On Strongly Convex Functions via Caputo–Fabrizio-Type Fractional Integral and Some Applications

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The theory of convex functions plays an important role in the study of optimization problems. The fractional calculus has been found the best to model physical and engineering processes. The aim of this paper is to study some properties of strongly convex functions via the Caputo–Fabrizio fractional integral operator. In this paper, we present Hermite–Hadamard-type inequalities for strongly convex functions via the Caputo–Fabrizio fractional integral operator. Some new inequalities of strongly convex functions involving the Caputo–Fabrizio fractional integral operator are also presented. Moreover, we present some applications of the proposed inequalities to special means.

1. Introduction

The theory of fractional calculus got rapid development, and it has brought the attention of many researchers from various disciplines [1–3]. In the last few years, it was observed that fractional calculus is very useful for modeling complicated problems of engineering, chemistry, mechanics, and many other branches. Various interesting notations of fractional calculus exist in the literature, for example, the Riemann–Liouville fractional integral and Caputo–Fabrizio fractional integral [4–14].

Among these notions, Riemann–Liouville and Caputo involve the following singular kernel [11]:

\[ K(\zeta, x) = \frac{(\zeta - x)^{-\varsigma}}{\Gamma(1 - \varsigma)}, \quad 0 < \varsigma < 1. \]  

However, it was observed by Caputo and Fabrizio in [8] that certain phenomena cannot be modelled by the already existing definition in the literature. That is why, they proposed a more general fractional derivative in [8] and named it as the Caputo–Fabrizio fractional integral operator. It mainly involves the following nonsingular kernel:

\[ K(\zeta, x) = e^{-\varsigma(\zeta - x)/(1 - \varsigma)}, \quad 0 < \varsigma < 1. \]  

Nowadays, many researchers of applied sciences are using the Caputo–Fabrizio fractional integral operator to model their problem. For more details about the fractional integral with a nonsingular kernel, we refer [15–19] to the readers.

The theory of inequalities also plays an important role in applied as well as in pure mathematics. The Hermite–Hadamard inequality is the most important inequality in the literature, and this inequality has been studied for different classes of convex functions, see [20–24]. The classical version of the Hermite–Hadamard inequality for convex functions is stated as follows:

If \( \varphi: I = [a, b] \subset \mathbb{R} \rightarrow \mathbb{R} \) is an integrable and continuous convex function, then its mean value remains between the value of \( \varphi \) at \( (a + b)/2 \) of interval \( I = [a, b] \) and arithmetic mean value of \( \varphi \) at the endpoints \( a, b \in I = [a, b] \). In other words, it means that

\[ \varphi \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b \varphi(x) dx \leq \frac{\varphi(a) + \varphi(b)}{2}. \]  


Inequality (3), in the literature, is generalized by several fractional integral operators to meet the desired results, see, for instance, [25–28]. In this paper, we present the Hermite–Hadamard inequality for a strongly convex function in the setting of the Caputo–Fabrizio fractional integral operator. We also present some new inequalities for strongly convex functions in the setting of the Caputo–Fabrizio fractional integral operator. We also give some applications of the presented inequalities in special mean.

2. Preliminaries

In this section, we present some definitions from the literature.

Definition 1. A function is convex if
\[ \phi(x + (1 - \zeta)y) \leq \zeta \phi(x) + (1 - \zeta) \phi(y), \]
for every \( x, y \in I \) and \( \zeta \in [0, 1] \).

Definition 2 (see [29]). Assume \( \lambda \geq 0 \). A function \( \phi : I \to \mathbb{R} \) is strongly convex if
\[ \phi((x + (1 - \zeta)y) \leq \zeta \phi(x) + (1 - \zeta) \phi(y) - \lambda \zeta (1 - \zeta)(x - y)^2, \]
for every \( x, y \in I \) and \( \zeta \in [0, 1] \).

Remark 1. Setting \( \lambda = 0 \) in inequality (5), we obtain convex function (4).

