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Research Article

On New χ-Fixed Point Results for $\lambda-(\Upsilon,\chi)$-Contractions in Complete Metric Spaces with Applications

Babak Mohammadi,[1] **Wutiphol Sintunavarat**,[2] **and Vahid Parvaneh**[3]

[1]*Department of Mathematics, Marand Branch, Islamic Azad University, Marand, Iran*
[2]*Department of Mathematics and Statistics, Faculty of Science and Technology, Thammasat University Rangsit Center, Pathum Thani 12120, Thailand*
[3]*Department of Mathematics, Gilan-E-Gharb Branch, Islamic Azad University, Gilan-E-Gharb, Iran*

Correspondence should be addressed to Vahid Parvaneh; zam.dalahoo@gmail.com

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The main aim of this work is to introduce the new concept of $\lambda-(\Upsilon,\chi)$-contraction self-mappings and prove the existence of χ-fixed points for such mappings in metric spaces. Our results generalize and improve some results in existing literature. Moreover, some fixed point results in partial metric spaces can be derived from our χ-fixed points results. Finally, the existence of solutions of nonlinear integral equations is investigated via the theoretical results in this work.

1. Introduction and Preliminaries

One of the most famous metrical fixed point theorem is the Banach contraction principle (BCP) which is the classical tool for solving several nonlinear problems. Based on the noncomplexity and the usefulness of this principle, many mathematicians have improved, extended, and generalized it into several directions. For instance, in [1], on the basis of the probabilistic metric space and the S-metric space, Hu and Gu introduced the concept of the probabilistic metric space, which is called the Menger probabilistic S-metric space. They also proved some fixed point theorems in the framework of Menger probabilistic S-metric spaces. In [2], using the notion of the cyclic representation of a nonempty set with respect to a pair of mappings, Mohanta and Biswas obtained coincidence points and common fixed points of a pair of self-mappings satisfying a type of contraction condition involving comparison functions and (w)-comparison functions in partial metric spaces.

Many researchers attempted to introduce the new idea on generalizations of a metric space and then they investigated fixed point results in new spaces.

In 1994, partial metric spaces were introduced initially by Matthews [3]. One of the important points in this space is the possibility of being nonzero the self-distance.

Definition 1 (see [3]). Let Π be a nonempty set. A mapping $\mathscr{W}\colon \Pi\times\Pi \longrightarrow [0,\infty)$ is called a partial metric if and only if

(p1) $\mathscr{W}(\mathscr{X},\mathscr{X}) = \mathscr{W}(\mathscr{Y},\mathscr{Y}) = \mathscr{W}(\mathscr{X},\mathscr{Y}) \Leftrightarrow \mathscr{X} = \mathscr{Y}$,

(p2) $\mathscr{W}(\mathscr{X},\mathscr{X}) \leq \mathscr{W}(\mathscr{X},\mathscr{Y})$,

(p3) $\mathscr{W}(\mathscr{X},\mathscr{Y}) = \mathscr{W}(\mathscr{Y},\mathscr{X})$,

(p4) $\mathscr{W}(\mathscr{X},\mathscr{Y}) \leq \mathscr{W}(\mathscr{X},\mathscr{Z}) + \mathscr{W}(\mathscr{Z},\mathscr{Y}) - \mathscr{W}(\mathscr{Z},\mathscr{Z})$,

for any $\mathscr{X},\mathscr{Y},\mathscr{Z} \in \Pi$. Moreover, the pair $(\Pi,\mathscr{W})$ will be a partial metric space.

Note that any metric space is a partial metric space but the reverse is not true, in general. An example of a partial metric space is the pair $([0,\infty),\mathscr{W})$, where $\mathscr{W}(\mathscr{X},\mathscr{Y}) = \max\{\mathscr{X},\mathscr{Y}\}$ for all $\mathscr{X},\mathscr{Y} \in [0,\infty)$. We see that $\mathscr{W}(\mathscr{X},\mathscr{X})$ may not to be zero for some $\mathscr{X} \in \Pi$. For further examples of a partial metric, we refer to [3].

Definition 2 (see [3]). Let $(\Pi,\mathscr{W})$ be a partial metric space.

(1) $\{\mathscr{X}_n\}\subseteq\Pi$ is said to be converging to a point $\mathscr{X} \in \Pi$ if and only if $\lim_{n\longrightarrow\infty}\mathscr{W}(\mathscr{X}_n,\mathscr{X}) = \mathscr{W}(\mathscr{X},\mathscr{X})$.

(2) $\{\mathscr{X}_n\}\subseteq\Pi$ is called a Cauchy sequence if and only if $\lim_{n,m\longrightarrow\infty}\mathscr{W}(\mathscr{X}_n,\mathscr{X}_m)$ exists and is finite.

(3) $(\Pi,\mathscr{W})$ is said to be complete if and only if every Cauchy sequence $\{\mathscr{X}_n\}\subseteq\Pi$ converges to some point $\mathscr{X} \in \Pi$ such that $\lim_{n,m\longrightarrow\infty}\mathscr{W}(\mathscr{X}_n,\mathscr{X}_m) = \lim_{n\longrightarrow\infty}\mathscr{W}(\mathscr{X}_n,\mathscr{X}) = \mathscr{W}(\mathscr{X},\mathscr{X})$.

Remark 1 (see [3]). If $(\Pi,\mathscr{W})$ is a partial metric space, then the pair $(\Pi,d_{\mathscr{W}})$ is a metric space where $d_{\mathscr{W}}\colon \Pi\times\Pi \longrightarrow [0,\infty)$ is defined by $d_{\mathscr{W}}(\mathscr{X},\mathscr{Y}) = 2\mathscr{W}(\mathscr{X},\mathscr{Y}) - \mathscr{W}(\mathscr{X},\mathscr{X}) - \mathscr{W}(\mathscr{Y},\mathscr{Y})$ for all $\mathscr{X},\mathscr{Y} \in \Pi$.

Lemma 1 (see [3]). *Let $(\Pi,\mathscr{W})$ be a partial metric space.*

(i) $\{\mathscr{X}_n\}$ is Cauchy in $(\Pi,\mathscr{W})$ if and only if $\{\mathscr{X}_n\}$ is Cauchy in $(\Pi,d_{\mathscr{W}})$.

(ii) The partial metric space $(\Pi,\mathscr{W})$ is complete if and only if the metric space $(\Pi,d_{\mathscr{W}})$ is complete.

