without loss of generality, we may assume that $y^{k_i} \rightharpoonup \overline{y}$ as $i \longrightarrow +\infty$. Replacing $y$ by $v^{k_i}$ in (71), we get

$$f\left(v^{k_i}, y^{k_i}\right) \leq -\frac{1}{\beta_{k_i}}\left\|y^{k_i} - v^{k_i}\right\|^2. \tag{75}$$

According to (29), for $m_{k_i} - 1$, we have

$$f\left(z^{k_i, m_{k_i}-1}, v^{k_i}\right) - f\left(z^{k_i, m_{k_i}-1}, y^{k_i}\right) < \frac{\alpha}{2\beta_{k_i}}\left\|y^{k_i} - v^{k_i}\right\|^2. \tag{76}$$

From (75) and (76), we obtain

$$f\left(v^{k_i}, y^{k_i}\right) \leq \frac{2}{\alpha}\left[f\left(z^{k_i, m_{k_i}-1}, y^{k_i}\right) - f\left(z^{k_i, m_{k_i}-1}, v^{k_i}\right)\right]. \tag{77}$$

Letting $i \longrightarrow +\infty$ in (77) and noting that $v^{k_i} \rightharpoonup p^\dagger$, $y^{k_i} \rightharpoonup \overline{y}$ and $z^{k_i, m_{k_i}-1} \rightharpoonup p^\dagger$ as $i \longrightarrow +\infty$, we obtain

$$f\left(p^\dagger, \overline{y}\right) \leq \frac{2}{\alpha} f\left(p^\dagger, \overline{y}\right). \tag{78}$$

Therefore, $f(p^\dagger, \overline{y}) = 0$ and $\lim_{i\longrightarrow+\infty}\|y^{k_i} - v^{k_i}\| = 0$. Consequently, by the similar argument as that in Case 1, we get $p^\dagger \in \mathrm{SEP}(C, f)$.

At the same time, from (61), $x^{k_i} \rightharpoonup p^\dagger$ and Lemma 2, we deduce that $p^\dagger \in \mathrm{Fix}(S)$. Therefore, $p^\dagger \in \mathrm{Fix}(S) \cap \mathrm{SEP}(C, f)$.

Next, we show that $p^\dagger \in \mathrm{Fix}(T) \cap \mathrm{SEP}(Q, g)$. First, by (39), we have

$$\begin{aligned}(\lambda_k - \zeta_k)\zeta_k\left\|T\left[(1-\lambda_k)w^k + \lambda_k T w^k\right] - w^k\right\|^2 &\le \left\|w^k - Ax^*\right\|^2 - \left\|q^k - Ax^*\right\|^2 \\ &\le \left\|w^k - q^k\right\|\left[\left\|w^k - Ax^*\right\| + \left\|q^k - Ax^*\right\|\right].\end{aligned} \tag{79}$$

Since $\liminf_{k\longrightarrow+\infty}(\lambda_k - \zeta_k)\zeta_k > 0$ and $\{w^k\}$ and $\{q^k\}$ are bounded, from (56) and (79), we deduce that

$$\lim_{k\longrightarrow+\infty}\left\|T\left[(1-\lambda_k)w^k + \lambda_k T w^k\right] - w^k\right\| = 0. \tag{80}$$

Observe that

$$\begin{aligned}\left\|Tw^k - w^k\right\| &\le \left\|Tw^k - T\left[(1-\lambda_k)w^k + \lambda_k T w^k\right]\right\| \\ &\quad + \left\|T\left[(1-\lambda_k)w^k + \lambda_k T w^k\right] - w^k\right\| \\ &\le L_2\lambda_k\left\|Tw^k - w^k\right\| + \left\|T\left[(1-\lambda_k)w^k + \lambda_k T w^k\right] - w^k\right\|.\end{aligned} \tag{81}$$

It follows that

$$\left\|Tw^k - w^k\right\| \le \frac{1}{1 - L_2\lambda_k}\left\|T\left[(1-\lambda_k)w^k + \lambda_k T w^k\right] - w^k\right\|. \tag{82}$$

This together with (80) implies that $\lim_{k\longrightarrow+\infty}\|Tw^k - w^k\| = 0$. Combining this with $w^{k_i} \rightharpoonup Ap^\dagger$ and the fact that $I - T$ is demiclosed at zero (Lemma 2), it is immediate that $Ap^\dagger \in \mathrm{Fix}(T)$.

By Lemma 3, we have

$$\left\|J^g_{\varsigma_k}\left(Au^k\right) - J^g_{\varsigma}\left(Au^k\right)\right\| \le \frac{\varsigma_k - \varsigma}{\varsigma_k}\left\|J^g_{\varsigma_k}\left(Au^k\right) - Au^k\right\|. \tag{83}$$

Hence,

$$\begin{aligned}\left\|J^g_{\varsigma}\left(Au^k\right) - Au^k\right\| &\le \left\|J^g_{\varsigma_k}\left(Au^k\right) - Au^k\right\| + \left\|J^g_{\varsigma_k}\left(Au^k\right) - J^g_{\varsigma}\left(Au^k\right)\right\| \\ &\le 2\left\|J^g_{\varsigma_k}\left(Au^k\right) - Au^k\right\|.\end{aligned} \tag{84}$$

It follows from (55) that $\lim_{k\longrightarrow\infty}\|J^g_{\varsigma}Au^k - Au^k\| = 0$. Since $J^g_{\varsigma}$ is nonexpansive and $Au^{k_i} \rightharpoonup Ap^\dagger$, we deduce that $Ap^\dagger \in \mathrm{Fix}(J^g_{\varsigma}) = \mathrm{SEP}(Q, g)$ by Lemma 3. So, $p^\dagger \in \Gamma$ and $\omega_w(x^k) \subset \Gamma$.