Definition 3 (see [8]). Let \( \phi \in H^1(a,b) \), \( a < b, \zeta \in [0, 1] \); then, the left Caputo–Fabrizio fractional derivative is defined by
\[ \left( \frac{\text{CF}}{a} D_{\zeta}^{\phi} \right)(\zeta) = \frac{B(\zeta)}{(1 - \zeta)} \phi^{\zeta}(x)e^{-\zeta(x - y)/(1 - \zeta)}dx, \]
and the left Caputo–Fabrizio fractional integral is defined by
\[ \left( \frac{\text{CF}}{a} I_{\zeta}^{\phi} \right)(\zeta) = \frac{(1 - \zeta)}{B(\zeta)} \phi(\zeta) + \zeta \psi(x)dx, \]
where \( B(\zeta) > 0 \) is a normalization of function with \( B(0) = B(1) = 1 \).

Definition 4 (see [8]). Let \( \phi \in H^1(a,b) \), \( a < b, \zeta \in [0, 1] \); then, the right Caputo–Fabrizio fractional derivative is defined by
\[ \left( \frac{\text{CF}}{b} D_{\zeta}^{\phi} \right)(\zeta) = \frac{-B(\zeta)}{(1 - \zeta)} \phi^{\zeta}(x)e^{-\zeta(x - y)/(1 - \zeta)}dx, \]
and the right Caputo–Fabrizio fractional integral is defined by
\[ \left( \frac{\text{CF}}{b} I_{\zeta}^{\phi} \right)(\zeta) = \frac{(1 - \zeta)}{B(\zeta)} \phi(\zeta) + \frac{\zeta}{B(\zeta)} \phi(x)dx, \]
where \( B(\zeta) > 0 \) is a normalization of function with \( B(0) = B(1) = 1 \).

3. Hermite–Hadamard-Type Inequalities via Caputo–Fabrizio Fractional Integrals for Strongly Convex Functions

Theorem 1. Assume \( \phi : I \to \mathbb{R} \) to be a strongly convex function with modulus \( \lambda \geq 0 \) and \( \phi \in L_1[a,b] \); then, the inequality
\[ \phi\left( \frac{a + b}{2} \right) + \frac{\lambda}{12}(b - a)^2 \]
\[ \leq \frac{B(\zeta)}{\zeta(b - a)} \left[ \left( \frac{\text{CF}}{a} I_{\zeta}^{\phi}(\zeta) \right)(\zeta) + \left( \frac{\text{CF}}{b} I_{\zeta}^{\phi}(\zeta) \right)(\zeta) - \frac{2(1 - \zeta)}{B(\zeta)} \phi(\zeta) \right] \]
\[ \leq \frac{\phi(a) + \phi(b)}{2} - \frac{\lambda}{6}(b - a)^2, \]
holds, where \( B(\zeta) > 0 \) is a normalization function, \( \zeta \in [0, 1] \), and \( \zeta \in [0, 1] \).

Proof. Since \( \phi \) is strongly convex function, we have
\[ \phi\left( \frac{a + b}{2} \right) - \frac{\lambda}{12}(b - a)^2 \leq \frac{1}{b - a} \int_{a}^{b} \phi(x)dx, \]
\[ \leq \frac{\phi(a) + \phi(b)}{2} - \frac{\lambda}{6}(b - a)^2. \]

The left side of inequality (11) yields
\[ 2\phi\left( \frac{a + b}{2} \right) - \frac{\lambda}{6}(b - a)^2 \leq \frac{2}{b - a} \int_{a}^{b} \phi(x)dx, \]
\[ \leq \frac{2}{b - a} \left[ \int_{a}^{b} \phi(x)dx + \int_{a}^{b} \phi(x)g(x)dx \right]. \]

Multiplying \( \zeta(b - a)2/B(\zeta) \) on both sides of the abovementioned inequality, adding \( (2 - \lambda)/B(\zeta)\phi(\zeta) \)
\( g(\zeta) \) and rearranging the terms, we obtain
\[ \phi\left( \frac{a + b}{2} \right) + \frac{\lambda}{12}(b - a)^2 \]
\[ \leq \frac{B(\zeta)}{\zeta(b - a)} \left[ \left( \frac{\text{CF}}{a} I_{\zeta}^{\phi}(\zeta) \right)(\zeta) + \left( \frac{\text{CF}}{b} I_{\zeta}^{\phi}(\zeta) \right)(\zeta) - \frac{2(1 - \zeta)}{B(\zeta)} \phi(\zeta) \right], \]
which is the left side of Theorem 1.