(iii) For each $\{\mathscr{X}_n\}\subseteq\Pi$ and $\mathscr{X} \in \Pi$, $\lim_{n\longrightarrow\infty}d_{\mathscr{W}}(\mathscr{X}_n,\mathscr{X}) = 0 \Leftrightarrow \lim_{n,m\longrightarrow\infty}\mathscr{W}(\mathscr{X}_n,\mathscr{X}_m) = \lim_{n\longrightarrow\infty}\mathscr{W}(\mathscr{X}_n,\mathscr{X}) = \mathscr{W}(\mathscr{X},\mathscr{X})$.

According to the published work of Matthews [3], fixed point results in partial metric spaces have been investigated widely by many mathematicians. In 2014, the new concepts of χ-fixed points, χ-Picard mappings, and weakly χ-Picard mappings have been introduced by Jleli et al. [4]. Several χ-fixed point results for mappings satisfying the generalized Banach contractive condition based on the idea of new control function are proved in [4]. Moreover, they also claimed that some fixed point results in partial metric spaces can be derived from these χ-fixed point results in metric spaces. Next, we recall the definitions of χ-fixed points, χ-Picard mappings, and weakly χ-Picard mappings. Before presenting these definitions, some notations are needed.

Let Π be a nonempty set, $\chi\colon \Pi \longrightarrow [0,\infty)$ be a given function, and $\Gamma\colon \Pi \longrightarrow \Pi$ be a mapping.

Throughout this paper, unless otherwise specified, the set of all fixed points of Γ is denoted by $F(\Gamma) = \{\mathscr{X} \in \Pi | \Gamma(\mathscr{X}) = \mathscr{X}\}$ and the set of all zeros of χ is denoted by $Z_\chi = \{\mathscr{X} \in \Pi | \chi(\mathscr{X}) = 0\}$.

Definition 3 (see [4]). Let Π be a nonempty set, $\chi\colon \Pi \longrightarrow [0,\infty)$ be a given function, and $\Gamma\colon \Pi \longrightarrow \Pi$ be a mapping. $\mathscr{Z} \in \Pi$ is called a χ-fixed point of Γ if and only if $\mathscr{Z}$ is a fixed point of Γ such that $\chi(\mathscr{Z}) = 0$, that is, $\mathscr{Z} \in F(\Gamma)\cap Z_\chi$.

Definition 4 (see [4]). Let Π be a nonempty set and $\chi\colon \Pi \longrightarrow [0,\infty)$ be a given function. A mapping $\Gamma\colon \Pi \longrightarrow \Pi$ is called a χ-Picard mapping if the following conditions hold:

(i) $F(\Gamma)\cap Z_\chi = \{\mathscr{Z}\}$

(ii) $\Gamma^n\mathscr{X} \longrightarrow \mathscr{Z}$ as $n \longrightarrow \infty$ for any $\mathscr{X} \in \Pi$

Definition 5 (see [4]). Let Π be a nonempty set and $\chi\colon \Pi \longrightarrow [0,\infty)$ be a given function. A mapping $\Gamma\colon \Pi \longrightarrow \Pi$ is called a weakly χ-Picard mapping if the following conditions hold:

(i) Γ has at least one χ-fixed point

(ii) The sequence $\{\Gamma^n\mathscr{X}\}$ converges to some χ-fixed point of Γ for any $\mathscr{X} \in \Pi$

A new control function $\Upsilon\colon [0,\infty)^3 \longrightarrow [0,\infty)$ has been introduced by Jleli et al. [4] where

(Υ1) $\max\{a,b\} \leq \Upsilon(a,b,c)$, for all $a,b,c \in [0,\infty)$

(Υ2) $\Upsilon(0,0,0) = 0$

(Υ3) Υ is continuous

Throughout this paper, unless otherwise is specified, the class of all functions satisfying the properties (Υ1) − (Υ3) is denoted by $\overline{\Upsilon}$.

Example 1 (see [4]). Suppose that the mappings $\Upsilon_1,\Upsilon_2,\Upsilon_3\colon [0,\infty)^3 \longrightarrow [0,\infty)$ are defined by $\Upsilon_1(a,b,c) = a+b+c$, $\Upsilon_2(a,b,c) = \max\{a,b\}+c$, $\Upsilon_3(a,b,c) = a+a^2+b+c$ for all $a,b,c \in [0,\infty)$. Then, $\Upsilon_1,\Upsilon_2,\Upsilon_3 \in \overline{\Upsilon}$.

Using the notion of control functions in Υ, Jleli et al. [4] introduced the ideas of (Υ,χ)-contractions and (Υ,χ)-weak contractions and proved existence of χ-fixed point for such mappings as follows.

Definition 6 (see [4]). Let (Π,d) be a metric space, $\chi\colon \Pi \longrightarrow [0,\infty)$ be a given function, and $\Upsilon \in \overline{\Upsilon}$. A mapping $\Gamma\colon \Pi \longrightarrow \Pi$ is called an (Υ,χ)-contraction if and only if there is $k \in (0,\infty)$ such that $\Upsilon(d(\Gamma\mathscr{X},\Gamma\mathscr{Y}),\chi(\Gamma\mathscr{X}),\chi(\Gamma\mathscr{Y})) \leq k\Upsilon(d(\mathscr{X},\mathscr{Y}),\chi(\mathscr{X}),\chi(\mathscr{Y}))$, for all $\mathscr{X},\mathscr{Y} \in \Pi$.

Definition 7 (see [4]). Let (Π,d) be a metric space, $\chi\colon \Pi \longrightarrow [0,\infty)$ be a given function, and $\Upsilon \in \overline{\Upsilon}$. A mapping $\Gamma\colon \Pi \longrightarrow \Pi$ is called an (Υ,χ)-weak contraction if and only if there are $k \in (0,\infty)$ and $L \geq 0$ such that $\Upsilon(d(\Gamma\mathscr{X},\Gamma\mathscr{Y}),\chi(\Gamma\mathscr{X}),\chi(\Gamma\mathscr{Y})) \leq k\Upsilon(d(\mathscr{X},\mathscr{Y}),\chi(\mathscr{X}),\chi(\mathscr{Y})) + L\,(\Upsilon(d(\mathscr{Y},\Gamma\mathscr{X}),\chi(\mathscr{Y}),\chi(\Gamma\mathscr{X})) - \Upsilon(0,\chi(\mathscr{Y}),\chi(\Gamma\mathscr{X})))$ for all $\mathscr{X},\mathscr{Y} \in \Pi$.