Replacing $x^* = P_C h(q^\dagger)$ in (45), we have

$$\begin{aligned}\left\|x^{k+1} - P_C h\left(q^\dagger\right)\right\|^2 &\le \left(1 - \frac{2(1-\kappa) - \mu_k}{1 - \kappa\mu_k}\mu_k\right)\left\|x^k - P_C h\left(q^\dagger\right)\right\|^2 \\ &\quad + \frac{2(1-\kappa) - \mu_k}{1 - \kappa\mu_k}\mu_k \times \frac{2}{2(1-\kappa) - \mu_k} \\ &\quad \cdot \left\langle h\left(P_C h\left(q^\dagger\right)\right) - P_C h\left(q^\dagger\right), x^{k+1} - P_C h\left(q^\dagger\right)\right\rangle.\end{aligned} \tag{85}$$

Noting that $\limsup_{k\longrightarrow+\infty}\langle h(P_C h(q^\dagger)) - P_C h(q^\dagger), x^{k+1} - P_C h(q^\dagger)\rangle \le 0$, applying Lemma 6 to the last inequality, we deduce that $x^k \longrightarrow P_C h(q^\dagger)$. This completes the proof. □

Next, we can apply Algorithm 1 and Theorem 1 for solving the split equilibrium problem (3). Setting $S = I$ and $T = I$ in Algorithm 1, we deduce that $v^k = x^k$ and $q^k = w^k$. Consequently, we have the following algorithm and corollary.

*Algorithm 2.* Fix an initial point $x^0 \in C$. Set $k = 0$.

Step 1: assume that $x^k$ is known and compute

$$y^k = \arg\min_{y^\dagger \in C}\left\{f\left(x^k, y^\dagger\right) + \frac{1}{2\beta_k}\left\|x^k - y^\dagger\right\|^2\right\}. \tag{86}$$

If $y^k = x^k$, then set $u^k = x^k$ and go to Step 4. Otherwise, go to Step 2.

Step 2: let $m_k = \min\{1, 2, \ldots, k, \ldots\}$ such that

$$f\left(z^{k,m_k}, x^k\right) - f\left(z^{k,m_k}, y^k\right) \ge \frac{\alpha}{2\beta_k}\left\|x^k - y^k\right\|^2, \tag{87}$$

where

$$z^{k,m_k} = (1 - \vartheta^{m_k})x^k + \vartheta^{m_k}y^k. \tag{88}$$

Write $\vartheta_k = \vartheta^{m_k}$ and $z^k = z^{k,m_k}$.

Step 3: compute

$$u^k = P_C\left(v^k - \tau_k\iota_k v^k\right), \tag{89}$$

where $v^k \in \partial_2 f(z^k, x^k)$ and $\iota_k = (f(z^k, x^k)/\|v^k\|^2)$.

Step 4: for any $v \in Q$, find $w^k$ such that

$$g\left(w^k, v\right) + \frac{1}{\varsigma_k}\langle v - w^k, w^k - Au^k\rangle \ge 0. \tag{90}$$

Step 5: compute

$$x^{k+1} = \mu_k h\left(x^k\right) + (1 - \mu_k)P_C\left[u^k + \gamma A^*\left(w^k - Au^k\right)\right]. \tag{91}$$

Step 6: set $k: = k + 1$ and return to Step 1.

**Corollary 1.** *Assume that $\Gamma_1 \ne \varnothing$. Then, the sequence $\{x^k\}$ generated by (91) strongly converges to a solution $q_1 = P_{\Gamma_1}h(q_1)$.*

Next, we can apply Algorithm 1 and Theorem 1 for solving the split fixed point problem (4). Setting $f = 0$ and $g = 0$ in Algorithm 1, we deduce that $y^k = x^k$ and $w^k = v^k$. Consequently, we have the following algorithm and corollary.

*Algorithm 3.* Fix an initial point $x^0 \in C$. Define the sequence $\{x^k\}$ iteratively by

$$\begin{cases} v^k = (1-\delta_k)x^k + \delta_k S\left[(1-\eta_k)x^k + \eta_k S x^k\right], \\ q^k = (1-\zeta_k)v^k + \zeta_k T\left[(1-\lambda_k)v^k + \lambda_k T v^k\right], \\ x^{k+1} = \mu_k h\left(x^k\right) + (1-\mu_k)P_C\left[x^k + \gamma A^*\left(q^k - Ax^k\right)\right], \quad k \geq 0. \end{cases} \tag{92}$$

**Corollary 2.** *Assume that $\Gamma_2 \neq \varnothing$. Then, the sequence $\{x^k\}$ generated by (92) strongly converges to a solution $q_2 = P_{\Gamma_2} h(q_2)$.*

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

## Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

## Acknowledgments

Li-Jun Zhu was supported by the National Natural Science Foundation of China (grant no. 11861003) and the Natural Science Foundation of Ningxia Province (grant nos. NZ17015 and NXYLXK2017B09). Ching-Feng Wen was supported by the grant of MOST 109-2115-M-037-001.

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