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Research Article

On Strongly Convex Functions via Caputo–Fabrizio-Type Fractional Integral and Some Applications

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The theory of convex functions plays an important role in the study of optimization problems. The fractional calculus has been found the best to model physical and engineering processes. The aim of this paper is to study some properties of strongly convex functions via the Caputo–Fabrizio fractional integral operator. In this paper, we present Hermite–Hadamard-type inequalities for strongly convex functions via the Caputo–Fabrizio fractional integral operator. Some new inequalities of strongly convex functions involving the Caputo–Fabrizio fractional integral operator are also presented. Moreover, we present some applications of the proposed inequalities to special means.

1. Introduction

The theory of fractional calculus got rapid development, and it has brought the attention of many researchers from various disciplines [1–3]. In the last few years, it was observed that fractional calculus is very useful for modeling complicated problems of engineering, chemistry, mechanics, and many other branches. Various interesting notations of fractional calculus exist in the literature, for example, the Riemann–Liouville fractional integral and Caputo–Fabrizio fractional integral [4–14].

Among these notions, Riemann–Liouville and Caputo involve the following singular kernal [11]:

$$K(\zeta, x) = \frac{(\zeta - x)^{-\varsigma}}{\Gamma(1-\varsigma)}, \quad 0 < \varsigma < 1. \tag{1}$$

However, it was observed by Caputo and Fabrizio in [8] that certain phenomena cannot be modelled by the already existing definition in the literature. That is why, they proposed a more general fractional derivative in [8] and named it as the Caputo–Fabrizio fractional integral operator. It mainly involves the following nonsingular kernal:

$$K(\zeta, x) = e^{-\varsigma(\zeta - x)/(1-\varsigma)}, \quad 0 < \varsigma < 1. \tag{2}$$

Nowadays, many researchers of applied sciences are using the Caputo–Fabrizio fractional integral operator to model their problem. For more details about the fractional integral with a nonsingular kernal, we refer [15–19] to the readers.

The theory of inequalities also plays an important role in applied as well as in pure mathematics. The Hermite–Hadamard inequality is the most important inequality in the literature, and this inequality has been studied for different classes of convex functions, see [20–24]. The classical version of the Hermite–Hadamard inequality for convex functions is stated as follows:

If $\varrho\colon I = [a,b] \subset \mathbb{R} \longrightarrow \mathbb{R}$ is an integrable and continuous convex function, then its mean value remains between the value of ϱ at $(a+b)/2$ of interval $I = [a,b]$ and arithmetic mean value of ϱ at the endpoints $a, b \in I = [a,b]$. In other words, it means that

$$\varrho\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \varrho(x) x \leq \frac{\varrho(a)+\varrho(b)}{2}. \tag{3}$$

Inequality (3), in the literature, is generalized by several fractional integral operators to meet the desired results, see, for instance, [25–28]. In this paper, we present the Hermite–Hadamard inequality for a strongly convex function in the setting of the Caputo–Fabrizio fractional integral operator. We also present some new inequalities for strongly convex functions in the setting of the Caputo–Fabrizio fractional integral operator. We also give some applications of the presented inequalities in special mean.

2. Preliminaries

In this section, we present some definitions from the literature.

Definition 1. A function is convex if

$$\varrho(\zeta x + (1-\zeta)y) \le \zeta\varrho(x) + (1-\zeta)\varrho(y), \tag{4}$$

for every $x, y \in I$ and $\zeta \in [0,1]$.

Definition 2 (see [29]). Assume $\lambda \ge 0$. A function $\varrho\colon I \longrightarrow \mathbb{R}$ is strongly convex if

$$\varrho(\zeta x + (1-\zeta)y) \le \zeta\varrho(x) + (1-\zeta)\varrho(y) - \lambda\zeta(1-\zeta)(x-y)^2, \tag{5}$$

for every $x, y \in I$ and $\zeta \in [0,1]$.

Remark 1. Setting $\lambda = 0$ in inequality (5), we obtain convex function (4).

Definition 3 (see [8]). Let $\varrho \in H^1(a,b)$, $a<b, \varsigma \in [0,1]$; then, the left Caputo–Fabrizio fractional derivative is defined by

$$\left({}_{a}^{\mathrm{CF}}D^{\varsigma}\varrho\right)(\zeta) = \frac{B(\varsigma)}{(1-\varsigma)} \in \zeta_a^{\zeta}\varrho'(x)e^{-\varsigma(\zeta-x)^{\varsigma}/(1-\varsigma)}\mathrm{d}x, \tag{6}$$

and the left Caputo–Fabrizio fractional integral is defined by

$$\left({}_{a}^{\mathrm{CF}}I^{\varsigma}\varrho\right)(\zeta) = \frac{(1-\varsigma)}{B(\varsigma)}\varrho(\zeta) + \frac{\varsigma}{B(\varsigma)} \in \zeta_a^{\zeta}\varrho(x)\mathrm{d}x, \tag{7}$$

where $B(\varsigma) > 0$ is a normalization of function with $B(0) = B(1) = 1$.

Definition 4 (see [8]). Let $\varrho \in H^1(a,b)$, $a<b, \varsigma \in [0,1]$; then, the right Caputo–Fabrizio fractional derivative is defined by

$$\left({}^{\mathrm{CF}}D_b^{\varsigma}\varrho\right)(\zeta) = \frac{-B(\varsigma)}{(1-\varsigma)} \in \zeta_{\zeta}^{b}\varrho'(x)e^{-\varsigma(x-\zeta)^{\varsigma}/(1-\varsigma)}\mathrm{d}x, \tag{8}$$

and the right Caputo–Fabrizio fractional integral is defined by

$$\left({}^{\mathrm{CF}}I_b^{\varsigma}\varrho\right)(\zeta) = \frac{(1-\varsigma)}{B(\varsigma)}\varrho(\zeta) + \frac{\varsigma}{B(\varsigma)} \in \zeta_{\zeta}^{b}\varrho(x)\mathrm{d}x, \tag{9}$$

where $B(\varsigma) > 0$ is a normalization of function with $B(0) = B(1) = 1$.