Theorem 1 (see [4]). *Let (Π,d) be a complete metric space, $\chi\colon \Pi \longrightarrow [0,\infty)$ be a given function, and $\Upsilon \in \overline{\Upsilon}$. Assume that*

(H1) χ is lower semicontinuous

(H2) $\Gamma\colon \Pi \longrightarrow \Pi$ is an (Υ,χ)-contraction mapping

Then, the following assertions hold:

(i) $F(\Gamma)\subseteq Z_\chi$

(ii) Γ is a χ-Picard mapping

(iii) If $\mathscr{X} \in \Pi$ and $\mathscr{Z}$ is a χ-fixed point of Γ, then $d(\Gamma^n \mathscr{X}, \mathscr{Z}) \leq (k^n/(1-k))\Upsilon(d(\Gamma\mathscr{X}, \mathscr{X}), \chi(\Gamma\mathscr{X}), \chi(\mathscr{X}))$, for all $n \in \mathbb{N}$

Theorem 2 (see [4]). *Let (Π, d) be a complete metric space, $\chi\colon \Pi \longrightarrow [0,\infty)$ be a given function, and $\Upsilon \in \overline{\Upsilon}$. Assume that*

(H1) χ is lower semicontinuous

(H2) $\Gamma\colon \Pi \longrightarrow \Pi$ is an (Υ, χ)-weak contraction mapping

Then, the following assertions hold:

(i) $F(\Gamma) \subseteq Z_\chi$

(ii) Γ is a weakly χ-Picard mapping

(iii) If $\mathscr{X} \in \Pi$ and $\mathscr{Z}$ is a χ-fixed point of Γ, then $d(\Gamma^n \mathscr{X}, \mathscr{Z}) \leq ((k^n)/(1-k))\Upsilon(d(\mathscr{X}, \Gamma\mathscr{X}), \chi(\mathscr{X}), \chi(\Gamma\mathscr{X}))$, for all $n \in \mathbb{N}$

Nowadays, many authors have extended the Banach contractive condition in the BCP into many ways by using various types of the control functions. In 2014, Jleli and Samet [5] presented the new idea of a control function and proved the fixed point results for mappings involving this new control function. Here, we restate the idea of the control function proposed in Jleli and Samet [5] and give the main work in [5] which is the main inspiration in this paper.

Let Λ be the set of all functions $\lambda\colon [0,\infty) \longrightarrow [1,\infty)$ so that

(i) λ is non-decreasing

(ii) For each sequence $\{\iota_n\} \subset [0,\infty)$, $\lim_{n\longrightarrow\infty}\lambda(\iota_n) = 1$ if and only if $\lim_{n\longrightarrow\infty}\iota_n = 0$

(iii) There exist $r \in (0,1)$ and $l \in (0,\infty]$ such that $\lim_{t\longrightarrow 0^+}((\lambda(t)-1)/t^r) = l$

Theorem 3 (see [5]). *Let (Π, d) be a complete metric space and $\Gamma\colon \Pi \longrightarrow \Pi$ be a given mapping. Suppose that there exist $\lambda \in \Lambda$ and $k \in (0,1)$ such that for all $\mathscr{X}, \mathscr{Y} \in \Pi$ with $\Gamma\mathscr{X} \neq \Gamma\mathscr{Y}$, one has $\lambda(d(\Gamma\mathscr{X}, \Gamma\mathscr{Y})) \leq [\lambda(d(\mathscr{X}, \mathscr{Y}))]^k$. Then, Γ possesses a unique fixed point.*

Recall that $\chi\colon \Pi \longrightarrow [0,\infty)$ is lower semicontinuous at x_0 if $\liminf_{x\longrightarrow x_0}\chi(x) \geq \chi(x_0)$.

Note that there is no discussion so far on the combination of several ideas of contraction mappings in the literature. The goal of this work is to present the new concept of a λ-(Υ,χ)-contraction self-mappings. The existence results of χ-fixed points for such contraction mappings in metric spaces are provided. The main results of Jleli and Samet [4] and Jleli et al. [5] are particular cases of our main results. Furthermore, we give some fixed point results in partial metric spaces which can be derived from our χ-fixed points results. Finally, we apply the theoretical results in this work to prove the existence of solutions of nonlinear integral equations.

2. Main Results

To present the main result in this paper, we start with the following definition which is larger than the idea of many contraction mappings in the literature.

Definition 8. Let (Π, d) be a metric space, $\chi\colon \Pi \longrightarrow [0,\infty)$ be a given function, $\Upsilon \in \overline{\Upsilon}$, and $\lambda \in \Lambda$. A mapping $\Gamma\colon \Pi \longrightarrow \Pi$ is called a λ-(Υ,χ)-contraction if and only if there exists $k \in (0,1)$ such that

$$\lambda(\Upsilon(d(\Gamma\mathscr{X}, \Gamma\mathscr{Y}), \chi(\Gamma\mathscr{X}), \chi(\Gamma\mathscr{Y}))) \leq [\lambda(\Upsilon(d(\mathscr{X}, \mathscr{Y}), \chi(\mathscr{X}), \chi(\mathscr{Y})))]^k, \tag{1}$$

for all $\mathscr{X}, \mathscr{Y} \in \Pi$.

Now, we present the main results in this paper.

Theorem 4. *Let (Π, d) be a complete metric space, $\chi\colon \Pi \longrightarrow [0,\infty)$ be a given function, $\Upsilon \in \overline{\Upsilon}$, and $\lambda \in \Lambda$. Assume that*

(i) χ is lower semicontinuous

(ii) $\Gamma\colon \Pi \longrightarrow \Pi$ is an λ-(Υ,χ)-contraction

Then, the following assertions hold:

(i) $F(\Gamma) \subseteq Z_\chi$

(ii) Γ is a χ-Picard mapping

Proof. Suppose that $\mathscr{G} \in \Pi$ is a fixed point of Γ. Appling (1) with $\mathscr{X} = \mathscr{Y} = \mathscr{G}$, we obtain $\lambda(\Upsilon(0, \chi(\mathscr{G}), \chi(\mathscr{G}))) \leq [\lambda(\Upsilon(0, \chi(\mathscr{G}), \chi(\mathscr{G})])]^k$. This implies $\lambda(\Upsilon(0, \chi(\mathscr{G}), \chi(\mathscr{G}))) = 1$ and so $\Upsilon(0, \chi(\mathscr{G}), \chi(\mathscr{G})) = 0$. Then, $\chi(\mathscr{G}) \leq \Upsilon(0, \chi(\mathscr{G}), \chi(\mathscr{G})) = 0$ which implies $\chi(\mathscr{G}) = 0$. Thus, we have proved (i). Now, let $\mathscr{X}$ be an arbitrary point. From (1), we obtain

$$\begin{aligned} \lambda\left(d\left(\Gamma^n\mathscr{X}, \Gamma^{n+1}\mathscr{X}\right)\right) &\leq \lambda\left(\Upsilon\left(d\left(\Gamma^n\mathscr{X}, \Gamma^{n+1}\mathscr{X}\right), \chi\left(\Gamma^n\mathscr{X}\right), \chi\left(\Gamma^{n+1}\mathscr{X}\right)\right)\right), \\ &\leq \lambda\left(\Upsilon\left(d\left(\Gamma^{n-1}\mathscr{X}, \Gamma^n\mathscr{X}\right), \chi\left(\Gamma^{n-1}\mathscr{X}\right), \chi\left(\Gamma^n\mathscr{X}\right)\right)\right)^k \\ &\;\;\vdots \\ &\leq [\lambda(\Upsilon(d(\mathscr{X}, \Gamma\mathscr{X}), \chi(\mathscr{X}), \chi(\Gamma\mathscr{X})))]^{k^n}, \end{aligned} \tag{2}$$

for all $n \in \mathbb{N}$. If $n \longrightarrow \infty$ in the above inequality, we obtain $\lim_{n\longrightarrow\infty}\lambda(d(\Gamma^n\mathscr{X}, \Gamma^{n+1}\mathscr{X})) = 1$ and so $\lim_{n\longrightarrow\infty}d(\Gamma^n\mathscr{X}, \Gamma^{n+1}\mathscr{X}) = 0$. Thus, there exist $r \in (0,1)$ and $l \in (0,\infty]$ such that

$$\lim_{n\longrightarrow\infty}\frac{\lambda\left(d\left(\Gamma^n\mathscr{X}, \Gamma^{n+1}\mathscr{X}\right)\right) - 1}{d\left(\Gamma^n\mathscr{X}, \Gamma^{n+1}\mathscr{X}\right)^r} = l. \tag{3}$$

Similar to the proof of Theorem 2.1 in [5], we deduce $\{\Gamma^n\mathcal{X}\}$ is a Cauchy sequence. Since (Π, d) is complete, there exists $\mathcal{Z} \in \Pi$ such that $\Gamma^n\mathcal{X} \longrightarrow \mathcal{Z}$ as $n \longrightarrow \infty$. From (2), we obtain $1 \leq \lambda(\chi(\Gamma^n\mathcal{X})) \leq [\lambda(\Upsilon(\mathrm{d}(\mathcal{X}, \Gamma\mathcal{X}), \chi(\mathcal{X}), \chi(\Gamma\mathcal{X})))]^{k^n}$ for all $n \in \mathbb{N}$. If $n \longrightarrow \infty$ in the above inequality, we obtain $\lim_{n\longrightarrow\infty}\lambda(\chi(\Gamma^n\mathcal{X})) = 1$ and so $\lim_{n\longrightarrow\infty}\chi(\Gamma^n\mathcal{X}) = 0$. Since χ is lower semicontinuous, we obtain $\chi(\mathcal{Z}) = 0$. Again, using (1), we obtain

$$1 \leq \lim_{n\longrightarrow\infty} \lambda\Big(\mathrm{d}\Big(\Gamma^{n+1}\mathcal{X}, \Gamma\mathcal{Z}\Big)\Big) \leq \lim_{n\longrightarrow\infty} [\lambda(\Upsilon(\mathrm{d}(\Gamma^n\mathcal{X}, \mathcal{Z}), \chi(\Gamma^n\mathcal{X}), \chi(\mathcal{Z})))]^k = [\lambda(\Upsilon(0,0,0))]^k = 1. \tag{4}$$

Thus, $\mathrm{d}(\mathcal{Z}, \Gamma\mathcal{Z}) = 0$, that is, $\mathcal{Z}$ is a fixed point of Γ. Therefore, $\mathcal{Z}$ is also a χ-fixed point of Γ.

To show the uniqueness of fixed point, let $\mathcal{Z}, \mathcal{Z}'$ be two χ-fixed points of Γ. Applying (1) for $\mathcal{X} = \mathcal{Z}, \mathcal{Y} = \mathcal{Z}'$, we get $\lambda(\Upsilon(\mathrm{d}(\mathcal{Z}, \mathcal{Z}'), 0, 0)) \leq [\lambda(\Upsilon(\mathrm{d}(\mathcal{Z}, \mathcal{Z}'), 0, 0))]^k$. This implies $\lambda(\Upsilon(\mathrm{d}(\mathcal{Z}, \mathcal{Z}'), 0, 0)) = 1$ and so $\Upsilon(\mathrm{d}(\mathcal{Z}, \mathcal{Z}'), 0, 0) = 0$. Therefore, $\mathrm{d}(\mathcal{Z}, \mathcal{Z}') \leq \Upsilon(\mathrm{d}(\mathcal{Z}, \mathcal{Z}'), 0, 0) = 0$ which gives us $\mathrm{d}(\mathcal{Z}, \mathcal{Z}') = 0$. Thus, $\mathcal{Z} = \mathcal{Z}'$. Therefore, we have proved *(ii)*.

Taking $\Upsilon(a, b, c) = a + b + c$ and $\chi \equiv 0$ in the above theorem, we have the following. □

Corollary 1. *Let (Π, d) be a complete metric space and $\lambda \in \Lambda$. Assume that*

(i) There exists $k \in (0, 1)$ such that

$$\lambda(\Upsilon(\mathrm{d}(\Gamma\mathcal{X}, \Gamma\mathcal{Y}))) \leq [\lambda(\Upsilon(\mathrm{d}(\mathcal{X}, \mathcal{Y})))]^k, \tag{5}$$

for all $\mathcal{X}, \mathcal{Y} \in \Pi$.

Then, the following assertions hold:

(i) $F(\Gamma) \subseteq Z_\chi$

(ii) Γ is a χ-Picard mapping

Taking $\lambda(t) = \sqrt{t}$ for all $t \geq 0$ in the above corollary, we obtain the BCP.

Next, we present the second idea of the new mappings satisfying the generalized contractive condition which is similar to the first idea and then we prove the existence of a χ-fixed point result for this mapping.

