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Research Article

Quantum Inequalities of Hermite–Hadamard Type for r-Convex Functions

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In this present study, we first establish Hermite–Hadamard type inequalities for r-convex functions via q^{κ_2}-definite integrals. Then, we prove some quantum inequalities of Hermite–Hadamard type for product of two r-convex functions. Finally, by using these established inequalities and the results given by (Brahim et al. 2015), we prove several quantum Hermite–Hadamard type inequalities for coordinated r-convex functions and for the product of two coordinated r-convex functions.

1. Introduction

Quantum calculus research is an unlimited analysis of calculus and is known as q-calculus. We get the initial mathematical formulas in q-calculus as q reaches 1^-. The commencement of the analysis of q-calculus was initiated by Euler (1707–1783). The aforementioned results lead to an intensive investigation on q-calculus in the twentieth century. The concept of q-calculus is used in many areas in mathematics and physics such as theory, orthogonal polynomials, integration, basic hypergeometric functions, mechanical theory, and quantum and relativity theory. For more information about q-calculus, one can refer to [1–10].

Mathematically, convexity is very simple and natural which plays a very important role in various fields of pure and applied science, such as in the field of practicality, engineering science, and management science. In the recent past, the classical concept of convexity has been extended and generalized in different directions. Another factor that makes the theory of the most popular convex works is its relationship to the concept of inequality. Many inequalities can be achieved using the definition of convex functions. One of the widely studied inequalities involving convex works is the Hermite–Hadamard inequality, which is the first basic result of convex design with natural geometric descriptions and multiple uses and has attracted great interest in elementary mathematics. Many mathematicians have devoted their efforts to generalization, refinement, modelling, and multiplication of various fields of work such as the use of convex mappings (see, e.g., [11], p.137, and [12]).

The classical Hermite–Hadamard inequality states that if $\mathcal{F}: I \longrightarrow \mathbb{R}$ is a convex function on the interval I of real numbers and $\kappa_1, \kappa_2 \in I$ with $\kappa_1 < \kappa_2$, then

$$\mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2}\right) \leq \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa) d\varkappa \leq \frac{\mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_2)}{2}. \quad (1)$$

The inequality holds in the reversed direction if $\mathcal{F}$ is concave. We see that the Hermite–Hadamard inequality can be regarded as a refinement of the concept of integration and is easily followed by Jensen's inequality. The Hermite–Hadamard inequality of convex works has received renewed attention in recent years and has been studied in significant and practical variations.

In [13], Pachpatte proved the following inequalities for products of convex functions.

Theorem 1. *Let $\mathcal{F}$ and $\mathcal{G}$ be real-valued, nonnegative, and convex functions on $[\kappa_1, \kappa_2]$. Then, we have*

$$\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} F(\varkappa)\mathcal{G}(\varkappa)d\varkappa \leq \frac{1}{3}\mathcal{A}(\kappa_1,\kappa_2)+\frac{1}{6}\mathcal{B}(\kappa_1,\kappa_2), \quad (2)$$

$$2F\left(\frac{\kappa_1+\kappa_2}{2}\right)\mathcal{G}\left(\frac{\kappa_1+\kappa_2}{2}\right) \leq \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} F(\varkappa)\mathcal{G}(\varkappa)d\varkappa + \frac{1}{6}\mathcal{A}(\kappa_1,\kappa_2)+\frac{1}{3}\mathcal{B}(\kappa_1,\kappa_2), \quad (3)$$

where $\mathcal{A}(\kappa_1,\kappa_2) = F(\kappa_1)\mathcal{G}(\kappa_1)+F(\kappa_2)\mathcal{G}(\kappa_2)$ *and* $\mathcal{B}(\kappa_1,\kappa_2) = F(\kappa_1)\mathcal{G}(\kappa_2)+F(\kappa_2)\mathcal{G}(\kappa_1)$.

A positive function is called r-convex on $[\kappa_1,\kappa_2]$, if for all $\varkappa, y \in [\kappa_1,\kappa_2]$ and $\xi \in [0,1]$,

$$F(\xi\varkappa+(1-\xi)y) \leq \begin{cases} \left(\xi(F(\varkappa))^r+(1-\xi)(F(y))^r\right)^{1/r}, & \text{if } r \neq 0, \\ F(\varkappa)^{\xi}F(y)^{(1-\xi)}, & \text{if } r = 0. \end{cases} \quad (4)$$

It is obvious if $r = 1$, then the inequality classical convex functions. It should be noted that if F is r-convex function, then F is convex function. We have that 0-convex functions are simply log-convex functions and 1-convex functions are ordinary convex functions [14].

In [15], the definition of r-convex functions on coordinates is given, such that

Definition 1. A function F: $\Delta := [\kappa_1,\kappa_2]\times[\kappa_3,\kappa_4] \longrightarrow \mathbb{R}_+$ will be called r-convex on Δ for all $\xi,\lambda \in [0,1]$ and $(\varkappa, y), (u,v) \in \Delta$, if the following inequality holds:

$$F(\xi\varkappa+(1-\xi)y, \lambda u+(1-\lambda)v) \leq \begin{cases} \left[\xi\lambda F^r(\varkappa,u)+\xi(1-\lambda)F^r(\varkappa,v)+(1-\xi)\lambda F^r(y,u)+(1-\xi)(1-\lambda)F^r(y,v)\right]^{1/r}, & \text{if } r \neq 0, \\ F^{\xi\lambda}(\varkappa,u)F^{\xi(1-\lambda)}(\varkappa,v)F^{(1-\xi)\lambda}(y,u)F^{(1-\xi)(1-\lambda)}(y,v), & \text{if } r = 0. \end{cases} \quad (5)$$

It is simply to see that if we choose $r = 0$, we have coordinated log-convex functions and if we choose $r = 1$, we have coordinated convex functions. In [15], Ekinci et al. also prove several Hermite–Hadamard type inequalities for coordinated r-convex functions. In literature, many studies have been done on r-convex functions. For some of them, one can see [16–23].

2. Preliminaries of q-Calculus and Some Inequalities

In this section, we present some required definitions and related inequalities about q-calculus. For more information about q-calculus, one can refer to [1–10, 24, 25]. Also, here and further, we use the following notation (see [5]):

$$[n]_q = \frac{1-q^n}{1-q} = 1+q+q^2+\cdots+q^{n-1}, \quad q \in (0,1). \quad (6)$$

In [4], Jackson gave the q-Jackson integral from 0 to κ_2 for $0<q<1$ as follows:

$$\int_0^{\kappa_2} F(\varkappa)d_q\varkappa = (1-q)\kappa_2\sum_{n=0}^{\infty} q^n F(\kappa_2 q^n), \quad (7)$$

provided the sum converges absolutely.

Jackson in [4] gave the q-Jackson integral in a generic interval $[\kappa_1,\kappa_2]$ as

$$\int_{\kappa_1}^{\kappa_2} F(\varkappa)d_q\varkappa = \int_0^{\kappa_2} F(\varkappa)d_q\varkappa - \int_0^{\kappa_1} F(\varkappa)d_q\varkappa. \quad (8)$$

Definition 2 (see [9]). For a continuous function F: $[\kappa_1,\kappa_2] \longrightarrow \mathbb{R}$, then q-derivative of F at $\varkappa \in [\kappa_1,\kappa_2]$ for $0<q<1$ is characterized by the expression

$$_{\kappa_1}D_q F(\varkappa) = \frac{F(\varkappa)-F(q\varkappa+(1-q)\kappa_1)}{(1-q)(\varkappa-\kappa_1)}, \quad \varkappa \neq \kappa_1. \quad (9)$$

Since F: $[\kappa_1,\kappa_2] \longrightarrow \mathbb{R}$ is a continuous function, thus we have $_{\kappa_1}D_q F(\kappa_1) = \lim_{\varkappa\longrightarrow\kappa_1} {_{\kappa_1}D_q F(\varkappa)}$. The function F is said to be q-differentiable on $[\kappa_1,\kappa_2]$ if $_{\kappa_1}D_q F(\xi)$ exists for all $\varkappa \in [\kappa_1,\kappa_2]$. If $\kappa_1 = 0$ in (9), then $_0D_q F(\varkappa) = D_q F(\varkappa)$, where $D_q F(\varkappa)$ is familiar q-derivative of F at $\varkappa \in [\kappa_1,\kappa_2]$ defined by the expression (see [5])

$$D_q F(\varkappa) = \frac{F(\varkappa)-F(q\varkappa)}{(1-q)\varkappa}, \quad \varkappa \neq 0. \quad (10)$$

Definition 3 (see [9]). Let F: $[\kappa_1,\kappa_2] \longrightarrow \mathbb{R}$ be a continuous function. Then, the q_{κ_1}-definite integral on $[\kappa_1,\kappa_2]$ and $0<q<1$ are defined as

$$\int_{\kappa_1}^{\kappa_2} F(\varkappa)_{\kappa_1} d_q \varkappa = (1-q)(\kappa_2-\kappa_1)\sum_{n=0}^{\infty} q^n F(q^n\kappa_2 + (1-q^n)\kappa_1) = (\kappa_2-\kappa_1)\int_0^1 F((1-\xi)\kappa_1 + \xi\kappa_2) d_q\xi. \tag{11}$$

In [26], Alp et al. proved the following q_{κ_1}-Hermite–Hadamard inequality for convex functions in the setting of quantum calculus.

