Hindawi
Journal of Mathematics
Volume 2021, Article ID 6629203, 10 pages
https://doi.org/10.1155/2021/6629203

Research Article

Generalized Complementarity Problem with Three Classes of Generalized Variational Inequalities Involving ⊕ Operation

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Received 11 November 2020; Accepted 8 January 2021; Published 25 January 2021

Academic Editor: Sun Young Cho

In this study, we introduce and study a generalized complementarity problem involving XOR operation and three classes of generalized variational inequalities involving XOR operation. Under certain appropriate conditions, we establish equivalence between them. An iterative algorithm is defined for solving one of the three generalized variational inequalities involving XOR operation. Finally, an existence and convergence result is proved, supported by an example.

1. Introduction

It is well known that the many unrelated free boundary value problems related to mathematical and engineering sciences can be solved by using the techniques of variational inequalities. In a variational inequality formulation, the location of the free boundary becomes an intrinsic part of the solution, and no special devices are needed to locate it. Complementarity theory is an equally important area of operations research and application oriented. The linear as well as nonlinear programs can be distinguished by a family of complementarity problems. The complementarity theory have been elongated for the purpose of studying several classes of problems occurring in fluid flow through porous media, economics, financial mathematics, machine learning, optimization, and transportation equilibrium, for example, [1–5].

The correlations between the variational inequality problem and complementarity problem were recognized by Lions [6] and Mancino and Stampacchia [7]. However, Karamardian [8, 9] showed that both the problems are equivalent if the convex set involved is a convex cone. For more details on variational inequalities and complementarity problems, refer to [6, 10–12].

The exclusive "XOR," sometimes also exclusive disjunction (short: XOR) or antivalence, is a Boolean operation which only outputs true if only exactly one of its both inputs is true (so, if both inputs differ). There are many applications of XOR terminology, that is, it is used in cryptography, gray codes, parity, and CRC checks. Commonly, the ⊕ symbol is used to denote the XOR operation. Some problems related to variational inclusions involving XOR operation were studied by [13–16].

Influenced by the applications of all the above discussed concepts in this study, we introduce and study a generalized complementarity problem involving XOR operation with three classes of generalized variational inequalities involving XOR operation. Some equivalence relations are established between them. An existence and convergence result is proved for one of the three types of generalized variational inequalities involving XOR operation. For illustration, an example is provided.

2. Some Basic Concepts and Formulation of the Problem

Throughout this study, we assume E to be real ordered Banach space with norm $\|\cdot\|$ and E^* be its dual space. Suppose that d is the metric induced by the norm, 2^E (respectively, $\mathrm{CB}(E)$) is the family of nonempty (respectively, closed and bounded) subsets of E. The Hausdorff metric $D(.,.)$ on $CB(E)$ is defined as

$$D(A,\mathscr{B}) = \max\left\{\sup_{x\in\mathscr{A}} d(x,\mathscr{B}), \sup_{y\in\mathscr{B}} d(\mathscr{A},y)\right\}, \quad \forall \mathscr{A},\mathscr{B} \in CB(E), \tag{1}$$

where $d(x,\mathscr{B}) = \inf_{y\in\mathscr{B}} d(x,y)$, and $d(\mathscr{A},y) = \inf_{x\in\mathscr{A}} d(x,y)$.

Let C be a pointed closed convex positive cone in E, and $\langle t,x\rangle$ denotes the value of the linear continuous function $t\in E^*$ at x.

The following definitions and concepts are required to achieve the goal of this study, and most of them can be found in [17, 18].

Definition 1. The relation "$\leq$" is called the partial order relation induced by the cone C, that is, $x\leq y$ if and only if $y-x\in C$.

Definition 2. For arbitrary elements $x,y\in E$, if $x\leq y$ (or $y\leq x$) holds, then x and y are said to be comparable to each other (denoted by $x\propto y$).

Definition 3. For arbitrary elements $x,y\in E$, lub$\{x,y\}$ and glb$\{x,y\}$ mean the least upper bound and the greatest upper bound of the set $\{x,y\}$. Suppose lub$\{x,y\}$ and glb$\{x,y\}$ exist, then some binary operations are defined as

(i) $x\vee y = \text{lub}\{x,y\}$
(ii) $x\wedge y = \text{glb}\{x,y\}$
(iii) $x\oplus y = (x-y)\vee(y-x)$
(iv) $x\odot y = (x-y)\wedge(y-x)$

The operations $\vee,\wedge,\oplus$, and $\odot$ are called OR, AND, XOR, and XNOR operations, respectively.

Proposition 1. *Let $\oplus$ be an XOR operation and $\odot$ be an XNOR operation. Then, the following relations hold:*

(i) $x\odot x = 0$, $x\odot y = y\odot x$
(ii) if $x\propto 0$, *then* $-x\oplus 0\leq x\leq x\oplus 0$
(iii) $0\leq x\oplus y$, *if* $x\propto y$
(iv) If $x\propto y$, *then* $x\oplus y = 0$ *if and only if* $x = y$
(v) $x\oplus y = y\oplus x$
(vi) $x\oplus x = 0$
(vii) $0\leq x\oplus 0$
(viii) If $x\leq y$ *and* $u\leq v$, *then* $(x+u)\leq(y+v)$
(ix) If $x\propto y$, *then* $(x\oplus 0)\oplus(y\oplus 0)\leq(x\oplus y)\oplus 0 = x\oplus y$, *for all* $x,y,u,v\in E$ *and* $\lambda\in\mathbb{R}$

Proposition 2. *Let C be a cone in E; then, for each $x,y\in E$, the following relations hold:*

(i) $\|0\oplus 0\| = \|0\| = 0$
(ii) $\|x\vee y\|\leq\|x\|\vee\|y\|\leq\|x\|+\|y\|$
(iii) $\|x\oplus y\|\leq\|x-y\|$
(iv) If $x\propto y$, *then* $\|x\oplus y\| = \|x-y\|$

Definition 4. Let $A\colon E\longrightarrow E$ be a single-valued mapping, then

(i) A is said to be a comparison mapping, if $x\propto y$, then $A(x)\propto A(y)$, $x\propto A(x)$, and $y\propto A(y)$, for all $x,y\in E$
(ii) A is said to be a strongly comparison mapping, if A is a comparison mapping and $A(x)\propto A(y)$, if and only if $x\propto y$, for any $x,y\in E$

Definition 5. Let $f\colon E\longrightarrow\mathbb{R}\cup\{+\infty\}$ be a proper functional. A vector $\omega^*\in E^*$ is called subgradient of f at $x\in$ domf, if

$$\langle\omega^*, y-x\rangle\leq f(y)-f(x), \quad \text{for all } y\in E. \tag{2}$$

The set of all subgradients of f at x is denoted by $\partial f(x)$. The mapping $\partial f: E\longrightarrow 2^{E^*}$ defined by

$$\partial f(x) = \{\omega^*\in E^*\colon \langle\omega^*, y-x\rangle\leq f(y)-f(x), \quad \text{for all } y\in E\} \tag{3}$$

is called subdifferential of f.

