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Research Article

Subdirect Sums of Doubly Strictly Diagonally Dominant Matrices

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In this paper, the question of when the subdirect sum of two doubly strictly diagonally dominant (DSDDs) matrices is addressed. Some sufficient conditions are given, and these sufficient conditions only depend on the elements of the given matrices. Moreover, examples are presented to illustrate the corresponding results.

1. Introduction

In 1999, the concept of subdirect sums of square matrices was introduced by Fallat and Johnson, which is a generalization of the usual sum of matrices [1], and arises in several contexts, such as matrix completion problems, overlapping subdomains in domain decomposition problems, and global stiffness matrices in finite element methods [2].

For a given class matrix, an important problem is that whether the k-subdirect sums of matrices belong to the same class or not, which has been widely concerned for differently classes of matrices, such as nonsingular M-matrices [3], S-strictly diagonally dominant matrices [4], $\sum$-strictly diagonally dominant matrices [5], doubly diagonally dominant matrices [6], Nekrasov matrices [7, 8], and SDD_1 matrices [9].

In this paper, we focus on the subdirect sum of doubly strictly diagonally dominant (shortly as DSDD) matrices, which is a subclass of H-matrices [10], and some sufficient conditions such that the k-subdirect sums of DSDD matrices belong to DSDD matrices are given, and these sufficient conditions only depend on the elements of the given matrices.

Now, some notations and definitions are listed, which can also be found in [1, 11–13].

Let n be an integer number. $C^{n\times n}$ is the set of complex matrices.

Definition 1 (see [1]). Let A and B be square matrices of orders n_1 and n_2, respectively, and k be an integer number such that $1 \le k \le \min\{n_1, n_2\}$. Let A and B be partitioned in a 2×2 block as follows:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \tag{1}$$

where A_{22} and B_{11} are the square matrices of order k. We call the following square matrix of order $n = n_1 + n_2 - k$,

$$C = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} + B_{11} & B_{12} \\ 0 & B_{21} & B_{22} \end{bmatrix}, \tag{2}$$

the k-subdirect sum of A and B, and we denote it by $C = A\oplus_k B$.

In order to more explicitly express each element of C in terms of the ones of A and B, we can define the following set of indices:

$$\begin{aligned} S_1 &= \{1, 2, \ldots, n_1 - k\}, \\ S_2 &= \{n_1 - k + 1, n_1 - k + 2, \ldots, n_1\}, \\ S_3 &= \{n_1 + 1, n_1 + 2, \ldots, n\}. \end{aligned} \tag{3}$$

Then, C can be expressed as follows:

$$C=\begin{bmatrix} a_{11} & \cdots & a_{1,t} & a_{1,p} & \cdots & a_{1n_1} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{t,1} & \cdots & a_{t,t} & a_{t,p} & \cdots & a_{t,n_1} & 0 & \cdots & 0 \\ a_{p,1} & \cdots & a_{p,t} & a_{p,p}+b_{11} & \cdots & a_{p,n_1}+b_{1,n_1-t} & b_{1,n_1-t+1} & \cdots & b_{1,n-t} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n_1 1} & \cdots & a_{n_1,t} & a_{n_1,p}+b_{n_1-t,1} & \cdots & a_{n_1,n_1}+b_{n_1-t,n_1-t} & b_{n_1-t,n_1-t+1} & \cdots & b_{n_1-t,n-t} \\ 0 & \cdots & 0 & b_{n_1-t+1,1} & \cdots & b_{n_1-t+1,n_1-t} & b_{n_1-t+1,n_1-t+1} & \cdots & b_{n_1-t+1,n-t} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & b_{n-t,1} & \cdots & b_{n-t,n_1-t} & b_{n-t,n_1-t+1} & \cdots & b_{n-t,n-t} \end{bmatrix},$$

$$c_{ij}=\begin{cases} a_{ij}, & i\in S_1,\ j\in S_1\cup S_2, \\ 0, & i\in S_1,\ j\in S_3, \\ a_{ij}, & i\in S_2,\ j\in S_1, \\ a_{ij}+b_{i-t,j-t}, & i\in S_2,\ j\in S_2, \\ b_{i-t,j-t}, & i\in S_2,\ j\in S_3, \\ 0, & i\in S_3,\ j\in S_1, \\ b_{i-t,j-t}, & i\in S_3,\ j\in S_2\cup S_3, \end{cases} \tag{4}$$

where $t=n_1-k$ and $p=t+1$.

Definition 2 (see [12]). Given a matrix $A=[a_{ij}]\in C^{n\times n}$, A is called (row) diagonally dominant (DD) if

$$|a_{ii}|\geq R_i(A),\quad i=1,2,\ldots,n, \tag{5}$$

where

$$R_i(A)=\sum_{\substack{j=1\\ j\neq i}}^{n}|a_{ij}|,\quad i=1,2,\ldots,n. \tag{6}$$

If the inequality in (5) holds strictly for all i, we say that A is strictly diagonally dominant (SDD).

