Research Article

New Inertial Relaxed CQ Algorithms for Solving Split Feasibility Problems in Hilbert Spaces

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The split feasibility problem (SFP) has received much attention due to its various applications in signal processing and image reconstruction. In this paper, we propose two inertial relaxed CQ algorithms for solving the split feasibility problem in real Hilbert spaces according to the previous experience of applying inertial technology to the algorithm. These algorithms involve metric projections onto half-spaces, and we construct new variable step size, which has an exact form and does not need to know a prior information norm of bounded linear operators. Furthermore, we also establish weak and strong convergence of the proposed algorithms under certain mild conditions and present a numerical experiment to illustrate the performance of the proposed algorithms.

1. Introduction

The split feasibility problem in finite-dimensional Hilbert spaces was first introduced by Censor and Elfving [1] in 1994, for modeling inverse problem that arises from the phase retrievals and in medical image reconstruction [2]. The split feasibility problem can also be used to model the intensity-modulated radiation therapy [3].

Let H_1 and H_2 be two real Hilbert spaces with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$. C and Q are nonempty closed and convex subsets of real Hilbert spaces H_1 and H_2 , respectively, and A is a linear bounded operator from H_1 into H_2 . The split feasibility problem (SFP) is formulated as follows: find a point $x \in H_1$ satisfying

$$\begin{aligned} x &\in C, \\ Ax &\in Q. \end{aligned} \quad (1)$$

The solution set of the problem (SFP) (1) is denoted by S ; that is,

$$S := \{x \in C: Ax \in Q\}. \quad (2)$$

A very successful method that solves the (SFP) seems to be the CQ algorithm of Byrne [4], which generates $\{x_n\}$ by the iterative procedure: for any initial guess $x_1 \in H$,

$$x_{n+1} = P_C(x_n - \gamma A^*(I - P_Q)Ax_n), \quad \forall n \geq 1, \quad (3)$$

where P_C and P_Q are the metric projections onto C and Q , respectively. A^* is the adjoint operator of the linear operator A , and the step size γ is chosen in the open interval $(0, 2/\|A\|^2)$. The step size selection depends on the operator norm (or the largest eigenvalue of A^*A), which also is not a simple work.

The CQ algorithm (3) for solving the problem (SFP) (1) can be obtained from optimization. If we introduce the convex objective function

$$f(x) := \frac{1}{2} \|(I - P_Q)Ax\|^2, \quad x \in H_1, \quad (4)$$

then the CQ algorithm (3) comes immediately as a special case of the gradient-projection algorithm (GPA), since the convex objective function f is differentiable and has a Lipschitz gradient given by

$$\nabla f(x) = A^*(I - P_Q)Ax. \quad (5)$$

To overcome the computational difficulties, many authors have constructed the variable step size that does not require the norm $\|A\|$; see, for example, [5–12]. In particular, Lopez et al. [7] introduced a new choice of the variable step size sequence τ_n as follows:

$$\tau_n := \frac{\rho_n f(x_n)}{\|\nabla f(x_n)\|^2}, \quad \forall n \geq 1, \quad (6)$$

where $\{\rho_n\}$ is a sequence of positive real numbers, take zero for the lower bound and four for the upper bound. The advantage of the choice (6) of step size is that there is neither prior information about the matrix norm A nor any other conditions on Q and A .

Now let us consider the case when C and Q are level subsets of convex functions, where C and Q are, respectively, given by

$$\begin{aligned} C &= \{x \in H_1 : c(x) \leq 0\}, \\ Q &= \{y \in H_2 : q(y) \leq 0\}, \end{aligned} \quad (7)$$

where $c: H_1 \rightarrow (-\infty, +\infty]$ and $q: H_2 \rightarrow (-\infty, +\infty]$ are two lower semicontinuous convex functions, and ∂c and ∂q are bounded operators. But the associated projections P_C and P_Q do not have closed-form expressions, and the CQ algorithm is that the iterative process cannot be performed. In order to keep it going, Yang [13] made improvements to these two-level subsets; here is how they are defined:

$$\widetilde{C}_n = \{x \in H_1 : c(x_n) + \langle \xi_n, x - x_n \rangle \leq 0\}, \quad (8)$$

with $\xi_n \in \partial c(x_n)$, and

$$\widetilde{Q}_n = \{y \in H_2 : q(Ax_n) + \langle \zeta_n, y - Ax_n \rangle \leq 0\}, \quad (9)$$

with $\zeta_n \in \partial q(Ax_n)$.

It is easy to see that $\widetilde{C}_n$ and $\widetilde{Q}_n$ are both half-spaces, and the projections $P_{\widetilde{C}_n}$ and $P_{\widetilde{Q}_n}$ have closed-form expressions. In what follows, for each $n \geq 1$, define

$$f_n(x) := \frac{1}{2} \left\| (I - P_{\widetilde{Q}_n}) Ax \right\|^2, \quad (10)$$

$$\nabla f_n(x) = A^* (I - P_{\widetilde{Q}_n}) Ax.$$

Since these projections are easy to calculate, the algorithm is very practical.

Afterwards, the inertial technique was developed by Alvarez and Attouch in order to improve the performance of proximal point algorithms [14]. Dang et al. [15] proposed an inertial relaxed CQ algorithm $\{x_n\}$ for solving the problem (SFP) in a real Hilbert space, which is generated as follows: for any $x_0, x_1 \in H$,

$$\begin{cases} w_n = x_n + \theta_n (x_n - x_{n-1}), \\ x_{n+1} = P_{\widetilde{C}_n} \left(w_n - \gamma A^T (I - P_{\widetilde{Q}_n}) A (w_n) \right), \end{cases} \quad (11)$$

where $0 < \gamma < (2/\|A\|^2)$, and $0 \leq \theta_n \leq \overline{\theta}_n$ with

$$\overline{\theta}_n = \min \left\{ \theta, \frac{1}{\max \{ n^2 \|x_n - x_{n-1}\|^2, n^2 \|x_n - x_{n-1}\|^2 \}} \right\}, \quad \forall n \geq 1, \theta \in [0, 1), \quad (12)$$

$$\overline{C}_n = \{x \in H_1 : c(w_n) + \langle \xi_n, x - w_n \rangle \leq 0\},$$

with $\xi_n \in \partial c(w_n)$, and

$$\overline{Q}_n = \{y \in H_2 : q(Aw_n) + \langle \zeta_n, y - Aw_n \rangle \leq 0\}, \quad (13)$$

with $\zeta_n \in \partial q(Aw_n)$. The algorithm $\{x_n\}$ converges weakly to a point of a solution set of the problem (SFP), where step size also depends on the matrix norm $\|A\|$. It is obvious that the calculation of operator norm is more complicated, so Gibali et al. [16] has changed the step size of (11).

$$\lambda_n = \frac{\rho_n f_n(w_n)}{\eta_n^2}, \quad (14)$$

$$\eta_n = \max \{1, \|\nabla f_n(w_n)\|\}, \quad 0 \leq \theta_n \leq \overline{\theta}_n,$$

where

$$\overline{\theta}_n = \begin{cases} \min \left\{ \theta, \frac{\varepsilon_n}{\|x_n - x_{n-1}\|^2} \right\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise.} \end{cases} \quad (15)$$

If $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\|^2 < \infty$, then the sequence $\{x_n\}$ generated by (11) with step size λ_n converges weakly to a point of a solution set of the problem (SFP). For recent results on inertial algorithms (see [17–24]).