3. Hermite–Hadamard-Type Inequalities via Caputo–Fabrizio Fractional Integrals for Strongly Convex Functions

Theorem 1. *Assume $\varrho\colon I \longrightarrow \mathbb{R}$ to be a strongly convex function with modulus $\lambda \ge 0$ and $\varrho \in L_1[a,b]$; then, the inequality*

$$\begin{aligned}
&\varrho\left(\frac{a+b}{2}\right) + \frac{\lambda}{12}(b-a)^2 \\
&\le \frac{B(\varsigma)}{\varsigma(b-a)}\left[\left({}_{a}^{\mathrm{CF}}I^{\varsigma}\varrho\right)(\zeta) + \left({}^{\mathrm{CF}}I_b^{\varsigma}\varrho\right)(\zeta) - \frac{2(1-\varsigma)}{B(\varsigma)}\varrho(\zeta)\right] \\
&\le \frac{\varrho(a)+\varrho(b)}{2} - \frac{\lambda}{6}(b-a)^2,
\end{aligned} \tag{10}$$

holds, where $B(\varsigma) > 0$ is a normalization function, $\varsigma \in [0,1]$, and $\zeta \in [0,1]$.

Proof. Since ϱ is strongly convex function, we have

$$\begin{aligned}
&\varrho\left(\frac{a+b}{2}\right) - \frac{\lambda}{12}(b-a)^2 \le \frac{1}{b-a}\int_a^b \varrho(x)\mathrm{d}x \\
&\le \frac{\varrho(a)+\varrho(b)}{2} - \frac{\lambda}{6}(b-a)^2.
\end{aligned} \tag{11}$$

The left side of inequality (11) yields

$$\begin{aligned}
&2\varrho\left(\frac{a+b}{2}\right) - \frac{\lambda}{6}(b-a)^2 \le \frac{2}{b-a}\int_a^b \varrho(x)\mathrm{d}x, \\
&= \frac{2}{b-a}\left[\int_a^{\zeta}\varrho(x)g(x)\mathrm{d}x + \int_{\zeta}^{b}\varrho(x)g(x)\mathrm{d}x\right].
\end{aligned} \tag{12}$$

Multiplying $\varsigma(b-a)/2B(\varsigma)$ on both sides of the abovementioned inequality, adding $(2(1-\varsigma)/B(\varsigma))\varrho(\zeta) g(\zeta)$ and rearranging the terms, we obtain

$$\begin{aligned}
&\varrho\left(\frac{a+b}{2}\right) + \frac{\lambda}{12}(b-a)^2 \\
&\le \frac{B(\varsigma)}{\varsigma(b-a)}\left[\left({}_{a}^{\mathrm{CF}}I^{\varsigma}\varrho\right)(\zeta) + \left({}^{\mathrm{CF}}I_b^{\varsigma}\varrho\right)(\zeta) - \frac{2(1-\varsigma)}{B(\varsigma)}\varrho(\zeta)\right],
\end{aligned} \tag{13}$$

which is the left side of Theorem 1.

Now, to prove the right side of Theorem 1, we use the right side of (11), which is

$$\frac{2}{b-a}\int_a^b \varrho(x)\mathrm{d}x \le \varrho(a) + \varrho(b) - \frac{\lambda}{3}(b-a)^2. \tag{14}$$

Applying the same operations on the abovementioned inequality as on (12) yields the right side of Theorem 1, which is

$$\leq \frac{B(\varsigma)}{\varsigma(b-a)}\left[\left({}_{a}^{\mathrm{CF}}I^{\varsigma}\varrho\right)(\zeta)+\left({}^{\mathrm{CF}}I_{b}^{\varsigma}\varrho\right)(\zeta)-\frac{2(1-\varsigma)}{B(\varsigma)}\varrho(\zeta)\right]$$
$$\leq \frac{\varrho(a)+\varrho(b)}{2}-\frac{\lambda}{6}(b-a)^{2}. \tag{15}$$

The combination of (13) and (15) completes the proof. □

Theorem 2. *Assume that* $\varrho, g\colon I \longrightarrow \mathbb{R}$ *are two strongly convex functions with modulus* $\lambda \geq 0$ *and* $f, g \in L_1[a,b]$*; then, the inequality*

$$\frac{2B(\varsigma)}{\varsigma(b-a)}\left[\left({}_{a}^{\mathrm{CF}}I^{\varsigma}\varrho g\right)(\zeta)+\left({}^{\mathrm{CF}}I_{b}^{\varsigma}\varrho g\right)(\zeta)-\frac{2(1-\varsigma)}{B(\varsigma)}\varrho(\zeta)g(\zeta)\right]$$
$$\leq \frac{2}{3}P(a,b)+\frac{1}{3}Q(a,b)-\frac{\lambda}{3}(b-a)^{2}\left[R(a,b)-\frac{\lambda}{5}(b-a)^{2}\right], \tag{16}$$

holds with normalization function $B(\varsigma)>0$, $\varsigma \in [0,1]$, *and* $\zeta \in [0,1]$, *where* $P(a,b)=\varrho(a)g(a)+\varrho(b)g(b)$, $Q(a,b)=\varrho(a)\ g(b)+\varrho(b)g(a)$, *and* $R(a,b)=\varrho(a)+g(a)+\varrho(b)+g(b)$.

Proof. Since ϱ and g are strongly convex functions defined on I, by definition, we have

$$\varrho(\zeta a+(1-\zeta)b)\leq \zeta\varrho(a)+(1-\zeta)\varrho(b)-\lambda\zeta(1-\zeta)(b-a)^{2}, \tag{17}$$

$$g(\zeta a+(1-\zeta)b)\leq \zeta g(a)+(1-\zeta)g(b)-\lambda\zeta(1-\zeta)(b-a)^{2}, \tag{18}$$

for all $a, b \in I$ and $\zeta \in [0,1]$.