Definition 3 (see [13]). The matrix $A=[a_{ij}]\in C^{n\times n}$ is a doubly strictly diagonally dominant (DSDD) matrix if

$$|a_{ii}||a_{jj}|>R_i(A)R_j(A),\quad i,j=1,2,\ldots,n,i\neq j. \tag{7}$$

2. Subdirect Sums of DSDD Matrices

In general, the subdirect sum of two DSDD matrices is not always a DSDD matrix. We show this in the following example.

Example 1. Let

$$A=\begin{bmatrix} 1 & 1 & 1 \\ -1 & 4 & 0 \\ -1 & 0 & 4 \end{bmatrix},\quad B=\begin{bmatrix} 3 & -2 & -1 \\ -1 & 2 & 0 \\ 1 & 1 & 4 \end{bmatrix} \tag{8}$$

be two DSDD matrices, but

$$C=A\oplus_1 B=\begin{bmatrix} 1 & 1 & 1 & 0 \\ -1 & 7 & -2 & -1 \\ -1 & -1 & 6 & 0 \\ 0 & 1 & 1 & 4 \end{bmatrix} \tag{9}$$

is not a DSDD matrix, since $|c_{11}||c_{22}|=7<R_1(C)R_2(C)=8$.

Example 1 shows that the subdirect sum of DSDD matrices is not a DSDD matrix; then, a meaningful discussion is concerned: under what conditions, the subdirect sum of DSDD matrices is in the class of DSDD matrices?

In order to obtain the main results, we need the following lemma.

Lemma 1. *Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be DSDD square matrices of orders n_1 and n_2 partitioned as in (1), respectively. And, let k be an integer number such that $1 \le k \le \min\{n_1, n_2\}$, S_1, S_2, S_3 be defined as in (4), and all diagonal entries of A_{22} and B_{11} are positive (or all negative), with $C = A \oplus_k B$, then,*

(a) For $i \in S_1$, we have $R_i(C) = R_i(A)$

(b) For $k = 1$, $i \in S_2$, we have $R_i(C) = R_{n_1}(C) = R_{n_1}(A) + R_1(B)$

(c) For $k \geqslant 2$, $i \in S_2$, we have $R_i(C) \leqslant R_i(A) + R_{i-t}(B)$

(d) For $i \in S_3$, we have $R_i(C) = R_{i-t}(B)$

Proof. For $i \in S_1$, we can write

$$R_i(C) = \sum_{j\in S_1 j\ne i} |a_{ij}| + \sum_{j\in S_2} |a_{ij}| + 0 = R_i(A). \quad (10)$$

For $k = 1$, $i \in S_2 = \{n_1\}$, we can obtain

$$R_i(C) = R_{n_1}(C) = \sum_{j\in S_1} |a_{ij}| + \sum_{\substack{j\in S_3\\ j-t\ne 1}} |b_{1,j-t}| = R_{n_1}(A) + R_1(B). \quad (11)$$

For $k > 2$, $i \in S_2$, we obtain

$$\begin{aligned} R_i(C) &= \sum_{j\in S_1} |a_{ij}| + \sum_{j\in S_2 j\ne i} |a_{ij} + b_{i-t,j-t}| + \sum_{j\in S_3} |b_{i-t,j-t}| \\ &\le \sum_{j\in S_1} |a_{ij}| + \sum_{j\in S_2 j\ne i} |a_{ij}| + \sum_{j\in S_2 j\ne i} |b_{i-t,j-t}| \\ &\quad + \sum_{j\in S_3} |b_{i-t,j-t}| = R_i(A) + R_{i-t}(B). \end{aligned} \quad (12)$$

For the rest case of $i \in S_3$, the proof is similar to the proof of $i \in S_1$.

Firstly, we study the 1-subdirect sum of DSDD matrices.

Theorem 1. *Let A and B be DSDD matrices of orders n_1 and n_2 partitioned as in (1), respectively, and $k = 1$. Then, $C = A \oplus_1 B$ is a DSDD matrix if all diagonal entries of A_{22} and B_{11} are positive (or all negative) and for $i \in S_1$,*

$$|a_{ii}| > |a_{n_1,n_1}|, \quad (13)$$

$$R_{n_1}(A) \ge R_i(A), \quad (14)$$

$$\max_{j\in S_2\cup S_3} \frac{R_{j-t}(B)}{|b_{j-t,j-t}|} R_{n_1}(A) \le |a_{n_1,n_1}|, \quad (t = n_1 - 1). \quad (15)$$

Proof. Since $k = 1$ and A and B are the DSDD matrices of orders n_1 and n_2 respectively, it is obvious that $S_1 = \{1, 2, \ldots, n_1 - 1\}$, $S_2 = \{n_1\}$, $S_3 = \{n_1 + 1, n_1 + 2, \ldots, n\}$.