Theorem 2. *If* F: $[\kappa_1, \kappa_2] \longrightarrow \mathbb{R}$ *is a convex differentiable function on* $[\kappa_1, \kappa_2]$ *and* $0 < q < 1$. *Then, q-Hermite–Hadamard inequalities are as follows:*

$$F\left(\frac{q\kappa_1 + \kappa_2}{1+q}\right) \leq \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} F(\varkappa)_{\kappa_1} d_q\varkappa \leq \frac{qF(\kappa_1) + F(\kappa_2)}{1+q}. \tag{12}$$

On the other hand, Bermudo et al. gave the following new definition and related Hermite–Hadamard type inequalities.

Definition 4 (see [27]). Let F: $[\kappa_1, \kappa_2] \longrightarrow \mathbb{R}$ be a continuous function. Then, the q^{κ_2}-definite integral on $[\kappa_1, \kappa_2]$ for $0 < q < 1$ is defined as

$$\begin{aligned}\int_{\kappa_1}^{\kappa_2} F(\varkappa)^{\kappa_2} d_q\varkappa &= (1-q)(\kappa_2-\kappa_1)\sum_{n=0}^{\infty} q^n F(q^n\kappa_1 + (1-q^n)\kappa_2) \\ &= (\kappa_2-\kappa_1)\int_0^1 F(\xi\kappa_1 + (1-\xi)\kappa_2) d_q\xi.\end{aligned} \tag{13}$$

Theorem 3 (see [27]). *If* F: $[\kappa_1, \kappa_2] \longrightarrow \mathbb{R}$ *is a convex differentiable function on* $[\kappa_1, \kappa_2]$ *and* $0 < q < 1$. *Then, q-Hermite–Hadamard inequalities are as follows:*

$$F\left(\frac{\kappa_1 + q\kappa_2}{1+q}\right) \leq \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} F(\varkappa)^{\kappa_2} d_q\varkappa \leq \frac{F(\kappa_1) + qF(\kappa_2)}{1+q}. \tag{14}$$

From Theorem 2 and Theorem 3, one can get the following inequalities.

Corollary 1 (see [27]). *For any convex function* F: $[\kappa_1, \kappa_2] \longrightarrow \mathbb{R}$ *and* $0 < q < 1$, *we have*

$$F\left(\frac{q\kappa_1 + \kappa_2}{1+q}\right) + F\left(\frac{\kappa_1 + q\kappa_2}{1+q}\right) \leq \frac{1}{\kappa_2-\kappa_1}\left\{\int_{\kappa_1}^{\kappa_2} F(\varkappa)_{\kappa_1} d_q\varkappa + \int_{\kappa_1}^{\kappa_2} F(\varkappa)^{\kappa_2} d_q\varkappa\right\} \leq F(\kappa_1) + F(\kappa_2), \tag{15}$$

$$F\left(\frac{\kappa_1 + \kappa_2}{2}\right) \leq \frac{1}{2(\kappa_2-\kappa_1)}\left\{\int_{\kappa_1}^{\kappa_2} F(\varkappa)_{\kappa_1} d_q\varkappa + \int_{\kappa_1}^{\kappa_2} F(\varkappa)^{\kappa_2} d_q\varkappa\right\} \leq \frac{F(\kappa_1) + F(\kappa_2)}{2}. \tag{16}$$

Brahim et al. prove the following lemma and theorem for r-convex functions.

Lemma 1 (see [28]). *For* $p \geq 1$ *and* $0 < q < 1$, *the following inequality is valid:*

$$\int_0^1 (1-\xi)^p d_q\xi \leq \frac{q}{[p+1]_q}. \tag{17}$$

Theorem 4 (see [28]). *Let* F: $[\kappa_1, \kappa_2] \longrightarrow \mathbb{R}_+$ *be* r_1*-convex on* $[\kappa_1, \kappa_2]$. *Then, the following inequality holds for* $0 < r_1 \leq 1$ *and* $0 < q < 1$:

$$\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} F(\varkappa)\kappa_1 d_q\varkappa \leq \frac{1}{[1/r_1 + 1]_q}\left([qF(\kappa_1)]^{r_1} + [F(\kappa_2)]^{r_1}\right)^{1/r_1}. \tag{18}$$

Theorem 5 (see [28]). *Let $F, \mathcal{G}\colon [\kappa_1, \kappa_2] \longrightarrow \mathbb{R}_+$ be r_1-convex and r_2-convex functions, respectively, on $[\kappa_1, \kappa_2]$. Then, the following inequality holds for $0 < r_1, r_2 \leq 2$ and $0 < q < 1$:*

$$\frac{2}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} F(\varkappa)\mathcal{G}(\varkappa)\kappa_1 d_q\varkappa \leq \frac{1}{[2/r_1+1]_q}\cdot\left(\left[q^{1/2}F(\kappa_1)\right]^{r_1}+\left[F(\kappa_2)\right]^{r_1}\right)^{2/r_1} + \frac{1}{[2/r_2+1]_q}\left(\left[q^{1/2}\mathcal{G}(\kappa_1)\right]^{r_2}+\left[\mathcal{G}(\kappa_2)\right]^{r_2}\right)^{2/r_2}. \tag{19}$$

Theorem 6 (see [28]). *Let $F, \mathcal{G}\colon [\kappa_1, \kappa_2] \longrightarrow \mathbb{R}_+$ be r_1-convex and r_2-convex functions, respectively, on $[\kappa_1, \kappa_2]$ and $0 < q < 1$. Then, the following inequality holds if $r_1 > 1$ and $1/r_1 + 1/r_2 = 1$:*

$$\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} F(\varkappa)\mathcal{G}(\varkappa)_{\kappa_1} d_q\varkappa \leq \left(\frac{\left[qF(\kappa_1)\right]^{r_1}+\left[F(\kappa_2)\right]^{r_1}}{[2]_q}\right)^{1/r_1}\cdot\left(\frac{\left[q\mathcal{G}(\kappa_1)\right]^{r_2}+\left[\mathcal{G}(\kappa_1)\right]^{r_2}}{[2]_q}\right)^{1/r_2}. \tag{20}$$

In [29], Latif defined the $q_{\kappa_1\kappa_3}$-integral and related properties for two variable functions as follows.

Definition 5. Suppose that $F\colon \Delta \longrightarrow \mathbb{R}$ is continuous function and $0 < q_1, q_2 < 1$. Then, the definite $q_{\kappa_1\kappa_3}$-integral on Δ is defined by

$$\int_{\kappa_1}^{\varkappa}\int_{\kappa_3}^{y} F(\xi,s)_{\kappa_3}d_{q_2}s\,_{\kappa_1}d_{q_1}\xi = (1-q_1)(1-q_2)(\varkappa-\kappa_1)(y-\kappa_3)\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m F\left(q_1^n\varkappa+(1-q_1^n)\kappa_1, q_2^m y+(1-q_2^m)\kappa_3\right), \tag{21}$$

for $(\varkappa, y) \in \Delta$.

In [29], Latif et al. also proved a q-Hermite–Hadamard inequality for coordinated convex functions.

By Definitions 4 and 5, Budak et al. defined the following $q_{\kappa_1}^{\kappa_4}$, $q_{\kappa_3}^{\kappa_2}$ and $q^{\kappa_2\kappa_4}$ integrals.

Definition 6 (see [30]). Suppose that $F\colon \Delta \longrightarrow \mathbb{R}$ is a continuous function and $0 < q_1, q_2 < 1$. Then, the following $q_{\kappa_1}^{\kappa_4}$, $q_{\kappa_3}^{\kappa_2}$, and $q^{\kappa_2\kappa_4}$ integrals on Δ are defined by

$$\int_{\kappa_1}^{\varkappa}\int_{y}^{\kappa_4} F(\xi,s)\,^{\kappa_4}d_{q_2}s\,_{\kappa_1}d_{q_1}\xi = (1-q_1)(1-q_2)(\varkappa-\kappa_1)(\kappa_4-y)\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m F\left(q_2^m\varkappa+(1-q_2^m)\kappa_1, q_1^n y+(1-q_1^n)\kappa_4\right),$$
$$\int_{\varkappa}^{\kappa_2}\int_{\kappa_3}^{y} F(\xi,s)\,_{\kappa_3}d_{q_2}s\,^{\kappa_2}d_{q_1}\xi = (1-q_1)(1-q_2)(\kappa_2-\varkappa)(y-\kappa_3)\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m F\left(q_2^m\varkappa+(1-q_2^m)\kappa_2, q_1^n y+(1-q_1^n)\kappa_3\right), \tag{22}$$

$$\int_{\varkappa}^{\kappa_2}\int_{y}^{\kappa_4} F(\xi,s)\,^{\kappa_4}d_{q_2}s\,^{\kappa_2}d_{q_1}\xi = (1-q_1)(1-q_2)(\kappa_2-\varkappa)(\kappa_4-y)\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m F\left(q_2^m\varkappa+(1-q_2^m)\kappa_2, q_1^n y+(1-q_1^n)\kappa_4\right), \tag{23}$$

respectively, for $(\varkappa, y) \in \Delta$.