On the other hand, the CQ algorithm is the gradient-projection method for the variational inequality problem. In [25], Xu gave weak convergence in the setting of Hilbert spaces. Wang and Xu [26] proposed the following algorithm:

$$x_{n+1} = P_C [(1 - \alpha_n)(x_n - \gamma \nabla f(x_n))], \quad (16)$$

where $\gamma \in (0, 2/\|A\|^2)$. Under some conditions, it is proved that the sequence generated by the algorithm (16) strongly converges to the minimum-norm solution of the (SFP).

Motivated and inspired by the work of [7, 27–29], the authors of [30] introduced a self-adaptive CQ-type algorithm for finding a solution of the (SFP) in the setting of infinite-dimensional real Hilbert spaces; the advantage of this algorithm lies in the fact that step sizes are dynamically chosen and do not depend on the operator norm. This algorithm can be formulated as follows:

$$x_{n+1} = P_{C_n}[(1 - \beta_n)(x_n - \lambda_n \nabla f_n(x_n))], \quad (17)$$

where $\lambda_n = (\rho_n f_n(x_n) / \|\nabla f_n(x_n)\|^2)$. It is also proved that the sequence generated by the algorithm (17) strongly converges to the minimum-norm solution of the (SFP) under some conditions.

Inspired by the works mentioned above, we propose a new relaxed CQ algorithm to solve the (SFP) in a real Hilbert space by using inertial technology. The new step size proposed in this algorithm is independent of the operator norm in this paper, and we also establish weak convergence theorem of the proposed algorithms under some mild conditions in [31]. We add the inertial term on the basis of the algorithm in [30] to construct a new iterative process, so that the new algorithm strongly converges to a point in the solution set under some conditions.

The remainder of the paper is organized as follows. Some useful definitions and results are collected in Section 2 for the convergence analysis of the proposed algorithm. In Section 3, new inertial algorithms of weak and strong convergence for solving SFP are proposed, followed by the convergence analysis. In Section 4, we provide a numerical experiment to illustrate the performance of the proposed algorithms. Finally, we end the paper with some conclusion.

2. Preliminaries

Let H be a Hilbert space and let C be a nonempty closed convex subset in H . The strong (weak) convergence of a sequence $\{x_n\}$ to x is denoted by $x_n \rightarrow x$ ($x_n \rightharpoonup x$), respectively. For any sequence $\{x_n\} \subset H$, $\omega_w(x_n)$ denotes the weak ω -limit set of $\{x_n\}$; that is,

$$\omega_w(x_n) := \left\{x \in H: x_{n_j} \rightharpoonup x\right\}, \text{ for some subsequence } \{n_j\} \text{ of } \{n\}. \quad (18)$$

Definition 1. An operator $T: C \rightarrow H$ is called the following:

(i) Nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \quad (19)$$

(ii) Firmly nonexpansive if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C. \quad (20)$$

(iii) ν -inverse strongly monotone (ν -ism) if there is $\nu > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq \nu \|Tx - Ty\|^2, \quad \forall x, y \in C. \quad (21)$$

For every element $x \in H$, there exists a unique nearest point in C denoted by $P_C x$, such that

$$\|x - P_C x\| = \min\{\|x - y\| \mid y \in C\}. \quad (22)$$

Then operator P_C is called the metric projection from H onto C .

The projection has the following well-known properties.

Lemma 1 (see [32, 33]). For all $x, y \in H$ and $z \in C$, we have

- (1) $\langle x - P_C x, z - P_C x \rangle \leq 0$
- (2) $\|P_C x - P_C y\| \leq \|x - y\|$
- (3) $\|P_C x - P_C y\|^2 \leq \langle x - y, P_C x - P_C y \rangle$
- (4) $\|P_C x - z\|^2 \leq \|x - z\|^2 - \|(I - P_C)x\|^2$

Lemma 2. Let H be a real Hilbert space and $x, y, z \in H$, $t \in \mathbb{R}$; then

- (1) $\|(1 - t)x + ty\|^2 = (1 - t)\|x\|^2 + t\|y\|^2 - t(1 - t)\|x - y\|^2$;
- (2) $\|x - y\|^2 = \|y - z\|^2 - \|x - z\|^2 + 2\langle x - y, x - z \rangle$.

Definition 2 (see [34]). Let H be a real Hilbert space and let $f: H \rightarrow (-\infty, \infty)$ be a convex function. An element $v \in H$ is called the subgradient of f at $\bar{x} \in H$ if

$$\langle v, x - \bar{x} \rangle \leq f(x) - f(\bar{x}), \quad \forall x \in H. \quad (23)$$

The collection of all the subgradients of f at $\bar{x}$ is called the subdifferential of the function f at this point, which is denoted by $\partial f(\bar{x})$; that is,

$$\partial f(\bar{x}) = \{v \in H: \langle v, x - \bar{x} \rangle \leq f(x) - f(\bar{x}), \forall x \in H\}. \quad (24)$$

Definition 3. Let $f: H \rightarrow (-\infty, +\infty]$ be a proper function.

(i) f is lower semicontinuous at x if $x_n \rightarrow x$ implies

$$f(x) \leq \liminf_{n \rightarrow \infty} f(x_n). \quad (25)$$

(ii) f is weakly lower semicontinuous at x if $x_n \rightharpoonup x$ implies

$$f(x) \leq \liminf_{n \rightarrow \infty} f(x_n). \quad (26)$$

(iii) f is lower semicontinuous on H if it is lower semicontinuous at every point $x \in H$; f is weakly lower semicontinuous on H if it is weakly lower semicontinuous at every point $x \in H$.

(iv) f is lower semicontinuous if and only if it is weakly lower semicontinuous.

Lemma 3 (see [34]). Let $f: H \rightarrow (-\infty, +\infty]$ be an α -strongly convex function. Then, for all $x, y \in H$,

$$f(y) \geq f(x) + \langle \xi, y - x \rangle + \frac{\alpha}{2} \|y - x\|^2, \quad \xi \in \partial f(x). \quad (27)$$

Lemma 4 (see [25]). Let $t > 0$ and $x^* \in H$. Then the following statements are equivalent:

- (1) The point x^* solves the problem (SFP).
- (2) The point x^* solves the fixed-point equation

$$x^* = P_C(x^* - tA^*(I - P_Q)Ax^*). \quad (28)$$

- (3) The point x^* solves the variational inequality problem with respect to the gradient of f ; that is, find a point $x \in C$ such that

$$\langle \nabla f(x), y - x \rangle \geq 0, \quad \forall y \in C. \quad (29)$$

Lemma 5 (see [16]). Let H be a real Hilbert space and let $\{x_n\}$ be a sequence in H such that there exists a nonempty closed and convex subset S of H satisfying the following conditions:

- (i) For all $z \in S$, $\lim_{n \rightarrow \infty} \|x_n - z\|$ exists
- (ii) Any weak cluster point of $\{x_n\}$ belongs to S

Then there exists $x^* \in S$ such that $\{x_n\}$ converges weakly to x^* .