Definition 6. The resolvent operator $\mathscr{J}_\rho^{\partial f}$ associated with ∂f is given by

$$\mathscr{J}_\rho^{\partial f}(x) = [I+\rho\partial f]^{-1}(x), \quad \text{for all } x\in E, \tag{4}$$

where $\rho>0$ is a constant, and I is the identity operator.

It is well known that the resolvent operator $\mathscr{J}_\rho^{\partial f}$ is single-valued as well as nonexpansive.

Definition 7. A mapping $f\colon C\longrightarrow\mathbb{R}$ is said to be

(i) Positive homogeneous if, for all $\alpha>0$ and $x\in C$, $f(\alpha x) = \alpha f(x)$
(ii) Convex, if $x,y\in C$ and all $\lambda\in[0,1]$

$$f(\lambda x+(1-\lambda)y)\leq\lambda f(x)+(1-\lambda)f(y). \tag{5}$$

Definition 8. A multivalued mapping $F\colon C\longrightarrow 2^{E^*}\smallsetminus\{\varnothing\}$ is said to be

(i) Upper semicontinuous at $x\in C$ if, for every open set V containing $F(x)$, there exists an open set U containing x such that $F(U)\subseteq V$, where E^* is equipped with ω^* topology
(ii) Upper semicontinuous on C if it is upper semicontinuous at every point of C

(iii) Upper hemicontinuous on C if its restriction to line segments of C is upper semicontinuous

(iv) Monotone if, for every $x, y \in C$

$$\langle t_1 - t_2, y - x\rangle \geq 0, \quad \text{for all } t_1 \in F(y), t_2 \in F(x). \tag{6}$$

Definition 9. A multivalued mapping $F\colon E \longrightarrow 2^E$ is said to be D-Lipschitz continuous, if there exists a constant $\lambda_{D_F} > 0$ such that

$$D(F(x), F(y)) \leq \lambda_{D_F}\|x - y\|, \quad \text{for all } x, y \in E. \tag{7}$$

Definition 10. A multivalued mapping $F\colon E \longrightarrow 2^E$ is said to be relaxed Lipschitz continuous, if there exists a constant $k > 0$ such that

$$\langle w_1 - w_2, x - y\rangle \leq -k\|x - y\|^2, \quad \text{for all } w_1 \in F(x), w_2 \in F(y). \tag{8}$$

Let $F\colon C \longrightarrow 2^{E^*}\setminus\{\varnothing\}$ be a multivalued mapping with nonempty values and $f\colon C \longrightarrow \mathbb{R}\cup\{+\infty\}$ be a proper functional. We consider the following generalized complementarity problem involving XOR operation.

Find $\overline{x} \in C$, $\overline{t} \in F(\overline{x})$ such that

$$\begin{gathered}\langle \overline{t}, t\overline{x}\rangle \oplus f(\overline{x}) = 0,\\ \langle \overline{t}, ty\rangle \oplus f(y) \geq 0,\\ \forall y \in C.\end{gathered} \tag{9}$$

We denote by $S_{C\oplus}$ the solution set of generalized complementarity problem involving XOR operation (9).

We mention some special cases of problem (9) as follows.

(i) If we replace $\oplus$ by $+$ and f by $f\colon C \longrightarrow \mathbb{R}$, then problem (9) reduces to the problem of finding $\overline{x} \in C$ and $\overline{t} \in F(\overline{x})$ such that

$$\begin{gathered}\langle \overline{t}, t\overline{x}\rangle + f(\overline{x}) = 0,\\ \langle \overline{t}, ty\rangle + f(y) \geq 0,\\ \forall y \in C.\end{gathered} \tag{10}$$

Problem (10) is called generalized f complementarity problem, introduced and studied by Huang et al. [19].

(ii) If $f \equiv 0$, then problems (9) as well as (10) reduce to the problem of finding $\overline{x} \in C$ and $\overline{t} \in F(\overline{x})$ such that

$$\begin{gathered}\langle \overline{t}, t\overline{x}\rangle = 0,\\ \langle \overline{t}, ty\rangle \geq 0,\\ \forall y \in C.\end{gathered} \tag{11}$$

Problem (11) can be found in [20, 21].

We remark that for suitable choices of operators involved in the formulation of (9), a number of known complementarity problems can be obtained easily, for example, [17, 22–24].

Simultaneously, we also study the following three types of generalized variational inequalities involving XOR operation.

(1) Find $\overline{x} \in C$ such that

$$\exists \overline{t} \in F(\overline{x}), \quad \forall y \in C\colon \langle \overline{t}, tyn - q\overline{x}\rangle \oplus (f(y) - f(\overline{x})) \geq 0; \tag{12}$$

(2) Find $\overline{x} \in C$ such that

$$\forall y \in C, \quad \exists \overline{t} \in F(\overline{x})\colon \langle \overline{t}, tyn - q\overline{x}\rangle \oplus (f(y) - f(\overline{x})) \geq 0; \tag{13}$$

(3) Find $\overline{x} \in C$ such that

$$\forall y \in C, \quad \forall t \in F(y)\colon \langle t, y - \overline{x}\rangle \oplus (f(y) - f(\overline{x})) \geq 0. \tag{14}$$

We denote the solution set of (12) by $S_{1\oplus}$, (13) by $S_{2\oplus}$, and (14) by $S_{3\oplus}$.

Many known variational inequality problems can be obtained from problems (12)–(14), for example, [25–29] and the references therein.

3. Equivalence Results

We establish the equivalence among problems (9), (12)–(14). First, we establish the equivalence between generalized complementarity problem involving XOR operation (9) and generalized variational inequality problem involving XOR operation (12).