Budak et al. also proved some quantum Hermite–Hadamard type inequalities for coordinated convex functions. For other similar quantum inequalities, please see [31,32].

In this paper, we first prove the new variant of results of Brahim et al. for q^{κ_2}-integrals. We also obtain quantum versions of the inequalities in [15].

3. Quantum Hermite–Hadamard Type Inequalities for r-Convex Functions

In this section, we obtain some quantum inequalities of Hermite–Hadamard type for r-convex functions and for product of two r-convex functions.

Theorem 7. *Let $F\colon [\kappa_1, \kappa_2] \longrightarrow \mathbb{R}_+$ be a r_1-convex function on $[\kappa_1, \kappa_2]$. Then, the following inequality holds for $0 < r_1 \leq 1$:*

$$\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} F(\varkappa)\kappa_2 d_q\varkappa \leq \frac{1}{[1/r_1+1]_q}\left(\left[F(\kappa_1)\right]^{r_1}+\left[qF(\kappa_2)\right]^{r_1}\right)^{1/r_1}, \tag{24}$$

where $0 < q < 1$.

Proof. According to definition r_1-convex, for all $\xi \in [0,1]$, we have

$$F(\xi\kappa_1+(1-\xi)\kappa_2) \leq \left(\xi\left[F(\kappa_1)\right]^{r_1}+(1-\xi)\left[F(\kappa_2)\right]^{r_1}\right)^{1/r_1}. \tag{25}$$

By integrating the inequality on $[0,1]$, we obtain

$$\int_0^1 F(\xi\kappa_1+(1-\xi)\kappa_2)d_q\xi \le \int_0^1 \left(\xi[F(\kappa_1)]^{r_1}+(1-\xi)[F(\kappa_2)]^{r_1}\right)^{1/r_1}d_q\xi. \tag{26}$$

From Definition 4, we get

$$\int_{\kappa_1}^{\kappa_2} F(\varkappa)^{\kappa_2}d_q\varkappa \le \int_0^1 \left(\xi[F(\kappa_1)]^{r_1}+(1-\xi)[F(\kappa_2)]^{r_1}\right)^{1/r_1}d_q\xi. \tag{27}$$

Using Minkowski's inequality for right side of inequality (26),

$$\int_0^1 \left(\xi[F(\kappa_1)]^{r_1}+(1-\xi)[F(\kappa_2)]^{r_1}\right)^{1/r_1}d_q\xi \le \left(\left(\int_0^1 \xi^{1/r_1}d_q\xi\right)^{r_1}[F(\kappa_1)]^{r_1}+\left(\int_0^1 (1-\xi)^{1/r_1}d_q\xi\right)^{r_1}[F(\kappa_2)]^{r_1}\right)^{1/r_1}. \tag{28}$$

By Lemma 1, we have

$$\left(\int_0^1 \xi^{1/r_1}d_q\xi\right)^{r_1} = \left(\frac{1}{[1/r_1+1]_q}\right)^{r_1}, \tag{29}$$

$$\left(\int_0^1 (1-\xi)^{1/r_1}d_q\xi\right)^{r_1} \le \left(\frac{q}{[1/r_1+1]_q}\right)^{r_1}. \tag{30}$$

Thus, by substituting (29) and (30) in (28), we obtain

$$\begin{aligned}\int_{\kappa_1}^{\kappa_2} F(\varkappa)^{\kappa_2}d_q\varkappa &\le \left(\left(\frac{1}{[1/r_1+1]_q}\right)^{r_1}[F(\kappa_1)]^{r_1}+\left(\frac{q}{[1/r_1+1]_q}\right)^{r_1}[F(\kappa_2)]^{r_1}\right)^{1/r_1}\\ &\le \frac{1}{[1/r_1+1]_q}\left([F(\kappa_1)]^{r_1}+q^{r_1}[F(\kappa_2)]^{r_1}\right)^{1/r_1}.\end{aligned} \tag{31}$$

The proof is completed. □

Remark 1. If we take the limit $q \longrightarrow 1^-$ in Theorem 7, then Theorem 7 reduces to Theorem 2.1 in [33].

Remark 2. If we choose $r_1 = 1$ in Theorem 7, then inequality (24) reduces to the second inequality in (14).

Theorem 8. *Let $F, \mathcal{G}$: $[\kappa_1,\kappa_2] \longrightarrow \mathbb{R}_+$ be r_1-convex and r_2-convex functions, respectively, on $[\kappa_1,\kappa_2]$. Then, the following inequality holds for $0<r_1,r_2\le 2$:*

$$\frac{2}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} F(\varkappa)\mathcal{G}(\varkappa)^{\kappa_2}d_q\varkappa \le \frac{1}{[2/r_1+1]_q}\left([F(\kappa_1)]^{r_1}+[q^{1/2}F(\kappa_2)]^{r_1}\right)^{2/r_1}+\frac{1}{[2/r_2+1]_q}\left([\mathcal{G}(\kappa_1)]^{r_2}+[q^{1/2}\mathcal{G}(\kappa_2)]^{r_2}\right)^{2/r_2}, \tag{32}$$

where $0<q<1$.

Proof. By the assumptions that F is an r_1-convex function and $\mathcal{G}$ is an r_2-convex function, we can write

$$F(\xi\kappa_1+(1-\xi)\kappa_2) \le \left(\xi[F(\kappa_1)]^{r_1}+(1-\xi)[F(\kappa_2)]^{r_1}\right)^{1/r_1}, \tag{33}$$

$$\mathcal{G}(\xi\kappa_1+(1-\xi)\kappa_2) \le \left(\xi[\mathcal{G}(\kappa_1)]^{r_2}+(1-\xi)[\mathcal{G}(\kappa_2)]^{r_2}\right)^{1/r_2}, \tag{34}$$

for all $\xi \in [0,1]$ and $r_1, r_2 > 0$.

Then,

$$\begin{aligned}&F(\xi\kappa_1+(1-\xi)\kappa_2)\mathcal{G}(\xi\kappa_1+(1-\xi)\kappa_2)\\ &\le \left(\xi[F(\kappa_1)]^{r_1}+(1-\xi)[F(\kappa_2)]^{r_1}\right)^{1/r_1}\\ &\quad\cdot\left(\xi[\mathcal{G}(\kappa_1)]^{r_2}+(1-\xi)[\mathcal{G}(\kappa_2)]^{r_2}\right)^{1/r_2}.\end{aligned} \tag{35}$$

Integrating both sides with respect to ξ on $[0,1]$ and from Definition 4, we obtain

$$\begin{aligned}&\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} F(\varkappa)\mathcal{G}(\varkappa)^{\kappa_2} d_q\varkappa\\&\leq \int_0^1 \left(\xi[F(\kappa_1)]^{r_1}+(1-\xi)[F(\kappa_2)]^{r_1}\right)^{1/r_1}\left(\xi[\mathcal{G}(\kappa_1)]^{r_2}+(1-\xi)[\mathcal{G}(\kappa_2)]^{r_2}\right)^{1/r_2} d_q\xi.\end{aligned} \tag{36}$$

Using Cauchy's inequality for right side of inequality (36), we obtain

$$\begin{aligned}&\int_0^1 \left(\xi[F(\kappa_1)]^{r_1}+(1-\xi)[F(\kappa_2)]^{r_1}\right)^{1/r_1}\left(\xi[\mathcal{G}(\kappa_1)]^{r_2}+(1-\xi)[\mathcal{G}(\kappa_2)]^{r_2}\right)^{1/r_2} d_q\xi\\&\leq \frac{1}{2}\int_0^1 \left(\xi[F(\kappa_1)]^{r_1}+(1-\xi)[F(\kappa_2)]^{r_1}\right)^{2/r_1} d_q\xi+\frac{1}{2}\int_0^1 \left(\xi[\mathcal{G}(\kappa_1)]^{r_2}+(1-\xi)[\mathcal{G}(\kappa_2)]^{r_2}\right)^{2/r_2} d_q\xi.\end{aligned} \tag{37}$$

By using Minkowski's inequality, we have

$$\begin{aligned}&\int_0^1 \left(\xi[F(\kappa_1)]^{r_1}+(1-\xi)[F(\kappa_2)]^{r_1}\right)^{2/r_1} d_q\xi\\&\leq \left(\left(\int_0^1 \xi^{2/r_1} d_q\xi\right)^{r_1/2}[F(\kappa_1)]^{r_1}+\left(\int_0^1 (1-\xi)^{2/r_1} d_q\xi\right)^{r_1/2}[F(\kappa_2)]^{r_1}\right)^{2/r_1}\\&=\left(\left(\frac{1}{[2/r_1+1]_q}\right)^{r_1/2}[F(\kappa_1)]^{r_1}+\left(\frac{q}{[2/r_1+1]_q}\right)^{r_1/2}[F(\kappa_2)]^{r_1}\right)^{2/r_1}.\end{aligned} \tag{38}$$

Similarly, we have

$$\begin{aligned}&\int_0^1 \left(\xi[\mathcal{G}(\kappa_1)]^{r_2}+(1-\xi)[\mathcal{G}(\kappa_2)]^{r_2}\right)^{2/r_2} d_q\xi\\&\leq \left(\left(\frac{1}{[2/r_2+1]_q}\right)^{r_2/2}[\mathcal{G}(\kappa_1)]^{r_2}\right.\\&\left.+\left(\frac{q}{[2/r_2+1]_q}\right)^{r_2/2}[\mathcal{G}(\kappa_2)]^{r_2}\right)^{2/r_2}.\end{aligned} \tag{39}$$

Thus, from the inequalities (36)–(39), we obtain the desired result. □

Remark 3. If we take the limit $q \longrightarrow 1^-$ in Theorem 8, then Theorem 8 reduces to Theorem 2.3 in [33].