Case 1: for $i, j \in S_1$, from (a) of Lemma 1, we have

$$\begin{aligned} &|c_{ii}| = |a_{ii}|, |c_{jj}| = |a_{jj}|, \\ &R_i(C) = R_i(A), \\ &R_j(C) = R_j(A). \end{aligned} \quad (16)$$

Since A is DSDD, we obtain that for $i, j \in S_1$,

$$|c_{ii}||c_{jj}| = |a_{ii}||a_{jj}| > R_i(A)R_j(A) = R_i(C)R_j(C). \quad (17)$$

Case 2: for $i \in S_1$ and $j \in S_2 = \{n_1\}$, from (a) of Lemma 1, it is easy to obtain

$$\begin{aligned} &|c_{ii}| = |a_{ii}|, \\ &R_i(C) = R_i(A). \end{aligned} \quad (18)$$

Since all diagonal entries of A_{22} and B_{11} are positive (or all negative), we obtain

$$|c_{jj}| = |a_{n_1,n_1}| + |b_{11}|. \quad (19)$$

Since A is DSDD, and from inequalities (13)–(15), we have

$$\begin{aligned} |c_{ii}||c_{jj}| &= |a_{ii}|\left(|a_{n_1,n_1}| + |b_{11}|\right) = |a_{ii}||a_{n_1,n_1}| + |a_{ii}||b_{11}| > R_i(A)R_{n_1}(A) + |a_{n_1,n_1}||b_{11}| \\ &\ge R_i(A)R_{n_1}(A) + |b_{11}| \max_{j\in S_2\cup S_3} \frac{R_{j-t}(B)}{|b_{j-t,j-t}|} R_{n_1}(A) \ge R_i(A)R_{n_1}(A) + R_{n_1}(A)R_1(B) \\ &\ge R_i(A)R_{n_1}(A) + R_i(A)R_1(B) = R_i(A)\left(R_{n_1}(A) + R_1(B)\right). \end{aligned} \quad (20)$$

From (b) of Lemma 1, it is easy to obtain that for $i \in S_1$, $j \in S_2 = \{n_1\}$,

$$|c_{ii}||c_{jj}| > R_i(C)R_j(C). \quad (21)$$

Case 3: for $i \in S_1$ and $j \in S_3$, from (a) and (d) of Lemma 1, we have

$$\begin{aligned} &|c_{ii}| = |a_{ii}|, \\ &R_i(C) = R_i(A), \\ &|c_{jj}| = |b_{j-t,j-t}|, \\ &R_j(C) = R_{j-t}(B). \end{aligned} \quad (22)$$

Then, from inequalities (13)–(15), we have that for $i \in S_1$ and $j \in S_3$,

$$\begin{aligned} |c_{ii}||c_{jj}| &= |a_{ii}||b_{j-t,j-t}| > |a_{n_1,n_1}||b_{j-t,j-t}| \geq |b_{j-t,j-t}| \max_{j\in S_2\cup S_3} \frac{R_{j-t}(B)}{|b_{j-t,j-t}|} R_{n_1}(A) \\ &\geq R_{n_1}(A) R_{j-t}(B) \geq R_i(A) R_{j-t}(B) = R_i(C) R_j(C). \end{aligned} \tag{23}$$

Case 4: for $i \in S_2 = \{n_1\}$ and $j \in S_3$, from (b) and (d) of Lemma 1, we obtain

$$\begin{aligned} |c_{ii}| &= |a_{n_1,n_1}| + |b_{11}|, \\ |c_{jj}| &= |b_{j-t,j-t}|. \end{aligned} \tag{24}$$

Since B is DSDD, and from inequality (15), for $i \in S_2 = \{n_1\}$ and $j \in S_3$, we have

$$\begin{aligned} |c_{ii}||c_{jj}| &= \left(|a_{n_1,n_1}| + |b_{11}|\right)|b_{j-t,j-t}| \\ &= |a_{n_1,n_1}||b_{j-t,j-t}| + |b_{11}||b_{j-t,j-t}| \\ &> |b_{j-t,j-t}| \max_{j\in S_2\cup S_3} \frac{R_{j-t}(B)}{|b_{j-t,j-t}|} R_{n_1}(A) + R_1(B) R_{j-t}(B) \\ &\geq R_{n_1}(A) R_{j-t}(B) + R_1(B) R_{j-t}(B) \\ &= \left(R_{n_1}(A) + R_1(B)\right) R_{j-t}(B) = R_i(C) R_j(C). \end{aligned} \tag{25}$$

Case 5: for $i, j \in S_3$, from (d) of Lemma 1, we obtain

$$\begin{aligned} |c_{ii}| &= |b_{i-t,i-t}|, \\ R_i(C) &= R_{i-t}(B). \end{aligned} \tag{26}$$

Since B is DSDD, we have

$$|c_{ii}||c_{jj}| = |b_{i-t,i-t}||b_{j-t,j-t}| > R_{i-t}(B) R_{j-t}(B) = R_i(C) R_j(C). \tag{27}$$

Therefore, we can draw a conclusion that for any $i, j \in \{1, 2, \ldots, n\}$, $|c_{ii}||c_{jj}| > R_i(C) R_j(C)$, that is, $C = A \oplus_1 B$ is a DSDD matrix.