Theorem 1. *Let $F\colon C \longrightarrow 2^{E^*}\setminus\{\varnothing\}$ be a multivalued mapping with nonempty values and $f\colon C \longrightarrow \mathbb{R}\cup\{+\infty\}$ be a proper functional. Then, the following statements are true:*

(i) If $\langle \overline{t}, t\overline{x}\rangle \propto f(\overline{x})$, then $S_{C\oplus} \subseteq S_{1\oplus}$

(ii) If f is positive homogeneous, then $S_{1\oplus} \subseteq S_{C\oplus}$

Proof

(i) Let $\overline{x} \in S_{C\oplus}$, then $\overline{x} \in C$, and there exists $\overline{t} \in F(\overline{x})$ such that

$$\begin{gathered}\langle \overline{t}, t\overline{x}\rangle \oplus f(\overline{x}) = 0,\\ \langle \overline{t}, ty\rangle \oplus f(y) \geq 0.\end{gathered} \tag{15}$$

Since $\langle \overline{t}, t\overline{x}\rangle \propto f(\overline{x})$, by (iv) of Proposition 1, we have

$$\begin{gathered}\langle \overline{t}, t\overline{x}\rangle = f(\overline{x}),\\ \text{Also as } \langle \overline{t}, ty\rangle \oplus f(y) \geq 0,\\ \langle \overline{t}, ty\rangle \oplus f(y) \oplus f(y) \geq 0 \oplus f(y),\end{gathered} \tag{16}$$

which implies that

$$\langle \bar{t}, ty\rangle \geq f(y). \tag{17}$$

By using (16) and (17), we have

$$\begin{aligned}&\langle \bar{t}, tyn - q\bar{x}\rangle = \langle \bar{t}\\&\langle \bar{t}, tyn - q\bar{x}\rangle \oplus (f(y) - f(\bar{x})) \geq (f(y) - f(\bar{x}))\oplus(f(y) - f(\bar{x})),\end{aligned} \tag{18}$$

that is,

$$\langle \bar{t}, tyn - q\bar{x}\rangle \oplus (f(y) - f(\bar{x})) \geq 0, \tag{19}$$

which implies that $\bar{x} \in S_{1\oplus}$. So, we have $S_{C\oplus} \subseteq S_{1\oplus}$.

(ii) Let $\bar{x} \in S_{1\oplus}$, then $\bar{x} \in C$, and there exists $\bar{t} \in F(\bar{x})$ such that

$$\langle \bar{t}, tyn - q\bar{x}\rangle \oplus (f(y) - f(\bar{x})) \geq 0, \quad \forall y \in C. \tag{20}$$

Since C is a pointed closed convex positive cone, clearly $y = 2\bar{x} \in C$ and $y = (1/2)\bar{x} \in C$. Putting $y = 2\bar{x}$ in generalized variational inequality involving XOR operation (12) and using positive homogenity of f, we get

$$\begin{aligned}&\langle \bar{t}, tyn - q\bar{x}\rangle \oplus (f(y) - f(\bar{x})) \geq 0,\\&\langle \bar{t}, t2n\bar{x}q - h\bar{x}\rangle \oplus (f(2\bar{x}) - f(\bar{x})) \geq 0,\\&\langle \bar{t}, t\bar{x}\rangle \oplus f(\bar{x}) \geq 0.\end{aligned} \tag{21}$$

Now, putting $y = (1/2)\bar{x}$ in generalized variational inequality involving XOR operation ((12)) and using positive homogenity of f, we get

$$\begin{aligned}&\langle \bar{t}, tyn - q\bar{x}\rangle \oplus (f(y) - f(\bar{x})) \geq 0,\\&\langle \bar{t}, tyn - q\bar{x}\rangle \oplus (f(y) - f(\bar{x}))\oplus(f(y) - f(\bar{x})) \geq 0\oplus(f(y) - f(\bar{x})),\end{aligned} \tag{22}$$

which implies that

$$\begin{aligned}&\langle \bar{t}, y - \bar{x}\rangle \geq (f(y) - f(\bar{x})),\\&\left\langle \bar{t}, \frac{1}{2}\bar{x} - \bar{x}\right\rangle \geq \left(f\left(\frac{1}{2}\bar{x}\right) - f(\bar{x})\right),\\&\left\langle \bar{t}, -\frac{1}{2}\bar{x}\right\rangle \geq -\frac{1}{2}f(\bar{x}),\end{aligned} \tag{23}$$

thus,

$$\begin{aligned}&\langle \bar{t}, t\bar{x}\rangle \leq f(\bar{x}),\\&\langle \bar{t}, t\bar{x}\rangle \oplus f(\bar{x}) \leq f(\bar{x})\oplus f(\bar{x}) = 0,\end{aligned} \tag{24}$$

that is,

$$\langle \bar{t}, t\bar{x}\rangle \oplus f(\bar{x}) \leq 0. \tag{25}$$

Combining (21) and (25), we have

$$\langle \bar{t}, t\bar{x}\rangle \oplus f(\bar{x}) = 0. \tag{26}$$

From generalized variational inequality involving XOR operations (12) and (16), we have

$$\begin{aligned}&\langle \bar{t}, tyn - q\bar{x}\rangle \oplus (f(y) - f(\bar{x})) \geq 0,\\&\langle \bar{t}, tyn - q\bar{x}\rangle \oplus ((f(y) - f(\bar{x}))\oplus(f(y) - f(\bar{x}))) \geq 0\oplus(f(y) - f(\bar{x})),\end{aligned} \tag{27}$$

which implies that

$$\begin{aligned}&\langle \bar{t}, tyn - q\bar{x}\rangle \oplus 0 \geq 0\oplus(f(y) - f(\bar{x})),\\&\langle \bar{t}, tyn - q\bar{x}\rangle \geq (f(y) - f(\bar{x})),\\&\langle \bar{t}, ty\rangle - \langle \bar{t}\\&\langle \bar{t}, ty\rangle - f(\bar{x}) \geq f(y) - f(\bar{x}),\\&\langle \bar{t}, ty\rangle \geq f(y),\\&\langle \bar{t}, ty\rangle \oplus f(y) \geq f(y)\oplus f(y) = 0,\end{aligned} \tag{28}$$

thus, we have $\langle \bar{t}, ty\rangle \oplus f(y) \geq 0$. So, we have $\bar{x} \in S_{C\oplus}$. That is, $S_{1\oplus} \subseteq S_{C\oplus}$. □

