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Research Article

A Novel of Ideals and Fuzzy Ideals of Gamma-Semigroups

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In this paper, we define new types of ideals, fuzzy ideals, almost ideals, and fuzzy almost ideals of Γ-semigroups by using the elements of Γ. We investigate properties of them.

1. Introduction and Preliminaries

Ideal theory in semigroups, like all other algebraic structures, plays an important role in studying them. Bi-ideals in semigroups were introduced by Good and Hughes [1] in 1952. Steinfeld [2] gave a notion and studied quasi-ideals in semigroups in 1956. In 1965, Zadeh introduced the concept of fuzzy subsets in [3]. Since then, fuzzy subsets are now applied in various fields. Kuroki [4] first studied fuzzy ideals in semigroups. Almost ideals (A-ideals) in semigroups were studied by Grošek and Satko [5–7] in 1980-1981. Next year, Bogdanović [8] studied almost bi-ideals in semigroups by using concepts of almost ideals and bi-ideals in semigroups. Sen [9] introduced Γ-semigroups where Γ is a set of their operations. This algebraic structure is generalization of semigroups. Many results in semigroup theory were generalized to the results in Γ-semigroup theory. In 2006, Chinram [10] studied quasi-Γ-ideals in Γ-semigroups, and Jirotkul joined with Chinram [11] to study bi-Γ-ideals in Γ-semigroups in 2007. Moreover, Iampan [12] gave remarkable notes on bi-Γ-ideals (or bi-ideals) in Γ-semigroups in 2009. Recently, left almost ideals (left A-ideals) in Γ-semigroups were studied by Wattanatripop and Changphas [13] in 2017. Wattanatripop et al. [14] studied fuzzy almost bi-ideals, almost quasi-ideals, and fuzzy almost ideals in semigroups in 2018. So, all of these give the inspiration to study about new types of ideals and fuzzy ideals in Γ-semigroups in this paper.

First, we recall the definition of Γ-semigroups which was defined by Sen and Saha [15].

Definition 1 (see [15]). Let S and Γ be nonempty sets. Then, S is called a Γ-semigroup if there exists a mapping $S \times \Gamma \times S \longrightarrow S$ written as $(a, \gamma, b) \longmapsto a\gamma b$ satisfying the axiom $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$.

Remark 1

(1) In case $|\Gamma| = 1$, the definition of Γ-semigroup is a semigroup
(2) Every semigroup $(S, \cdot)$ can be considered to be a Γ-semigroup where $\Gamma := \{\cdot\}$
(3) If S is a Γ-semigroup, then for each $\alpha \in \Gamma$, (S, α) is a semigroup

Let S be a Γ-semigroup. For nonempty subsets A, B of S, let

$$A\Gamma B = \{a\alpha b \mid a \in A, b \in B, \alpha \in \Gamma\}. \quad (1)$$

If $x \in S$ and $\alpha \in \Gamma$, we let $A\Gamma x := A\Gamma\{x\}$, $x\Gamma A := \{x\}\Gamma A$, and $A\alpha B := A\{\alpha\}B$.

Definition 2. Let S be a Γ-semigroup. A nonempty subset A of S is called

(1) A sub-Γ-semigroup of S if $A\Gamma A \subseteq A$
(2) A left ideal of S if $S\Gamma A \subseteq A$
(3) A right ideal of S if $A\Gamma S \subseteq A$
(4) An ideal of S if it is both a left ideal and a right ideal of S

(5) A quasi-ideal of S if $S\Gamma A\cap A\Gamma S\subseteq A$

(6) A bi-ideal of S if A is a sub-Γ-semigroup of S and $A\Gamma S\Gamma A\subseteq A$

Recently, Wattanatripop and Changphas [13] defined the concepts of left almost ideals and right almost ideals of Γ-semigroups. A Γ-semigroup containing no proper left (respectively, right) almost ideals was characterized.