Example 2. The matrices

$$\begin{aligned} A &= \begin{bmatrix} 3 & 0 & -1 \\ -1 & 4 & 0 \\ -2 & 0 & 2 \end{bmatrix}, \\ B &= \begin{bmatrix} 3 & 1 & -1 \\ -1 & -2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \end{aligned} \tag{28}$$

are two DSDD matrices, and from Theorem 1, it is easy to verify that

$$C = A \oplus_1 B = \begin{bmatrix} 3 & 0 & -1 & 0 & 0 \\ -1 & 4 & 0 & 0 & 0 \\ -2 & 0 & 5 & 1 & -1 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 2 \end{bmatrix} \tag{29}$$

is a DSDD matrix since

$3 = |a_{11}| > |a_{33}| = 2, 4 = |a_{22}| > |a_{33}| = 2, 2 = R_3(A) > R_1(A) = 1, 2 = R_3(A) > R_2(A) = 1,$ and $\max_{j\in S_2\cup S_3}(R_{j-t}(B)/|b_{j-t,j-t}|) R_{n_1}(A) = (R_3(B)/|b_{33}|) R_3(A) = (4/7) \times 3 < |a_{33}| = 2$.

However,

$$C = A \oplus_2 B = \begin{bmatrix} 3 & 0 & -1 & 0 \\ -1 & 7 & 1 & -1 \\ -2 & -1 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \tag{30}$$

is not DSDD since $|c_{11}||c_{33}| = 0 < 3 = R_1(C) R_3(C)$.

Examples 2 motivates the search for other conditions such that $C = A \oplus_k B$ $(k \geqslant 2)$ is also a DSDD matrix, where A is a DSDD matrix and B is a DSDD matrix.

Next, some sufficient conditions ensuring that the k-subdirect sum of DSDD matrices is a DSDD matrix are given.

Theorem 2. *Let A and B be matrices of orders n_1 and n_2 partitioned as in (1), respectively, and k is an integer number such that $2 \le k \le \min\{n_1, n_2\}$. Let A and B be DSDD matrices, if all diagonal entries of A_{22} and B_{11} are positive (or all negative) and for any $i \in S_1 \cup S_2$,*

$$\max_{j \in S_2 \cup S_3} \frac{R_{j-t}(B)}{|b_{j-t,j-t}|} R_i(A) < |a_{ii}|, \quad (t = n_1 - k), \tag{31}$$

and then the k-subdirect sum $C = A \oplus_k B$ is DSDD.

Proof. Let A and B be DSDD matrices of orders n_1 and n_2 respectively; thus, it is obvious that $S_1 = \{1, 2, \ldots, n_1 - k\}$, $S_2 = \{n_1 - k + 1, n_1 - k + 2, \ldots, n_1\}$, and $S_3 = \{n_1 + 1, n_1 + 2, \ldots, n\}$.

Case 1: for $i, j \in S_1$, from (a) of Lemma 1, we have

$$\begin{aligned} |c_{ii}| &= |a_{ii}|, \\ R_i(C) &= R_i(A). \end{aligned} \tag{32}$$

Since A is DSDD, we obtain

$$|c_{ii}||c_{jj}| = |a_{ii}||a_{jj}| > R_i(A) R_j(A) = R_i(C) R_j(C). \tag{33}$$

Case 2: for $i \in S_1$ and $j \in S_2$, from (a) and (c) of Lemma 1, we obtain

$$\begin{aligned} |c_{ii}| &= |a_{ii}|, \\ R_i(C) &= R_i(A), \\ R_j(C) &\le R_j(A) + R_{j-t}(B). \end{aligned} \tag{34}$$

Since all diagonal entries of A_{22} and B_{11} are positive (or all negative), we obtain

$$|c_{jj}| = |a_{jj} + b_{j-t,j-t}| = |a_{jj}| + |b_{j-t,j-t}|. \tag{35}$$

Since A is DSDD, and from inequality (31), we have

$$\begin{aligned} |c_{ii}||c_{jj}| &= |a_{ii}|\left(|a_{jj}| + |b_{j-t,j-t}|\right) = |a_{ii}||a_{jj}| + |a_{ii}||b_{j-t,j-t}| \\ &> R_i(A) R_j(A) + |b_{j-t,j-t}| \max_{j \in S_2 \cup S_3} \frac{R_{j-t}(B)}{|b_{j-t,j-t}|} R_i(A) \\ &\ge R_i(A) R_j(A) + R_i(A) R_{j-t}(B) \\ &= R_i(A)\left(R_j(A) + R_{j-t}(B)\right). \end{aligned} \tag{36}$$

From (c) of Lemma 1, it is easy to obtain that

$$|c_{ii}||c_{jj}| > R_i(C) R_j(C). \tag{37}$$

Case 3: for $i \in S_1$ and $j \in S_3$, from (a) and (d) of Lemma 1, we conclude

$$\begin{aligned} |c_{ii}| &= |a_{ii}|, \\ R_i(C) &= R_i(A), \\ |c_{jj}| &= |b_{j-t,j-t}|, \\ R_j(C) &= R_{j-t}(B). \end{aligned} \tag{38}$$

Then, from inequality (31), we have

$$\begin{aligned} |c_{ii}||c_{jj}| &= |a_{ii}||b_{j-t,j-t}| \\ &> |b_{j-t,j-t}| \max_{j \in S_2 \cup S_3} \frac{R_{j-t}(B)}{|b_{j-t,j-t}|} R_i(A) \\ &\ge R_i(A) R_{j-t}(B) \\ &= R_i(C) R_j(C). \end{aligned} \tag{39}$$