Theorem 2. *The following statements are true.*

(i) $S_{1\oplus} \subseteq S_{2\oplus}$

(ii) If F is monotone, then $S_{2\oplus} \subseteq S_{3\oplus}$

(iii) If F is upper hemicontinuous and f is convex, then $S_{3\oplus} \subseteq S_{2\oplus}$

Proof

(i) Is trivial

(ii) Let $\bar{x} \in S_{2\oplus}$. Then, for all $y \in C$, there exists $\bar{t} \in F(\bar{x})$ such that

$$\langle \bar{t}, tyn - q\bar{x}\rangle \oplus (f(y) - f(\bar{x})) \geq 0. \tag{29}$$

Since F is monotone, for every $y \in C, t \in F(y)$, and using the above inequality, we have

$$\begin{aligned}&\langle t - \bar{t}, y - \bar{x}\rangle \geq 0,\\&\langle t, y - \bar{x}\rangle \geq \langle \bar{t}, tyn - q\bar{x}\rangle,\\&\langle t, y - \bar{x}\rangle \oplus (f(y) - f(\bar{x})) \geq \langle \bar{t}, tyn - q\bar{x}\rangle \oplus (f(y) - f(\bar{x})) \geq 0,\end{aligned} \tag{30}$$

which implies that $\langle t, y - \bar{x}\rangle \oplus (f(y) - f(\bar{x})) \geq 0$. Thus, $\bar{x} \in S_{3\oplus}$.

(iii) Suppose that the conclusion is not true. Then, there exists $\bar{x} \in C$ such that $\bar{x} \in S_{3\oplus}$ and $\bar{x} \notin S_{2\oplus}$. Then, for some $y \in C$ and $t \in F(\bar{x})$, we have

$$\langle t, y - \bar{x}\rangle \oplus (f(y) - f(\bar{x})) < 0. \tag{31}$$

Since F is upper hemicontinuous and f is convex, setting $x_\lambda = \lambda y + (1 - \lambda)\bar{x}$ and taking $\lambda \longrightarrow 0$, we have

$$\begin{aligned}&\langle t_\lambda, y - \bar{x}\rangle \oplus (f(y) - f(\bar{x})) < 0, \quad \forall t_\lambda \in F(x_\lambda),\\&\langle t_\lambda, y - \bar{x}\rangle \oplus ((f(y) - f(\bar{x}))\oplus(f(y) - f(\bar{x}))) < 0\oplus(f(y) - f(\bar{x})),\end{aligned} \tag{32}$$

which implies that

$$\begin{aligned}\langle t_\lambda, y-\overline{x}\rangle &< (f(y)-f(\overline{x})),\\ \langle t_\lambda, x_\lambda-\overline{x}\rangle &< (f(x_\lambda)-f(\overline{x})),\\ \langle t_\lambda, x_\lambda-\overline{x}\rangle\oplus(f(x_\lambda)-f(\overline{x})) &< (f(x_\lambda)-f(\overline{x}))\oplus(f(x_\lambda)-f(\overline{x})),\end{aligned}\tag{33}$$

thus,

$$\langle t_\lambda, x_\lambda-\overline{x}\rangle\oplus(f(x_\lambda)-f(\overline{x}))<0,\tag{34}$$

which contradicts that $\overline{x}\in S_{3\oplus}$. Thus, $\overline{x}\in S_{2\oplus}$, and (iii) is true. □

Remark 1. If we replace ⊕ by + and dropping the concepts related to ⊕ operation, then with slight modification in Theorems 1 and 2, one can obtain some results of Huang et al. [19]. Additionally, for suitable choices of operators in Theorems 1 and 2, one can obtain some results of Farajzadeh and Harandi [30].

4. Existence and Convergence Result

In this section, we first establish the equivalence between the generalized variational inequality problem involving XOR operation (12) and a nonlinear equation. Based on this equivalence, we construct an iterative algorithm for solving generalized variational inequality problem involving XOR operation (12).

Lemma 1. *The generalized variational inequality problem involving XOR operation (12) admits a solution $(\overline{x}, t\overline{t})$, $\overline{x}\in C$ and $\overline{t}\in F(\overline{x})$, if and only if the following relation is satisfied:*

$$\overline{x}=\mathcal{J}_\rho^{\partial f}[\overline{x}+t\rho n\overline{t}],\tag{35}$$

where $\rho>0$ is a constant, $\mathcal{J}_\rho^{\partial f}=[I+\rho\partial f]^{-1}$ is the resolvent operator associated with f, and I is the identity operator.

Proof. From the definition of resolvent operator $\mathcal{J}_\rho^{\partial f}$ associated with f and relation (35), we have

$$\begin{aligned}\overline{x}&=\mathcal{J}_\rho^{\partial f}[\overline{x}\\ &=[I+\rho\partial f]^{-1}[\overline{x}\end{aligned}\tag{36}$$

which implies that $\overline{x}+\rho\overline{t}\in\overline{x}+\rho\partial f(\overline{x})$, that is,

$$\overline{t}\in\partial f(\overline{x}).\tag{37}$$

By the definition of subdifferential operator $\partial f(\overline{x})$ and (37), we have

$$(f(y)-f(\overline{x}))\geq\langle\overline{t}, tyn-q\overline{x}\rangle.\tag{38}$$

Using (vi) of Proposition 1, we have

$$\begin{aligned}&\langle\overline{t}, tyn-q\overline{x}\rangle\oplus(f(y)-f(\overline{x}))\geq\langle\overline{t}\\ &\langle\overline{t}, tyn-q\overline{x}\rangle\oplus(f(y)-f(\overline{x}))\geq 0.\end{aligned}\tag{39}$$

Thus, the generalized variational inequality problem involving XOR operation (12) is satisfied.

Conversely, suppose that generalized variational inequality problem involving XOR operation (12) is satisfied. That is,

$$\begin{aligned}&\langle\overline{t}, tyn-q\overline{x}\rangle\oplus(f(y)-f(\overline{x}))\geq 0,\\ &\langle\overline{t}, tyn-q\overline{x}\rangle\oplus\langle\overline{t}\end{aligned}\tag{40}$$

that is, $(f(y)-f(\overline{x}))\geq\langle\overline{t}, tyn-q\overline{x}\rangle$, which implies that

$$\begin{aligned}\overline{t}&\in\partial f(\overline{x}),\\ \rho\overline{t}&\in\rho\partial f(\overline{x}),\\ \overline{x}+\rho\overline{t}&\in\overline{x}+\rho\partial f\\ \overline{x}+\rho\overline{t}&\in[I+\rho\partial f](\overline{x}),\\ \overline{x}&=[I+\rho\partial f]^{-1}[\overline{x}\\ \overline{x}&=\mathcal{J}_\rho^{\partial f}[\overline{x}+\rho\overline{t}],\end{aligned}\tag{41}$$

that is, the relation (35) is satisfied.