Corollary 2. *If we choose $r_1 = r_2 = 2$ in Theorem 8, then we have the inequality*

$$\begin{aligned}&\frac{2}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} F(\varkappa)\mathcal{G}(\varkappa)\kappa_2 d_q\varkappa\\&\leq \frac{[F(\kappa_1)]^2+q[F(\kappa_2)]^2}{[2]_q}+\frac{[\mathcal{G}(\kappa_1)]^2+q[\mathcal{G}(\kappa_2)]^2}{[2]_q}.\end{aligned} \tag{40}$$

Particularly, if $F(\varkappa) = \mathcal{G}(\varkappa)$ for all $\varkappa \in [\kappa_1, \kappa_2]$, then we get

$$\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} [F(\varkappa)]^2\kappa_2 d_q\varkappa \leq \frac{[F(\kappa_1)]^2+q[F(\kappa_2)]^2}{[2]_q}. \tag{41}$$

Theorem 9. *Let $F, \mathcal{G}: [\kappa_1, \kappa_2] \longrightarrow \mathbb{R}_+$ be r_1-convex and r_2-convex functions, respectively, on $[\kappa_1, \kappa_2]$. Then, we get the following inequality:*

$$\begin{aligned}&\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} F(\varkappa)\mathcal{G}(\varkappa)^{\kappa_2} d_q\varkappa\\&\leq \left(\frac{[F(\kappa_1)]^{r_1}+[qF(\kappa_2)]^{r_1}}{[2]_q}\right)^{1/r_1}\left(\frac{[\mathcal{G}(\kappa_1)]^{r_2}+[q\mathcal{G}(\kappa_2)]^{r_2}}{[2]_q}\right)^{1/r_2},\end{aligned} \tag{42}$$

where $0 < q < 1$ and $1/r_1 + 1/r_2 = 1$ with $r_1 > 1$.

Proof. From (36), we have

$$\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} F(\varkappa)\mathcal{G}(\varkappa)^{\kappa_2} d_q\varkappa \leq \int_0^1 \left(\xi[F(\kappa_1)]^{r_1}+(1-\xi)[F(\kappa_2)]^{r_1}\right)^{1/r_1}\left(\xi[\mathcal{G}(\kappa_1)]^{r_2}+(1-\xi)[\mathcal{G}(\kappa_2)]^{r_2}\right)^{1/r_2} d_q\xi. \tag{43}$$

Using Hölder inequality for quantum integrals, we have

$$\begin{aligned}\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} F(\varkappa)\mathcal{G}(\varkappa)\kappa_2 d_q\varkappa &\leq \left(\int_0^1 \left(\xi[F(\kappa_1)]^{r_1}+(1-\xi)[F(\kappa_2)]^{r_1}\right)d_q\xi\right)^{1/r_1}\left(\int_0^1 \left(\xi[\mathcal{G}(\kappa_1)]^{r_2}+(1-\xi)[\mathcal{G}(\kappa_2)]^{r_2}\right)d_q\xi\right)^{1/r_2}\\ &= \left(\frac{[F(\kappa_1)]^{r_1}+[qF(\kappa_2)]^{r_1}}{[2]_q}\right)^{1/r_1}\left(\frac{[\mathcal{G}(\kappa_1)]^{r_2}+[q\mathcal{G}(\kappa_2)]^{r_2}}{[2]_q}\right)^{1/r_2}.\end{aligned} \tag{44}$$

This completes the proof. □

Remark 4. If we take the limit $q \longrightarrow 1^-$ in Theorem 9, then Theorem 9 reduces to Theorem 2.6 in [33].

Corollary 3. *If we choose $r_1 = r_2 = 2$ in Theorem 9, then we have the inequality*

$$\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} F(\varkappa)\mathcal{G}(\varkappa)\kappa_2 d_q\varkappa \leq \sqrt{\frac{[F(\kappa_1)]^2+[qF(\kappa_2)]^2}{[2]_q}}\sqrt{\frac{[\mathcal{G}(\kappa_1)]^2+[q\mathcal{G}(\kappa_2)]^2}{[2]_q}}. \tag{45}$$

Particularly, if $F(\varkappa)=\mathcal{G}(\varkappa)$ for all $\varkappa \in [\kappa_1,\kappa_2]$, then we get

$$\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} [F(\varkappa)]^{2\kappa_2} d_q\varkappa \leq \frac{[F(\kappa_1)]^2+[qF(\kappa_2)]^2}{[2]_q}. \tag{46}$$

4. Quantum Hermite–Hadamard Type Inequalities for Coordinated r-Convex Functions

In this section, we present several Hermite–Hadamard type inequalities for coordinated r-convex functions via $q^{\kappa_2\kappa_4}$, $q_{\kappa_1}^{\kappa_4}$, $q_{\kappa_3}^{\kappa_2}$, and $q_{\kappa_1\kappa_3}$ integrals. We also prove some quantum inequalities of Hermite–Hadamard type for the product of two coordinated r-convex functions.where $0<r_1\leq 1$ and $0<q_1,q_2<1$.

Theorem 10. *Suppose that $F\colon \Delta \longrightarrow \mathbb{R}_+$ is a positive coordinated r_1-convex function on Δ. Then, one has the inequality*

$$\begin{aligned}&\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4} F(\varkappa,y)^{\kappa_2} d_{q_1}\varkappa\kappa_4 d_{q_2}y\\ &\leq \frac{1}{[1/r_1+1]_{q_2}}\frac{1}{2(\kappa_2-\kappa_1)}\\ &\int_{\kappa_1}^{\kappa_2}\left([F(\varkappa,\kappa_3)]^{r_1}+[q_2F(\varkappa,\kappa_4)]^{r_1}\right)^{1/r_1\kappa_2} d_{q_1}\varkappa\\ &+\frac{1}{[1/r_1+1]_{q_1}}\frac{1}{2(\kappa_4-\kappa_3)}\\ &\int_{\kappa_3}^{\kappa_4}\left([F(\kappa_1,y)]^{r_1}+[q_1F(\kappa_2,y)]^{r_1}\right)^{1/r_1\kappa_4} d_{q_2}\varkappa,\end{aligned} \tag{47}$$

Proof. Since $F\colon \Delta \longrightarrow \mathbb{R}_+$ is a coordinated r_1-convex function, then the partial mappings,

$$F_\varkappa\colon [\kappa_3,\kappa_4] \longrightarrow \mathbb{R}_+, F_\varkappa(v)=F(\varkappa,v), \tag{48}$$

$$F_y\colon [\kappa_1,\kappa_2] \longrightarrow \mathbb{R}_+, F_y(u)=F(u,y), \tag{49}$$

are r_1-convex. By inequality (24), we can write

$$\frac{1}{\kappa_4-\kappa_3}\int_{\kappa_3}^{\kappa_4} F_\varkappa(y)\,{}_{\kappa_4}d_{q_2}y \leq \frac{1}{[1/r_1+1]_{q_2}}\left(\left[F_\varkappa(\kappa_3)\right]^{r_1}+\left[q_2F_\varkappa(\kappa_4)\right]^{r_1}\right)^{1/r_1}, \tag{50}$$

i.e.,

$$\frac{1}{\kappa_4-\kappa_3}\int_{\kappa_3}^{\kappa_4} F(\varkappa,y)\,{}_{\kappa_4}d_{q_2}y \leq \frac{1}{[1/r_1+1]_{q_2}}\left(\left[F(\varkappa,\kappa_3)\right]^{r_1}+\left[q_2F(\varkappa,\kappa_4)\right]^{r_1}\right)^{1/r_1}. \tag{51}$$

Dividing both sides of the inequality $(\kappa_2-\kappa_1)$ and q^{κ_2}-integrating with respect to $\varkappa$ on $[\kappa_1,\kappa_2]$, we get

$$\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4} F(\varkappa,y)\,{}^{\kappa_2}d_{q_1}\varkappa\,{}_{\kappa_4}d_{q_2}y \leq \frac{1}{[1/r_1+1]_{q_2}}\left[\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\left(\left[F(\varkappa,\kappa_3)\right]^{r_1}+\left[q_2F(\varkappa,\kappa_4)\right]^{r_1}\right)^{1/r_1}\,{}^{\kappa_2}d_{q_1}\varkappa\right]. \tag{52}$$

By a similar argument for the mapping

$$F_y\colon [\kappa_1,\kappa_2]\longrightarrow \mathbb{R}_+, F_y(u)=F(u,y), \tag{53}$$

we have

$$\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4} F(\varkappa,y)\,{}^{\kappa_2}d_{q_1}\varkappa\,{}^{\kappa_4}d_{q_2}y \leq \frac{1}{[1/r_1+1]_{q_1}}\left[\frac{1}{\kappa_4-\kappa_3}\int_{\kappa_3}^{\kappa_4}\left(\left[F(\kappa_1,y)\right]^{r_1}+\left[q_1F(\kappa_2,y)\right]^{r_1}\right)^{1/r_1}\,{}^{\kappa_4}d_{q_2}y\right]. \tag{54}$$

By adding inequalities (52) and (54), we can obtain inequality (47). □

Remark 5. If we take the limit $q_1\longrightarrow 1^-$ and $q_2\longrightarrow 1^-$ in Theorem 10, then Theorem 10 reduces to Theorem 5 in [15].