Case 4: for $i, j \in S_2$, from (c) of Lemma 1, we obtain

$$\begin{aligned} |c_{ii}| &= |a_{ii}| + |b_{i-t,i-t}|, \\ R_i(C) &\le R_i(A) + R_{i-t}(B). \end{aligned} \tag{40}$$

Since A and B are DSDD, and from inequality (31), we conclude

$$\begin{aligned}
|c_{ii}||c_{jj}| &= \left(|a_{ii}| + |b_{i-t,i-t}|\right)\left(|a_{jj}| + |b_{j-t,j-t}|\right) = |a_{ii}||a_{jj}| + |a_{ii}||b_{j-t,j-t}| + |a_{jj}||b_{i-t,i-t}| + |b_{i-t,i-t}||b_{j-t,j-t}| \\
&> R_i(A)R_j(A) + |b_{j-t,j-t}| \max_{j\in S_2\cup S_3} \frac{R_{j-t}(B)}{|b_{j-t,j-t}|} R_i(A) + |b_{i-t,i-t}| \max_{i\in S_2\cup S_3} \frac{R_{i-t}(B)}{|b_{i-t,i-t}|} R_j(A) + R_{i-t}(B)R_{j-t}(B) \\
&\geq R_i(A)R_j(A) + R_i(A)R_{j-t}(B) + R_j(A)R_{i-t}(B) + R_{i-t}(B)R_{j-t}(B) \\
&= \left(R(A) + R_{i-t}(B)\right)\left(R_j(A) + R_{j-t}(B)\right) \geq R_i(C)R_j(C).
\end{aligned} \tag{41}$$

Case 5: for $i \in S_2$ and $j \in S_3$, from (c) and (d) of Lemma 1, we obtain

$$\begin{aligned}
|c_{ii}| &= |a_{ii}| + |b_{i-t,i-t}|, \\
R_i(C) &\leq R_i(A) + R_{i-t}(B), \\
|c_{jj}| &= |b_{j-t,j-t}|, \\
R_j(C) &= R_{j-t}(B).
\end{aligned} \tag{42}$$

Since B is DSDD, and from inequality (31), we obtain

$$\begin{aligned}
|c_{ii}||c_{jj}| &= \left(|a_{ii}| + |b_{i-t,i-t}|\right)|b_{j-t,j-t}| \\
&= |a_{ii}||b_{j-t,j-t}| + |b_{i-t,i-t}||b_{j-t,j-t}| \\
&> |b_{j-t,j-t}| \max_{j\in S_2\cup S_3} \frac{R_{j-t}(B)}{|b_{j-t,j-t}|} R_i(A) + R_{i-t}(B)R_{j-t}(B) \\
&\geq R_i(A)R_{j-t}(B) + R_{i-t}(B)R_{j-t}(B) \\
&= \left(R_i(A) + R_{i-t}(B)\right)R_{j-t}(B) \geq R_i(C)R_j(C).
\end{aligned} \tag{43}$$

Case 6: for $i, j \in S_3$, from (d) of Lemma 1, we obtain

$$\begin{aligned}
|c_{ii}| &= |b_{i-t,i-t}|, \\
R_i(C) &= R_{i-t}(B).
\end{aligned} \tag{44}$$

Since B is DSDD, we obtain

$$|c_{ii}||c_{jj}| = |b_{i-t,i-t}||b_{j-t,j-t}| > R_{i-t}(B)R_{j-t}(B) = R_i(C)R_j(C). \tag{45}$$

In conclusion, for any $i, j \in S_1 \cup S_2 \cup S_3$, $|c_{ii}||c_{jj}| > R_i(C)R_j(C)$. Therefore, $C = A\oplus_k B$ is a DSDD matrix. □

Example 3. Let

$$\begin{aligned}
A &= \begin{bmatrix} 4 & 1 & 4 \\ 1 & 5 & -2 \\ 0 & -1 & 2 \end{bmatrix}, \\
B &= \begin{bmatrix} 2 & 0 & 1 \\ 3 & 8 & 0 \\ -1 & 2 & 5 \end{bmatrix}
\end{aligned} \tag{46}$$

be two DSDD matrices. And from Definition 1, we obtain that

$$C = A\oplus_2 B = \begin{bmatrix} 4 & 1 & 4 & 0 \\ 1 & 7 & -2 & 1 \\ 0 & 2 & 10 & 0 \\ 0 & -1 & 2 & 5 \end{bmatrix}, \tag{47}$$

and $S_1 = \{1\}, S_2 = \{2,3\}$, and $S_3 = \{4\}$. From Theorem 2, $C = A\oplus_2 B$ is a DSDD matrix since

$$\begin{aligned}
\max_{j\in S_2\cup S_3} \frac{R_{j-t}(B)}{|b_{j-t,j-t}|} R_1(A) &= \frac{R_3(B)}{|b_{33}|} R_1(A) = \frac{3}{5} \times 5 < |a_{11}|, \\
\max_{j\in S_2\cup S_3} \frac{R_{j-t}(B)}{|b_{j-t,j-t}|} R_2(A) &= \frac{R_3(B)}{|b_{33}|} R_2(A) = \frac{3}{5} \times 3 < |a_{22}|, \\
\max_{j\in S_2\cup S_3} \frac{R_{j-t}(B)}{|b_{j-t,j-t}|} R_3(A) &= \frac{R_3(B)}{|b_{33}|} R_3(A) = \frac{3}{5} \times 1 < |a_{33}|.
\end{aligned} \tag{48}$$

From Definitions 2 and 3, it is easy to show that SDD matrices are contained into DSDD matrices. Therefore, from Theorem 2, we obtain the following corollaries, which present sufficient conditions such that k-subdirect sum $C = A\oplus_k B$ is DSDD.