Lemma 6 (see [35]). Let $\{\phi_n\} \subset [0, \infty)$ and $\{\delta_n\} \subset [0, \infty)$ be two nonnegative real sequences satisfying the following conditions:

- (1) $\phi_{n+1} - \phi_n \leq \theta_n(\phi_n - \phi_{n-1}) + \delta_n$
- (2) $\sum_{n=1}^{\infty} \delta_n < \infty$
- (3) $\{\theta_n\} \subset [0, \theta]$, where $\theta \in [0, 1)$

Then, $\{\phi_n\}$ is a converging sequence and $\sum_{n=1}^{\infty} [\phi_{n+1} - \phi_n]_+ < \infty$, where $[t]_+ = \max\{t, 0\}$ for any $t \in \mathbb{R}$.

Lemma 7 (see [36, 37]). Let $\{a_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ be sequences of nonnegative real numbers such that

$$a_{n+1} \leq (1 - \beta_n)a_n + \delta_n + \gamma_n, \quad n \geq 1, \quad (30)$$

where $\{\beta_n\}_{n=0}^{\infty}$ is a sequence in $(0, 1)$ and $\{\delta_n\}_{n=0}^{\infty}$ is a real sequence. Assume $\sum_{n=1}^{\infty} \gamma_n < \infty$. Then the following results hold:

- (1) If $\delta_n \leq \beta_n M$ for some $M \geq 0$, then $\{a_n\}_{n=0}^{\infty}$ is a bounded sequence
- (2) If $\sum_{n=1}^{\infty} \beta_n = \infty$ and $\limsup_{n \rightarrow \infty} (\delta_n / \beta_n) \leq 0$, then $\lim_{n \rightarrow \infty} a_n = 0$

Lemma 8 (see [38]). Assume that $\{s_n\}$ is a sequence of nonnegative real numbers such that

$$\begin{aligned} s_{n+1} &\leq (1 - \alpha_n)s_n + \alpha_n \delta_n, \quad n \geq 1, \\ s_{n+1} &\leq s_n - \lambda_n + \gamma_n, \quad n \geq 1, \end{aligned} \quad (31)$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$, $\{\lambda_n\}$ is a sequence of nonnegative real numbers, and $\{\delta_n\}$ and $\{\gamma_n\}$ are two sequences in $\mathfrak{R}$ such that

- (1) $\sum_{n=1}^{\infty} \alpha_n = \infty$
- (2) $\lim_{n \rightarrow \infty} \gamma_n = 0$
- (3) $\lim_{k \rightarrow \infty} \lambda_{n_k} = 0$ implies $\limsup_{k \rightarrow \infty} \delta_{n_k} \leq 0$ for any subsequence $\{n_k\}$ of $\{n\}$

Then $\lim_{n \rightarrow \infty} s_n = 0$.

3. Convergence Analysis

In this section, we consider the (SFP) in which C is given by

$$C = \{x \in H_1 | c(x) \leq 0\}, \quad (32)$$

where $c: H_1 \rightarrow (-\infty, +\infty]$ is an α -strongly convex function; the set Q is given by

$$Q = \{y \in H_2 | q(y) \leq 0\}, \quad (33)$$

where $q: H_2 \rightarrow (-\infty, +\infty]$ is a β -strongly convex function. We assume that the solution set S of the (SFP) is nonempty, and c and q are lower semicontinuous convex functions; furthermore, we also assume that ∂c and ∂q are bounded operators (i.e., bounded on bounded sets).

We agree to build the following sets in our algorithms according to [39]; that is, given the n -th iterative point w_n , we construct C_n as

$$C_n = \left\{ x \in H_1 | c(w_n) + \langle \xi_n, x - w_n \rangle + \frac{\alpha}{2} \|x - w_n\|^2 \leq 0 \right\}, \quad (34)$$

where $\xi_n \in \partial c(w_n)$.

$$Q_n = \left\{ y \in H_2 | q(Aw_n) + \langle \zeta_n, y - Aw_n \rangle + \frac{\beta}{2} \|y - Aw_n\|^2 \leq 0 \right\}, \quad (35)$$

where $\zeta_n \in \partial q(Aw_n)$.

If $\alpha = 0$ and $\beta = 0$, then C_n and Q_n are reduced to the half-spaces $\overline{C_n}$ and $\overline{Q_n}$, respectively. If $\alpha > 0$ and $\beta > 0$, then C_n and Q_n are nonempty closed ball of radius $(1/\alpha)\sqrt{\|\xi_n\|^2 - 2\alpha c(w_n)}$ centred at $w_n - (1/\alpha)\xi_n$ and $(1/\beta)\sqrt{\|\zeta_n\|^2 - 2\beta q(Aw_n)}$ centred at $Aw_n - (1/\beta)\zeta_n$, respectively.

In addition, for each $n \geq 0$, we define the following functions:

$$f_n(x) = \frac{1}{2} \|(I - P_{Q_n})Ax\|^2, \quad (36)$$

$$\nabla f_n(x) = A^*(I - P_{Q_n})Ax,$$

where Q_n is given as in (35), f_n is weakly lower semicontinuous, convex, and differentiable, and its gradient ∇f_n

is Lipschitz continuous. Now we propose new relaxed CQ algorithms for solving the (SFP).

Next, two inertial relaxed CQ algorithms will be introduced. The weak convergence of Algorithm 1 and the strong convergence of Algorithm 2 will be proved under different step sizes.

Algorithm 1. Choose positive sequence $\{\varepsilon_n\}$ satisfying $\sum_{n=0}^{\infty} \varepsilon_n < \infty$.

Let $x_0, x_1 \in C$ be arbitrary. Given x_n, x_{n-1} , update the next iteration via

$$\begin{aligned} w_n &= x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} &= P_{C_n}(w_n - \tau_n \nabla f_n(w_n)), \end{aligned} \quad (37)$$

where $0 \leq \theta_n < \overline{\theta}_n$, and

$$\overline{\theta}_n = \begin{cases} \min \left\{ \theta, \frac{\varepsilon_n}{\max \{ \|x_n - x_{n-1}\|^2, \|x_n - x_{n-1}\| \}} \right\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{if } x_n = x_{n-1}, \end{cases} \quad (38)$$

and C_n and Q_n are given as in (34) and (35).

$$\tau_n = \begin{cases} \frac{\sigma_n}{\|\nabla f_n(w_n)\|}, & \text{if } \|\nabla f_n(w_n)\| \neq 0, \\ 0, & \text{if } \|\nabla f_n(w_n)\| = 0, \end{cases} \quad (39)$$

where $\sum_{n=1}^{\infty} \sigma_n = \infty$, $\sum_{n=1}^{\infty} \sigma_n^2 < \infty$.