Definition 9. Let (Π, d) be a metric space, $\chi\colon \Pi \longrightarrow [0, \infty)$ be a given function, $\Upsilon \in \overline{\Upsilon}$, and $\lambda \in \Lambda$. A mapping $\Gamma\colon \Pi \longrightarrow \Pi$ is called a λ-(Υ, χ)-weak contraction if and only if there exist $k \in (0, 1)$ and $L \geq 0$ such that

$$\begin{aligned}\lambda(\Upsilon(\mathrm{d}(\Gamma\mathcal{X}, \Gamma\mathcal{Y}), \chi(\Gamma\mathcal{X}), \chi(\Gamma\mathcal{Y}))) &\leq [\lambda(\Upsilon(\mathrm{d}(\mathcal{X}, \mathcal{Y}), \chi(\mathcal{X}), \chi(\mathcal{Y})))]^k \\ &\quad [\lambda(\Upsilon(\mathrm{d}(\mathcal{Y}, \Gamma\mathcal{X}), \chi(\mathcal{Y}), \chi(\Gamma\mathcal{X})) - \Upsilon(0, \chi(\mathcal{Y}), \chi(\Gamma\mathcal{X})))]^L,\end{aligned} \tag{6}$$

for all $\mathcal{X}$, $\mathcal{Y} \in \Pi$.

Theorem 5. *Let (Π, d) be a complete metric space, $\chi\colon \Pi \longrightarrow [0, \infty)$ be a given function, $\Upsilon \in \overline{\Upsilon}$, and $\lambda \in \Lambda$. Assume that*

(i) χ is lower semicontinuous

(ii) $\Gamma\colon \Pi \longrightarrow \Pi$ is a λ-(Υ, χ)-weak contraction

Then, the following assertions hold:

(i) $F(\Gamma) \subseteq Z_\chi$

(ii) Γ is a weakly χ-Picard mapping

Proof. Suppose that $\mathcal{G} \in \Pi$ is a fixed point of Γ. Appling (6) with $\mathcal{X} = \mathcal{Y} = \mathcal{G}$, we obtain

$$\begin{aligned}\lambda(\Upsilon(0, \chi(\mathcal{G}), \chi(\mathcal{G}))) &\leq [\lambda(\Upsilon(0, \chi(\mathcal{G}), \chi(\mathcal{G})]]]^k [\lambda(\Upsilon(0, \chi(\mathcal{G}), \chi(\mathcal{G})) - \Upsilon\Upsilon(0, \chi(\mathcal{G}), \chi(\mathcal{G})))]^L \\ &= [\lambda(\Upsilon(0, \chi(\mathcal{G}), \chi(\mathcal{G})))]^k.\end{aligned} \tag{7}$$

This implies that $\lambda(\Upsilon(0, \chi(\mathcal{G}), \chi(\mathcal{G}))) = 1$ and so $\Upsilon(0, \chi(\mathcal{G}), \chi(\mathcal{G})) = 0$. Then, $\chi(\mathcal{G}) \leq \Upsilon(0, \chi(\mathcal{G}), \chi(\mathcal{G})) = 0$ which implies $\chi(\mathcal{G}) = 0$. Thus, we have proved *(i)*. Now, let $\mathcal{X}$ be an arbitrary point. From (6), we obtain

$$\begin{aligned}\lambda\left(\mathrm{d}\left(\Gamma^n\mathcal{X}, \Gamma^{n+1}\mathcal{X}\right)\right) &\leq \lambda\left(\Upsilon\left(\mathrm{d}\left(\Gamma^n\mathcal{X}, \Gamma^{n+1}\mathcal{X}\right), \chi\left(\Gamma^n\mathcal{X}\right), \chi\left(\Gamma^{n+1}\mathcal{X}\right)\right)\right), \\ &\leq \left[\lambda\left(\Upsilon\left(\mathrm{d}\left(\Gamma^{n-1}\mathcal{X}, \Gamma^n\mathcal{X}\right), \chi\left(\Gamma^{n-1}\mathcal{X}\right), \chi\left(\Gamma^n\mathcal{X}\right)\right)\right)\right]^k \\ &\quad [\lambda((\Upsilon(\mathrm{d}(\Gamma^n\mathcal{X}, \Gamma^n\mathcal{X})), \chi(\Gamma^n\mathcal{X}), \chi(\Gamma^n\mathcal{X})) - \Upsilon(0, \chi(\Gamma^n\mathcal{X}), \chi(\Gamma^n\mathcal{X})))]^L \\ &= \left[\lambda\left(\Upsilon\left(d\left(\Gamma^{n-1}\mathcal{X}, \Gamma^n\mathcal{X}\right), \chi\left(\Gamma^{n-1}\mathcal{X}\right), \chi\left(\Gamma^n\mathcal{X}\right)\right)\right)\right]^k \\ &\vdots \\ &\leq [\lambda(\Upsilon(\mathrm{d}(\mathcal{X}, \Gamma\mathcal{X}), \chi(\mathcal{X}), \chi(\Gamma\mathcal{X})))]^{k^n}.\end{aligned} \tag{8}$$

Taking the limit as $n \longrightarrow \infty$ in the above inequality, it gives us $\lim_{n\longrightarrow\infty}\lambda(d(\Gamma^n\mathcal{X},\Gamma^{n+1}\mathcal{X})) = 1$ and so $\lim_{n\longrightarrow\infty}d(\Gamma^n\mathcal{X},\Gamma^{n+1}\mathcal{X}) = 0$. Thus, there exist $r \in (0,1)$ and $l \in (0,\infty]$ such that

$$\lim_{n\longrightarrow\infty}\frac{\lambda\left(d\left(\Gamma^n\mathcal{X},\Gamma^{n+1}\mathcal{X}\right)\right)-1}{d\left(\Gamma^n\mathcal{X},\Gamma^{n+1}\mathcal{X}\right)^r} = l. \tag{9}$$

Similar to proof of Theorem 2.1 in [5], we deduce $\{\Gamma^n\mathcal{X}\}$ is a Cauchy sequence. Since (Π,d) is complete, there exists $\mathcal{Z} \in \Pi$ such that $\Gamma^n\mathcal{X} \longrightarrow \mathcal{Z}$ as $n \longrightarrow \infty$. From (8), we obtain. $1 \le \lambda(\chi(\Gamma^n\mathcal{X})) \le [\lambda(\Upsilon(d(\mathcal{X},\Gamma\mathcal{X}),\chi(\mathcal{X}),\chi(\Gamma\mathcal{X})))]^{k^n}$.