Now, we recall the definitions and some notations of fuzzy subsets. A fuzzy subset of a set S is a function from S into the closed interval $[0,1]$. For any two fuzzy subsets f and g of a set S,

(1) $f\cap g$ is a fuzzy subset of S defined by

$$(f\cap g)(x)=\min\{f(x),g(x)\},\quad \text{for all } x\in S. \tag{2}$$

(2) $f\cup g$ is a fuzzy subset of S defined by

$$(f\cup g)(x)=\max\{f(x),g(x)\},\quad \text{for all } x\in S. \tag{3}$$

(3) $f\circ g$ is a fuzzy subset of S defined by

$$(f\circ g)(x)=\begin{cases}\sup\limits_{x=ab}\{\min\{f(a),g(b)\}\}, & \text{if } x\in S^2,\\ 0, & \text{otherwise.}\end{cases} \tag{4}$$

(4) $f\subseteq g$ if $f(x)\leq g(x)$ for all $x\in S$.

For a fuzzy subset f of a set S, the support of f is defined by

$$\operatorname{supp}(f)=\{x\in S\,|\,f(x)\neq 0\}. \tag{5}$$

The characteristic mapping of a subset A of S is a fuzzy subset of S defined by

$$C_A(x)=\begin{cases}1, & \text{if } x\in A,\\ 0, & \text{otherwise.}\end{cases} \tag{6}$$

The definition of fuzzy points was given by Pu and Liu [16]. For $x\in S$ and $t\in(0,1]$, a fuzzy point x_t of a set S is a fuzzy subset of S defined by

$$x_t(y)=\begin{cases}t, & \text{if } y=x,\\ 0, & \text{otherwise.}\end{cases} \tag{7}$$

Some basic concepts of fuzzy semigroup theory can be seen in [17].

For a Γ-semigroup S, let $\mathcal{F}(S)$ be the set of all fuzzy subsets of S. For each $\alpha\in\Gamma$, define a binary operation $\circ_\alpha$ on $\mathcal{F}(S)$ by

$$(f\circ_\alpha g)(x)=\begin{cases}\sup\limits_{x=a\alpha b}\{\min\{f(a),g(b)\}\}, & \text{if } x\in S\alpha S,\\ 0, & \text{otherwise.}\end{cases} \tag{8}$$

Let $\Gamma^*:=\{\circ_\alpha\,|\,\alpha\in\Gamma\}$. Then, $(\mathcal{F}(S),\Gamma^*)$ is a Γ-semigroup.

In 2017, Wattanatripop and Changphas [13] defined the concepts of left A-ideals and right A-ideals (almost left ideals and almost right ideals) of a Γ-semigroup as follows.

A nonempty subset G_L $[G_R]$ of a Γ-semigroup S is called a left (right) A-ideal of S if

$$sG_L\cap G_L\neq\varnothing\,[G_Rs\cap G_R\neq\varnothing],\quad \text{for all } s\in S. \tag{9}$$

In 1981, Bogdanović [8] gave the definition of almost bi-ideals of semigroups as follows.

A nonempty subset B of a semigroup S is called an almost bi-ideal of S if

$$BsB\cap B\neq\varnothing,\quad \text{for all } s\in S. \tag{10}$$

In 2018, Wattanatripop et al. [14, 18] introduced the notions of almost quasi-ideals, fuzzy almost bi-ideals, fuzzy almost left (right) ideals, and fuzzy almost quasi-ideals in semigroups as follows.

A nonempty subset Q of a semigroup S is called an almost quasi-ideal of S if

$$(sQ\cap Qs)\cap Q\neq\varnothing,\quad \text{for all } s\in S. \tag{11}$$

Let f be a fuzzy subset of a semigroup S such that $f\neq 0$. Then, f is called

(1) A fuzzy almost bi-ideal of S if for all $s\in S$,

$$f\circ C_s\circ f\cap f\neq 0. \tag{12}$$

(2) A fuzzy almost left ideal of S if for all $s\in S$,

$$C_s\circ f\cap f\neq 0. \tag{13}$$

(3) A fuzzy almost right ideal of S if for all $s\in S$,

$$f\circ C_s\cap f\neq 0. \tag{14}$$

(4) A fuzzy almost quasi-ideal of S if for all $s\in S$,

$$(C_s\circ f\cap f\circ C_s)\cap f\neq 0. \tag{15}$$

In 1981, Kuroki [4] introduced the notion of fuzzy ideals of semigroups as follows: A fuzzy subset f of a semigroup S is called:

(1) A fuzzy left ideal of S if $f(ab)\geq f(b)$ for all $a,b\in S$.