Now, to prove the right side of Theorem 1, we use the right side of (11), which is
\[ \frac{2}{b - a} \int_{a}^{b} \phi(x)dx \leq \phi(a) + \phi(b) - \frac{\lambda}{3}(b - a)^2. \]

Applying the same operations on the abovementioned inequality as on (12) yields the right side of Theorem 1, which is
Theorem 2. Assume that \( \varphi, g : I \to \mathbb{R} \) are two strongly convex functions with modulus \( \lambda \geq 0 \) and \( f, g \in L_1[a,b] \); then, the inequality
\[
\frac{2B(c)}{c(b-a)} \left[ \frac{B(c)}{c} \right]^2 \varphi(\zeta)g(\zeta) + \lambda \frac{(b-a)^2}{2} \varphi(\zeta)g(\zeta) - \frac{2(1-c)}{B(c)} \varphi(\zeta)g(\zeta) \leq \frac{\varphi(a) + \varphi(b)}{2} - \lambda \frac{(b-a)^2}{6}.
\]
(15)

holds with normalization function \( B(c) > 0, c \in [0,1] \), and \( \zeta \in [0,1] \), where \( P(a,b) = \varphi(a)g(a) + \varphi(b)g(b) \), \( Q(a,b) = \varphi(a)g(b) + \varphi(b)g(a) \), and \( R(a,b) = \varphi(a)g(a) + \varphi(b)g(b) \).

Proof. Since \( \varphi \) and \( g \) are strongly convex functions defined on \( I \), by definition, we have
\[
\varphi(\zeta a + (1-\zeta)b) \leq \varphi(a) + (1-\zeta)\varphi(b) - \lambda \zeta (1-\zeta)(b-a)^2,
\]
(17)
\[
g(\zeta a + (1-\zeta)b) \leq g(a) + (1-\zeta)g(b) - \lambda \zeta (1-\zeta)(b-a)^2,
\]
(18)
for all \( a,b \in I \) and \( \zeta \in [0,1] \).

Multiplying (17) and (18), we have
\[
\varphi(\zeta a + (1-\zeta)b)g(\zeta a + (1-\zeta)b) \leq \zeta^2 \varphi(a)g(a) + (1-\zeta)^2 \varphi(b)g(b)
\]
\[
+ \zeta(1-\zeta)[\varphi(a)g(b) + \varphi(b)g(a)]
\]
\[
- \lambda \zeta (1-\zeta)(b-a)^2[\varphi(b) + g(b)]
\]
\[
- \lambda (1-\zeta)^2(g(a)(b-a)^2
\]
\[
- \lambda \zeta^2(1-\zeta)(b-a)^2[\varphi(a) + g(a)]
\]
\[
+ \lambda (1-\zeta)^2(b-a)^4.
\]
Integrating the abovementioned inequality w.r.t \( \zeta \) over \([0,1]\), we obtain
\[
\frac{2}{b-a} \int_a^b \varphi(x)g(x)dx \leq \frac{2}{3} [\varphi(a)g(a) + \varphi(b)g(b)] + \frac{1}{3} [\varphi(a)g(b) + \varphi(b)g(a)]
\]
\[
- \frac{\lambda}{3}(b-a)^2[\varphi(a) + \varphi(b) + g(a) + g(b)] - \frac{\lambda}{5}(b-a)^2,
\]
(20)
\[
\frac{2}{b-a} \int_a^b \varphi(x)g(x)dx \leq \frac{2}{3} [P(a,b)] + \frac{1}{3} [Q(a,b)] - \frac{\lambda}{3}(b-a)^2[R(a,b)] - \frac{\lambda}{5}(b-a)^2.
\]