Multiplying (17) and (18), we have

$$\begin{aligned}
&\varrho(\zeta a+(1-\zeta)b)g(\zeta a+(1-\zeta)b)\\
&\leq \zeta^{2}\varrho(a)g(a)+(1-\zeta)^{2}\varrho(b)g(b)\\
&\quad+\zeta(1-\zeta)[\varrho(a)g(b)+\varrho(b)g(a)]\\
&\quad-\lambda\zeta(1-\zeta)^{2}(b-a)^{2}[\varrho(b)+g(b)]\\
&\quad-\lambda\zeta^{2}(1-\zeta)g(a)(b-a)^{2}\\
&\quad-\lambda\zeta^{2}(1-\zeta)(b-a)^{2}[\varrho(a)+g(a)]\\
&\quad+\lambda^{2}\zeta^{2}(1-\zeta)^{2}(b-a)^{4}.
\end{aligned} \tag{19}$$

Integrating the abovementioned inequality w.r.t "ζ" over $[0,1]$, we obtain

$$\begin{aligned}
&\frac{2}{b-a}\int_{a}^{b}\varrho(x)g(x)\mathrm{d}x\leq \frac{2}{3}[\varrho(a)g(a)+\varrho(b)g(b)]+\frac{1}{3}[\varrho(a)g(b)+\varrho(b)g(a)]\\
&-\frac{\lambda}{3}(b-a)^{2}[\varrho(a)+\varrho(b)+g(a)+g(b)]-\frac{\lambda}{5}(b-a)^{2},\\
&\frac{2}{b-a}\int_{a}^{b}\varrho(x)g(x)\mathrm{d}x\\
&\leq \frac{2}{3}[P(a,b)]+\frac{1}{3}[Q(a,b)]-\frac{\lambda}{3}(b-a)^{2}[R(a,b)]-\frac{\lambda}{5}(b-a)^{2}.
\end{aligned} \tag{20}$$

Multiplying $\varsigma(b-a)/2B(\varsigma)$ on both sides and adding $(2(1-\varsigma)/B(\varsigma))\varrho(\zeta)g(\zeta)$, we obtain

$$\begin{aligned}
&\frac{\varsigma}{B(\varsigma)}\left[\int_{a}^{\zeta}\varrho(x)g(x)\mathrm{d}x+\int_{\zeta}^{b}\varrho(x)g(x)\mathrm{d}x\right]+\frac{2(1-\varsigma)}{B(\varsigma)}\varrho(\zeta)g(\zeta)\\
&\leq \frac{\varsigma(b-a)}{2B(\varsigma)}\left[\frac{2}{3}[P(a,b)]+\frac{1}{3}[Q(a,b)]-\frac{\lambda}{3}(b-a)^{2}[R(a,b)]-\frac{\lambda}{5}(b-a)^{2}\right]\\
&\quad+\frac{2(1-\varsigma)}{B(\varsigma)}\varrho(\zeta)g(\zeta).
\end{aligned} \tag{21}$$

Now, the use of (7) and (9) and rearrangements of the terms of abovementioned inequality complete the proof. □

Theorem 3. *Assume that* $f, g\colon I \longrightarrow \mathbb{R}$ *are two strongly convex functions with modulus* $\lambda \geq 0$ *and* $f, g \in L_1[a,b]$*; then, the inequality*

$$\frac{2\varsigma}{B(\varsigma)}\varrho\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)-\frac{1}{(b-a)}\left[\left({}_{a}^{\mathrm{CF}}I^{\varsigma}\varrho g\right)(\zeta)+\left({}^{\mathrm{CF}}I_{b}^{\varsigma}\varrho g\right)(\zeta)\right]$$
$$+\frac{2(1-\varsigma)}{B(\varsigma)(b-a)}\varrho(\zeta)g(\zeta) \tag{22}$$
$$\leq\frac{\varsigma}{2B(\varsigma)}\left[\frac{2}{3}Q(a,b)+\frac{1}{3}P(a,b)-\frac{\lambda}{3}(b-a)^{2}\left[R(a,b)-\frac{\lambda}{5}(b-a)^{2}-\frac{(b-a)^{2}}{2}-\frac{1}{5}\right]\right]$$

holds with normalization function $B(\varsigma)>0$, $\varsigma\in[0,1]$, and $\zeta\in[0,1]$, where $P(a,b)=\varrho(a)g(a)+\varrho(b)g(b)$, $Q(a,b)=\varrho(a)g(b)+\varrho(b)\,g(a)$, and $R(a,b)=\varrho(a)+g(a)+\varrho(b)+g(b)$.

Proof. Since ϱ and g be the two strongly convex functions, so for $\zeta=1/2$, we have

$$\varrho\left(\frac{a+b}{2}\right)\leq\frac{\varrho(\zeta a+(1-\zeta)b)+\varrho(\zeta a+(1-\zeta)a)}{2}-\frac{\lambda}{4}(2\zeta-1)(b-a)^{2}, \tag{23}$$

$$g\left(\frac{a+b}{2}\right)\leq\frac{g(\zeta a+(1-\zeta)b)+g(\zeta a+(1-\zeta)a)}{2}-\frac{\lambda}{4}(2\zeta-1)(b-a)^{2}, \tag{24}$$

for all $a,b\in I$ and $\zeta\in[0,1]$.

Multiplying (23) and (24), we obtain

$$\begin{aligned}&\varrho\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\\&\leq\frac{1}{4}[\varrho(\zeta a+(1-\zeta)b)g(\zeta a+(1-\zeta)b)+\varrho(\zeta a+(1-\zeta)a)g(\zeta a+(1-\zeta)a)\\&\quad+\left(\zeta^{2}+(1-\zeta)^{2}\right)[\varrho(a)g(b)+(b)g(a)]+2\zeta(1-\zeta)[\varrho(a)g(a)+\varrho(b)g(b)]\\&\quad-\lambda(b-a)^{2}\left(\zeta^{2}(1-\zeta)+\zeta(1-\zeta)^{2}\right)[\varrho(a)+g(b)+\varrho(b)+g(a)]\\&\quad+2\lambda^{2}\zeta^{2}(1-\zeta)^{2}(b-a)^{2}+2\lambda^{2}\zeta(1-\zeta)(2\zeta-1)^{2}(b-a)^{4}+\frac{\lambda}{2}(2\zeta-1)^{2}(b-a)^{4}\\&\quad-\frac{\lambda}{2}(2\zeta-1)^{2}(b-a)^{2}R(a,b).\end{aligned} \tag{25}$$