Corollary 1. *Let A and B be square matrices of orders* n_1 *and* n_2 *partitioned as in (1), respectively, and k is an integer number such that* $1 \le k \le min\{n_1, n_2\}$. *We assume that A is a DSDD matrix and B is a SDD matrix. If there exists an* $i_0 \in S_1 \cup S_2$ *such that*

$$\max_{j\in S_2\cup S_3} \frac{R_{j-t}(B)}{|b_{j-t,j-t}|} R_{i_0}(A) < |a_{i_0,i_0}| < R_{i_0}(A), \quad (49)$$

$$|a_{ii}| \geq R_i(A), \quad i \in S_1 \cup S_2 \backslash \{i_0\}. \quad (50)$$

And, all diagonal entries of A_{22} and B_{11} are positive (or all negative), then the k-subdirect sum $C = A\oplus_k B$ is a DSDD matrix.

Proof. Without loss of generality, we can assume $i_0 = 1$ such that $\max_{j\in S_2\cup S_3}(R_{j-t}(B)/|b_{j-t,j-t}|)R_1(A) < |a_{11}| < R_1(A)$ and $|a_{ii}| \ge R_i(A), \quad i = 2, 3, \ldots, n_1$.

Case 1: for $i, j \in S_1$, from (a) of Lemma 1, we have

$$\begin{aligned} |c_{ii}| &= |a_{ii}|, \\ |c_{jj}| &= |a_{jj}|, \\ R_i(C) &= R_i(A), \\ R_j(C) &= R_j(A). \end{aligned} \quad (51)$$

Since A is DSDD, we obtain that for $i, j \in S_1$,

$$|c_{ii}||c_{jj}| = |a_{ii}||a_{jj}| > R_i(A)R_j(A) = R_i(C)R_j(C). \quad (52)$$

Case 2: for $i \in S_1$ and $j \in S_2$, from (a) of Lemma 1, it is easy to obtain

$$\begin{aligned} |c_{ii}| &= |a_{ii}|, \\ R_i(C) &= R_i(A). \end{aligned} \quad (53)$$

Since all diagonal entries of A_{22} and B_{11} are positive (or all negative), we obtain

$$|c_{jj}| = |a_{jj}| + |b_{j-t,j-t}|. \quad (54)$$

If $i = 1$, since A is DSDD, and from inequality (49), we can obtain

$$\begin{aligned} |c_{11}||c_{jj}| &= |a_{11}|\left(|a_{jj}| + |b_{j-t,j-t}|\right) = |a_{11}||a_{jj}| + |a_{11}||b_{j-t,j-t}| \\ &> R_1(A)R_j(A) + |b_{j-t,j-t}| \max_{j\in S_2\cup S_3} \frac{R_{j-t}(B)}{|b_{j-t,j-t}|} R_1(A) \geq R_1(A)R_j(A) + |b_{j-t,j-t}| \frac{R_{j-t}(B)}{|b_{j-t,j-t}|} R_1(A) \\ &= R_1(A)R_j(A) + R_{j-t}(B)R_1(A) = R_1(A)\left(R_j(A) + R_{j-t}(B)\right) \geq R_1(C)R_j(C). \end{aligned} \quad (55)$$

If $i = 2, 3, \ldots, n_1 - k$, since B is SDD, and from inequality (50), we can write

$$|c_{ii}||c_{jj}| = |a_{ii}|\left(|a_{jj}| + |b_{j-t,j-t}|\right) > R_i(A)\left(R_j(A) + R_{j-t}(B)\right) \geq R_i(C)R_j(C). \quad (56)$$

Case 3: for $i \in S_1$ and $j \in S_3$, from (a) and (d) of Lemma 1, we have

$$\begin{aligned} |c_{ii}| &= |a_{ii}|, \\ R_i(C) &= R_i(A), \\ |c_{jj}| &= |b_{j-t,j-t}|, \\ R_j(C) &= R_{j-t}(B). \end{aligned} \quad (57)$$

If $i = 1$, from inequality (49), we have

$$\begin{aligned} |c_{11}||c_{jj}| &= |a_{11}||b_{j-t,j-t}| > |b_{j-t,j-t}| \max_{j\in S_2\cup S_3} \frac{R_{j-t}(B)}{|b_{j-t,j-t}|} R_1(A) \\ &\geq |b_{j-t,j-t}| \frac{R_{j-t}(B)}{|b_{j-t,j-t}|} R_1(A) = R_1(A)R_{j-t}(B) \\ &= R_1(C)R_j(C). \end{aligned} \quad (58)$$