Multiplying \( c(b-a)/2B(c) \) on both sides and adding \( (2(1-c)/B(c))\varphi(\zeta)g(\zeta) \), we obtain
\[
\frac{c}{B(c)} \left[ \int_a^\zeta \varphi(x)g(x)dx + \int_\zeta^b \varphi(x)g(x)dx \right] + \frac{2(1-c)}{B(c)} \varphi(\zeta)g(\zeta)
\]
\[
\leq \frac{c(b-a)}{2B(c)} \left[ \frac{2}{3} [P(a,b)] + \frac{1}{3} [Q(a,b)] - \frac{\lambda}{3}(b-a)^2[R(a,b)] - \frac{\lambda}{5}(b-a)^2 \right]
\]
\[
+ \frac{2(1-c)}{B(c)} \varphi(\zeta)g(\zeta).
\]
(21)

Theorem 3. Assume that \( f, g : I \to \mathbb{R} \) are two strongly convex functions with modulus \( \lambda \geq 0 \) and \( f, g \in L_1[a,b] \); then, the inequality
\[ \frac{2\zeta}{B(\zeta)} \varphi \left( \frac{a+b}{2} \right) g \left( \frac{a+b}{2} \right) - \frac{1}{(b-a)} \left[ \frac{\varphi (\zeta) g (\zeta)}{\varphi (\zeta) + g (\zeta)} \right] \]

\[ + \frac{2 (1-\zeta)}{B(\zeta)(b-a)} \varphi (\zeta) g (\zeta) \]

holds with normalization function \( B(\zeta) > 0, \zeta \in [0,1], \) and \( \zeta \in [0,1], \) where \( P(a,b) = \varphi (a)g (a) + \varphi (b)g (b), \) \( Q(a,b) = \varphi (a)g (b) + \varphi (b)g (a), \) and \( R(a,b) = \varphi (a) + g (a) + \varphi (b) + g (b). \)

Proof: Since \( \varphi \) and \( g \) be the two strongly convex functions, so for \( \zeta = 1/2, \) we have

\[ \varphi \left( \frac{a+b}{2} \right) \leq \frac{\varphi (\zeta a + (1-\zeta)b) + \varphi (\zeta a + (1-\zeta))}{2} - \frac{\lambda}{4} (2\zeta - 1)(b-a)^2,\]  

(23)

\[ g \left( \frac{a+b}{2} \right) \leq \frac{g (\zeta a + (1-\zeta)b) + g (\zeta a + (1-\zeta))}{2} - \frac{\lambda}{4} (2\zeta - 1)(b-a)^2,\]  

(24)

for all \( a,b \in I \) and \( \zeta \in [0,1]. \)

Multiplying (23) and (24), we obtain

\[ \varphi \left( \frac{a+b}{2} \right) g \left( \frac{a+b}{2} \right) \leq \frac{1}{4} \left[ \varphi (\zeta a + (1-\zeta)b)g (\zeta a + (1-\zeta)b) + \varphi (\zeta a + (1-\zeta))g (\zeta a + (1-\zeta)) \right] \]

\[ + \left( \zeta^2 + (1-\zeta)^2 \right) \left[ \varphi (a)g (b) + \varphi (b)g (a) \right] + 2\zeta \left( 1-\zeta \right) \left[ \varphi (a)g (a) + \varphi (b)g (b) \right] \]

\[ - \lambda (b-a)^2 \left( \zeta^2 (1-\zeta) + (1-\zeta)^2 \right) [\varphi (a) + g (b) + \varphi (b) + g (a)] \]

\[ + 2\lambda^2 \zeta^2 (1-\zeta)^2 (b-a)^2 + 2\lambda^2 \zeta (1-\zeta) (2\zeta - 1)^2 (b-a)^4 + \frac{\lambda}{2} (2\zeta - 1)^2 (b-a)^4 \]

\[ - \frac{\lambda}{2} (2\zeta - 1)^2 (b-a)^2 R(a,b). \]  