Integrating the abovementioned inequality w.r.t "ζ" over $[0,1]$ and using the technique of change of variable, we obtain

$$\begin{aligned}4\varrho\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)&\leq\frac{2}{b-a}\int_{a}^{b}\varrho(x)g(x)\mathrm{d}x+\frac{1}{3}P(a,b)+\frac{2}{3}Q(a,b)\\&\quad-\frac{\lambda}{3}(b-a)^{2}\left[R(a,b)-\frac{\lambda}{5}-\frac{(b-a)^{2}}{2}-\frac{1}{5}\right].\end{aligned} \tag{26}$$

Multiplying $\varsigma(b-a)/2B(\varsigma)$ on both sides and subtracting $(2(1-\varsigma)/B(\varsigma))\varrho(\zeta)g(\zeta)$, we obtain

$$\begin{aligned}&\frac{2\varsigma(b-a)}{2B(\varsigma)}\varrho\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\\&\leq\frac{\varsigma}{B(\varsigma)}\left[\int_a^\zeta\varrho(x)g(x)\mathrm{d}x+\int_\zeta^b\varrho(x)g(x)\mathrm{d}x\right]-\frac{2(1-\varsigma)}{B(\varsigma)}\varrho(\zeta)g(\zeta)\\&+\frac{\varsigma(b-a)}{2B(\varsigma)}\left[\frac{1}{3}P(a,b)+\frac{2}{3}Q(a,b)-\frac{\lambda}{3}(b-a)^2\left[R(a,b)-\frac{\lambda}{5}-\frac{(b-a)^2}{2}-\frac{1}{5}\right]\right]\\&-\frac{2(1-\varsigma)}{B(\varsigma)}\varrho(\zeta)g(\zeta).\end{aligned}\tag{27}$$

Now, the use of (7) and (9) and rearrangements of the terms of the abovementioned inequality complete the proof. □

4. Some New Caputo–Fabrizio Fractional Integral Inequalities for Strongly Convex Functions

Lemma 1 (see [28, 30]). *Assume that $\varrho\colon I\longrightarrow\mathbb{R}$ is a differentiable mapping on I°, where $a,b\in I$ with $a<b$. If $\varrho'\in L_1[a,b]$, then the inequality*

$$\begin{aligned}&\frac{b-a}{2}\in\zeta_1^0(1-2\zeta)\varrho'(\zeta a+(1-\zeta)b)\mathrm{d}\zeta-\frac{2(1-\varsigma)}{\varsigma(b-a)}\varrho(\zeta)\\&=\frac{\varrho(a)+\varrho(b)}{2}-\frac{B(\varsigma)}{\varsigma(b-a)}\left[\left({}_a^{\mathrm{CF}}I^\varsigma\varrho g\right)(\zeta)+\left({}^{\mathrm{CF}}I_b^\varsigma\varrho g\right)(\zeta)\right]\end{aligned}\tag{28}$$

holds, where $B(\varsigma)>0$ is a normalization function, $\varsigma\in[0,1]$, and $\zeta\in[0,1]$.

Theorem 4. *Assume that $\varrho\colon I\longrightarrow\mathbb{R}$ is a differentiable positive mapping on I°, where $a,b\in I$ with $a<b$. If $\varrho'\in L_1[a,b]$ and $|\varrho'|$ are two strongly convex functions, then the inequality*

$$\begin{aligned}&\left|\frac{\varrho(a)+\varrho(b)}{2}+\frac{2(1-\varsigma)}{\varsigma(b-a)}\varrho(\zeta)-\frac{B(\varsigma)}{\varsigma(b-a)}\right.\\&\left.\left[\left({}_a^{\mathrm{CF}}I^\varsigma\varrho g\right)(\zeta)+\left({}^{\mathrm{CF}}I_b^\varsigma\varrho g\right)(\zeta)\right]\right|\\&\leq\frac{(b-a)\left(\left|\varrho'(a)\right|+\left|\varrho'(b)\right|\right.}{8}-\frac{\lambda}{32}(b-a)^3\end{aligned}\tag{29}$$

holds, where $B(\varsigma)>0$ is a normalization function, $\varsigma\in[0,1]$, and $\zeta\in[0,1]$.

Proof. By using Lemma 1, convexity of $|\varrho'|$, and the property of absolute value, we get

$$\begin{aligned}&\left|\frac{\varrho(a)+\varrho(b)}{2}+\frac{2(1-\varsigma)}{\varsigma(b-a)}\varrho(\zeta)-\frac{B(\varsigma)}{\varsigma(b-a)}\left[\left({}_a^{\mathrm{CF}}I^\varsigma\varrho g\right)(\zeta)+\left({}^{\mathrm{CF}}I_b^\varsigma\varrho g\right)(\zeta)\right]\right|,\\&=\left|\frac{b-a}{2}\in\zeta_0^1(1-2\zeta)\varrho'(\zeta a+(1-\zeta)b)\mathrm{d}\zeta\right|\\&\leq\frac{b-a}{2}\in\zeta_0^1(1-2\zeta)\left|\varrho'(\zeta a+(1-\zeta)b)\right|\mathrm{d}\zeta\\&\leq\frac{b-a}{2}\in\zeta_0^1|(1-2\zeta)|\left[t\left|\varrho'(a)\right|+(1-\zeta)\left|\varrho'(b)\right|-\lambda\zeta(1-\zeta)(b-a)^2\right]\mathrm{d}\zeta,\\&=\frac{(b-a)\left|\varrho'(a)\right|+\left|\varrho'(b)\right|}{8}-\frac{\lambda}{32}(b-a)^3.\end{aligned}\tag{30}$$

This completes the proof. □

Theorem 5. *Assume* $\varrho\colon I \longrightarrow \mathbb{R}$ *to be a differentiable positive mapping on* I°, $a, b \in I$ *with* $a < b$, *and* $(1/p) + (1/q) = 1$. *If* $\varrho' \in L_1[a,b]$ *and* $|\varrho'|^q$ *is a strongly convex function, then the inequality*

$$\left|\frac{\varrho(a)+\varrho(b)}{2} + \frac{2(1-\varsigma)}{\varsigma(b-a)}\varrho(\zeta) - \frac{B(\varsigma)}{\varsigma(b-a)}\right.$$
$$\left.\left[\left({}_{a}^{\mathrm{CF}}I^{\varsigma}\varrho g\right)(\zeta) + \left({}^{\mathrm{CF}}I_{b}^{\varsigma}\varrho g\right)(\zeta)\right]\right|$$
$$\leq \frac{b-a}{2(p+1)^{1/p}}\left[\frac{|\varrho'(a)|^{p/(p-1)} + |\varrho'(b)|^{p/(p-1)}}{2} - \frac{\lambda}{6}(b-a)^2\right]^{(p-1)/p} \tag{31}$$

holds, where $B(\varsigma) > 0$ *is a normalization function,* $\varsigma \in [0,1]$, *and* $\zeta \in [0,1]$.