If $x_{n+1} = w_n$, then stop; otherwise, set $n := n + 1$ and go to the next iteration.

By assuming $\overline{\theta}_n$, we know

which means

$$\begin{aligned} \sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\|^2 &< \infty, \\ \sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| &< \infty, \end{aligned} \quad (40)$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \theta_n \|x_n - x_{n-1}\|^2 &= 0, \\ \lim_{n \rightarrow \infty} \theta_n \|x_n - x_{n-1}\| &= 0. \end{aligned} \quad (41)$$

From $\xi_n \in \partial c(w_n)$, applying Lemma 3, we get $C \subseteq C_n$; and a similar way is used to get $Q \subseteq Q_n$.

Now let us show that our proposed algorithm has a very important property: if $x_{n+1} = w_n$ for some $n > 0$, then w_n is a solution of (SFP). Indeed, $x_{n+1} \in C_n$, so that $w_n \in C_n$ as $w_n = x_{n+1}$ by assumption. So we get $c(w_n) \leq 0$ from (34), that is, $w_n \in C$. On the other hand, according to the algorithm, we have $w_n = P_{C_n}(w_n - \tau_n A^*(I - P_{Q_n})Aw_n)$, which together with Lemma 4 implies that $Aw_n \in Q_n$. It also implies that $q(Aw_n) \leq 0$ from (35); then $Aw_n \in Q$. The conclusion is tenable.

Lemma 9. Let $\{x_n\}$ and $\{w_n\}$ be the sequences generated by Algorithm 1. Then, for any $z \in S$, it follows that

$$\|x_{n+1} - z\|^2 \leq \|w_n - z\|^2 + \sigma_n^2 - \frac{4\sigma_n f_n(w_n)}{\|\nabla f_n(w_n)\|}. \quad (42)$$

Proof. For $z \in S$, we have $z \in C, Az \in Q$; and we have $z = P_C z = P_{C_n} z, Az = P_Q Az = P_{Q_n} Az$.

It follows from Lemma 1 that

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \|P_{C_n}(w_n - \tau_n \nabla f_n(w_n)) - z\|^2 \\ &\leq \|(w_n - z) - \tau_n \nabla f_n(w_n)\|^2 - \|(w_n - x_{n+1}) - \tau_n \nabla f_n(w_n)\|^2 \\ &= \|w_n - z\|^2 - \|w_n - x_{n+1}\|^2 - 2\tau_n \langle \nabla f_n(w_n), w_n - z \rangle + 2\tau_n \langle \nabla f_n(w_n), w_n - x_{n+1} \rangle, \end{aligned} \quad (43)$$

where

$$\begin{aligned} 2\tau_n \langle \nabla f_n(w_n), w_n - z \rangle &= 2\tau_n \langle (I - P_{Q_n})Aw_n - (I - P_{Q_n})Az, Aw_n - Az \rangle \\ &\geq 2\tau_n \|(I - P_{Q_n})Aw_n\|^2 \\ &= 4\tau_n f_n(w_n), \\ 2\tau_n \langle \nabla f_n(w_n), w_n - x_{n+1} \rangle &\leq 2\tau_n \|\nabla f_n(w_n)\| \cdot \|w_n - x_{n+1}\| \\ &\leq \|w_n - x_{n+1}\|^2 + \tau_n^2 \|\nabla f_n(w_n)\|^2. \end{aligned} \quad (44)$$

Hence, we have

$$\|x_{n+1} - z\|^2 \leq \|w_n - z\|^2 - \|w_n - x_{n+1}\|^2 - 4\tau_n f_n(w_n) + \|w_n - x_{n+1}\|^2 + \tau_n^2 \|\nabla f_n(w_n)\|^2. \quad (45)$$

If $\|\nabla f_n(w_n)\| = 0$, then $\tau_n = 0$, so that

$$\|x_{n+1} - z\|^2 \leq \|w_n - z\|^2. \quad (46)$$

If $\|\nabla f_n(w_n)\| \neq 0$, we have

$$\begin{aligned} \|x_{n+1} - z\|^2 &\leq \|w_n - z\|^2 + \tau_n^2 \|\nabla f_n(w_n)\|^2 - 4\tau_n f_n(w_n) \\ &= \|w_n - z\|^2 + \sigma_n^2 - \frac{4\sigma_n f_n(w_n)}{\|\nabla f_n(w_n)\|}. \end{aligned} \quad (47)$$

The proof is complete.

Theorem 1. Assume that θ_n satisfies the assumption. Then there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ generated by Algorithm 1 which weakly converges to a solution of (SFP).

Proof. We first show that, for any $z \in S$, the limit of $\{\|x_n - z\|\}$ exists. By applying Lemma 9, we have

$$\|x_{n+1} - z\|^2 \leq \|w_n - z\|^2 + \sigma_n^2 - \frac{4\sigma_n f_n(w_n)}{\|\nabla f_n(w_n)\|}. \quad (48)$$

From the construction of w_n and Lemma 2, we have

$$\begin{aligned} \|w_n - z\|^2 &= \|(1 + \theta_n)(x_n - z) - \theta_n(x_{n-1} - z)\|^2 \\ &= (1 + \theta_n)\|x_n - z\|^2 - \theta_n\|x_{n-1} - z\|^2 \\ &\quad + \theta_n(1 + \theta_n)\|x_n - x_{n-1}\|^2, \end{aligned} \quad (49)$$

$$\leq (1 + \theta_n)\|x_n - z\|^2 - \theta_n\|x_{n-1} - z\|^2 + 2\theta_n\|x_n - x_{n-1}\|^2. \quad (50)$$

Combining (48) and (50) immediately, we get

$$\begin{aligned} \|x_{n+1} - z\|^2 &\leq (1 + \theta_n)\|x_n - z\|^2 - \theta_n\|x_{n-1} - z\|^2 \\ &\quad + 2\theta_n\|x_n - x_{n-1}\|^2 + \sigma_n^2. \end{aligned} \quad (51)$$

Denote $\phi_n = \|x_n - z\|^2$; from (51), we have

$$\phi_{n+1} - \phi_n \leq \theta_n(\phi_n - \phi_{n-1}) + 2\theta_n\|x_n - x_{n-1}\|^2 + \sigma_n^2, \quad (52)$$

where

$$\begin{aligned} \sum_{n=1}^{\infty} \theta_n\|x_n - x_{n-1}\|^2 &< \infty, \\ \sum_{n=1}^{\infty} \sigma_n^2 &< \infty. \end{aligned} \quad (53)$$

Using Lemma 6, the limit of ϕ_n exists, and $\sum_{n=1}^{\infty} (\|x_{n+1} - z\|^2 - \|x_n - z\|^2)_+ < \infty$, which implies that $\sum_{n=1}^{\infty} (\|x_{n+1} - z\|^2 - \|x_n - z\|^2) < \infty$, $(\|x_{n+1} - z\|^2 - \|x_n - z\|^2)_+ = \max\{\|x_{n+1} - z\|^2 - \|x_n - z\|^2, 0\}$. This also implies that the sequence $\{x_n\}$ is bounded, so $\{w_n\}$ is bounded.