Based on Lemma 1, we develop the following iterative algorithm for solving the generalized variational inequality problem involving XOR operation (12). □

Iterative Algorithm 1. Let $C\subset E$ be a pointed closed convex positive cone. Suppose that $\overline{t}_n\propto\overline{t}_{n-1}$, for $n=1,2,\ldots$. Let for $\overline{x}_0\in C$, there exists $t_0\in F(\overline{x}_0)$, such that

$$\overline{x}_1=(1-\alpha)\overline{x}_0+\alpha\mathcal{J}_\rho^{\partial f}[\overline{x}_0+\rho\overline{t_0}].\tag{42}$$

Since $\overline{t}_0\in F(\overline{x}_0)\in CB(E)$, by Nadler [31], there exists $\overline{t}_1\in F(\overline{x}_1)$, using (iv) of Proposition 2, and as $\overline{t}_0\propto\overline{t}_1$, we have

$$\|\overline{t}_0\oplus\overline{t}_1\|=\|\overline{t}_0-\overline{t}_1\|\leq D(F(\overline{x}_0), F(\overline{x}_1)).\tag{43}$$

Continuing this way, compute the sequences $\{\overline{x}_n\}$ and $\{\overline{t}_n\}$ by the following scheme:

$$\overline{x}_{n+1}=(1-\alpha)\overline{x}_n+\alpha\mathcal{J}_\rho^{\partial f}[\overline{x}_n+\rho\overline{t}_n],\tag{44}$$

$$\|\overline{t}_n\oplus\overline{t}_{n-1}\|=\|\overline{t}_n-\overline{t}_{n-1}\|\leq D(F(\overline{x}_n), F(\overline{x}_{n-1})),\tag{45}$$

for $n=1,2,\ldots$, where $\overline{x}_n\in C$, $\overline{t}_n\in F(\overline{x}_n)$ can be chosen arbitrarily, $\alpha\in[0,1]$, $D(.,.)$ is the Hausdorff metric on $CB(E)$, and $\rho>0$ is a constant.

Now, we prove our main result.

Theorem 3. *Let E be a real ordered Banach space and C be a pointed closed convex positive cone in E with partial ordering "$\leq$." Let $f\colon C\longrightarrow\mathbb{R}\cup\{+\infty\}$ be a functional such that the resolvent operator $\mathcal{J}_\rho^{\partial f}$ associated with f is strongly comparison and continuous. Suppose that $F\colon C\longrightarrow 2^{E^*}\setminus\{\varnothing\}$ is a multivalued mapping such that F is the relaxed Lipschitz continuous with constant $k>0$ and D-Lipschitz continuous with constant $\lambda_{D_F}>0$. Let $\overline{x}_n\propto\overline{x}_{n-1}$ and $\overline{t}_n\propto\overline{t}_{n-1}$, where $\overline{t}_n\in F(\overline{x}_n)$ and $\overline{t}_{n-1}\in F(\overline{x}_{n-1})$, $n=1,2,\ldots$, such that for $\rho>0$, the following condition is satisfied:*

$$\left|\rho-\frac{k}{\lambda_{D_F}^2}\right|<\frac{k}{\lambda_{D_F}^2}. \tag{46}$$

Then, the sequences $\{\bar{x}_n\}$ and $\{\bar{t}_n\}$ strongly converge to x^* and t^*, respectively, the solutions of generalized variational inequality problem involving XOR operation (12).

Proof. Since $\bar{x}_{n+1}\propto\bar{x}_n$, for $n=1,2,\ldots$, using (iii) of Proposition 1, we evaluate

$$\begin{aligned}0&\leq\bar{x}_{n+1}\oplus\bar{x}_n\\&=\left[(1-\alpha)\bar{x}_n+\alpha\mathcal{J}_\rho^{\partial f}\left[\bar{x}_n+\rho\bar{t}_n\right]\right]\oplus\left[(1-\alpha)\bar{x}_{n-1}+\alpha\mathcal{J}_\rho^{\partial f}\left[\bar{x}_{n-1}+\rho\bar{t}_{n-1}\right]\right]\\&\leq(1-\alpha)\left(\bar{x}_n\oplus\bar{x}_{n-1}\right)+\alpha\left[\mathcal{J}_\rho^{\partial f}\left[\bar{x}_n+\rho\bar{t}_n\right]\oplus\mathcal{J}_\rho^{\partial f}\left[\bar{x}_{n-1}+\rho\bar{t}_{n-1}\right]\right].\end{aligned} \tag{47}$$

From (47), it follows that

$$\begin{aligned}\left\|\bar{x}_{n+1}\oplus\bar{x}_n\right\|&=\left\|(1-\alpha)\left(\bar{x}_n\oplus\bar{x}_{n-1}\right)+\alpha\left[\mathcal{J}_\rho^{\partial f}\left[\bar{x}_n+\rho\bar{t}_n\right]\oplus\mathcal{J}_\rho^{\partial f}\left[\bar{x}_{n-1}+\rho\bar{t}_{n-1}\right]\right]\right\|\\&\leq(1-\alpha)\left\|\bar{x}_n\oplus\bar{x}_{n-1}\right\|+\alpha\left\|\mathcal{J}_\rho^{\partial f}\left[\bar{x}_n+\rho\bar{t}_n\right]\oplus\mathcal{J}_\rho^{\partial f}\left[\bar{x}_{n-1}+\rho\bar{t}_{n-1}\right]\right\|.\end{aligned} \tag{48}$$

As $\bar{x}_n\propto\bar{x}_{n-1}$, $\bar{t}_n\propto\bar{t}_{n-1}$, obviously, $\bar{x}_n+\rho\bar{t}_n\propto\bar{x}_{n-1}+\rho\bar{t}_{n-1}$, for $n=1,2,\ldots$. Since the resolvent operator $\mathcal{J}_\rho^{\partial f}$ is strongly comparison, we have

$$\mathcal{J}_\rho^{\partial f}\left[\bar{x}_n+\rho\bar{t}_n\right]\propto\mathcal{J}_\rho^{\partial f}\left[\bar{x}_{n-1}+\rho\bar{t}_{n-1}\right]. \tag{49}$$