Proof. We start the proof by using Lemma 1, convexity of $|\varrho'|^q$, the property of absolute value, where $(1/p) + (1/q) = 1$, and Holder's inequality to obtain

$$\begin{aligned}
&\left|\frac{\varrho(a)+\varrho(b)}{2} + \frac{2(1-\varsigma)}{\varsigma(b-a)}\varrho(\zeta) - \frac{B(\varsigma)}{\varsigma(b-a)}\left[\left({}_{a}^{\mathrm{CF}}I^{\varsigma}\varrho g\right)(\zeta) + \left({}^{\mathrm{CF}}I_{b}^{\varsigma}\varrho g\right)(\zeta)\right]\right|,\\
&= \left|\frac{b-a}{2} \in \zeta_0^1 (1-2\zeta)\varrho'(\zeta a + (1-\zeta)b)\mathrm{d}\zeta\right|\\
&\leq \frac{b-a}{2} \in \zeta_0^1 |(1-2\zeta)|\left|\varrho'(\zeta a + (1-\zeta)b)\right|\mathrm{d}\zeta\\
&\leq \frac{b-a}{2}\left(\in \zeta_0^1 |(1-2\zeta)|^p \mathrm{d}\zeta\right)^{1/p}\left(\int_0^1 \left|\varrho'(\zeta a + (1-\zeta)b)\right|^q \mathrm{d}\zeta\right)^{1/q}\\
&\leq \frac{b-a}{2}\left(\in \zeta_0^1 |(1-2\zeta)|^p \mathrm{d}\zeta\right)^{1/p}\left(\int_0^1 \left[t\left|\varrho'(a)\right| + (1-\zeta)\left|\varrho'(b)\right| - \lambda\zeta(1-\zeta)(b-a)^2\right]\mathrm{d}\zeta\right)^{1/q},\\
&= \frac{b-a}{2(p+1)^{1/p}}\left(\frac{|\varrho'(a)|^q + |\varrho'(b)|^q}{2} - \frac{\lambda}{6}(b-a)^3\right)^{1/q},\\
&= \frac{b-a}{2(p+1)^{1/p}}\left(\frac{|\varrho'(a)|^{p/(p-1)} + |\varrho'(b)|^{p/(p-1)}}{2} - \frac{\lambda}{6}(b-a)^3\right)^{(p-1)/p},
\end{aligned} \tag{32}$$

where $\int_0^1 |1-2\zeta|^p \mathrm{d}\zeta = \int_0^{1/2} (1-2\zeta)\ ^p \mathrm{d}\zeta + \int_{1/2}^1 (1-2\zeta)^p \mathrm{d}\zeta = 2\int_0^{1/2} (1-2\zeta)^p \mathrm{d}\zeta = 1/(p+1)$. This completes the proof. □

5. Some Applications of Caputo–Fabrizio Fractional Integral Inequalities to Special Means

Means are important in applied and pure mathematics; especially, they are used frequently in numerical approximation. In the literature, they are ordered in the following way:

$$H \leq G \leq L \leq I \leq A. \tag{33}$$

The special means of two numbers a and b in the order of $b > a$ are known as arithmetic mean, geometric mean, harmonic mean, power mean, logarithmic mean, p-logarithmic mean, and identric mean. They are listed below from (34)–(40), respectively.

$$A(a,b) = \frac{a+b}{2}, \tag{34}$$

$$G(a,b) = \sqrt{ab}, \tag{35}$$

$$H(a,b) = \frac{2ab}{a+b}, \tag{36}$$

$$M_p(a,b) = \left[\left(\frac{a^p + b^p}{2}\right)^{1/p}\right], \quad p \neq 0, \tag{37}$$

$$L(a,b) = \frac{b-a}{\ln(b) - \ln(a)}, \tag{38}$$

$$L_p(a,b) = \left[\left(\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right)^{1/p}\right], \quad p \in \mathbb{R} \setminus -1, 0, \tag{39}$$

$$I(a,b) = \frac{1}{e}\left(\frac{b^p}{a^p}\right)^{1/(b-a)}. \tag{40}$$

There are several results connecting these means, see [31] for some new relations; however, very few results are known for arbitrary real numbers. For this, it is clear that we can extend some of the abovementioned means as follows:

$$A(a,b) = \frac{a+b}{2}, a, b \in \mathbb{R},$$

$$\overline{L}(a,b) = \frac{b-a}{\ln|b| - \ln|a|}, \quad a, b \in \mathbb{R} \setminus \{0\},$$

$$L_n(a,b) = \left[\left(\frac{b^{n+1} - a^{n+1}}{(n+1)(b-a)}\right)^{1/n}\right]. \quad n \in \mathbb{N}, n \geq 1, a, b \in \mathbb{R}, a < b. \tag{41}$$

Now, we shall use the results of Sections 3 and 4 to prove the following new inequalities connecting the abovementioned means for arbitrary real numbers.