We next show that $\omega_w(x_n) \subset S$. Since $\{w_n\}$ is bounded, from the Lipschitz continuity of ∇f_n , we get that $\{\|\nabla f_n(w_n)\|\}$ is bounded. From (48) and (50), we get

$$\frac{4\sigma_n f_n(w_n)}{\|\nabla f_n(w_n)\|} \leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + \theta_n(\|x_n - z\|^2 - \|x_{n-1} - z\|^2) + 2\theta_n\|x_n - x_{n-1}\|^2 + \sigma_n^2, \quad (54)$$

where $\sum_{n=1}^{\infty} (\|x_{n+1} - z\|^2 - \|x_n - z\|^2) < \infty$, $\sum_{n=1}^{\infty} \theta_n\|x_n - x_{n-1}\|^2 < \infty$, and $\sum_{n=1}^{\infty} \sigma_n^2 < \infty$, so we have

$$\sum_{n=1}^{\infty} \frac{4\sigma_n f_n(w_n)}{\|\nabla f_n(w_n)\|} < \infty. \quad (55)$$

But $\sum_{n=1}^{\infty} \sigma_n = \infty$, so

$$\begin{aligned} \liminf_{n \rightarrow \infty} f_n(w_n) &= 0, \\ \text{i.e., } \liminf_{n \rightarrow \infty} \|(I - P_{Q_n})Aw_n\| &= 0. \end{aligned} \quad (56)$$

On the other hand, since $\{x_n\}$ is bounded, the set $\omega_w(x_n)$ is nonempty. Let $x^* \in \omega_w(x_n)$; then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^*$. Furthermore,

$$\|w_n - x_n\|^2 = \theta_n^2\|x_n - x_{n-1}\|^2 \leq \theta_n\|x_n - x_{n-1}\|^2 \longrightarrow 0. \quad (57)$$

Let $\{w_{n_j}\}$ be a subsequence of the sequence $\{w_n\}$ such that

$$\liminf_{n \rightarrow \infty} \|(I - P_{Q_n})Aw_n\|^2 = \lim_{j \rightarrow \infty} \|(I - P_{Q_{n_j}})Aw_{n_j}\|^2 = 0. \quad (58)$$

Since $\{w_{n_j}\}$ is bounded, there exists a subsequence $\{w_{n_{j_m}}\}$ of $\{w_{n_j}\}$, which converges weakly to x^* . Without loss of generality, we can assume that $w_{n_j} \rightharpoonup x^*$, and A is a bounded linear operator, so $Aw_{n_j} \rightharpoonup Ax^*$.

From Lemma 1, we conclude that

$$\langle w_n - \tau_n \nabla f_n(w_n) - x_{n+1}, z - x_{n+1} \rangle \leq 0. \quad (59)$$

Since $\tau_n \longrightarrow 0$ and $\{\|\nabla f_n(w_n)\|\}$ is bounded, we have $\tau_n \nabla f_n(w_n) \longrightarrow 0$. Hence, we get

$$\langle x_{n+1} - w_n, x_{n+1} - z \rangle \leq \langle \tau_n \nabla f_n(w_n), z - x_{n+1} \rangle \longrightarrow 0. \quad (60)$$

Since $\sum_{n=1}^{\infty} (\|x_{n+1} - z\|^2 - \|x_n - z\|^2) < \infty$ and $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\|^2 < \infty$, from (50), we obtain

$$\begin{aligned} \|x_{n+1} - w_n\|^2 &= \|w_n - z\|^2 - \|x_{n+1} - z\|^2 + 2\langle x_{n+1} - w_n, x_{n+1} - z \rangle \\ &\leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + \theta_n (\|x_n - z\|^2 - \|x_{n-1} - z\|^2) + 2\theta_n \|x_n - x_{n-1}\|^2 + 2\langle x_{n+1} - w_n, x_{n+1} - z \rangle \longrightarrow 0. \end{aligned} \quad (61)$$

Thus,

$$\|x_{n+1} - x_n\| \leq \|x_{n+1} - w_n\| + \|w_n - x_n\| \longrightarrow 0. \quad (62)$$

Since $P_{Q_{n_j}} A w_{n_j} \in Q_{n_j}$, by the definition of Q_{n_j} ,

$$q(Aw_{n_j}) + \langle \zeta_{n_j}, P_{Q_{n_j}} A w_{n_j} - A w_{n_j} \rangle + \frac{\beta}{2} \|P_{Q_{n_j}} A w_{n_j} - A w_{n_j}\|^2 \leq 0, \quad (63)$$

where $\zeta_{n_j} \in \partial q(Aw_{n_j})$. From the boundedness assumption of ∂q and $\lim_{j \rightarrow \infty} \|(I - P_{Q_{n_j}}) A w_{n_j}\|^2 = 0$, we have

$$q(Aw_{n_j}) \leq \|\zeta_{n_j}\| \cdot \|(I - P_{Q_{n_j}}) A w_{n_j}\| - \frac{\beta}{2} \|(I - P_{Q_{n_j}}) A w_{n_j}\|^2 \longrightarrow 0. \quad (64)$$

From the weak lower semicontinuity of the convex function q , it follows that

$$q(Ax^*) \leq \liminf_{j \rightarrow \infty} q(Aw_{n_j}) \leq 0, \quad (65)$$

which means that $Ax^* \in Q$.

Furthermore, $x_{n_j+1} \in C_{n_j}$, and, by the definition of C_{n_j} ,

$$c(w_{n_j}) + \langle \xi_{n_j}, x_{n_j+1} - w_{n_j} \rangle + \frac{\alpha}{2} \|x_{n_j+1} - w_{n_j}\|^2 \leq 0, \quad (66)$$

where $\xi_{n_j} \in \partial c(w_{n_j})$. From the boundedness assumption of ∂c and $\|x_{n_j+1} - w_{n_j}\| \longrightarrow 0$, we have

$$c(w_{n_j}) \leq \|\xi_{n_j}\| \cdot \|w_{n_j} - x_{n_j+1}\| - \frac{\alpha}{2} \|x_{n_j+1} - w_{n_j}\|^2 \longrightarrow 0. \quad (67)$$

From the weak lower semicontinuity of the convex function c , it follows that

$$c(x^*) \leq \liminf_{j \rightarrow \infty} c(w_{n_j}) \leq 0, \quad (68)$$

which means that $x^* \in C$. Therefore, $x_{n_j} \rightharpoonup x^* \in S$. The proof is complete.

Algorithm 2. Choose positive sequence $\{\varepsilon_n\}$ satisfying $\sum_{n=0}^{\infty} \varepsilon_n < \infty$.