Integrating the abovementioned inequality w.r.t “\( \zeta \)” over \([0,1]\) and using the technique of change of variable, we obtain

\[ 4 \varphi \left( \frac{a+b}{2} \right) g \left( \frac{a+b}{2} \right) \leq \frac{2}{b-a} \int_a^b \varphi (x)g (x)dx + \frac{1}{3} P(a,b) + \frac{2}{3} Q(a,b) \]

\[ - \frac{\lambda}{3} (b-a)^2 \left[ R(a,b) - \frac{\lambda}{5} \frac{(b-a)^2}{2} - \frac{1}{5} \right]. \]  

(26)
Multiplying $\zeta(b-a)/2B(\zeta)$ on both sides and subtracting $(2(1-\zeta)/B(\zeta))q(\zeta)g(\zeta)$, we obtain

$$
\frac{2\zeta(b-a)}{2B(\zeta)} \left\{ \frac{a+b}{2} \right\} \left( \frac{a+b}{2} \right) \\
\leq \frac{\zeta}{B(\zeta)} \left[ \int_a^\zeta \varphi(x)g(x)dx + \int_\zeta^b \varphi(x)g(x)dx \right] - \frac{2(1-\zeta)}{B(\zeta)} \varphi(\zeta)g(\zeta) \\
+ \frac{\zeta(b-a)}{2B(\zeta)} \left[ \frac{1}{3} P(a,b) + \frac{2}{3} Q(a,b) - \frac{2}{3} (b-a)^2 \right] \\
- \frac{2(1-\zeta)}{B(\zeta)} \varphi(\zeta)g(\zeta).
$$

(27)

Now, the use of (7) and (9) and rearrangements of the terms of the abovementioned inequality complete the proof. \square

4. Some New Caputo–Fabrizio Fractional Integral Inequalities for Strongly Convex Functions

Lemma 1 (see [28, 30]). Assume that $\varphi: I \rightarrow \mathbb{R}$ is a differentiable mapping on $I^0$, where $a, b \in I$ with $a < b$. If $\varphi' \in L_1[a,b]$, then the inequality

$$
\frac{b-a}{2} \in \zeta \left( 1 - 2\zeta \right) \varphi' \left( \zeta a + (1-\zeta) b \right) d\zeta - \frac{2(1-\zeta)}{\zeta(b-a)} \varphi(\zeta) \\
\leq \frac{\varphi(a) + \varphi(b)}{2} - \frac{B(\zeta)}{\zeta(b-a)} \left[ \left( \frac{CF}{a} \right) g(x) \right] (\zeta) + \left( \frac{CF}{b} \right) g(x) \right] (\zeta) \\
$$

(28)

holds, where $B(\zeta) > 0$ is a normalization function, $\zeta \in [0, 1]$, and $\zeta \in [0, 1]$. 

Theorem 4. Assume that $\varphi: I \rightarrow \mathbb{R}$ is a differentiable positive mapping on $I^0$, where $a, b \in I$ with $a < b$. If $\varphi' \in L_1[a,b]$ and $|\varphi'|$ are two strongly convex functions, then the inequality

$$
\left| \frac{\varphi(a) + \varphi(b)}{2} + \frac{2(1-\zeta)}{\zeta(b-a)} \varphi(\zeta) - \frac{B(\zeta)}{\zeta(b-a)} \left[ \left( \frac{CF}{a} \right) g(x) \right] (\zeta) + \left( \frac{CF}{b} \right) g(x) \right] (\zeta) \\
\leq \frac{(b-a)(|\varphi'(a)| + |\varphi'(b)|)}{8} - \frac{\lambda}{32} (b-a)^3
$$

holds, where $B(\zeta) > 0$ is a normalization function, $\zeta \in [0, 1]$, and $\zeta \in [0, 1]$. 