Proposition 1. *Suppose $a, b \in \mathbb{R}^+$, $a < b$ and $n \in \mathbb{N}$, $n \geq 2$. Then, the following inequality holds:*

$$\left|A(a^n, b^n) - L_n^n(a,b)\right| \leq \frac{b-a}{8}\left[nA\left(|a|^{n-1}, |b|^{n-1}\right) - \frac{\lambda}{4}(b-a)^2\right]. \tag{42}$$

Proof. Insertion of $\varrho(x) = x^n$, where $n \in \mathbb{N}, n \geq 2$, with $\varsigma = 1$ and $B(\varsigma) = B(1) = 1$ in Theorem 4 completes the proof. □

Proposition 2. *Suppose $a, b \in \mathbb{R}^+$, $a < b$ and $n \in \mathbb{N}, n \geq 2$. Then, the following inequality holds:*

$$\begin{aligned} &\left|A(a^n, b^n) - L_n^n(a,b)\right| \\ &\leq \frac{n(b-a)}{2(p+1)^{1/p}}\left[nA\left(|a|^{(n-1)(p/(p-1))}, |b|^{(n-1)(p/(p-1))}\right)\right. \\ &\quad \left. - \frac{\lambda(b-a)^2}{6n^{p/(p-1)}}\right]^{(p-1)/p}. \end{aligned} \tag{43}$$

Proof. Insertion of $\varrho(x) = x^n$, where $n \in \mathbb{N}, n \geq 2$, with $\varsigma = 1$ and $B(\varsigma) = B(1) = 1$ in Theorem 5 completes the proof. □

Proposition 3. *Suppose $a, b \in \mathbb{R}^+$, $a < b$ and $n \in \mathbb{N}, n \geq 2$. Then, the following inequality holds:*

$$\begin{aligned} &A^{-1}(a,b) + \frac{\lambda}{12}(b-a)^2 \leq \overline{L}^{-1}(a,b) \\ &\leq A\left(a^{-1}, b^{-1}\right) + \frac{\lambda}{6}(b-a)^2. \end{aligned} \tag{44}$$

Proof. Insertion of $\varrho(x) = x^n$, where $n \in \mathbb{N}, n \geq 2$, with $\varsigma = 1$ and $B(\varsigma) = B(1) = 1$ in Theorem 1 completes the proof. □

Proposition 4. *Suppose $a, b \in \mathbb{R}^+$, $a < b$ and $n \in \mathbb{N}, n \geq 2$. Then, the following inequality holds:*

$$\begin{aligned} &\left|A\left(a^{-1}, b^{-1}\right) - \overline{L}^{-1}(a,b)\right| \\ &\leq \frac{b-a}{4}A\left(|a|^{-2}, |b|^{-2}\right) - \frac{\lambda}{32}(b-a)^3. \end{aligned} \tag{45}$$

Proof. Insertion of $\varrho(x) = x^{-1}$, where $x \in [a,b]$, with $\varsigma = 1$ and $B(\varsigma) = B(1) = 1$ in Theorem 4 completes the proof. □

Proposition 5. *Suppose $a, b \in \mathbb{R}^+$, $a < b$ and $n \in \mathbb{N}, n \geq 2$. Then, the following inequality holds:*

$$\begin{aligned} &\left|A\left(a^{-1}, b^{-1}\right) - \overline{L}^{-1}(a,b)\right| \\ &\leq \frac{b-a}{2(p+1)^{1/p}}\left[A\left(|a|^{-2p/(p-1)}, |b|^{-2p/(p-1)}\right) - \frac{\lambda}{6}(b-a)^2\right]^{(p-1)/p}. \end{aligned} \tag{46}$$

Proof. Insertion of $\varrho(x) = x^{-1}$, where $x \in [a,b]$, with $\varsigma = 1$ and $B(\varsigma) = B(1) = 1$ in Theorem 4 completes the proof. □

6. Some Applications of Caputo–Fabrizio Fractional Integral Inequalities to the Trapezoidal Formula

Suppose d is the division of interval $[a,b]$, d: $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$, and consider the trapezoidal formula

$$T(\varrho, d) = \sum_{i=0}^{i=1} \frac{\varrho(x_i) + \varrho(x_{i+1})}{2}(x_{i+1} - x_i). \tag{47}$$

It is well known that if the mapping ϱ: $I \longrightarrow \mathbb{R}$ is twice differentiable on (a,b) and $M = \max_{x \in (a,b)}[\varrho''(x)] < \infty$, then

$$\int_a^b \varrho(x)\mathrm{d}x = T(\varrho, d) + E(\varrho, d), \tag{48}$$

where the approximation error $E(\varrho, d)$ of the integral $\int_a^b \varrho(x)\mathrm{d}x$ by the trapezoidal formula $T(\varrho, d)$ satisfies

$$|E(\varrho, d)| \leq \frac{M}{12}\sum_{i=0}^{n-1}(x_{i+1} - x_i)^3. \tag{49}$$

It is clear that if the mapping f is not twice differentiable or the second derivative is not bounded on (a,b), then (49) cannot be applied. In recent studies [30, 32–35], Dragomir and Wang showed that the remainder term $E(\varrho, d)$ can be estimated in terms of the first derivative only. These estimates have a wider range of applications. Here, we shall propose some new estimates of the remainder term $E(\varrho, d)$ which supplement, in a sense, those established in [30, 32–35].

Proposition 6. *Assume that* $\varrho\colon I \longrightarrow \mathbb{R}$ *is a differentiable positive mapping on* I°, $a, b \in I$ *with* $a < b$. *If* $\varrho' \in L_1[a,b]$ *and* $|\varrho'|$ *is a strongly convex function, then for every division* d *of* $[a,b]$, *the following inequality holds:*

$$\begin{aligned}|E(\varrho,d)| &\leq \frac{1}{8}\sum_{n-1}^{i=1}(x_{i+1}-x_i)^2\left[\left(|\varrho'(x_i)|+|\varrho'(x_{i+1})|\right)-\frac{\lambda}{4}\sum_{n-1}^{i=1}(x_{i+1}-x_i)^2\right]\\ &\leq \frac{1}{4}\sum_{n-1}^{i=1}(x_{i+1}-x_i)^2\left[\max\left\{\left(|\varrho'(a)|,|\varrho'(b)|\right)\right\}-\frac{\lambda}{8}\sum_{n-1}^{i=1}(x_{i+1}-x_i)^2\right].\end{aligned} \tag{50}$$