Using above facts, (iv) of Proposition 2 and non-expansiveness of $\mathcal{J}_\rho^{\partial f}$, (48) becomes

$$\begin{aligned}\left\|\bar{x}_{n+1}-\bar{x}_n\right\|&\leq(1-\alpha)\left\|\bar{x}_n-\bar{x}_{n-1}\right\|+\alpha\left\|\mathcal{J}_\rho^{\partial f}\left[\bar{x}_n+\rho\bar{t}_n\right]-\mathcal{J}_\rho^{\partial f}\left[\bar{x}_{n-1}+\rho\bar{t}_{n-1}\right]\right\|\\&\leq(1-\alpha)\left\|\bar{x}_n-\bar{x}_{n-1}\right\|+\alpha\left\|\left[\bar{x}_n+\rho\bar{t}_n\right]-\left[\bar{x}_{n-1}+\rho\bar{t}_{n-1}\right]\right\|\\&=(1-\alpha)\left\|\bar{x}_n-\bar{x}_{n-1}\right\|+\alpha\left\|\bar{x}_n-\bar{x}_{n-1}+\rho\left(\bar{t}_n-\bar{t}_{n-1}\right)\right\|.\end{aligned} \tag{50}$$

Since the multivalued mapping F is the relaxed Lipschitz continuous with constant $k>0$, D-Lipschitz continuous with constant $\lambda_{D_F}>0$, and using (45) of Iterative Algorithm 1, we have

$$\begin{aligned}\left\|\bar{x}_n-\bar{x}_{n-1}+\rho\left(\bar{t}_n-\bar{t}_{n-1}\right)\right\|^2&=\left\|\bar{x}_n-\bar{x}_{n-1}\right\|^2+2\rho\langle\bar{t}_n-\bar{t}_{n-1},\bar{x}_n-\bar{x}_{n-1}\rangle+\rho^2\left\|\bar{t}_n-\bar{t}_{n-1}\right\|^2\\&\leq\left\|\bar{x}_n-\bar{x}_{n-1}\right\|^2-2\rho k\left\|\bar{x}_n-\bar{x}_{n-1}\right\|^2+\rho^2\lambda_{D_F}^2\left\|\bar{x}_n-\bar{x}_{n-1}\right\|^2\\&=\left(1-2\rho k+\rho^2\lambda_{D_F}^2\right)\left\|\bar{x}_n-\bar{x}_{n-1}\right\|^2,\end{aligned} \tag{51}$$

thus,

$$\begin{aligned}\left\|\bar{x}_n-\bar{x}_{n-1}+\rho\left(\bar{t}_n-\bar{t}_{n-1}\right)\right\|&\leq\sqrt{\left(1-2\rho k+\rho^2\lambda_{D_F}^2\right)}\left\|\bar{x}_n-\bar{x}_{n-1}\right\|\\&=\theta\left\|\bar{x}_n-\bar{x}_{n-1}\right\|,\end{aligned} \tag{52}$$

where $\theta=\sqrt{1-2\rho k+\rho^2\lambda_{D_F}^2}$.

Combining (50) and (52), we have

$$\begin{aligned}\left\|\bar{x}_{n+1}-\bar{x}_n\right\|&\leq(1-\alpha)\left\|\bar{x}_n-\bar{x}_{n-1}\right\|+\alpha\theta\left\|\bar{x}_n-\bar{x}_{n-1}\right\|\\&\leq(1-\alpha+\alpha\theta)\left\|\bar{x}_n-\bar{x}_{n-1}\right\|,\end{aligned} \tag{53}$$

thus, we have

$$\left\|\bar{x}_{n+1}-\bar{x}_n\right\|\leq\gamma^n\left\|\bar{x}_1-\bar{x}_0\right\|, \tag{54}$$

where $\gamma=(1-\alpha+\alpha\theta)$. Hence, for $m>n>0$, we have

$$\left\|\bar{x}_n-\bar{x}_m\right\|\leq\sum_{i=n}^{m-1}\left\|\bar{x}_{i+1}-\bar{x}_i\right\|\leq\left\|\bar{x}_1-\bar{x}_0\right\|\sum_{i=n}^{m-1}\gamma^i. \tag{55}$$

It is clear from condition (46) that $0<\gamma<1$, and consequently, we have $\|\bar{x}_n-\bar{x}_m\|\longrightarrow 0$, as $n\longrightarrow\infty$. Thus, $\{\bar{x}_n\}$ is a Cauchy sequence in E, and as E is complete, $\bar{x}_n\longrightarrow x^*\in E$, as $n\longrightarrow\infty$. From (45) of Iterative Algorithm 1, we have

$$\begin{aligned}\left\|\bar{t}_n\oplus\bar{t}_{n-1}\right\|&=\left\|\bar{t}_n-\bar{t}_{n-1}\right\|\\&\leq D\left(F\left(\bar{x}_n\right),F\left(\bar{x}_{n-1}\right)\right)\\&\leq\lambda_{D_F}\left\|\bar{x}_n-\bar{x}_{n-1}\right\|,\end{aligned} \tag{56}$$

thus, $\{\bar{t}_n\}$ is also a Cauchy sequence in E such that $\bar{t}_n\longrightarrow t^*\in E$, as $n\longrightarrow\infty$. Now, we will show that (x^*,t^*) is a solution of generalized variational inequality problem involving XOR operation (12). As $\bar{x}_n\longrightarrow x^*,\bar{t}_n\longrightarrow t^*$, and resolvent operator $\mathcal{J}_\rho^{\partial f}$ is continuous, we can write

$$
\begin{aligned}
x^* &= \lim_{n\longrightarrow\infty} \overline{x}_{n+1} \\
&= \lim_{n\longrightarrow\infty} \left[(1-\alpha)\overline{x}_n + \alpha \mathcal{J}_\rho^{\partial f}\left[\overline{x}_n + \rho \overline{t}_n\right]\right] \\
&= (1-\alpha) \lim_{n\longrightarrow\infty} \overline{x}_n + \alpha \mathcal{J}_\rho^{\partial f}\left[\lim_{n\longrightarrow\infty} \overline{x}_n + \rho \lim_{n\longrightarrow\infty} \overline{t}_n\right] \\
&= (1-\alpha)x^* + \alpha \mathcal{J}_\rho^{\partial f}\left[x^* + \rho t^*\right].
\end{aligned} \tag{57}
$$