Remark 6. If we choose $r_1=1$ in Theorem 10, then inequality (47) reduces to the third inequality of Theorem 3.6 in [30].

Theorem 11. *Suppose that $F\colon \Delta\longrightarrow\mathbb{R}_+$ is a positive coordinated r_1-convex function on Δ. Then, one has the inequality*

$$\begin{aligned}\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4} F(\varkappa,y)\,{}_{\kappa_1}d_{q_1}\varkappa\,{}_{\kappa_3}d_{q_2}y &\leq \frac{1}{[1/r_1+1]_{q_2}}\frac{1}{2(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}\\ &\cdot\left(\left[q_2F(\varkappa,\kappa_3)\right]^{r_1}+\left[F(\varkappa,\kappa_4)\right]^{r_1}\right)^{1/r_1}\,{}_{\kappa_1}d_{q_1}\varkappa\\ &+\frac{1}{[1/r_1+1]_{q_1}}\frac{1}{2(\kappa_4-\kappa_3)}\int_{\kappa_3}^{\kappa_4}\\ &\cdot\left(\left[q_1F(\kappa_1,y)\right]^{r_1}+\left[F(\kappa_2,y)\right]^{r_1}\right)^{1/r_1}\,{}_{\kappa_3}d_{q_2}y,\end{aligned} \tag{55}$$

where $0<r_1\leq 1$ and $0<q_1,q_2<1$.

Proof. The proof is similar to the proof of Theorem 10 by using Theorem 4. □

Theorem 12. *Suppose that $F\colon \Delta\longrightarrow\mathbb{R}_+$ is a positive coordinated convex function on Δ. Then, one has the inequality*

$$\begin{aligned}\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4} F(\varkappa,y)\,{}_{\kappa_1}d_{q_1}\varkappa\,{}_{\kappa_4}d_{q_2}y &\leq \frac{1}{[1/r_1+1]_{q_2}}\frac{1}{2(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}\\ &\cdot\left(\left[q_2F(\varkappa,\kappa_3)\right]^{r_1}+\left[F(\varkappa,\kappa_4)\right]^{r_1}\right)^{1/r_1}\,{}_{\kappa_1}d_{q_1}\varkappa\\ &+\frac{1}{[1/r_1+1]_{q_1}}\frac{1}{2(\kappa_4-\kappa_3)}\int_{\kappa_3}^{\kappa_4}\\ &\cdot\left(\left[F(\kappa_1,y)\right]^{r_1}+\left[q_1F(\kappa_2,y)\right]^{r_1}\right)^{1/r_1}\,{}_{\kappa_4}d_{q_2}y,\end{aligned} \tag{56}$$

where $0<r_1\leq 1$ and $0<q_1,q_2<1$.

Proof. The proof is similar to the proof of Theorem 10 by using Theorems 4 and 7. □

Theorem 13. *Suppose that* $F: \Delta \longrightarrow \mathbb{R}_+$ *is a positive coordinated* r_1*-convex function on* Δ. *Then, one has the inequality*

$$\begin{aligned}
&\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4} F(\chi,y)^{\kappa_2}d_{q_1}\chi^{\kappa_3}d_{q_2}y\\
&\leq \frac{1}{[1/r_1+1]_{q_2}}\frac{1}{2(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}\\
&\left([F(\chi,\kappa_3)]^{r_1}+[q_2F(\chi,\kappa_4)]^{r_1}\right)^{1/r_1\,\kappa_2}d_{q_1}\chi\\
&+\frac{1}{[1/r_1+1]_{q_1}}\frac{1}{2(\kappa_4-\kappa_3)}\int_{\kappa_3}^{\kappa_4}\\
&\left([q_1F(\kappa_1,y)]^{r_1}+[F(\kappa_2,y)]^{r_1}\right)^{1/r_1}_{\kappa_3}d_{q_2},
\end{aligned}\tag{57}$$

where $0<r_1\leq 1$ *and* $0<q_1,q_2<1$.

Proof. The proof is similar to the proof of Theorem 10 by using Theorems 4 and 7. □

Theorem 14. *Suppose that* $F,\mathcal{G}: \Delta \longrightarrow \mathbb{R}_+$ *is a coordinated* r_1*-convex function and coordinated* r_2*-convex function, respectively, on* Δ. *Then, we have the inequality*

$$\begin{aligned}
&\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4} F(\varkappa,y)\mathcal{G}(\varkappa,y)^{\kappa_4}d_{q_2}y^{\kappa_2}d_{q_1}\varkappa\\
&\leq\frac{1}{4[2/r_1+1]_{q_2}}\frac{1}{\kappa_2-\kappa_1}\\
&\int_{\kappa_1}^{\kappa_2}\left([F(\varkappa,\kappa_3)]^{r_1}+\left[q_2^{1/2}F(\varkappa,\kappa_4)\right]^{r_1}\right)^{2/r_1}d_{q_1}\varkappa\\
&+\frac{1}{4[2/r_2+1]_{q_2}}\frac{1}{\kappa_2-\kappa_1}\\
&\int_{\kappa_1}^{\kappa_2}\left([\mathcal{G}(\varkappa,\kappa_3)]^{r_2}+\left[q_2^{1/2}\mathcal{G}(\varkappa,\kappa_4)\right]^{r_2}\right)^{2/r_2\,\kappa_2}d_{q_1}\varkappa\\
&+\frac{1}{4[2/r_1+1]_{q_1}}\frac{1}{\kappa_4-\kappa_3}\\
&\int_{\kappa_3}^{\kappa_4}\left([F(\kappa_1,y)]^{r_1}+\left[q_1^{1/2}F(\kappa_2,y)\right]^{r_1}\right)^{2/r_1\,\kappa_4}d_{q_2}y\\
&+\frac{1}{4[2/r_2+1]_{q_1}}\frac{1}{\kappa_4-\kappa_3}\\
&\int_{\kappa_3}^{\kappa_4}\left([\mathcal{G}(\kappa_1,y)]^{r_2}+\left[q_1^{1/2}\mathcal{G}(\kappa_2,y)\right]^{r_2}\right)^{2/r_2\,\kappa_4}d_{q_2}y,
\end{aligned}\tag{58}$$

where $0<r_1,\ r_2\leq 2$, *and* $0<q_1,q_2<1$.

Proof. Since $F: \Delta \longrightarrow \mathbb{R}_+$ is a coordinated r_1-convex on Δ, then the partial mappings,

$$\begin{aligned}
&F_\varkappa: [\kappa_3,\kappa_4]\longrightarrow\mathbb{R}_+, F_\varkappa(v)=F(\varkappa,v),\\
&F_y: [\kappa_1,\kappa_2]\longrightarrow\mathbb{R}_+, F_y(u)=F(u,y),
\end{aligned}\tag{59}$$

are r_1-convex on Δ. On the other hand, if $\mathcal{G}$ is a coordinated r_2-convex function, then the partial mappings,

$$\begin{aligned}
&\mathcal{G}_\varkappa: [\kappa_3,\kappa_4]\longrightarrow\mathbb{R}_+, \mathcal{G}_\varkappa(v)=\mathcal{G}(\varkappa,v),\\
&\mathcal{G}_y: [\kappa_1,\kappa_2]\longrightarrow\mathbb{R}_+, \mathcal{G}_y(u)=\mathcal{G}(u,y),
\end{aligned}\tag{60}$$

are r_2-convex on Δ. From inequality (32), we get

$$\begin{aligned}
&\frac{2}{\kappa_4-\kappa_3}\int_{\kappa_3}^{\kappa_4}F_\varkappa(y)\mathcal{G}_\varkappa(y)^{\kappa_4}d_{q_2}y\leq\frac{1}{[2/r_1+1]_{q_2}}\\
&\left([F_\varkappa(\kappa_3)]^{r_1}+\left[q_2^{1/2}F_\varkappa(\kappa_4)\right]^{r_1}\right)^{2/r_1}\\
&+\frac{1}{[2/r_2+1]_{q_2}}\left([\mathcal{G}_\varkappa(\kappa_3)]^{r_2}+\left[q_2^{1/2}\mathcal{G}_\varkappa(\kappa_4)\right]^{r_2}\right)^{2/r_2},
\end{aligned}\tag{61}$$

i.e.,

$$\begin{aligned}
&\frac{2}{\kappa_4-\kappa_3}\int_{\kappa_3}^{\kappa_4}F(\varkappa,y)\mathcal{G}(\varkappa,y)\kappa_4 d_{q_2}y\\
&\leq\frac{1}{[2/r_1+1]_{q_2}}\left([F(\varkappa,\kappa_3)]^{r_1}+\left[q_2^{1/2}F(\varkappa,\kappa_4)\right]^{r_1}\right)^{2/r_1}\\
&+\frac{1}{[2/r_2+1]_{q_2}}\left([\mathcal{G}(\varkappa,\kappa_3)]^{r_2}+\left[q_2^{1/2}\mathcal{G}(\varkappa,\kappa_4)\right]^{r_2}\right)^{2/r_2}.
\end{aligned}\tag{62}$$