Taking $n \longrightarrow \infty$, we obtain $\lim_{n\longrightarrow\infty}\lambda(\chi(\Gamma^n\mathcal{X})) = 1$ and so $\lim_{n\longrightarrow\infty}\chi(\Gamma^n\mathcal{X}) = 0$. Since χ is lower semicontinuous, we obtain $\chi(\mathcal{Z}) = 0$. Again using (11), we obtain

$$\begin{aligned}
1 &\le \lim_{n\longrightarrow\infty}\lambda\left(d\left(\Gamma^{n+1}\mathcal{X},\Gamma\mathcal{Z}\right)\right)\\
&\le \lim_{n\longrightarrow\infty}[\lambda(\Upsilon(d(\Gamma^n\mathcal{X},\mathcal{Z}),\chi(\Gamma^n\mathcal{X}),\chi(\mathcal{Z})))]^k\\
&[\lambda(\Upsilon(d(\Gamma^n\mathcal{X},\mathcal{Z}),\chi(\Gamma^n\mathcal{X}),\chi(\mathcal{Z}))-\Upsilon(0,\chi(\Gamma^n\mathcal{X}),\chi(\mathcal{Z})))]^L\\
&= [\lambda(\Upsilon(0,0,0))]^k\\
&= 1.
\end{aligned} \tag{10}$$

Therefore, $\lambda(d(\mathcal{Z},\Gamma\mathcal{Z})) = 1$ which implies $d(\mathcal{Z},\Gamma\mathcal{Z}) = 0$, that is, $\mathcal{Z}$ is a fixed point of Γ.

Taking $\Upsilon(a,b,c) = a+b+c$ and $\chi \equiv 0$ in the above theorem, we have the following. □

Corollary 2. *Let (Π,d) be a complete metric space and $\lambda \in \Lambda$. Assume that*

(i) There exist $k \in (0,1)$ and $L \ge 0$ such that

$$\begin{aligned}\lambda(\Upsilon(d(\Gamma\mathcal{X},\Gamma\mathcal{Y}),\chi(\Gamma\mathcal{X}),\chi(\Gamma\mathcal{Y}))) &\le [\lambda(\Upsilon(d(\mathcal{X},\mathcal{Y})))]^k\\
&\cdot[\lambda(\Upsilon(d(\mathcal{Y},\Gamma\mathcal{X})))]^L,\end{aligned} \tag{11}$$

for all $\mathcal{X},\mathcal{Y} \in \Pi$.

Then, the following assertions hold:

(i) $F(\Gamma)\subseteq Z_\chi$

(ii) Γ is a χ-Picard mapping

Remark 2. Note that the advantage of Corollary 2 is that we can choose the power $L = 0$ to obtain Corollary 1. That is, Corollary 2 is more general than Corollary 1. Also, by taking different functions λ, we can obtain many contractive conditions in Theorems 4 and 5.

Next illustrative example is furnished which demonstrates the validity of the hypotheses and degree of utility of Theorem 6 while previous results in the literature are not applicable.

Example 2. Let $\Pi = \{\tau_n | n \in \mathbb{N}\cup\{0\}\}$, where $\tau_n = ((n(n+1))/2)$ for all $n \in \mathbb{N}$ and $\tau_0 = 0$. Obviously, (Π,d) is a complete metric space with the metric $d\colon \Pi\times\Pi \longrightarrow [0,\infty)$ defined by $d(\mathcal{X},\mathcal{Y}) = |\mathcal{X}-\mathcal{Y}|$ for all $\mathcal{X},\mathcal{Y} \in \Pi$. Define a mapping $\Gamma\colon \Pi \longrightarrow \Pi$ by $\Gamma\tau_n = \tau_{n-1}$ for all $n \in \mathbb{N}$ and $\Gamma\tau_0 = \tau_0 = 0$. Then, Γ is not a Banach contraction mapping, since

$$\lim_{n\longrightarrow\infty}\frac{d(\Gamma\tau_n,\Gamma\tau_1)}{d(\tau_n,\tau_1)} = \lim_{n\longrightarrow\infty}\frac{\tau_{n-1}-\tau_0}{\tau_n-\tau_1} = \lim_{n\longrightarrow\infty}\frac{((n(n-1))/2)}{((n(n+1))/2)-1} = 1. \tag{12}$$

Therefore, the BCP cannot be applied in this example.

Now, we define a function $\lambda \in \Lambda$ by $\lambda(t) = e^{\sqrt{te^t}}$ for all $t \in [0,\infty)$ and a function $\Upsilon \in \overline{\Upsilon}$ by $\Upsilon(a,b,c) = a+b+c$ for all $a,b,c \in [0,\infty)$. Also, we define a function $\chi\colon \Pi \longrightarrow [0,\infty)$ by $\chi(\mathcal{X}) = \mathcal{X}$ for all $\mathcal{X} \in \Pi$. We shall show that Γ is a λ-(Υ,χ)-contraction mapping. For any $m,n \in \mathbb{N}$ with $n > m$, we have

$$\begin{aligned}
&\frac{d(\Gamma\tau_n,\Gamma\tau_m)+\Gamma\tau_n+\Gamma\tau_m}{d(\tau_n,\tau_m)+\tau_n+\tau_m}e^{d(\Gamma\tau_n,\Gamma\tau_m)+\Gamma\tau_n+\Gamma\tau_m-[d(\tau_n,\tau_m)+\tau_n+\tau_m]}\\
&=\frac{((n(n-1))/2)-((m(m-1))/2)+((n(n-1))/2)+((m(m-1))/2)}{((n(n+1))/2)-((m(m+1))/2)+((n(n+1))/2)+((m(m+1))/2)}\\
&\quad\cdot e^{((n(n-1))/2)-((m(m-1))/2)+((n(n-1))/2)+((m(m-1))/2)-[((n(n+1))/2)-((m(m+1))/2)+((n(n+1))/2)+((m(m+1))/2)]}\\
&=\frac{n(n-1)}{n(n+1)}e^{-2n}\\
&\le e^{-2}.
\end{aligned} \tag{13}$$