Let $x_0, x_1 \in C$ be arbitrary. Given x_n, x_{n-1} , update the next iteration via

$$\begin{aligned} w_n &= x_n + \theta_n (x_n - x_{n-1}), \\ x_{n+1} &= \beta_n u + P_{C_n} [(1 - \beta_n)(w_n - \tau_n \nabla f_n(w_n))], \\ \tau_n &= \frac{\rho_n f_n(w_n)}{\|\nabla f_n(w_n)\|^2}, \end{aligned} \quad (69)$$

where $0 \leq \theta_n < \overline{\theta}_n$, and

$$\overline{\theta}_n = \begin{cases} \min \left\{ \theta, \frac{\varepsilon_n}{\max \{ \|x_n - x_{n-1}\|^2, \|x_n - x_{n-1}\| \}} \right\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{if } x_n = x_{n-1}, \end{cases} \quad (70)$$

and C_n and Q_n are given as in (34) and (35), $\{\beta_n\} \subset (0, 1)$, $\lim_{n \rightarrow \infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$, and $\inf_n \rho_n (4 - \rho_n) > 0$.

If $x_{n+1} = w_n$, then stop; otherwise, set $n := n + 1$ and go to the next iteration.

Theorem 2. Assume that $\inf_n \rho_n (4 - \rho_n) > 0$ and $\varepsilon_n = o(\beta_n)$. Then the sequence x_n generated by Algorithm 2 converges strongly to $z = P_S u$.

Proof. First, we show that, for any $z \in S$, the sequence $\{x_n\}$ is bounded. From the construction of w_n , we have

$$\begin{aligned}\|w_n - z\| &= \|x_n + \theta_n(x_n - x_{n-1}) - z\| \\ &\leq \|x_n - z\| + \theta_n\|x_n - x_{n-1}\|,\end{aligned}\tag{71}$$

$$\begin{aligned}\|w_n - \tau_n \nabla f_n(w_n) - z\|^2 &= \|w_n - z\|^2 + \tau_n^2 \|\nabla f_n(w_n)\|^2 \\ &\quad - 2\tau_n \langle \nabla f_n(w_n), w_n - z \rangle \\ &\leq \|w_n - z\|^2 + \tau_n^2 \|\nabla f_n(w_n)\|^2 - 4\tau_n f_n(w_n) \\ &= \|w_n - z\|^2 + \frac{\rho_n^2 f_n^2(w_n)}{\|\nabla f_n(w_n)\|^2} - \frac{4\rho_n f_n^2(w_n)}{\|\nabla f_n(w_n)\|^2} \\ &= \|w_n - z\|^2 - \rho_n(4 - \rho_n) \frac{f_n^2(w_n)}{\|\nabla f_n(w_n)\|^2} \\ &\leq \|w_n - z\|^2.\end{aligned}\tag{72}$$

So, combining (71) and (72), we get

$$\begin{aligned}\|x_{n+1} - z\| &= \|\beta_n u + P_{C_n}[(1 - \beta_n)(w_n - \tau_n \nabla f_n(w_n))] - z\| \\ &= \|P_{C_n}[(1 - \beta_n)(w_n - \tau_n \nabla f_n(w_n))] - (1 - \beta_n)z + \beta_n(u - z)\| \\ &\leq \|P_{C_n}[(1 - \beta_n)(w_n - \tau_n \nabla f_n(w_n))] - (1 - \beta_n)z\| + \beta_n\|u - z\| \\ &\leq \|(1 - \beta_n)(w_n - \tau_n \nabla f_n(w_n)) - (1 - \beta_n)z\| + \beta_n\|u - z\| \\ &\leq (1 - \beta_n)\|w_n - \tau_n \nabla f_n(w_n) - z\| + \beta_n\|u - z\| \\ &\leq (1 - \beta_n)\|w_n - z\| + \beta_n\|u - z\| \\ &\leq (1 - \beta_n)[\|x_n - z\| + \theta_n\|x_n - x_{n-1}\|] + \beta_n\|u - z\| \\ &\leq (1 - \beta_n)\|x_n - z\| + \beta_n[\sigma_n + \|u - z\|],\end{aligned}\tag{73}$$

where $\sigma_n = (1 - \beta_n)(\theta_n/\beta_n)\|x_n - x_{n-1}\|$. According to hypothesis θ_n ,

$$\theta_n \leq \frac{\varepsilon_n}{\|x_n - x_{n-1}\|} \Rightarrow \frac{\theta_n}{\beta_n} \|x_n - x_{n-1}\| \leq \frac{\varepsilon_n}{\beta_n} \longrightarrow 0.\tag{74}$$

Note that

$$\lim_{n \rightarrow \infty} \sigma_n = \lim_{n \rightarrow \infty} (1 - \beta_n) \frac{\theta_n}{\beta_n} \|x_n - x_{n-1}\| = 0,\tag{75}$$

which implies that the sequence $\{\sigma_n\}$ is bounded. Setting

$$M = \max \left\{ \sup_{n \in \mathbb{N}} \sigma_n, \|u - z\| \right\},\tag{76}$$

as well as using Lemma 7, we conclude that the sequence $\{\|x_n - z\|\}$ is bounded. This shows that the sequence $\{x_n\}$ is bounded and so is $\{w_n\}$.

Since $\{\|x_n - z\|\}$ is bounded, assume that there exists a constant M_1 such that $\|x_n - z\| \leq M_1$. Thus,

$$\begin{aligned}\|w_n - z\|^2 &\leq (\|x_n - z\| + \theta_n\|x_n - x_{n-1}\|)^2 \\ &= \|x_n - z\|^2 + \theta_n^2\|x_n - x_{n-1}\|^2 + 2\theta_n\|x_n - x_{n-1}\| \cdot \|x_n - z\| \\ &\leq \|x_n - z\|^2 + \theta_n\|x_n - x_{n-1}\|^2 + 2M_1 \cdot \theta_n\|x_n - x_{n-1}\|,\end{aligned}\tag{77}$$

and we get

$$\begin{aligned}
& \|x_{n+1} - z\|^2 \\
&= \|\beta_n u + P_{C_n}[(1 - \beta_n)(w_n - \tau_n \nabla f_n(w_n))] - z\|^2 \\
&= \|P_{C_n}[(1 - \beta_n)(w_n - \tau_n \nabla f_n(w_n))] - (1 - \beta_n)z + \beta_n(u - z)\|^2 \\
&\leq \|P_{C_n}[(1 - \beta_n)(w_n - \tau_n \nabla f_n(w_n))] - (1 - \beta_n)z\|^2 + 2\beta_n \langle u - z, x_{n+1} - z \rangle \\
&\leq \|(1 - \beta_n)(w_n - \tau_n \nabla f_n(w_n)) - (1 - \beta_n)z\|^2 + 2\beta_n \langle u - z, x_{n+1} - z \rangle \\
&\leq (1 - \beta_n) \left[\|w_n - z\|^2 - \rho_n(4 - \rho_n) \frac{f_n^2(w_n)}{\|\nabla f_n(w_n)\|^2} \right] + 2\beta_n \langle u - z, x_{n+1} - z \rangle \\
&\leq (1 - \beta_n) \left[\|x_n - z\|^2 + \theta_n \|x_n - x_{n-1}\|^2 + 2M_1 \cdot \theta_n \|x_n - x_{n-1}\| - \rho_n(4 - \rho_n) \frac{f_n^2(w_n)}{\|\nabla f_n(w_n)\|^2} \right] + 2\beta_n \langle u - z, x_{n+1} - z \rangle.
\end{aligned} \tag{78}$$