Dividing both sides of the inequality $(\kappa_2-\kappa_1)$ and q^{κ_2}-integrating with respect to $\varkappa$ on $[\kappa_1,\kappa_2]$, we have

$$\begin{aligned}
&\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}F(\varkappa,y)\mathcal{G}(\varkappa,y)^{\kappa_4}d_{q_2}y^{\kappa_2}d_{q_1}\varkappa\\
&\leq\frac{1}{2}\frac{1}{[2/r_1+1]_{q_2}}\left[\frac{1}{\kappa_2-\kappa_1}\right.\\
&\left.\int_{\kappa_1}^{\kappa_2}\left([F(\varkappa,\kappa_3)]^{r_1}+\left[q_2^{1/2}F(\varkappa,\kappa_4)\right]^{r_1}\right)^{2/r_1\,\kappa_2}d_{q_1}\varkappa\right]\\
&+\frac{1}{2}\frac{1}{[2/r_2+1]_{q_2}}\left[\frac{1}{\kappa_2-\kappa_1}\right.\\
&\left.\int_{\kappa_1}^{\kappa_2}\left([\mathcal{G}(\varkappa,\kappa_3)]^{r_2}+\left[q_2^{1/2}\mathcal{G}(\varkappa,\kappa_4)\right]^{r_2}\right)^{2/r_2\,\kappa_2}d_{q_1}\varkappa\right].
\end{aligned}\tag{63}$$

By a similar argument, we obtain

$$\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4} F(\varkappa,y)\mathscr{G}(\varkappa,y)^{\kappa_4}d_{q_2}y^{\kappa_2}d_{q_1}\varkappa$$
$$\leq\frac{1}{2}\frac{1}{[2/r_1+1]_{q_1}}\left[\frac{1}{\kappa_4-\kappa_3}\int_{\kappa_3}^{\kappa_4}\left(\left[F(\kappa_1,y)\right]^{r_1}+\left[q_1^{1/2}F(\kappa_2,y)\right]^{r_1}\right)^{2/r_1\,\kappa_4}d_{q_2}y\right]$$
$$+\frac{1}{2}\frac{1}{[2/r_2+1]_{q_1}}\left[\frac{1}{\kappa_4-\kappa_3}\int_{\kappa_3}^{\kappa_4}\left(\left[\mathscr{G}(\kappa_1,y)\right]^{r_2}+\left[q_1^{1/2}\mathscr{G}(\kappa_2,y)\right]^{r_2}\right)^{2/r_2\,\kappa_4}d_{q_2}y\right]. \tag{64}$$

By adding inequalities (63) and (64), we obtain the required result. □

Remark 7. If we take the limit $q_1 \longrightarrow 1^-$ and $q_2 \longrightarrow 1^-$ in Theorem 14, then Theorem 14 reduces to Theorem 6 in [15].

Corollary 4. *If we choose $r_1 = r_2 = 2$ in Theorem 14, then we have the inequality*

$$\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4} F(\varkappa,y)\mathscr{G}(\varkappa,y)^{\kappa_4}d_{q_2}y^{\kappa_2}d_{q_1}\varkappa$$
$$\leq\frac{1}{4(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}\left(\frac{\left[F(\varkappa,\kappa_3)\right]^2+q_2\left[F(\varkappa,\kappa_4)\right]^2}{[2]_{q_2}}\right)^{\kappa_2}d_{q_1}\varkappa$$
$$+\frac{1}{4(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}\left(\frac{\left[\mathscr{G}(\varkappa,\kappa_3)\right]^2+q_2\left[\mathscr{G}(\varkappa,\kappa_4)\right]^2}{[2]_{q_2}}\right)^{\kappa_2}d_{q_1}\varkappa$$
$$+\frac{1}{4(\kappa_4-\kappa_3)}\int_{\kappa_3}^{\kappa_4}\left(\frac{\left[F(\kappa_1,y)\right]^2+q_1\left[F(\kappa_2,y)\right]^2}{[2]_{q_1}}\right)^{\kappa_4}d_{q_2}y$$
$$+\frac{1}{4(\kappa_4-\kappa_3)}\int_{\kappa_3}^{\kappa_4}\left(\frac{\left[\mathscr{G}(\kappa_1,y)\right]^2+q_1\left[\mathscr{G}(\kappa_2,y)\right]^2}{[2]_{q_1}}\right)^{\kappa_4}d_{q_2}y. \tag{65}$$

Particularly, if $F(\varkappa,y)=\mathscr{G}(\varkappa,y)$ for all $(\varkappa,y)\in\Delta$, then we get

$$\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\left[F(\varkappa,y)\right]^{2\,\kappa_4}d_{q_2}y^{\kappa_2}d_{q_1}\varkappa$$
$$\leq\frac{1}{2(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}\left(\frac{\left[F(\varkappa,\kappa_3)\right]^2+q_2\left[F(\varkappa,\kappa_4)\right]^2}{[2]_{q_2}}\right)^{\kappa_2}d_{q_1}\varkappa$$
$$+\frac{1}{2(\kappa_4-\kappa_3)}\int_{\kappa_3}^{\kappa_4}\left(\frac{\left[F(\kappa_1,y)\right]^2+q_1\left[F(\kappa_2,y)\right]^2}{[2]_{q_1}}\right)^{\kappa_4}d_{q_2}y. \tag{66}$$

Theorem 15. *Suppose that $F,\mathscr{G}\colon \Delta \longrightarrow \mathbb{R}_+$ is a coordinated r_1-convex function and coordinated r_2-convex function, respectively, on Δ. Then, we have the inequality*

$$\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4} F(\varkappa,y)\mathscr{G}(\varkappa,y)_{\kappa_3}d_{q_2}y^{\kappa_2}d_{q_1}\varkappa$$
$$\leq\frac{1}{4[2/r_1+1]_{q_2}}\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\left(\left[F(\varkappa,\kappa_3)\right]^{r_1}+\left[q_2^{1/2}F(\varkappa,\kappa_4)\right]^{r_1}\right)^{2/r_1\,\kappa_2}d_{q_1}\varkappa$$
$$+\frac{1}{4[2/r_2+1]_{q_2}}\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\left(\left[\mathscr{G}(\varkappa,\kappa_3)\right]^{r_2}+\left[q_2^{1/2}\mathscr{G}(\varkappa,\kappa_4)\right]^{r_2}\right)^{2/r_2\,\kappa_2}d_{q_1}\varkappa \tag{67}$$
$$+\frac{1}{4[2/r_1+1]_{q_1}}\frac{1}{\kappa_4-\kappa_3}\int_{\kappa_3}^{\kappa_4}\left(\left[q_1^{1/2}F(\kappa_1,y)\right]^{r_1}+\left[F(\kappa_2,y)\right]^{r_1}\right)^{2/r_1}{}_{\kappa_3}d_{q_2}y$$
$$+\frac{1}{4[2/r_2+1]_{q_1}}\frac{1}{\kappa_4-\kappa_3}\int_{\kappa_3}^{\kappa_4}\left(\left[q_1^{1/2}\mathscr{G}(\kappa_1,y)\right]^{r_2}+\left[\mathscr{G}(\kappa_2,y)\right]^{r_2}\right)^{2/r_2}{}_{\kappa_3}d_{q_2}y,$$

where $0 < r_1,\ r_2 \leq 2$, *and* $0 < q_1, q_2 < 1$.