Proof. Applying subinterval $[x_i, x_{i+1}]$, $i = 0, \ldots, n-1$, of the division d from Theorem 4, we obtain

$$\begin{aligned}&\left|\frac{\varrho(x_i)+\varrho(x_{i+1})}{2}(x_{i+1}-x_i)-\int_{x_i}^{x_{i+1}}\varrho(x)\mathrm{d}x\right|\\ &\leq \frac{(x_{i+1}-x_i)^2}{4}\left[\frac{\left(|\varrho'(x_i)|+|\varrho'(x_{i+1})|\right)}{2}-\frac{\lambda}{16}(x_{i+1}-x_i)^2\right].\end{aligned} \tag{51}$$

Summing over $i = 0, \ldots, n-1$ and taking that $|\varrho'|$ is a strongly convex function, then by using (47), (48), and triangular inequality, we complete the proof. □

Proposition 7. *Assume that* $\varrho\colon I \longrightarrow \mathbb{R}$ *is a differentiable positive mapping on* I°, $a, b \in I$ *with* $a < b$ *and* $(1/p) + (1/q) = 1$. *If* $\varrho' \in L_1[a,b]$ *and* $|\varrho'|^q$ *is a strongly convex function, then for every division* d *of* $[a,b]$, *the following inequality holds:*

$$\begin{aligned}|E(\varrho,d)| &\leq \frac{1}{2(p+1)^{1/p}}\sum_{n-1}^{i=1}(x_{i+1}-x_i)^2\left[\left(\frac{|\varrho'(x_i)|^{p/(p-1)}+|\varrho'(x_{i+1})|^{p/(p-1)}}{2}\right)-\frac{\lambda}{6}\sum_{n-1}^{i=1}(x_{i+1}-x_i)^2\right]^{p/(p-1)}\\ &\leq \frac{1}{2(p+1)^{1/p}}\sum_{n-1}^{i=1}(x_{i+1}-x_i)^2\left[\max\left\{\left(|\varrho'(a)|,|\varrho'(b)|\right)\right\}-\frac{\lambda}{6}\sum_{n-1}^{i=1}(x_{i+1}-x_i)^2\right].\end{aligned} \tag{52}$$

Proof. Applying subinterval $[x_i, x_{i+1}]$, $i = 0, \ldots, n-1$, of the division d, we obtain from Theorem 5

$$\begin{aligned}&\left|\frac{\varrho(x_i)+\varrho(x_{i+1})}{2}(x_{i+1}-x_i)-\int_{x_i}^{x_{i+1}}\varrho(x)\mathrm{d}x\right|\\ &\leq \frac{(x_{i+1}-x_i)^2}{2(p+1)^{1/p}}\left[\frac{\left(|\varrho'(x_i)|^{p/(p-1)}+|\varrho'(x_{i+1})|^{p/(p-1)}\right)}{2}-\frac{\lambda}{6}(x_{i+1}-x_i)^2\right]^{(p-1)/p}.\end{aligned} \tag{53}$$

Summing over $i = 0, \ldots, n-1$ and taking that $|\varrho'|^q$, where $(1/p) + (1/q) = 1$, is a strongly convex function, then by using (47), (48), and triangular inequality, we complete the proof. □

7. Conclusions

The convex functions play an important role in approximation theory, and the fractional calculus has been found the best to model physical and engineering processes. Some properties of strongly convex functions via the Caputo–Fabrizio fractional integral operator have been studied in this paper. Precisely speaking, Hermite–Hadamard-type and some new inequalities for strongly convex functions via the Caputo–Fabrizio fractional integral operator are proved, and applications of the proposed inequalities to special means are also presented in this paper.

Data Availability

All data required for this research are available within this paper.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Qi Li added the application section to justify the novelty of the paper, Muhammad Shoaib Saleem designed the problem and supervised the work, Peiyu Yan wrote the literature review and arranged the funding for this paper, Muhammad Sajid Zahoor proved the main results, and Muhammad Imran wrote the first draft of the paper. All authors read and approved the final version of this paper.

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References

[1] A. Fernandez and P. Mohammed, "Hermite-Hadamard inequalities in fractional calculus defined using mittag-leffler kernels," *Mathematical Methods in the Applied Sciences*, vol. 2020, pp. 1–18, 2020.

[2] X. Yang, G. Farid, G. Farid, W. Nazeer, Y.-M. Chu, and C. Dong, "Fractional generalized hadamard and fejér-hadamard inequalities for m-convex functions," *AIMS Mathematics*, vol. 5, no. 6, pp. 6325–6340, 2020.

[3] P. O. Mohammed, M. Z. Sarikaya, and D. Baleanu, "On the generalized hermite-hadamard inequalities via the tempered fractional integrals," *Symmetry*, vol. 12, no. 4, p. 595, 2020.

[4] M. E. Gordji, M. R. Delavar, and M. D. L. Sen, "On φ-convex functions," *Journal of Mathematical Inequalities*, vol. 10, no. 1, pp. 173–183, 2016.

[5] Y. C. Kwun, G. Farid, S. Ullah, W. Nazeer, K. Mahreen, and S. M. Kang, "Inequalities for a unified integral operator and associated results in fractional calculus," *IEEE Access*, vol. 7, pp. 126283–126292, 2019.

[6] İ. İşcan, "Generalization of different type integral inequalities for s-convex functions via fractional integrals," *Applicable Analysis*, vol. 93, no. 9, pp. 1846–1862, 2014.

[7] Y. Y. Gambo, F. Jarad, D. Baleanu, and T. Abdeljawad, "On Caputo modification of the hadamard fractional derivatives," *Advances in Difference Equations*, vol. 2014, no. 1, p. 10, 2014.

[8] M. Caputo and M. Fabrizio, "A new definition of fractional derivative without singular kernel," *Progress in Fractional Differentiation and Applications*, vol. 1, no. 2, pp. 1–13, 2015.

[9] F. Song, Z. Yu, and H. Yang, "Modeling and analysis of fractional neutral disturbance waves in arterial vessels," *Mathematical Modelling of Natural Phenomena*, vol. 14, no. 3, p. 301, 2019.