From (78),

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\leq (1 - \beta_n) \|x_n - z\|^2 + \beta_n \left[\frac{\theta_n}{\beta_n} \|x_n - x_{n-1}\|^2 + 2M_1 \right. \\
&\quad \left. \cdot \frac{\theta_n}{\beta_n} \|x_n - x_{n-1}\| + 2\langle u - z, x_{n+1} - z \rangle \right],
\end{aligned}$$

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\leq \|x_n - z\|^2 - \rho_n(4 - \rho_n) \frac{f_n^2(w_n)}{\|\nabla f_n(w_n)\|^2} \\
&\quad + \theta_n \|x_n - x_{n-1}\|^2 + 2M_1 \cdot \theta_n \|x_n - x_{n-1}\| \\
&\quad + 2\beta_n \langle u - z, x_{n+1} - z \rangle.
\end{aligned} \tag{79}$$

Let

$$\begin{aligned}
s_n &= \|x_n - z\|^2; \\
\delta_n &= \frac{\theta_n}{\beta_n} \|x_n - x_{n-1}\|^2 + 2M_1 \cdot \frac{\theta_n}{\beta_n} \|x_n - x_{n-1}\| \\
&\quad + 2\langle u - z, x_{n+1} - z \rangle; \\
\eta_n &= \rho_n(4 - \rho_n) \frac{f_n^2(w_n)}{\|\nabla f_n(w_n)\|^2}; \\
\gamma_n &= \theta_n \|x_n - x_{n-1}\|^2 + 2M_1 \cdot \theta_n \|x_n - x_{n-1}\| \\
&\quad + 2\beta_n \langle u - z, x_{n+1} - z \rangle.
\end{aligned} \tag{80}$$

Then (78) can reduce to the inequalities

$$\begin{aligned}
s_{n+1} &\leq (1 - \beta_n) s_n + \beta_n \delta_n, \quad n \geq 1, \\
s_{n+1} &\leq s_n - \eta_n + \gamma_n.
\end{aligned} \tag{81}$$

Furthermore, we know that

$$\sum_{n=0}^{\infty} \beta_n = \infty, \tag{82}$$

$$\begin{aligned}
\lim_{n \rightarrow \infty} \gamma_n &= \lim_{n \rightarrow \infty} \left[\theta_n \|x_n - x_{n-1}\|^2 + 2M_1 \right. \\
&\quad \left. \cdot \theta_n \|x_n - x_{n-1}\| + 2\beta_n \langle u - z, x_{n+1} - z \rangle \right] = 0.
\end{aligned} \tag{83}$$

Let $\{n_k\}$ be a subsequence of $\{n\}$ and suppose that

$$\lim_{k \rightarrow \infty} \eta_{n_k} = 0. \tag{84}$$

Then, we have

$$\lim_{k \rightarrow \infty} \rho_{n_k}(4 - \rho_{n_k}) \frac{f_{n_k}^2(w_{n_k})}{\|\nabla f_{n_k}(w_{n_k})\|^2} = 0, \tag{85}$$

which implies, by our assumption, that

$$\frac{f_{n_k}^2(w_{n_k})}{\|\nabla f_{n_k}(w_{n_k})\|^2} \longrightarrow 0, \quad \text{as } k \longrightarrow \infty. \tag{86}$$

Since $\{\|\nabla f_{n_k}(w_{n_k})\|\}$ is bounded, it follows that $f_{n_k}(w_{n_k}) \longrightarrow 0$ as $k \longrightarrow \infty$, so we get $\lim_{k \rightarrow \infty} \|(I - P_{Q_{n_k}})Aw_{n_k}\| = 0$.

We next show that $\omega_w(x_n) \subset S$. Since $\{x_n\}$ is bounded, the set $\omega_w(x_n)$ is nonempty. Let $x^* \in \omega_w(x_n)$; then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^*$.

$$\|w_n - x_n\| = \|x_n + \theta_n(x_n - x_{n-1}) - x_n\| = \theta_n \|x_n - x_{n-1}\| \longrightarrow 0, \tag{87}$$

and then $w_{n_j} \rightharpoonup x^*$, and A is a bounded linear operator, so $Aw_{n_j} \rightharpoonup Ax^*$.

Since $P_{Q_{n_j}}Aw_{n_j} \in Q_{n_j}$, we have

$$q(Aw_{n_j}) + \langle \zeta_{n_j}, P_{Q_{n_j}}Aw_{n_j} - Aw_{n_j} \rangle + \frac{\beta}{2} \|P_{Q_{n_j}}Aw_{n_j} - Aw_{n_j}\|^2 \leq 0, \tag{88}$$

where $\zeta_{n_j} \in \partial q(Aw_{n_j})$, and, by the boundedness of ∂q , we get

$$q(Aw_{n_j}) \leq \|\zeta_{n_j}\| \cdot \|(I - P_{Q_{n_j}})Aw_{n_j}\| - \frac{\beta}{2} \|(I - P_{Q_{n_j}})Aw_{n_j}\|^2 \longrightarrow 0, \quad (89)$$

and, using the weak lower semicontinuity of q ,

$$q(Ax^*) \leq \liminf_{j \rightarrow \infty} q(Aw_{n_j}) \leq 0. \quad (90)$$

Thus, $Ax^* \in Q$.

On the other hand,

$$\begin{aligned} \|x_{n+1} - w_n\| &\leq (1 - \beta_n) \|w_n - \tau_n \nabla f_n(w_n) - w_n\| + \beta_n \|u - z\| \\ &= (1 - \beta_n) \cdot \frac{\rho_n f_n(w_n)}{\|\nabla f_n(w_n)\|} + \beta_n \|u - z\| \longrightarrow 0. \end{aligned} \quad (91)$$

Since $(x_{n_j+1} - \beta_{n_j}u) \in C_{n_j}$, we have

$$c(w_{n_j}) + \langle \xi_{n_j}, x_{n_j+1} - \beta_{n_j}u - w_{n_j} \rangle + \frac{\alpha}{2} \|x_{n_j+1} - \beta_{n_j}u - w_{n_j}\|^2 \leq 0, \quad (92)$$

where $\xi_{n_j} \in \partial c(w_{n_j})$, and, by the boundedness of ∂c , we get

$$\begin{aligned} c(w_{n_j}) &\leq \|\xi_{n_j}\| \cdot \|w_{n_j} - x_{n_j+1} + \beta_{n_j}u\| - \frac{\alpha}{2} \|x_{n_j+1} - w_{n_j} - \beta_{n_j}u\|^2 \\ &\leq \|\xi_{n_j}\| \cdot [\|w_{n_j} - x_{n_j+1}\| + \beta_{n_j} \|u\|] \\ &\quad - \frac{\alpha}{2} [\|x_{n_j+1} - w_{n_j}\|^2 + \beta_{n_j}^2 \|u\|^2 - 2 \langle x_{n_j+1} - w_{n_j}, \beta_{n_j}u \rangle] \longrightarrow 0, \end{aligned} \quad (93)$$

and, using the weak lower semicontinuity of c ,

$$c(x^*) \leq \liminf_{j \rightarrow \infty} c(w_{n_j}) \leq 0. \quad (94)$$

Thus, $x^* \in C$; then $x^* \in S$, that is, $\omega_w(x_n) \subset S$.

