

## Research Article

# Line Graphs of Monogenic Semigroup Graphs

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The concept of monogenic semigroup graphs  $\Gamma(\mathcal{S}_M)$  is firstly introduced by Das et al. (2013) based on zero divisor graphs. In this study, we mainly discuss the some graph properties over the line graph  $L(\Gamma(\mathcal{S}_M))$  of  $\Gamma(\mathcal{S}_M)$ . In detail, we prove the existence of graph parameters, namely, radius, diameter, girth, maximum degree, minimum degree, chromatic number, clique number, and domination number over  $L(\Gamma(\mathcal{S}_M))$ .

## 1. Introduction

The history of studying zero-divisor graphs has began over commutative rings by Beck's paper [1], and then, it is followed over commutative and noncommutative rings by some of the joint papers (cf. [2–4]). After that, DeMeyer et al. [5, 6] studied these graphs over commutative and noncommutative semigroups. Since zero-divisor graphs have taken so much attention, the researchers added a huge number of studies to improve the literature. In [7], the authors introduced monogenic semigroup graphs  $\Gamma(\mathcal{S}_M)$  based on actually zero-divisor graphs. In detail, to define  $\Gamma(\mathcal{S}_M)$ , the authors first considered a finite multiplicative monogenic semigroup with zero as the set:

$$\mathcal{S}_M = \{0, x, x^2, x^3, \dots, x^n\}. \quad (1)$$

By considering the definition given in [5], it has been obtained an undirected (zero-divisor) graph  $\Gamma(\mathcal{S}_M)$  associated to  $\mathcal{S}_M$  as in the following. The vertices of the graph are labeled by the nonzero zero divisors (in other words, all nonzero element) of  $\mathcal{S}_M$ , and any two distinct vertices  $x^i$  and  $x^j$ , where  $(1 \leq i, j \leq n)$ , are connected by an edge in case  $x^i x^j = 0$  with the rule  $x^i x^j = x^{i+j} = 0$  if and only if  $i + j \geq n + 1$ . The fundamental spectral properties of graph  $\Gamma(\mathcal{S}_M)$ , such as the diameter, girth, maximum and minimum degree, chromatic number, clique number, degree sequence, irregularity index, and dominating number, are

presented in [7]. Furthermore, in [8], first and second Zagreb indices, Randić index, geometric-arithmetic index, and atom-bond connectivity index, Wiener index, Harary index, first and second Zagreb eccentricity indices, eccentric connectivity index, and the degree distance which emphasize the importance of graph  $\Gamma(\mathcal{S}_M)$  have been studied.

It is known that the line graph  $L(G)$  of  $G$  is a graph whose vertices are the edges of  $G$ , and any two vertices are incident if and only if they have a common end vertex in  $G$ . Although line graphs were firstly introduced by the papers [9, 10], the details of these studies started by Harary [11] and after that by Harary (Chapter 8 in [12]). In fact the line graph is an active topic of research studies at this moment. For example, some topological indices of line graphs have been considered in [13, 14].

In Section 2, we will mainly deal with the special parameters, namely, radius, diameter, girth, maximum and minimum degree, domination number, and chromatic and clique numbers over the line graph  $L(\Gamma(\mathcal{S}_M))$  of monogenic semigroup graph  $\Gamma(\mathcal{S}_M)$  associated with  $\mathcal{S}_M$  as given in (1). We note that [15] will be followed for unexplained terminology and notation in this paper.

## 2. Main Results

In this part, our aim is to reach previously mentioned goals. At this point, we remind that, for any simple graph  $G$ , the

graph properties such as radius, diameter, and girth are obtained by calculating the distance between any two vertices or the total number of whole vertices. So, our proofs will be based on this idea.

The *eccentricity* of a vertex  $v$ , denoted by  $\text{ecc}(v)$ , in a connected graph  $G$  is the maximum distance between  $v$  and any other vertex  $u$  of  $G$  (for a disconnected graph, all vertices are defined to have infinite eccentricity). It is clear that  $\text{diam}(G)$  is equal to the *maximum eccentricity* among all vertices of  $G$ . On the contrary, the *minimum eccentricity* is called the *radius* [16, 17] of  $G$  and denoted by

$$\text{rad}(G) = \min_{u \in E(G)} \left\{ \max_{v \in E(G)} \{d_G(u, v)\} \right\}. \quad (2)$$

**Theorem 1.** *Let  $G = \Gamma(S_M)$  be a monogenic semigroup graph. Then, the radius of the line graph  $L(G)$  is given by*

$$r(L(G)) = \begin{cases} 1, & 3 \leq n \leq 5, \\ 2, & n \geq 6. \end{cases} \quad (3)$$

*Proof.* We can easily see that the result is true for  $n = 2, 3, 4$  by Figure 1. So, let us consider  $n \geq 6$ , and let us take into account any two vertices  $\{x^{i_1}, x^{j_1}\}$  and  $\{x^{i_2}, x^{j_2}\}$  from the graph  $L(G)$ . By the definition of  $G$ , we have  $i_1 + j_1 > n$  and  $i_2 + j_2 > n$ , since  $x^{i_1}x^{j_1}, x^{i_2}x^{j_2} \in E(G)$ .