If $i = 2, \ldots, n_1 - k$, since B is SDD, and from inequality (50), we can write

$$|c_{ii}||c_{jj}| = |a_{ii}||b_{j-t,j-t}| > R_i(A)R_{j-t}(B) = R_i(C)R_j(C). \quad (59)$$

Case 4: for $i, j \in S_2$, from (b) and (c) of Lemma 1, we obtain

$$\begin{aligned} |c_{ii}| &= |a_{ii}| + |b_{i-t,i-t}|, \\ R_i(C) &\le R_i(A) + R_{i-t}(B). \end{aligned} \quad (60)$$

Since B is SDD, and from inequality (50), we can obtain

$$\begin{aligned} |c_{ii}||c_{jj}| &= \left(|a_{ii}| + |b_{i-t,i-t}|\right)\left(|a_{jj}| + |b_{j-t,j-t}|\right) \\ &> \left(R_i(A) + R_{i-t}(B)\right)\left(R_j(A) + R_{j-t}(B)\right) \\ &\ge R_i(C)R_j(C). \end{aligned} \quad (61)$$

Case 5: for $i \in S_2$ and $j \in S_3$, from (c) and (d) of Lemma 1, we obtain

$$\begin{aligned} |c_{ii}| &= |a_{ii}| + |b_{i-t,i-t}|, \\ R_i(C) &\le R_i(A) + R_{i-t}(B), \\ |c_{jj}| &= |b_{j-t,j-t}|, \\ R_j(C) &= R_{j-t}(B). \end{aligned} \quad (62)$$

Since B is SDD, and from inequality (50), we obtain

$$|c_{ii}||c_{jj}| = \left(|a_{ii}| + |b_{i-t,i-t}|\right)|b_{j-t,j-t}| > \left(R_i(A) + R_{i-t}(B)\right)R_{j-t}(B) \ge R_i(C)R_j(C). \quad (63)$$

Case 6: for $i, j \in S_3$, from (d) of Lemma 1, we obtain

$$\begin{aligned} |c_{ii}| &= |b_{i-t,i-t}|, \\ R_i(C) &= R_{i-t}(B). \end{aligned} \quad (64)$$

Since B is SDD, we can obtain

$$|c_{ii}||c_{jj}| = |b_{i-t,i-t}||b_{j-t,j-t}| > R_{i-t}(B)R_{j-t}(B) = R_i(C)R_j(C). \quad (65)$$

Therefore, we can draw a conclusion that for any $i, j \in \{1, 2, \ldots, n\}$, $|c_{ii}||c_{jj}| > R_i(C)R_j(C)$; that is, $C = A \oplus_k B$ is a DSDD matrix. □

Corollary 2. *Let A and B be square matrices of orders n_1 and n_2 partitioned as in (1), respectively, and k is an integer number such that $1 \le k \le \min\{n_1, n_2\}$. We assume that A is a SDD matrix and B is a DSDD matrix. If there exists a $j_0 \in S_2 \cup S_3$ such that*

$$\max_{i \in S_1 \cup S_2} \frac{R_i(A)}{|a_{ii}|} R_{j_0-t}(B) < |b_{j_0-t,j_0-t}| < R_{j_0-t}(B), \quad (66)$$

$$|a_{ii}| \ge R_{j-t}(B), \quad j \in S_2 \cup S_3 \backslash \{j_0\}. \quad (67)$$

And, all diagonal entries of A_{22} and B_{11} are positive (or all negative), then the k-subdirect sum $C = A \oplus_k B$ is a DSDD matrix.

Proof. Without loss of generality, we can assume $j_0 = n$ such that $\max_{i \in S_1 \cup S_2} (R_i(A)/|a_{ii}|) R_{n-t}(B) < |b_{n-t,n-t}| < R_{n-t}(B)$ and $|b_{j-t,j-t}| \ge R_{i-t}(B)$, $j = n_1 - k + 1, n_1 - k + 2, \ldots, n - 1$.

Case 1: for $i, j \in S_1$, from (a) of Lemma 1, we have

$$\begin{aligned} |c_{ii}| &= |a_{ii}|, \\ |c_{jj}| &= |a_{jj}|, \\ R_i(C) &= R_i(A), \\ R_j(C) &= R_j(A). \end{aligned} \quad (68)$$

Since A is SDD, we obtain that for $i, j \in S_1$,

$$|c_{ii}||c_{jj}| = |a_{ii}||a_{jj}| > R_i(A)R_j(A) = R_i(C)R_j(C). \quad (69)$$

Case 2: for $i \in S_1$ and $j \in S_2$, from (a) of Lemma 1, it is easy to obtain

$$\begin{aligned} |c_{ii}| &= |a_{ii}|, \\ R_i(C) &= R_i(A). \end{aligned} \quad (70)$$

Since all diagonal entries of A_{22} and B_{11} are positive (or all negative), we obtain

$$|c_{jj}| = |a_{jj}| + |b_{j-t,j-t}|. \quad (71)$$

Since A is SDD and from inequality (67), we have

$$|c_{ii}||c_{jj}| = |a_{ii}|\left(|a_{jj}| + |b_{j-t,j-t}|\right) > R_i(A)\left(R_j(A) + R_{j-t}(B)\right) \geq R_i(C)R_j(C). \quad (72)$$