Proof. By using Lemma 1, convexity of $|\varphi'|$, and the property of absolute value, we get

$$
\left| \frac{\varphi(a) + \varphi(b)}{2} + \frac{2(1-\zeta)}{\zeta(b-a)} \varphi(\zeta) - \frac{B(\zeta)}{\zeta(b-a)} \left[ \left( \frac{CF}{a} \right) g(x) \right] (\zeta) + \left( \frac{CF}{b} \right) g(x) \right] (\zeta) \\
= \left| \frac{b-a}{2} \in \zeta \left( 1 - 2\zeta \right) \varphi' \left( \zeta a + (1-\zeta) b \right) d\zeta \\
\leq \frac{b-a}{2} \in \zeta \left( 1 - 2\zeta \right) |\varphi' \left( \zeta a + (1-\zeta) b \right) d\zeta \\
\leq \frac{b-a}{2} \in \zeta \left( 1 - 2\zeta \right) \left[ t|\varphi'(a)| + (1-\zeta)|\varphi'(b)| - \lambda \zeta (1-\zeta) (b-a)^2 \right] d\zeta, \\
= \frac{(b-a)|\varphi'(a)| + |\varphi'(b)|}{8} - \frac{\lambda}{32} (b-a)^3.
$$

(30)
This completes the proof. □

**Theorem 5.** Assume \( \varphi : I \rightarrow \mathbb{R} \) to be a differentiable positive mapping on \( I^* \), \( a, b \in I \) with \( a < b \), and \((1/p) + (1/q) = 1\). If \( \varphi' \in L^1[a, b] \) and \(|\varphi'|^q\) is a strongly convex function, then the inequality

\[
\left| \frac{\varphi(a) + \varphi(b)}{2} + \frac{2(1 - \varsigma)}{\varsigma(b - a)} \varrho(\varsigma) - \frac{B(\varsigma)}{\varsigma(b - a)} \right| \leq \frac{b - a}{2(p + 1)^{1/p}} \left[ |\varphi'(a)|^{p/(p-1)} + |\varphi'(b)|^{p/(p-1)} - \frac{\lambda}{6}(b - a)^3 \right]^{(p-1)/p} \quad (31)
\]

holds, where \( B(\varsigma) > 0 \) is a normalization function, \( \varsigma \in [0, 1] \), and \( \varsigma \in [0, 1] \).

**Proof.** We start the proof by using Lemma 1, convexity of \(|\varphi'|^q\), the property of absolute value, where \((1/p) + (1/q) = 1\), and Holder’s inequality to obtain

\[
\left| \frac{\varphi(a) + \varphi(b)}{2} + \frac{2(1 - \varsigma)}{\varsigma(b - a)} \varrho(\varsigma) - \frac{B(\varsigma)}{\varsigma(b - a)} \right| \leq \frac{b - a}{2(p + 1)^{1/p}} \left[ |\varphi'(a)|^{p/(p-1)} + |\varphi'(b)|^{p/(p-1)} - \frac{\lambda}{6}(b - a)^3 \right]^{(p-1)/p} \quad (32)
\]

where, \( \int_0^1 |1 - 2\varsigma|^p d\varsigma = \int_0^{1/2} (1 - 2\varsigma)^p d\varsigma + \int_{1/2}^1 (1 - 2\varsigma)^p d\varsigma = \frac{1}{(p + 1)} \) and \( \varsigma \in [0, 1] \). This completes the proof. □

5. Some Applications of Caputo–Fabrizio Fractional Integral Inequalities to Special Means

Means are important in applied and pure mathematics; especially, they are used frequently in numerical approximation. In the literature, they are ordered in the following way:

\[ H \leq G \leq L \leq I \leq A. \quad (33) \]

The special means of two numbers \( a \) and \( b \) in the order of \( b > a \) are known as arithmetic mean, geometric mean,
There are several results connecting these means, see [31] for some new relations; however, very few results are known for arbitrary real numbers. For this, it is clear that we can extend some of the abovementioned means as follows:

\[ A(a,b) = \frac{a+b}{2}, a, b \in \mathbb{R}, \]

\[ T(a,b) = \frac{b-a}{\ln(b) - \ln(a)}, a, b \in \mathbb{R} \setminus \mathbb{N}, \]

\[ L_n(a,b) = \left[ \frac{b^{n+1} - a^{n+1}}{(n+1)(b-a)} \right]^{1/n}, n \in \mathbb{N}, n \geq 1, a, b \in \mathbb{R}, a < b. \]  

Now, we shall use the results of Sections 3 and 4 to prove the following new inequalities connecting the abovementioned means for arbitrary real numbers.