[10] X.-J. Yang, M. Abdel-Aty, and C. Cattani, "A new general fractional-order derivataive with Rabotnov fractional-exponential kernel applied to model the anomalous heat transfer," *Thermal Science*, vol. 23, no. 3 Part A, pp. 1677–1681, 2019.

[11] J. T. Machado, V. Kiryakova, and F. Mainardi, "Recent history of fractional calculus," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 3, pp. 1140–1153, 2011.

[12] F. Ali, M. Saqib, I. Khan, and N. A. Sheikh, "Application of caputo-fabrizio derivatives to MHD free convection flow of generalized walters'-B fluid model," *The European Physical Journal Plus*, vol. 131, no. 10, p. 377, 2016.

[13] A. Atangana, "On the new fractional derivative and application to nonlinear fisher's reaction-diffusion equation," *Applied Mathematics and Computation*, vol. 273, pp. 948–956, 2016.

[14] M. Caputo and M. Fabrizio, "Applications of new time and spatial fractional derivatives with exponential kernels," *Progress in Fractional Differentiation and Applications*, vol. 2, no. 1, pp. 1–11, 2016.

[15] G. Farid, A. Ur Rehman, S. Bibi, A. U. Rehman, S. Bibi, and Y.-M. Chu, "Refinements of two fractional versions of hadamard inequalities for caputo fractional derivatives and related results," *Open Journal of Mathematical Sciences*, vol. 5, no. 1, pp. 1–10, 2021.

[16] B. Meftah, A. Souahi, and A. Souahi, "Cebyšev inequalities for co-ordinated \(QC\)-convex and \((s, QC)\)-convex," *Engineering and Applied Science Letters*, vol. 4, no. 1, pp. 14–20, 2021.

[17] T. Abdeljawad and D. Baleanu, "On fractional derivatives with exponential kernel and their discrete versions," *Reports on Mathematical Physics*, vol. 80, no. 1, pp. 11–27, 2017.

[18] T. Abdeljawad, "Fractional operators with exponential kernels and a Lyapunov type inequality," *Advances in Difference Equations*, vol. 2017, no. 1, pp. 1–11, 2017.

[19] S. E. Fadugba, "Development of a new numerical scheme for the solution of exponential growth and decay models," *Open Journal of Mathematical Sciences*, vol. 4, no. 1, pp. 18–26, 2021.

[20] N. Merentes and K. Nikodem, "Remarks on strongly convex functions," *Aequationes Mathematicae*, vol. 80, no. 1-2, pp. 193–199, 2010.

[21] M. Eshaghi, S. S. Dragomir, and M. Rostamian Delavar, "An inequality related to η-convex functions (II)," *International Journal of Nonlinear Analysis and Applications*, vol. 6, no. 2, pp. 27–33, 2015.

[22] M. Z. Sarikaya, A. Saglam, and H. Yildirim, "On some Hadamard-type inequalities for h-convex functions," *Journal of Mathematical Inequalities*, vol. 2, no. 3, pp. 335–341, 2008.

[23] H. Angulo, J. Giménez, A. Milena Moros, and K. Nikodem, "On strongly \$h\$-convex functions h-convex functions," *Annals of Functional Analysis*, vol. 2, no. 2, pp. 85–91, 2011.

[24] M. A. Latif, S. Rashid, S. S. Dragomir, and Y. M. Chu, "Hermite-Hadamard type inequalities for co-ordinated convex and qausi-convex functions and their applications," *Journal of Inequalities and Applications*, vol. 2019, no. 1, pp. 1–33, 2019.

[25] J. Wang, X. Li, M. Fekan, and Y. Zhou, "Hermite-hadamard-type inequalities for Riemann-liouville fractional integrals via two kinds of convexity," *Applicable Analysis*, vol. 92, no. 11, pp. 2241–2253, 2013.

[26] A. A. Al-Gonah and W. K. Mohammed, "A new forms of extended hypergeometric functions and their properties,"

Engineering and Applied Science Letters, vol. 4, no. 1, pp. 30–41, 2021.

[27] G. Farid, K. Mahreen, and Y.-M. Chu, "Study of inequalities for unified integral operators of generalized convex functions," *Open Journal of Mathematical Sciences*, vol. 5, no. 1, pp. 80–93, 2021.

[28] S. Foschi and D. Ritelli, "The lambert function, the quintic equation and the proactive discovery of the implicit function theorem," *Open Journal of Mathematical Sciences*, vol. 5, no. 1, pp. 94–114, 2021.

[29] B. T. Polyak, "Existence theorems and convergence of minimizing sequences in extremum problems with restrictions," *Soviet Mathematics-Doklady*, vol. 7, pp. 72–75, 1966.

[30] S. S. Dragomir and R. P. Agarwal, "Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula," *Applied Mathematics Letters*, vol. 11, no. 5, pp. 91–95, 1998.

[31] R. P. Agarwal and S. S. Dragomir, "An application of Hayashi's inequality for differentiable functions," *Computers & Mathematics with Applications*, vol. 32, no. 6, pp. 95–99, 1996.

[32] S. S. Dragomir and S. Wang, "Applications of Ostrowski's inequality to the estimation of error bounds for some special means and for some numerical quadrature rules," *Applied Mathematics Letters*, vol. 11, no. 1, pp. 105–109, 1998.

[33] G. Hong, G. Farid, W. Nazeer et al., "Boundedness of fractional integral operators containing mittag-leffler function via exponentially-convex functions," *Journal of Mathematics*, vol. 2020, Article ID 3584105, 7 pages, 2020.

[34] Y. C. Kwun, M. S. Saleem, M. Ghafoor, W. Nazeer, and S. M. Kang, "Hermitehadamard-type inequalities for functions whose derivatives are ?-convex via fractional integrals," *Journal of Inequalities and Applications*, vol. 2019, no. 1, pp. 1–16, 2019.

[35] J. Zhao, S. I. Butt, J. Nasir, Z. Wang, I. Tlili, and Y. Liu, "Hermite-jensen-mercer type inequalities for caputo fractional derivatives," *Journal of Function Spaces*, vol. 2020, no. 1, 11 pages, Article ID 7061549, 2020.