Next, we have

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq (1 - \beta_n) \|w_n - \tau_n \nabla f_n(w_n) - x_n\| + \beta_n \|u - x_n\| \\ &\leq (1 - \beta_n) [\|w_n - x_n\| + \tau_n \|\nabla f_n(w_n)\|] + \beta_n \|x_n - u\| \\ &\leq \|w_n - x_n\| + \frac{\rho_n f_n(w_n)}{\|\nabla f_n(w_n)\|} + \beta_n \|x_n - u\| \longrightarrow 0. \end{aligned} \quad (95)$$

For $z = P_S u$ and $x_{n_k} \rightharpoonup x^* \in S$, using Lemma 1, $\langle u - z, x^* - z \rangle \leq 0$, so

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle u - z, x_n - z \rangle &= \limsup_{k \rightarrow \infty} \langle u - z, x_{n_k} - z \rangle \\ &= \langle u - z, x^* - z \rangle \leq 0, \end{aligned} \quad (96)$$

and then

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle u - z, x_{n+1} - z \rangle \\ = \limsup_{n \rightarrow \infty} (\langle u - z, x_{n+1} - x_n \rangle + \langle u - z, x_n - z \rangle) \leq 0, \end{aligned} \quad (97)$$

and thus

$$\begin{aligned} \limsup_k \delta_{n_k} &= \limsup_k \left[\frac{\theta_{n_k}}{\beta_{n_k}} \|x_{n_k} - x_{n_k-1}\|^2 + 2M_1 \right. \\ &\quad \left. \cdot \frac{\theta_{n_k}}{\beta_{n_k}} \|x_{n_k} - x_{n_k-1}\| + 2 \langle u - z, x_{n_k+1} - z \rangle \right] \leq 0. \end{aligned} \quad (98)$$

From (82), (83), (98), and Lemma 8, we conclude that the sequence $\{x_n\}$ converges strongly to $z = P_S u$. The proof is complete.

4. Numerical Experiments

In this section, we present a numerical experiment to illustrate the performance of the proposed algorithms. Our numerical experiments are coded in MATLAB R2007 running on personal computer with 3.50 GHz Intel Core i3 and 4 GB RAM. In what follows, we apply our algorithms to solve the problem of least absolute shrinkage and selection operator, which requires solving a convex optimization problem as

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} \|Ax - y\|^2, \quad (99)$$

$$\text{s.t., } \|x\|_1 \leq t_0,$$

where $A \in \mathbb{R}^{m \times n}$, $y \in \mathbb{R}^m$, and $t_0 > 0$ are given elements. In our experiment, we first generate an $m \times n$ matrix A randomly by a standardized normal distribution, and x is a sparse signal with n elements, only K of which is nonzero, which is also generated randomly. The observation y is generated as $y = Ax$. The parameters in this experiment are set with $n = 512$, $m = 256$, $\varepsilon = 10^{-4}$, and $t_0 = K$. In this situation, it is readily seen that $C = \{x \in \mathbb{R}^n: c(x) \leq 0\}$ with $c(x) = \|x\|_1 - t_0$ and $Q = \{y\}$, which in turn implies that

$$C_n = \{x \in \mathbb{R}^n: \langle \xi_n, x \rangle \leq \langle \xi_n, w_n \rangle - \|w_n\|_1 + t_0\}, \quad (100)$$

where ξ_n is defined by

$$(\xi_n)_i = \begin{cases} 1, & \text{if } (\xi_n)_i > 0; \\ [-1, 1], & \text{if } (\xi_n)_i = 0; \\ -1, & \text{if } (\xi_n)_i < 0, \end{cases} \quad (101)$$

standing for the subdifferential of $\|\cdot\|_1$. As a half-space, the associated projection onto C_n takes the following form:

FIGURE 1: Iterative results with $K = 50$. FIGURE 2: Iterative results with $K = 40$.

$$P_{C_n}(x) = \begin{cases} x + \frac{\langle \xi_n, w_n - x \rangle - \|w_n\|_1 + t_0 \xi_n}{\|\xi_n\|^2} \xi_n, & \text{if } x \notin C_n \\ x, & \text{if } x \in C_n. \end{cases} \quad (102)$$

To show the efficiency of our algorithm, we compare it with the algorithm proposed in [40]. The only difference of these two algorithms is that there are no inertial terms in the algorithm proposed in [40]. For the convenience, we denote Algorithm 1 by Algo. I and the algorithm in [40] by Algo. II, respectively. In Algorithm 1, we set

$$\theta_n = \begin{cases} \min \left\{ 0.8, \frac{1}{n^2 \|x_n - x_{n-1}\|^2} \right\}, & \text{if } x_n \neq x_{n-1}, \\ 0.8, & \text{if } x_n = x_{n-1}, \end{cases}$$

$$\tau_n = \begin{cases} \frac{1}{n \|A^*(Aw_n - y)\|}, & \text{if } \|A^*(Aw_n - y)\| \neq 0, \\ 0, & \text{if } \|A^*(Aw_n - y)\| = 0. \end{cases} \quad (103)$$

In Algo. II, we set $\theta_n \equiv 0$ and τ_n is chosen the same as above. The stopping criterion is that $\|x^{k+1} - x^k\| < \varepsilon$. The initial points are $x_0 = (0, 0, \dots, 0)^T$ and $x_1 = 100(1, 1, \dots, 1)^T$. The numerical results of these two algorithms with different choices of the sparsity number K are listed in Figures 1–4. It is easy to see that Algo. I converges faster than Algo. II does, which indicates that our modified algorithm indeed accelerates the convergence of the original algorithm.

FIGURE 3: Iterative results with $K = 30$. FIGURE 4: Iterative results with $K = 20$.

5. Conclusions

In this paper, we present two inertial relaxed CQ algorithms for solving split feasibility problems in Hilbert spaces by adopting variable step size. These algorithms adopt the new convex subset form, and it is easy to calculate the projections onto these sets. Furthermore, step size selection in the algorithms does not depend on the operator norm. The convergence theorems are established under some mild conditions and a numerical experiment is given to illustrate the performance of the proposed algorithms.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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