Case 3: for $i \in S_1$ and $j \in S_3$, from (a) and (d) of Lemma 1, we have

$$\begin{aligned} |c_{ii}| &= |a_{ii}|, \\ R_i(C) &= R_i(A), \\ |c_{jj}| &= |b_{j-t,j-t}|, \\ R_j(C) &= R_{j-t}(B). \end{aligned} \quad (73)$$

If $j = n$, from inequality (66), we have

$$\begin{aligned} |c_{ii}||c_{nn}| &= |a_{ii}||b_{n-t,n-t}| > |a_{ii}| \max_{i \in S_1 \cup S_2} \frac{R_i(A)}{|a_{ii}|} R_{n-t}(B) \\ &\geq |a_{ii}| \frac{R_i(A)}{|a_{ii}|} R_{n-t}(B) \\ &= R_i(A)R_{n-t}(B) = R_i(C)R_n(C). \end{aligned} \quad (74)$$

If $j = n_1 - k + 1, n_1 - k + 2, \ldots, n - 1$, since A is SDD, and from inequality (67), we can write

$$|c_{ii}||c_{jj}| = |a_{ii}||b_{j-t,j-t}| > R_i(A)R_{j-t}(B) = R_i(C)R_j(C). \quad (75)$$

Case 4: for $i, j \in S_2$, from (c) of Lemma 1, we obtain

$$\begin{aligned} |c_{ii}| &= |a_{ii}| + |b_{i-t,i-t}|, \\ R_i(C) &\leq R_i(A) + R_{i-t}(B). \end{aligned} \quad (76)$$

Since A is SDD, and from inequality (67), we can obtain

$$\begin{aligned} |c_{ii}||c_{jj}| &= \left(|a_{ii}| + |b_{i-t,i-t}|\right)\left(|a_{jj}| + |b_{j-t,j-t}|\right) \\ &> \left(R_i(A) + R_{i-t}(B)\right)\left(R_j(A) + R_{j-t}(B)\right) \geq R_i(C)R_j(C). \end{aligned} \quad (77)$$

Case 5: for $i \in S_2,\ j \in S_3$, from (c) and (d) of Lemma 1, we obtain

$$\begin{aligned} |c_{ii}| &= |a_{ii}| + |b_{i-t,i-t}|, \\ R_i(C) &\leq R_i(A) + R_{i-t}(B), \\ |c_{jj}| &= |b_{j-t,j-t}|, \\ R_j(C) &= R_{j-t}(B). \end{aligned} \quad (78)$$

If $j = n$, since B is DSDD, and from inequality (66), we have

$$\begin{aligned} |c_{ii}||c_{nn}| &= \left(|a_{ii}| + |b_{i-t,i-t}|\right)|b_{n-t,n-t}| \\ &= |a_{ii}||b_{n-t,n-t}| + |b_{i-t,i-t}||b_{n-t,n-t}| \\ &> |a_{ii}| \max_{i \in S_1 \cup S_2} \frac{R_i(A)}{|a_{ii}|} R_{n-t}(B) + R_{i-t}(B)R_{n-t}(B) \\ &\geq |a_{ii}| \frac{R_i(A)}{|a_{ii}|} R_{n-t}(B) + R_{i-t}(B)R_{n-t}(B) \\ &= R_i(A)R_{n-t}(B) + R_{i-t}(B)R_{n-t}(B) \\ &= \left(R_i(A) + R_{i-t}(B)\right)R_{n-t}(B) \geq R_i(C)R_n(C). \end{aligned} \quad (79)$$

If $j = n_1 - k + 1, n_1 - k + 2, \ldots, n - 1$, since A is SDD, and from inequality (67), we can write

$$|c_{ii}||c_{jj}| = \left(|a_{ii}| + |b_{i-t,i-t}|\right)|b_{j-t,j-t}| > \left(R_i(A) + R_{i-t}(B)\right)R_{j-t}(B) \geq R_i(C)R_j(C). \quad (80)$$

Case 6: for $i, j \in S_3$, from (d) of Lemma 1, we obtain

$$\begin{aligned}|c_{ii}| &= |b_{i-t,i-t}|,\\ R_i(C) &= R_{i-t}(B).\end{aligned} \quad (81)$$

Since B is DSDD, we can obtain

$$|c_{ii}||c_{jj}| = |b_{i-t,i-t}||b_{j-t,j-t}| > R_{i-t}(B)R_{j-t}(B) = R_i(C)R_j(C). \quad (82)$$

In conclusion, for any $i, j \in S_1 \cup S_2 \cup S_3$, $|c_{ii}||c_{jj}| > R_i(C)R_j(C)$. Therefore, $C = A \oplus_k B$ is a DSDD matrix. □

3. Conclusions

In this paper, some sufficient conditions such that the subdirect sum of DSDD matrices is in the class of DSDD matrices are given. Moreover, numerical examples are also presented to illustrate the corresponding results.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors jointly worked on the results, and they read and approved the final manuscript.

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