Putting $k = e^{-1}$, the above inequality is equivalent to

$$\left(\mathrm{d}\left(\Gamma\tau_n, \Gamma\tau_m\right) + \Gamma\tau_n + \Gamma\tau_m\right)e^{\mathrm{d}\left(\Gamma\tau_n,\Gamma\tau_m\right)+\Gamma\tau_n+\Gamma\tau_m} \le k^2\left(\mathrm{d}\left(\tau_n, \tau_m\right) + \tau_n + \tau_m\right)e^{\mathrm{d}\left(\tau_n,\tau_m\right)+\tau_n+\tau_m}, \tag{14}$$

or equivalently

$$e^{\sqrt{\left(\mathrm{d}\left(\Gamma\tau_n,\Gamma\tau_m\right)+\Gamma\tau_n+\Gamma\tau_m\right)e^{\mathrm{d}\left(\Gamma\tau_n,\Gamma\tau_m\right)+\Gamma\tau_n+\Gamma\tau_m}}} \le e^{k\sqrt{\mathrm{d}\left(\tau_n,\tau_m\right)+\tau_n+\tau_m e^{\mathrm{d}\left(\tau_n,\tau_m\right)+\tau_n+\tau_m}}}. \tag{15}$$

Therefore, we obtain

$$\lambda\left(\Upsilon\left(\mathrm{d}\left(\Gamma\tau_n, \Gamma\tau_m\right), \chi\left(\Gamma\tau_n\right), \chi\left(\Gamma\tau_m\right)\right)\right) \le \left[\lambda\left(\Upsilon\left(\mathrm{d}\left(\tau_n, \tau_m\right), \chi\left(\tau_n\right), \chi\left(\tau_m\right)\right)\right)\right]^k. \tag{16}$$

Then, all hypotheses of Theorem 7 hold and so Γ has a unique χ-fixed point. Here, $\tau_0 = 0$ is the unique χ-fixed point of Γ.

3. Applications of Theoretical Results

In this section, we give two applications of our main results in the previous section. These applications consist of two parts. The first part is related to the fixed point results in partial metric spaces. The second part shows the application of theoretical results to solve the nonlinear integral equation.

Theorem 6. *Let $(\Pi, \mathscr{W})$ be a complete partial metric space and $\Gamma\colon \Pi \longrightarrow \Pi$ be a mapping such that*

$$\lambda\left(\mathscr{W}\left(\Gamma\mathscr{X}, \Gamma\mathscr{Y}\right)\right) \le \left[\lambda\left(\mathscr{W}\left(\mathscr{X}, \mathscr{Y}\right)\right)\right]^k, \tag{17}$$

for all $\mathscr{X}, \mathscr{Y} \in \Pi$, where $k \in (0, 1)$. Then, Γ has a unique fixed point $\mathscr{Z}$. Moreover, $\chi(\mathscr{Z}) = 0$ implies $\mathscr{W}(\mathscr{Z}, \mathscr{Z}) = 0$.

Proof. Define a metric $d_{\mathscr{W}}\colon \Pi \times \Pi \longrightarrow [0, \infty)$ by

$$\mathrm{d}_{\mathscr{W}}\left(\mathscr{X}, \mathscr{Y}\right) = 2\mathscr{W}\left(\mathscr{X}, \mathscr{Y}\right) - \mathscr{W}\left(\mathscr{X}, \mathscr{X}\right) - \mathscr{W}\left(\mathscr{Y}, \mathscr{Y}\right), \tag{18}$$

for all $\mathscr{X}, \mathscr{Y} \in \Pi$. In addition, we define a new metric $d\colon \Pi \times \Pi \longrightarrow [0, \infty)$ by

$$\mathrm{d}\left(\mathscr{X}, \mathscr{Y}\right) = \left(\left(d_{\mathscr{W}}\left(\mathscr{X}, \mathscr{Y}\right)\right)/2\right), \tag{19}$$

for all $\mathscr{X}, \mathscr{Y} \in \Pi$. Also, we set a function $\chi\colon \Pi \longrightarrow [0, \infty)$ and a function $\Upsilon \in \overline{\Upsilon}$ by

$$\begin{aligned} \chi\left(\mathscr{X}\right) &= \frac{\mathscr{W}\left(\mathscr{X}, \mathscr{X}\right)}{2}, \quad \text{for all } \mathscr{X} \in \Pi, \\ \Upsilon\left(a, b, c\right) &= a + b + c, \quad \text{for all } a, b, c \in [0, \infty). \end{aligned} \tag{20}$$

Then, from (17), we have

$$\begin{aligned} &\lambda\left(\mathrm{d}\left(\Gamma\mathscr{X}, \Gamma\mathscr{Y}\right) + \mathscr{W}\left(\Gamma\mathscr{X}, \Gamma\mathscr{X}\right) + \mathscr{W}\left(\Gamma\mathscr{Y}, \Gamma\mathscr{Y}\right)\right) \\ &\quad \le \left[\lambda\left(\mathrm{d}\left(\mathscr{X}, \mathscr{Y}\right) + \mathscr{W}\left(\mathscr{X}, \mathscr{X}\right) + \mathscr{W}\left(\mathscr{Y}, \mathscr{Y}\right)\right)\right]^k, \end{aligned} \tag{21}$$

for all $\mathscr{X}, \mathscr{Y} \in \Pi$. It yields that

$$\lambda\left(\Upsilon\left(\mathrm{d}\left(\Gamma\mathscr{X}, \Gamma\mathscr{Y}\right), \chi\left(\Gamma\mathscr{X}\right), \chi\left(\Gamma\mathscr{Y}\right)\right)\right) \le \left[\lambda\left(\Upsilon\left(\mathrm{d}\left(\mathscr{X}, \mathscr{Y}\right), \chi\left(\mathscr{X}\right), \chi\left(\mathscr{Y}\right)\right)\right)\right]^k, \tag{22}$$

for all $\mathscr{X}, \mathscr{Y} \in \Pi$. By Theorem 7, Γ has a unique χ-fixed point $\mathscr{Z}$. It implies that Γ has a unique fixed point $\mathscr{Z} \in \Pi$. Moreover, $\chi(\mathscr{Z}) = 0$ implies $\mathscr{W}(\mathscr{Z}, \mathscr{Z}) = 0$.