**Proposition 1.** Suppose \( a, b \in \mathbb{R}^+, a < b \) and \( n \in \mathbb{N}, n \geq 2 \). Then, the following inequality holds:

\[ |A(a^n, b^n) - L_n^n(a,b)| \leq \frac{b-a}{8} \left\{ nA\left(\frac{a^{n-1}}{|b|^{n-1}}, \frac{b^{n-1}}{|a|^{n-1}}\right) - \frac{\lambda}{4}(b-a)^2 \right\}. \]  

**Proof.** Insertion of \( \varphi(x) = x^n \), where \( n \in \mathbb{N}, n \geq 2 \), with \( \zeta = 1 \) and \( B(\zeta) = B(1) = 1 \) in Theorem 4 completes the proof. □

**Proposition 2.** Suppose \( a, b \in \mathbb{R}^+, a < b \) and \( n \in \mathbb{N}, n \geq 2 \). Then, the following inequality holds:

\[ |A(a^n, b^n) - L_n^n(a,b)| \leq \frac{n(b-a)}{2(2n+1)} \left\{ nA\left(\frac{a^{n-1}}{|b|^{n-1}}, \frac{b^{n-1}}{|a|^{n-1}}\right) \right\}. \]  

**Proof.** Insertion of \( \varphi(x) = x^n \), where \( n \in \mathbb{N}, n \geq 2 \), with \( \zeta = 1 \) and \( B(\zeta) = B(1) = 1 \) in Theorem 5 completes the proof. □

**Proposition 3.** Suppose \( a, b \in \mathbb{R}^+, a < b \) and \( n \in \mathbb{N}, n \geq 2 \). Then, the following inequality holds:

\[ A^{-1}(a,b) + \frac{\lambda}{12}(b-a)^2 \leq L^{-1}_n(a,b) \leq A^{-1}(a^{-1},b^{-1}) + \frac{\lambda}{6}(b-a)^2. \]  

**Proof.** Insertion of \( \varphi(x) = x^n \), where \( n \in \mathbb{N}, n \geq 2 \), with \( \zeta = 1 \) and \( B(\zeta) = B(1) = 1 \) in Theorem 1 completes the proof. □

**Proposition 4.** Suppose \( a, b \in \mathbb{R}^+, a < b \) and \( n \in \mathbb{N}, n \geq 2 \). Then, the following inequality holds:

\[ |A(a^{-1}, b^{-1}) - L^{-1}_n(a,b)| \leq \frac{b-a}{4} A\left(\frac{|a|^{-2} \cdot |b|^{-2}}{b-a} \right) - \frac{\lambda}{32}(b-a)^3. \]  

**Proof.** Insertion of \( \varphi(x) = x^{-1} \), where \( x \in [a,b] \), with \( \zeta = 1 \) and \( B(\zeta) = B(1) = 1 \) in Theorem 4 completes the proof. □

**Proposition 5.** Suppose \( a, b \in \mathbb{R}^+, a < b \) and \( n \in \mathbb{N}, n \geq 2 \). Then, the following inequality holds:

\[ |A(a^{-1}, b^{-1}) - L^{-1}_n(a,b)| \leq \frac{b-a}{4n} A\left(\frac{|a|^{-2} \cdot |b|^{-2}}{b-a} \right) - \frac{\lambda}{6}(b-a)^3. \]  

**Proof.** Insertion of \( \varphi(x) = x^{-1} \), where \( x \in [a,b] \), with \( \zeta = 1 \) and \( B(\zeta) = B(1) = 1 \) in Theorem 4 completes the proof. □

6. Some Applications of Caputo–Fabrizio Fractional Integral Inequalities to the Trapezoidal Formula

Suppose \( d \) is the division of interval \([a,b]\), \( d = a = x_0 < x_1 < \cdots < x_n-1 < x_n = b \), and consider the trapezoidal formula

\[ T(\varphi,d) = \frac{1}{2} \sum_{i=0}^{n-1} \varphi(x_i) + \frac{\varphi(x_{i+1})}{2}(x_{i+1} - x_i). \]  