Proof. The proof is similar to the proof of Theorem 14 by using Theorems 5 and 8. ☐

Theorem 16. *Suppose that* $F, \mathcal{G}\colon \Delta \longrightarrow \mathbb{R}_+$ *is a coordinated* r_1*-convex function and coordinated* r_2*-convex function, respectively, on* Δ. *Then, we have the inequality*

$$\begin{aligned}
&\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4} F(\varkappa,y)\mathcal{G}(\varkappa,y)\,{}_{\kappa_3}d_{q_2}y\,{}_{k_1}d_{q_1}\varkappa\\
&\leq \frac{1}{4[2/r_1+1]_{q_2}}\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\left(\left[q_2^{1/2}F(\varkappa,\kappa_3)\right]^{r_1}+\left[F(\varkappa,\kappa_4)\right]^{r_1}\right)^{2/r_1}{}_{k_1}d_{q_1}\varkappa\\
&+\frac{1}{4[2/r_2+1]_{q_2}}\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\left(\left[q_2^{1/2}\mathcal{G}(\varkappa,\kappa_3)\right]^{r_2}+\left[\mathcal{G}(\varkappa,\kappa_4)\right]^{r_2}\right)^{2/r_2}{}_{k_1}d_{q_1}\varkappa\\
&+\frac{1}{4[2/r_1+1]_{q_1}}\frac{1}{\kappa_4-\kappa_3}\int_{\kappa_3}^{\kappa_4}\left(\left[q_1^{1/2}F(\kappa_1,y)\right]^{r_1}+\left[F(\kappa_2,y)\right]^{r_1}\right)^{2/r_1}{}_{\kappa_3}d_{q_2}y\\
&+\frac{1}{4[2/r_2+1]_{q_1}}\frac{1}{\kappa_4-\kappa_3}\int_{\kappa_3}^{\kappa_4}\left(\left[q_1^{1/2}\mathcal{G}(\kappa_1,y)\right]^{r_2}+\left[\mathcal{G}(\kappa_2,y)\right]^{r_2}\right)^{2/r_2}{}_{\kappa_3}d_{q_2}y,
\end{aligned}\tag{68}$$

where $0 < r_1,\ r_2 \leq 2$, *and* $0 < q_1, q_2 < 1$.

Proof. The proof is similar to the proof of Theorem 14 by using Theorem 5. ☐

Theorem 17. *Suppose that* $F, \mathcal{G}\colon \Delta \longrightarrow \mathbb{R}_+$ *is a coordinated* r_1*-convex function and coordinated* r_2*-convex function, respectively, on* Δ. *Then, we have the inequality*

$$\begin{aligned}
&\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4} F(\varkappa,y)\mathcal{G}(\varkappa,y)\,{}^{\kappa_4}d_{q_2}y\,{}_{k_1}d_{q_1}\varkappa\\
&\leq \frac{1}{4[2/r_1+1]_{q_2}}\frac{1}{\kappa_2-\kappa_1}\\
&\int_{\kappa_1}^{\kappa_2}\left(\left[q_2^{1/2}F(\varkappa,\kappa_3)\right]^{r_1}+\left[F(\varkappa,\kappa_4)\right]^{r_1}\right)^{2/r_1}{}_{k_1}d_{q_1}\varkappa\\
&+\frac{1}{4[2/r_2+1]_{q_2}}\frac{1}{\kappa_2-\kappa_1}\\
&\int_{\kappa_1}^{\kappa_2}\left(\left[q_2^{1/2}\mathcal{G}(\varkappa,\kappa_3)\right]^{r_2}+\left[\mathcal{G}(\varkappa,\kappa_4)\right]^{r_2}\right)^{2/r_2}{}_{k_1}d_{q_1}\varkappa\\
&+\frac{1}{4[2/r_1+1]_{q_1}}\frac{1}{\kappa_4-\kappa_3}\\
&\int_{\kappa_3}^{\kappa_4}\left(\left[F(\kappa_1,y)\right]^{r_1}+\left[q_1^{1/2}F(\kappa_2,y)\right]^{r_1}\right)^{2/r_1}{}^{\kappa_4}d_{q_2}y\\
&+\frac{1}{4[2/r_2+1]_{q_1}}\frac{1}{\kappa_4-\kappa_3}\\
&\int_{\kappa_3}^{\kappa_4}\left(\left[\mathcal{G}(\kappa_1,y)\right]^{r_2}+\left[q_1^{1/2}\mathcal{G}(\kappa_2,y)\right]^{r_2}\right)^{2/r_2}{}^{\kappa_4}d_{q_2}y,
\end{aligned}\tag{69}$$

where $0 > r_1,\ r_2 \leq 2$, *and* $0 < q_1, q_2 < 1$.

Proof. The proof is similar to the proof of Theorem 14 by using Theorems 5 and 8. ☐

Theorem 18. *Suppose that* $F, \mathcal{G}\colon \Delta \longrightarrow \mathbb{R}_+$ *is a coordinated* r_1*-convex function and coordinated* r_2 *-convex function, respectively, on* Δ. *Then, we have the inequality*

$$\begin{aligned}
&\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4} F(\varkappa,y)\mathcal{G}(\varkappa,y)\,{}^{\kappa_4}d_{q_2}\,{}^{\kappa_2}d_{q_1}\varkappa\\
&\leq\frac{1}{2}\left(\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\left(\frac{\left[F(\varkappa,\kappa_3)\right]^{r_1}+\left[q_2F(\varkappa,\kappa_4)\right]^{r_1}}{[2]_{q_2}}\right)^{1/r_1}{}^{\kappa_2}d_{q_1}\varkappa\right)\\
&\times\left(\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\left(\frac{\left[\mathcal{G}(\varkappa,\kappa_3)\right]^{r_2}+\left[q_2\mathcal{G}(\varkappa,\kappa_4)\right]^{r_2}}{[2]_{q_2}}\right)^{1/r_2}{}^{\kappa_2}d_{q_1}\varkappa\right)\\
&+\frac{1}{2}\left(\frac{1}{\kappa_4-\kappa_3}\int_{\kappa_3}^{\kappa_4}\left(\frac{\left[F(\kappa_1,y)\right]^{r_1}+\left[q_1F(\kappa_2,y)\right]^{r_1}}{[2]_{q_1}}\right)^{1/r_1}{}^{\kappa_4}d_{q_2}\right)\\
&\times\left(\frac{1}{\kappa_4-\kappa_3}\int_{\kappa_3}^{\kappa_4}\left(\frac{\left[\mathcal{G}(\kappa_1,y)\right]^{r_2}+\left[q_1\mathcal{G}(\kappa_2,y)\right]^{r_2}}{[2]_{q_1}}\right)^{1/r_2}{}^{\kappa_4}d_{q_2}\right).
\end{aligned}\tag{70}$$

where $0 < q_1, q_2 < 1$, *and* $1/r_1 + 1/r_2 = 1$ *with* $r_1 > 1$.

Proof. By applying inequality (42) for the partial mapping $F_\varkappa$ and $\mathcal{G}_\varkappa$, we can write

$$\frac{1}{(\kappa_4-\kappa_3)}\int_{\kappa_3}^{\kappa_4}\mathcal{F}(\varkappa,y)\mathcal{G}(\varkappa,y)^{\kappa_4}d_{q_2}y$$
$$\le\left(\frac{[\mathcal{F}(\varkappa,\kappa_3)]^{r_1}+[q_2\mathcal{F}(\varkappa,\kappa_4)]^{r_1}}{[2]_{q_2}}\right)^{1/r_1} \quad (71)$$
$$\cdot\left(\frac{[\mathcal{G}(\varkappa,\kappa_3)]^{r_2}+[q_2\mathcal{G}(\varkappa,\kappa_4)]^{r_2}}{[2]_{q_2}}\right)^{1/r_2}.$$

By using q^{κ_2}-integral, we obtain

$$\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\mathcal{F}(\varkappa,y)\mathcal{G}(\varkappa,y)^{\kappa_4}d_{q_2}y^{\kappa_2}d_{q_1}\varkappa$$
$$\le\left(\frac{1}{(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}\left(\frac{[\mathcal{F}(\varkappa,\kappa_3)]^{r_1}+[q_2\mathcal{F}(\varkappa,\kappa_4)]^{r_1}}{[2]_{q_2}}\right)^{1/r_1}{}^{\kappa_2}d_{q_1}\varkappa\right)$$
$$\times\left(\frac{1}{(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}\left(\frac{[\mathcal{G}(\varkappa,\kappa_3)]^{r_2}+[q_2\mathcal{G}(\varkappa,\kappa_4)]^{r_2}}{[2]_{q_2}}\right)^{\kappa_2}d_{q_1}\varkappa\right). \quad (72)$$

Similarly, by applying inequality (42) for the partial mapping $\mathcal{F}_y$ and $\mathcal{G}_y$, we can write

$$\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\mathcal{F}(\varkappa,y)\mathcal{G}(\varkappa,y)^{\kappa_4}d_{q_2}y^{\kappa_2}d_{q_1}\varkappa$$
$$\le\left(\frac{1}{(\kappa_4-\kappa_3)}\int_{\kappa_3}^{\kappa_4}\left(\frac{[\mathcal{F}(\kappa_1,y)]^{r_1}+[q_1\mathcal{F}(\kappa_2,y)]^{r_1}}{[2]_{q_1}}\right)^{1/r_1}{}^{\kappa_4}d_{q_2}y\right)$$
$$\times\left(\frac{1}{(\kappa_4-\kappa_3)}\int_{\kappa_3}^{\kappa_4}\left(\frac{[\mathcal{G}(\kappa_1,y)]^{r_2}+[q_1\mathcal{G}(\kappa_2,y)]^{r_2}}{[2]_{q_1}}\right)^{1/r_2}{}^{\kappa_4}d_{q_2}y\right). \quad (73)$$

By adding inequalities (72) and (73), we obtain the desired result (70). □

Remark 8. If we take the limit $q_1\longrightarrow 1^-$ and $q_2\longrightarrow 1^-$ in Theorem 18, then Theorem 18 reduces to Theorem 7 in [15].