Thus, the relation (35) is satisfied. It remains to show that $t^* \in F(x^*)$. Since $\overline{t}_n \in F(\overline{x}_n)$, we have

$$
\begin{aligned}
d(t^*, F(x^*)) &\le \left\|t^* - \overline{t}_n\right\| + d(\overline{t}_n, F(x^*)) \\
&\le \left\|t^* - \overline{t}_n\right\| + D(F(\overline{x}_n), F(x^*)) \\
&\le \left\|t^* - \overline{t}_n\right\| + \lambda_{D_F}\left\|\overline{x}_n - x^*\right\| \longrightarrow 0, \quad \text{as}\, n \longrightarrow \infty.
\end{aligned} \tag{58}
$$

Hence $d(t^*, F(x^*)) \longrightarrow 0$, $t^* \in F(x^*)$ as $F(x^*) \in CB(E)$. By Lemma 1, $x^* \in C$, $t^* \in F(x^*)$ is a solution of generalized variational inequality problem involving XOR operation (12). This completes the proof. □

Remark 2. Combining Theorems 1 and 3, we assert that the solution $\overline{x} \in C, \overline{t} \in F(\overline{x})$ of generalized variational inequality involving XOR operation (12) is also a solution of generalized complementarity problem involving XOR operation (9).

5. Numerical Example

In this section, we construct a numerical example in support of Theorem 3. Finally, the convergence graphs and the computation tables are provided for the sequences generated by Iterative Algorithm 1.

Example 1. Let $E = E^* = \mathbb{R}$ with the usual inner product and norm. Let $C = \{\overline{x} \in t\mathbb{R}n\colon q0h \le_{\overline{x}x} \le 71\}$ be a pointed closed convex positive cone in $\mathbb{R}$. Let $f\colon C \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a functional, $\partial f\colon \mathbb{R} \longrightarrow 2^{\mathbb{R}}$ be the subdifferential of f, $F\colon C \longrightarrow 2^{\mathbb{R}} \setminus \{\varnothing\}$ be a multivalued mapping, and $\mathcal{J}_\rho^{\partial f}$ be the resolvent operator associated with f such that

$$
\begin{aligned}
f(\overline{x}) &= 2\overline{x}^2 + 1, \\
F(\overline{x}) &= \left\{-\frac{\overline{x}}{7}\right\}, \quad \forall \overline{x} \in C.
\end{aligned} \tag{59}
$$

Then,

$$
\begin{aligned}
\partial f(\overline{x}) &= \{4\overline{x}\}, \\
\mathcal{J}_\rho^{\partial f}(\overline{x}) &= \left\{\frac{\overline{x}}{1+4\rho}\right\}, \quad \forall \overline{x} \in C.
\end{aligned} \tag{60}
$$

One can easily verify that the resolvent operator $\mathcal{J}_\rho^{\partial f}$ is a strongly comparison mapping and continuous.

For $\overline{x}, y \in C$, $w_1 \in F(\overline{x})$, and $w_2 \in F(y)$, we have

$$
\begin{aligned}
\langle w_1 - w_2, \overline{x} - y\rangle &= \left\langle -\frac{\overline{x}}{7} + \frac{y}{7}, \overline{x} - y\right\rangle \\
&= -\frac{1}{7}\|\overline{x} - ty\|^2 \\
&\le -\frac{1}{10}\|\overline{x} - ty\|^2,
\end{aligned} \tag{61}
$$

that is,

$$
\langle w_1 - w_2, \overline{x} - y\rangle \le -\frac{1}{10}\|\overline{x} - ty\|^2. \tag{62}
$$

Thus, F is the relaxed Lipschitz continuous with constant $k = (1/10)$.

Also,

$$
\begin{aligned}
D(F(\overline{x}), F(y)) &= \max\left\{\sup_{\overline{x}\in F(\overline{x})} d(\overline{x}, tFn(y)), \sup_{y\in F(y)} d(F(\overline{x}), y)\right\} \\
&\le \max\left\{\left\|-\frac{\overline{x}}{7} + \frac{y}{7}\right\|, \left\|-\frac{y}{7} + \frac{\overline{x}}{7}\right\|\right\} \\
&= \frac{1}{7}\max\{\|\overline{x} - ty\|, \|\overline{x} - y\|\} \\
&\le \frac{1}{7}\|\overline{x} - y\| \\
&\le \frac{1}{5}\|\overline{x} - y\|,
\end{aligned} \tag{63}
$$

that is,

$$
D(F(\overline{x}), F(y)) \le \frac{1}{5}\|\overline{x} - ty\|. \tag{64}
$$

Thus, F is the D-Lipschitz continuous with constant $\lambda_{D_F} = (1/5)$.

Let us take $\rho = 1$, then for $k = (1/10)$ and $\lambda_{D_F} = (1/5)$, the condition (46)

$$
\left|\rho - \frac{k}{\lambda_{D_F}^2}\right| < \frac{k}{\lambda_{D_F}^2}, \tag{65}
$$

is satisfied.

Furthermore, for $\rho = 1$ and $\alpha = (1/3)$, we obtain the sequences $\{\overline{x}_n\}$ and $\{\overline{t}_n\}$ generated by the Iterative Algorithm 1 as

$$
\begin{aligned}
\overline{x}_{n+1} &= (1-\alpha)\overline{x}_n + \alpha \mathcal{J}_\rho^{\partial f}\left[\overline{x}_n + \rho \overline{t}_n\right] \\
&= \frac{2}{3}\overline{x}_n + \frac{1}{15}\left[\overline{x}_n + \overline{t}_n\right],
\end{aligned} \tag{66}
$$

where $\overline{t}_n \in F(\overline{x}_n)$, and thus, $\overline{t}_n = -(\overline{x}_n/7)$. It is clear that the sequence $\{\overline{x}_n\}$ converges to $x^* = 0$, and consequently, the sequence $\{\overline{t}_n\}$ also converges to $t^* = 0$.

For initial values $\overline{x}_0 = 5, 10$, and 15, we have the following convergence graphs, which ensure that the sequences $\{\overline{x}_n\}$ and $\{\overline{t}_n\}$ converge to 0. Two computation tables are

Table 1: The values of x_n with initial values $x_0 = 5$, $x_0 = 10$, and $x_0 = 15$.