If  $|\{x^{i_1}, x^{j_1}\} \cap \{x^{i_2}, x^{j_2}\}| = 1$ , then  $d(\{x^{i_1}, x^{j_1}\}, \{x^{i_2}, x^{j_2}\}) = 1$ .

If  $|\{x^{i_1}, x^{j_1}\} \cap \{x^{i_2}, x^{j_2}\}| = 0$ , then  $d(\{x^{i_1}, x^{j_1}\}, \{x^{i_2}, x^{j_2}\}) = 1$  since  $\{x^{i_1}, x^{j_1}\} \sim \{x^{i_1}, x^{j_2}\} \sim \{x^{i_2}, x^{j_2}\}$ .

So, we obtain  $\text{ecc}(\{x^i, x^j\}) = 2$  for all  $\{x^i, x^j\} \in V(L(G))$ . Thus,  $L(G)$  is a 2-self-centered graph for  $n \geq 6$ .

Hence, the result is obtained.  $\square$

It is known that the *diameter* of  $G$  is defined by the set

$$\text{diam}(G) = \sup\{d(x, y) : x \text{ and } y \text{ are vertices of } G\}. \quad (4)$$

**Theorem 2.** *Let  $G = \Gamma(S_M)$  be a monogenic semigroup graph. Then, the diameter of  $L(G)$  is*

$$D(L(G)) = \begin{cases} 1, & n = 3, \\ 2, & n \geq 4. \end{cases} \quad (5)$$

*Proof.* The proof can be obtained in a similar way as in the proof of Theorem 1.  $\square$

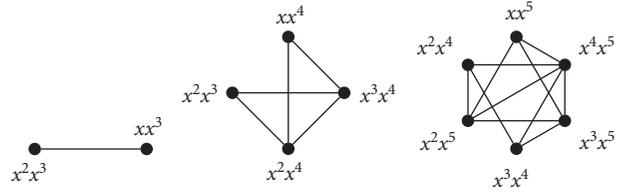


FIGURE 1: Line graphs  $\Gamma(S_M)$  for  $n = 3, 4, 5$ .

We recall that the *girth* of a graph  $G$  is the length of a shortest cycle contained in  $G$ . Moreover, the girth is defined to be infinity if  $G$  does not contain any cycle.

**Theorem 3.** *For a monogenic semigroup graph  $G = \Gamma(S_M)$ , the girth of line graph  $L(G)$  is 3.*

*Proof.* Since  $L(G) \cong K_2$  for  $n = 3$ , we have  $\text{girth}(L(G)) = \infty$ . So, we assume that  $n \geq 4$  which implies the set  $A = \{x^n, x^{n-1}\}, \{x^n, x^{n-2}\}, \{x^{n-1}, x^{n-2}\}$  is a complete subgraph of  $L(G)$ . Hence,  $\text{girth}(L(G)) = 3$ , as required.  $\square$

The *maximum degree*  $\Delta$  of  $G$  is the number of the largest degree in  $G$ , and the *minimum degree*  $\delta$  of  $G$  is the number of the smallest degree in  $G$  (see, for instance, [15]). According to these reminders, we can state and prove the following theorem in terms of the line graph.

**Theorem 4.** *Let  $G = \Gamma(S_M)$  be a monogenic semigroup graph. Then,  $\Delta(L(G)) = 2n - 5$ .*

*Proof.* We know that  $d(e) = d(u) + d(v) - 2$  for  $e = uv \in E(G)$ . From definition, the monogenic semigroup graph  $\Gamma(S_M)$ , we have  $1 = d(x) < d(x^2) < \dots < d(x^{\lfloor n/2 \rfloor}) = d(x^{\lfloor n/2 \rfloor + 1}) < \dots < d(x^n) = n - 1$ . Therefore, the vertex of maximum degree in  $L(G)$  must be the vertex  $\{x^n, x^{n-1}\}$ . As a result, we have

$$\begin{aligned} \Delta(L(G)) &= d(\{x^n, x^{n-1}\}) = d(x^n) + d(x^{n-1}) - 2 \\ &= n - 1 + n - 2 - 2 = 2n - 5, \end{aligned} \quad (6)$$

as required.  $\square$

**Theorem 5.** *Let  $G = \Gamma(S_M)$  be a monogenic semigroup graph. Then,  $\delta(L(G)) = n - 2$ .*

*Proof.* By definition of the degree sequences of monogenic semigroup graphs in the studies [7, 8], we obtain that the set vertices with minimum degree are