Corollary 5. *If we choose $r_1=r_2=2$ in Theorem 18, then we have the inequality*

$$\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\mathcal{F}(\varkappa,y)\mathcal{G}(\varkappa,y)^{\kappa_4}d_{q_2}y^{\kappa_2}d_{q_1}\varkappa$$
$$\le\frac{1}{2}\left(\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\sqrt{\frac{[\mathcal{F}(\varkappa,\kappa_3)]^2+[q_2\mathcal{F}(\varkappa,\kappa_4)]^2}{[2]_{q_2}}}{}^{\kappa_2}d_{q_1}\varkappa\right)$$
$$\times\left(\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\sqrt{\frac{[\mathcal{G}(\varkappa,\kappa_3)]^2+[q_2\mathcal{G}(\varkappa,\kappa_4)]^2}{[2]_{q_2}}}{}^{\kappa_2}d_{q_1}\varkappa\right)$$
$$+\frac{1}{2}\left(\frac{1}{\kappa_4-\kappa_3}\int_{\kappa_3}^{\kappa_4}\sqrt{\frac{[\mathcal{F}(\kappa_1,y)]^2+[q_1\mathcal{F}(\kappa_2,y)]^2}{[2]_{q_1}}}{}^{\kappa_4}d_{q_2}y\right)$$
$$\times\left(\frac{1}{\kappa_4-\kappa_3}\int_{\kappa_3}^{\kappa_4}\sqrt{\frac{[\mathcal{G}(\kappa_1,y)]^2+[q_1\mathcal{G}(\kappa_2,y)]^2}{[2]_{q_1}}}{}^{\kappa_4}d_{q_2}y\right). \quad (74)$$

Theorem 19. *Suppose that $\mathcal{F},\mathcal{G}\colon\Delta\longrightarrow\mathbb{R}_+$ is a coordinated r_1-convex function and coordinated r_2-convex function, respectively, on Δ. Then, we have the inequality*

$$\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\mathcal{F}(\varkappa,y)\mathcal{G}(\varkappa,y)_{k_3}d_{q_2}y^{\kappa_2}d_{q_1}\varkappa$$
$$\le\frac{1}{2}\left(\frac{1}{(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}\left(\frac{[\mathcal{F}(\varkappa,\kappa_3)]^{r_1}+[q_2\mathcal{F}(\varkappa,\kappa_4)]^{r_1}}{[2]_{q_2}}\right)^{1/r_1}{}^{\kappa_2}d_{q_1}\varkappa\right)$$
$$\times\left(\frac{1}{(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}\left(\frac{[\mathcal{G}(\varkappa,\kappa_3)]^{r_2}+[q\mathcal{G}(\varkappa,\kappa_4)]^{r_2}}{[2]_{q_2}}\right)^{1/r_2}{}^{\kappa_2}d_{q_1}\varkappa\right)$$
$$+\frac{1}{2}\left(\frac{1}{(\kappa_4-\kappa_3)}\int_{\kappa_3}^{\kappa_4}\left(\frac{[q_1\mathcal{F}(\kappa_1,y)]^{r_1}+[\mathcal{F}(\kappa_2,y)]^{r_1}}{[2]_{q_1}}\right)^{1/r_1}{}_{k_3}d_{q_2}y\right)$$
$$\times\left(\frac{1}{(\kappa_4-\kappa_3)}\int_{\kappa_3}^{\kappa_4}\left(\frac{[q_1\mathcal{G}(\kappa_1,y)]^{r_2}+[\mathcal{G}(\kappa_2,y)]^{r_2}}{[2]_{q_1}}\right)^{1/r_2}{}_{k_3}d_{q_2}y\right). \quad (75)$$

where $0<q_1,q_2<1$, and $1/r_1+1/r_2=1$ with $r_1>1$.

Proof. The proof is similar to the proof of Theorem 18 by using Theorems 6 and 9. □

Theorem 20. *Suppose that $\mathcal{F},\mathcal{G}\colon\Delta\longrightarrow\mathbb{R}_+$ is a coordinated r_1-convex function and coordinated r_2-convex function, respectively, on Δ. Then, we have the inequality*

$$\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\mathcal{F}(\varkappa,y)\mathcal{G}(\varkappa,y)^{\kappa_4}d_{q_2}y_{\kappa_1}d_{q_1}\varkappa$$
$$\le\frac{1}{2}\left(\frac{1}{(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}\left(\frac{[q_2\mathcal{F}(\varkappa,\kappa_3)]^{r_1}+[\mathcal{F}(\varkappa,\kappa_4)]^{r_1}}{[2]_{q_2}}\right)^{1/r_1}{}_{\kappa_1}d_{q_1}\varkappa\right)$$
$$\times\left(\frac{1}{(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}\left(\frac{[q_2\mathcal{G}(\varkappa,\kappa_3)]^{r_2}+[\mathcal{G}(\varkappa,\kappa_4)]^{r_2}}{[2]_{q_2}}\right)^{1/r_2}{}_{\kappa_1}d_{q_1}\varkappa\right)$$
$$+\frac{1}{2}\left(\frac{1}{(\kappa_4-\kappa_3)}\int_{\kappa_3}^{\kappa_4}\left(\frac{[\mathcal{F}(\kappa_1,y)]^{r_1}+[q_1\mathcal{F}(\kappa_2,y)]^{r_1}}{[2]_{q_1}}\right)^{1/r_1}{}^{\kappa_4}d_{q_2}y\right)$$
$$\times\left(\frac{1}{(\kappa_4-\kappa_3)}\int_{\kappa_3}^{\kappa_4}\left(\frac{[\mathcal{G}(\kappa_1,y)]^{r_2}+[q_1\mathcal{G}(\kappa_2,y)]^{r_2}}{[2]_{q_1}}\right)^{1/r_2}{}^{\kappa_4}d_{q_2}y\right), \quad (76)$$

where $0<q_1,q_2<1$, and $1/r_1+1/r_2=1$ with $r_1>1$.

Proof. The proof is similar to the proof of Theorem 18 by using Theorems 6 and 9. □

Theorem 21. *Suppose that $\mathcal{F},\mathcal{G}\colon\Delta\longrightarrow\mathbb{R}_+$ is a coordinated r_1-convex function and coordinated r_2 -convex function, respectively, on Δ. Then, we have the inequality*

$$\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4} F(\varkappa,y)\mathscr{G}(\varkappa,y)\,{}_{\kappa_3}d_{q_2}y\,{}_{\kappa_1}d_{q_1}\varkappa$$
$$\le\frac{1}{2}\left(\frac{1}{(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}\left(\frac{[q_2F(\varkappa,\kappa_3)]^{r_1}+[F(\varkappa,\kappa_4)]^{r_1}}{[2]_{q_2}}\right)^{1/r_1}{}_{\kappa_1}d_{q_1}\varkappa\right)$$
$$\times\left(\frac{1}{(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}\left(\frac{[q_2\mathscr{G}(\varkappa,\kappa_3)]^{r_2}+[\mathscr{G}(\varkappa,\kappa_4)]^{r_2}}{[2]_{q_2}}\right)^{1/r_2}{}_{\kappa_1}d_{q_1}\varkappa\right)$$
$$+\frac{1}{2}\left(\frac{1}{(\kappa_4-\kappa_3)}\int_{\kappa_3}^{\kappa_4}\left(\frac{[q_1F(\kappa_1,y)]^{r_1}+[F(\kappa_2,y)]^{r_1}}{[2]_{q_1}}\right)^{1/r_1}{}_{\kappa_3}d_{q_2}y\right)$$
$$\times\left(\frac{1}{(\kappa_4-\kappa_3)}\int_{\kappa_3}^{\kappa_4}\left(\frac{[q_1\mathscr{G}(\kappa_1,y)]^{r_2}+[\mathscr{G}(\kappa_2,y)]^{r_2}}{[2]_{q_1}}\right)^{1/r_2}{}_{\kappa_3}d_{q_2}y\right). \tag{77}$$

where $0<q_1,q_2<1$, *and* $1/r_1+1/r_2=1$ *with* $r_1>1$.

Proof. The proof is similar to the proof of Theorem 18 by using Theorem 6. □

5. Conclusions

In this study, we present several quantum Hermite–Hadamard type inequalities for r-convex functions and coordinated r-convex functions. We also give some quantum inequalities for the product of two r-convex functions and for the product of two coordinated r-convex functions. In the future work, we can establish the similar quantum inequalities by using generalized r-convex functions.

Data Availability

No datasets were generated or analysed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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