No. of Iteration	For $x_0 = 5$ x_n	For $x_0 = 10$ x_n	For $x_0 = 15$ x_n
$n = 1$	5	10	15
$n = 2$	3.61904761904762	7.23809523809524	10.8571428571429
$n = 3$	2.61950113378685	5.23900226757370	7.85850340136055
$n = 4$	1.89601986826477	3.79203973652953	5.68805960479430
$n = 5$	1.37235723798212	2.74471447596423	4.11707171394635
$n = 6$	0.993325238920389	1.98665047784078	2.97997571676117
$n = 7$	0.718978268170948	1.43795653634190	2.15693480451284
$n = 10$	0.272639416260542	0.545278832521084	0.817918248781626
$n = 14$	0.0748317352528748	0.149663470505750	0.224495205758624
$n = 18$	0.0205391747010088	0.0410783494020177	0.0616175241030265
$n = 21$	0.00778853666217476	0.0155770733243495	0.0233656099865243
$n = 25$	0.00213773093232492	0.00427546186464984	0.00641319279697477
$n = 26$	0.00154731000815899	0.00309462001631798	0.00464193002447697
$n = 27$	0	0	0
$n = 28$	0	0	0

Table 2: The values of t_n with initial values $x_0 = 5$, $x_0 = 10$, and $x_0 = 15$.

No. of Iteration	For $x_0 = 5$ t_n	For $x_0 = 10$ t_n	For $x_0 = 15$ t_n
$n = 1$	−0.714285714285714	−1.42857142857143	−2.14285714285714
$n = 2$	0.102040816326531	0.204081632653061	0.306122448979592
$n = 3$	−0.0145772594752187	−0.0291545189504373	−0.0437317784256560
$n = 4$	0.00208246563931695	0.00416493127863390	0.00624739691795085
$n = 5$	−0.000297495091330993	−0.000594990182661986	−0.000892485273992979
$n = 6$	4.24992987615704e − 05	8.49985975231408e − 05	0.000127497896284711
$n = 7$	−6.07132839451006e − 06	−1.21426567890201e-05	−1.82139851835302e − 05
$n = 10$	−1.23904661112450e − 07	3.54013317464143e − 08	5.31019976196215e − 08
$n = 14$	−5.16054398635777e − 11	1.47444113895936e − 11	2.21166170843905e − 11
$n = 18$	−2.14933110635476e − 14	6.14094601815645e − 15	9.21141902723467e − 15
$n = 21$	6.26627144709842e − 17	−1.79036327059955e − 17	−2.68554490589932e − 17
$n = 25$	2.60985899504307e − 20	5.21971799008614e − 20	7.82957698512922e − 20
$n = 26$	−3.72836999291867e − 21	−7.45673998583735e − 21	−1.11851099787560e − 20
$n = 27$	0	0	0
$n = 28$	0	0	0

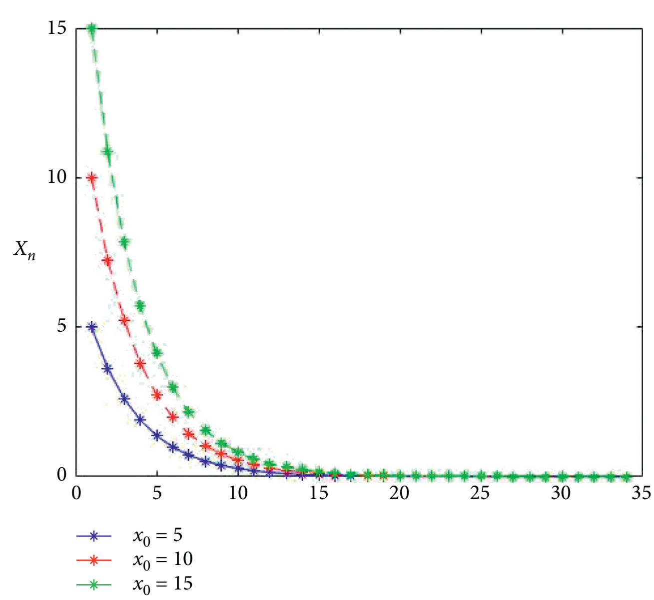

Figure 1: The convergence graph of the sequence $\{\overline{x}_n\}$ with initial values $x_0 = 5$, $x_0 = 10$, and $x_0 = 15$.

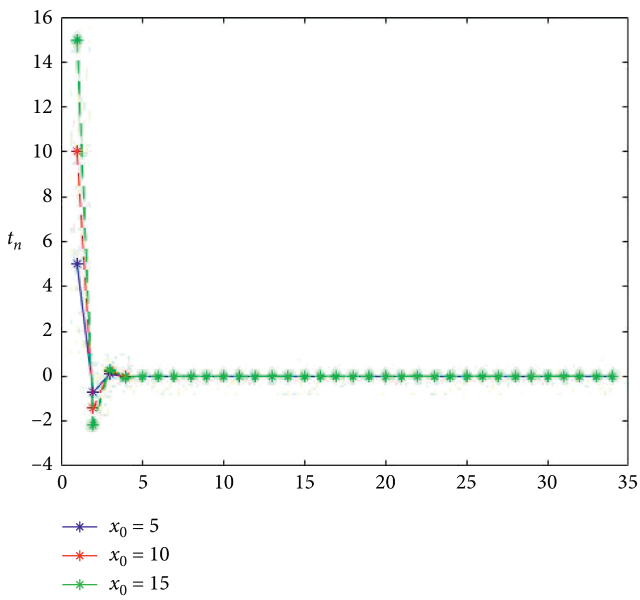

Figure 2: The convergence graph of the sequence $\{\bar{t}_n\}$ with initial values $x_0 = 5$, $x_0 = 10$, and $x_0 = 15$.

provided for the iterations (Tables 1 and 2) of the sequences $\{\bar{x}_n\}$ and $\{\bar{t}_n\}$ (Figures 1, and 2).

6. Conclusion

In this study, we introduce and study a generalized complementarity problem involving XOR operation with three classes of generalized variational inequalities involving XOR operation. Some equivalence relations are established between them. Finally, a generalized variational inequality problem involving XOR operation (12) is solved in real ordered Banach spaces. A numerical example is constructed with convergence graphs and computation tables for illustration of our main result.

We remark that our results may be further extended using other tools of functional analysis.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (Grant no. 11671365) and the Natural Science Foundation of Zhejiang Province (Grant no. LY14A010011).

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