$$\begin{cases} \{\{x, x^n\}, \{x^2, x^{n-1}\}, \dots, \{x^{\lfloor n/2 \rfloor}, x^{\lfloor n/2 \rfloor + 1}\}\}, & n \text{ is even,} \\ \{\{x, x^n\}, \{x^2, x^{n-1}\}, \dots, \{x^{\lfloor n/2 \rfloor}, x^{\lfloor n/2 \rfloor + 2}\}, \{x^{\lfloor n/2 \rfloor + 1}, x^{\lfloor n/2 \rfloor + 2}\}\}, & n \text{ is odd.} \end{cases} \quad (7)$$

Therefore, we have

$$\begin{aligned} \delta(L(G)) &= d(\{x, x^n\}) = d(x) + d(x^n) - 2 \\ &= 1 + n - 1 - 2 = n - 2. \end{aligned} \tag{8}$$

Hence, we get the proof.  $\square$

A subset  $D$  of the vertex set  $V(G)$  of any graph  $G$  is called a *dominating set* if every vertex  $V(G) \setminus D$  is joined to at least one vertex of  $D$  by an edge. Additionally, the *domination number*  $\gamma(G)$  is the number of vertices in a smallest dominating set for  $G$  (we may refer [15] for the fundamentals of domination number).

**Theorem 6.** Let  $G = \Gamma(S_M)$  be a monogenic semigroup graph. The domination number of  $L(G)$  is

$$\gamma(L(G)) = \begin{cases} 1, & 3 \leq n \leq 5, \\ k + 1, & 4k + 2 \leq n \leq 4k + 5 (k \in \mathbb{Z}^+). \end{cases} \tag{9}$$

*Proof.* Let us consider the set  $A = \{\{x^n, x^{n-1}\}, \{x^{n-2}, x^{n-3}\}, \dots, \{x^{m+1}, x^m\}\} \subset V(L(G))$ , where  $m = \lceil n/2 \rceil$  or  $m = \lceil n/2 \rceil + 1$ . In fact,  $A$  is the domination set in  $V(L(G))$ .

Case 1: suppose  $m = \lceil n/2 \rceil$ :

$$\begin{aligned} n = 4k + 2 \Rightarrow |A| &= \frac{n - ((n/2) + 1)}{2} + 1 = \frac{4k + 2 - (2k + 2)}{2} + 1 = k + 1, \\ n = 4k + 3 \Rightarrow |A| &= \frac{n - ((n/2) + 1)}{2} + 1 = \frac{4k + 3 - (2k + 3)}{2} + 1 = k + 1, \\ n = 4k + 4 \Rightarrow |A| &= \frac{n - ((n/2) + 1)}{2} + 1 = \frac{4k + 4 - (2k + 3)}{2} + 1 = k + \frac{3}{2} \notin \mathbb{Z}, \\ n = 4k + 5 \Rightarrow |A| &= \frac{n - ((n/2) + 1)}{2} + 1 = \frac{4k + 5 - (2k + 4)}{2} + 1 = k + \frac{3}{2} \notin \mathbb{Z}. \end{aligned} \tag{10}$$

Case 2: now, suppose  $m = \lceil n/2 \rceil + 1$ :

$$\begin{aligned} n = 4k + 4 \Rightarrow |A| &= \frac{n - ((n/2) + 2)}{2} + 1 = \frac{4k + 4 - (2k + 4)}{2} + 1 = k + 1, \\ n = 4k + 5 \Rightarrow |A| &= \frac{n - ((n/2) + 1)}{2} + 1 = \frac{4k + 5 - (2k + 5)}{2} + 1 = k + 1. \end{aligned} \tag{11}$$

The above steps complete the proof.  $\square$

Basically, the coloring of any graph  $G$  is to be an assignment of colors (elements of some set) to the vertices of  $G$ , one color to each vertex, so that adjacent vertices are assigned distinct colors. If  $n$  different colors are used, then the coloring is referred to as an  $n$ -coloring. If there exists an  $n$ -coloring of  $G$ , then  $G$  is called  $n$ -colorable. The minimum number  $n$  for which  $G$  is  $n$ -colorable is called the *chromatic number* of  $G$  and is denoted by  $\chi(G)$ .

In addition, there exists another graph parameter, namely, the *clique* of a graph  $G$ . In fact, depending on the vertices, each of the maximal complete subgraphs of  $G$  is called a clique. Moreover, the largest number of vertices in any clique of  $G$  is called the *clique number* and denoted by  $\omega(G)$ . In general, by [15], it is well known that  $\chi(G) \geq \omega(G)$  for any graph  $G$ . For every induced subgraph  $K$  of  $G$ , if  $\chi(K) = \omega(K)$  holds, then  $G$  is called a *perfect graph* [18].