It is well known that if the mapping \( \varphi \): \( I \rightarrow \mathbb{R} \) is twice differentiable on \((a,b)\) and \( M = \max_{x \in (a,b)} |\varphi''(x)| < \infty \), then

\[ \int_a^b \varphi(x)dx = T(\varphi,d) + E(\varphi,d), \]  

where the approximation error \( E(\varphi,d) \) of the integral \( \int_a^b \varphi(x)dx \) by the trapezoidal formula \( T(\varphi,d) \) satisfies

\[ |E(\varphi,d)| \leq \frac{M}{12} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^3. \]  

It is clear that if the mapping \( f \) is not twice differentiable or the second derivative is not bounded on \((a,b)\), then (49)
cannot be applied. In recent studies [30, 32–35], Dragomir and Wang showed that the remainder term $E(q, d)$ can be estimated in terms of the first derivative only. These estimates have a wider range of applications. Here, we shall propose some new estimates of the remainder term $E(q, d)$ which supplement, in a sense, those established in [30, 32–35].

**Proof.** Applying subinterval $[x_i, x_{i+1}]$, $i = 0, \ldots, n - 1$, of the division $d$ from Theorem 4, we obtain

$$|E(q, d)| \leq \frac{1}{2} \sum_{i=1}^{n-1} (x_{i+1} - x_i)^2 \left[ \left( |q'(x_i)| + |q'(x_{i+1})| \right) - \frac{\lambda}{4} \sum_{i=1}^{n-1} (x_{i+1} - x_i)^2 \right].$$

(50)

Proving over $i = 0, \ldots, n - 1$ and taking that $|q'|$ is a strongly convex function, then by using (47), (48), and triangular inequality, we complete the proof. □

**Proposition 6.** Assume that $q: I \rightarrow \mathbb{R}$ is a differentiable positive mapping on $I^*$, $a, b \in I$ with $a < b$. If $q' \in L_1[a, b]$ and $|q'|$ is a strongly convex function, then for every division $d$ of $[a, b]$, the following inequality holds:

$$|E(q, d)| \leq \frac{1}{2} \sum_{i=1}^{n-1} (x_{i+1} - x_i)^2 \left[ \left( |q'(x_i)|^{p/(p-1)} + |q'(x_{i+1})|^{p/(p-1)} \right) - \frac{\lambda}{6} \sum_{i=1}^{n-1} (x_{i+1} - x_i)^2 \right]^{p/(p-1)}.$$

(52)

**Proof.** Applying subinterval $[x_i, x_{i+1}]$, $i = 0, \ldots, n - 1$, of the division $d$, we obtain from Theorem 5

$$|E(q, d)| \leq \frac{1}{2} \sum_{i=1}^{n-1} (x_{i+1} - x_i)^2 \left[ \left( |q'(x_i)|^{p/(p-1)} + |q'(x_{i+1})|^{p/(p-1)} \right) - \frac{\lambda}{6} \sum_{i=1}^{n-1} (x_{i+1} - x_i)^2 \right]^{p/(p-1)}.$$

(53)

Summing over $i = 0, \ldots, n - 1$ and taking that $|q'|^p$, where $(1/p) + (1/q) = 1$, is a strongly convex function, then by using (47), (48), and triangular inequality, we complete the proof. □

7. Conclusions

The convex functions play an important role in approximation theory, and the fractional calculus has been found the best to model physical and engineering processes. Some properties of strongly convex functions via the Caputo–Fabrizio fractional integral operator have been studied in this paper. Precisely speaking, Hermite–Hadamard-type and some new inequalities for strongly convex functions via the Caputo–Fabrizio fractional integral operator are proved, and applications of the proposed inequalities to special means are also presented in this paper.
Data Availability

All data required for this research are available within this paper.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors’ Contributions

Qi Li added the application section to justify the novelty of the paper, Muhammad Shoaib Saleem designed the problem and supervised the work, Peiyu Yan wrote the literature review and arranged the funding for this paper, Muhammad Sajid Zahoor proved the main results, and Muhammad Imran wrote the first draft of the paper. All authors read and approved the final version of this paper.

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