Based on the proof of the above theorem and Theorem 5, we get the following result. □

Theorem 7. *Let $(\Pi, \mathscr{W})$ be a compete partial metric space and $\Gamma\colon \Pi \longrightarrow \Pi$ be a mapping such that*

$$\lambda\left(\mathscr{W}\left(\Gamma\mathscr{X}, \Gamma\mathscr{Y}\right)\right) \le \left[\lambda\left(\mathscr{W}\left(\mathscr{X}, \mathscr{Y}\right)\right)\right]^k \left[\lambda\left(\mathscr{W}\left(\mathscr{Y}, \Gamma\mathscr{X}\right) - \mathscr{W}\left(\mathscr{Y}, \mathscr{Y}\right) - \mathscr{W}\left(\Gamma\mathscr{X}, \Gamma\mathscr{X}\right)\right)\right]^L, \tag{23}$$

for all $\mathscr{X}, \mathscr{Y} \in \Pi$, where $k \in (0, 1)$. Then, Γ has a fixed point $\mathscr{Z}$. Moreover, $\mathscr{W}(\mathscr{Z}, \mathscr{Z}) = 0$.

Next, we will consider the following nonlinear integral equation:

$$\mathcal{X}(t) = \phi(t) + \int_a^b \mathcal{Q}(t, s, \mathcal{X}(s))\mathrm{d}s, \tag{24}$$

where $a, b \in \mathbb{R}$, $\mathcal{X} \in C[a,b]$ (the set of all continuous functions from $[a,b]$ to $\mathbb{R}$), and $\phi\colon [a,b] \longrightarrow \mathbb{R}$ and $\mathcal{Q}\colon [a,b] \times [a,b] \times \mathbb{R} \longrightarrow \mathbb{R}$ are two given functions.

Theorem 8. *Consider integral equation (24). Suppose that the following conditions hold:*

(i) $\mathcal{Q}\colon [a,b] \times [a,b] \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and

$$\lambda\left(\int_a^b |\mathcal{Q}(t,s,\alpha)\mathrm{d}s - \mathcal{Q}(t,s,\beta)|\mathrm{d}s\right) \leq \int_a^b \lambda(|\mathcal{Q}(t,s,\alpha) - \mathcal{Q}(t,s,\beta)|)\mathrm{d}s, \tag{25}$$

for all $t, s \in [a,b]$ and for all $\alpha, \beta \in \mathbb{R}$.

(ii) There exist $\lambda \in \Lambda$ and $k \in (0,1)$ such that

$$\lambda(|\mathcal{Q}(t,s,\alpha) - \mathcal{Q}(t,s,\beta)|) \leq \frac{[\lambda(|\alpha - \beta|)]^k}{b - a}, \tag{26}$$

for all $t, s \in [a,b]$ and for all $\alpha, \beta \in \mathbb{R}$.

Then, integral equation (24) has a unique solution.

Proof. Let $\Pi = C[a,b]$. Define the metric d on Π by $\mathrm{d}(\mathcal{X}, \mathcal{Y}) = \sup_{t \in [a,b]} |\mathcal{X}(t) - \mathcal{Y}(t)|$ for all $\mathcal{X}, \mathcal{Y} \in \Pi$. Then, (Π, d) is a complete metric space. Consider a mapping $\Gamma\colon \Pi \longrightarrow \Pi$ defined by $(\Gamma\mathcal{X})(t) = \phi(t) + \int_a^t \mathcal{Q}(t,s,\mathcal{X}(s))\mathrm{d}s$ for all $\mathcal{X} \in \Pi$. Define the control function $\Upsilon\colon [0,\infty)^3 \longrightarrow [0,\infty)$ by $\Upsilon(a,b,c) = a + b + c$ for all $a, b, c \in [0,\infty)$. Also, define $\chi\colon \Pi \longrightarrow [0,\infty)$ by $\chi(\mathcal{X}) = 0$ for all $\mathcal{X} \in \Pi$. Let $\mathcal{X}, \mathcal{Y} \in \Pi$ and $t \in [a,b]$. Then, we have

$$\begin{aligned}
\lambda(|\Gamma\mathcal{X}(t) - \Gamma\mathcal{Y}(t)|) &= \lambda\left(\left|\int_a^t \mathcal{Q}(t,s,\mathcal{X}(s))\mathrm{d}s - \int_a^t \mathcal{Q}(t,s,\mathcal{Y}(s))\mathrm{d}s\right|\right) \\
&\leq \int_a^b \lambda(|\mathcal{Q}(t,s,\mathcal{X}(s)) - \mathcal{Q}(t,s,\mathcal{Y}(s))|)\mathrm{d}s \\
&\leq \int_a^b \frac{[\lambda(|\mathcal{X}(s) - \mathcal{Y}(s)|)]^k}{b-a}\mathrm{d}s \\
&\leq \frac{1}{b-a}\int_a^b [\lambda(d(\mathcal{X},\mathcal{Y}))]^k \mathrm{d}s \\
&\leq [\lambda(d(\mathcal{X},\mathcal{Y}))]^k.
\end{aligned} \tag{27}$$

Since $\chi(\mathcal{X}) = 0$ for all $\mathcal{X} \in \Pi$, we get

$$\lambda(\mathrm{d}(\Gamma\mathcal{X}, \Gamma\mathcal{Y}) + \chi(\Gamma\mathcal{X}) + \chi(\Gamma\mathcal{Y})) \leq [\lambda(\mathrm{d}(\mathcal{X},\mathcal{Y}) + \chi(\mathcal{X}) + \chi(\mathcal{Y}))]^k. \tag{28}$$

Therefore,

$$\lambda(\Upsilon(\mathrm{d}(\Gamma\mathcal{X}, \Gamma\mathcal{Y}), \chi(\Gamma\mathcal{X}), \chi(\Gamma\mathcal{Y}))) \leq [\lambda(\Upsilon(\mathrm{d}(\mathcal{X},\mathcal{Y}), \chi(\mathcal{X}), \chi(\mathcal{Y})))]^k. \tag{29}$$

Thus, $\Gamma\colon \Pi \longrightarrow \Pi$ is a λ-(Υ, χ)-contraction mapping. By Theorem 7, Γ has a unique χ-fixed point $\mathcal{X} \in \Pi$, that is, $(\Gamma\mathcal{X})(t) = \mathcal{X}(t)$ for all $t \in [a,b]$ and $\chi(\mathcal{X}) = 0$ which means that integral equation (24) has a unique solution. □

4. Conclusions

In this paper, we obtained some fixed point results first in a metric space and then in a partial metric space as results. The famous Banach contraction principle is a special case of our results. There are other terms such as $\mathrm{d}(\mathcal{X}, \Gamma\mathcal{Y})$, $\mathrm{d}(\mathcal{Y}, \Gamma\mathcal{Y})$, and $\mathrm{d}(\mathcal{X}, \Gamma\mathcal{X})$ which we can consider in future research. But, certainly, we should also work with other control functions. For more details in this direction, the readers can refer to [6].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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