**Theorem 7.** Let  $G = \Gamma(S_M)$  be a monogenic semigroup graph. Then,  $\omega(L(G)) = n - 1$ .

*Proof.* Due to the definition of  $L(G)$ , if the vertex  $\{x^i, x^j\}$  is adjacent to the vertex  $\{x^t, x^l\}$ , then  $|\{x^i, x^j\} \cap \{x^t, x^l\}| = 1$ . Therefore, the vertex sets of complete graphs in  $L(G)$  must be the form of

$$A = \{\{x^i, x^j\}: j \in \mathbb{Z}^+, i + j \geq n + 1\}, \quad (x^i \in V(G)). \tag{12}$$

So, since the vertex with maximum degree in  $G$  is  $x^n$ , the maximum complete subgraph in  $L(G)$  is the subgraph with vertex set  $A$ , where

$$A = \{\{x^n, x\}, \{x^n, x^2\}, \dots, \{x^n, x^{n-1}\}\}, \tag{13}$$

such that the number of elements in it is  $n - 1$ . Thus,  $\omega(L(G)) = n - 1$ , as required.  $\square$

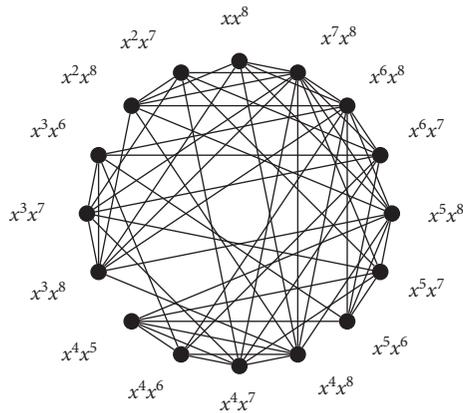


FIGURE 2: Line graph of  $\Gamma(S_M)$  for  $n = 8$ .

**Theorem 8.** For a monogenic semigroup graph  $G = \Gamma(S_M)$ , the chromatic number of line graph of  $G$  is determined by  $\chi(L(G)) = n - 1$ .

*Proof.* Since the set  $S = \{\{x^n, x\}, \{x^n, x^2\}, \dots, \{x^n, x^{n-1}\}\}$  is the complete subgraph of  $L(G)$ , we must paint each vertex in this set with a different color. That means we need to use  $n - 1$  colors for this set  $S$ . The graph  $L(G)$  has at least one vertex  $\{x^n, x^k\}$  that is not adjacent to  $\{x^i, x^j\}$  for all of the vertices in  $L(G)$ , where  $\{x^i, x^j\} \notin S$ . Therefore, we can use one of the colors which used in the set  $S$  for the vertex  $\{x^i, x^j\} \notin S$ . Thus,  $\chi(L(G)) = n - 1$ , as required.  $\square$

*Example 1.* The graph is given in Figure 2.

Let us consider the semigroup

$$S_M = \{x, x^2, x^3, x^4, x^5, x^6, x^7, x^8\}. \quad (14)$$

Now, by considering the graph  $L(\Gamma(S_M))$  as drawn in Figure 2, we can list the following results as example:

- (i)  $\text{rad}(L(\Gamma(S_M))) = 2$  (obtained by Theorem 1)
- (ii)  $\text{diam}(L(\Gamma(S_M))) = 2$  (obtained by Theorem 2)
- (iii)  $\text{girth}(L(\Gamma(S_M))) = 3$  (obtained by Theorem 3)
- (iv)  $\Delta(L(\Gamma(S_M))) = 11$  (obtained by Theorem 4)
- (v)  $\delta(L(\Gamma(S_M))) = 6$  (obtained by Theorem 5)
- (vi)  $\gamma(L(\Gamma(S_M))) = 2$  (obtained by Theorem 6)
- (vii)  $\chi(L(\Gamma(S_M))) = 7$  (obtained by Theorem 7)
- (viii)  $\omega(L(\Gamma(S_M))) = 7$  (obtained by Theorem 8)

### 3. Conclusions

The aim of the study is to investigate the concept of monogenic semigroup graphs  $\Gamma(\mathcal{S}_M)$ , which is firstly introduced by Das et al. [7], based on zero-divisor graphs. We examine the some graph properties over the line graph  $L(\Gamma(\mathcal{S}_M))$  of  $\Gamma(\mathcal{S}_M)$ . The existences of graph parameters, namely, radius, diameter, girth, maximum degree, minimum degree, chromatic number, clique number, and domination number over  $L(\Gamma(\mathcal{S}_M))$  are proved, respectively.

### Data Availability

No data were used to support this study.

### Conflicts of Interest

The authors declare that they have no conflicts of interest.

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