(2) A fuzzy right ideal of S if $f(ab)\geq f(a)$ for all $a,b\in S$.

(3) A fuzzy ideal of S if it is both a fuzzy left ideal and a fuzzy right ideal of S.

The aim of this paper is to define new types of ideals and fuzzy ideals of a Γ-semigroup S by using elements in Γ. In Section 2, we consider new types of ideals of S. In Section 3, we study new types of fuzzy ideals of S. In Section 4, we consider new types of almost ideals of S. In Section 5, we study new types of fuzzy almost ideals of S.

2. New Types of Ideals

In this section, we will focus on (α,β)-ideals, (α,β)-quasi-ideals, and (α,β)-bi-ideals of Γ-semigroups for $\alpha,\beta\in\Gamma$.

2.1. (α, β)-Ideals. First, we will define (α, β)-ideals of Γ-semigroups as follows.

Definition 3. Let S be a Γ-semigroup, A be a nonempty subset of S, and $\alpha, \beta \in \Gamma$. Then, A is called

(1) A left α-ideal of S if $S\alpha A \subseteq A$.
(2) A right β-ideal of S if $A\beta S \subseteq A$.
(3) An (α, β)-ideal of S if it is both a left α-ideal and a right β-ideal of S.
(4) An α-ideal of S if it is an (α, α)-ideal of S.

Remark 2

(1) Every left ideal of a Γ-semigroup S is a left α-ideal of S for all $\alpha \in \Gamma$.
(2) Every right ideal of a Γ-semigroup S is a right β-ideal of S for all $\beta \in \Gamma$.
(3) Every ideal of a Γ-semigroup S is an (α, β)-ideal of S for all $\alpha, \beta \in \Gamma$.

However, the converse of Example 1 is not generally true. We can see in the following example.

Example 1. Let $S = \Gamma = \mathbb{N}$ and $(a, \gamma, b) \longmapsto a + \gamma + b$ for all $a, b \in S$ and $\gamma \in \Gamma$. Then, S is a Γ-semigroup. Let $A = \{1\} \cup \{6, 7, 8, 9, \ldots\}$. It is easy to show that A is a left 4-ideal but not a left ideal of S.

Theorem 1. *The following statements are true:*

(1) If L is a left α-ideal of a Γ-semigroup S, then L is a left ideal of a semigroup (S, α).
(2) If R is a right β-ideal of a Γ-semigroup S, then R is a right ideal of a semigroup (S, β).
(3) If I is an α-ideal of a Γ-semigroup S, then I is an ideal of a semigroup (S, α).

For a nonempty subset A of a Γ-semigroup S, let $(A)_{l(\alpha)}$, $(A)_{r(\beta)}$, and $(A)_{i(\alpha,\beta)}$ be the left α-ideal, the right β-ideal, and the (α, β)-ideal of S generated by A, respectively.

Theorem 2. *Let A be a nonempty subset of a Γ-semigroup S. Then,*

(1) $(A)_{l(\alpha)} = A \cup S\alpha A$.
(2) $(A)_{r(\beta)} = A \cup A\beta S$.
(3) $(A)_{i(\alpha,\beta)} = A \cup S\alpha A \cup A\beta S \cup S\alpha A\beta S$.

Proof

(1) Let A be a nonempty subset of a Γ-semigroup S. Let $L = A \cup S\alpha A$. Clearly, $A \subseteq L$. Since S is a Γ-semigroup, $S\alpha L = S\alpha(A \cup S\alpha A) = S\alpha A \cup S\alpha S\alpha A = S\alpha A \subseteq L$. Therefore, L is a left α-ideal of S. Next, let C be any left α-ideal of S containing A. Since C is a left α-ideal of S and $A \subseteq C$, $S\alpha A \subseteq C$. Therefore, $L = A \cup S\alpha A \subseteq C$. Hence, L is the smallest left α-ideal of S containing A. Therefore, $(A)_{l(\alpha)} = L = A \cup S\alpha A$, as required.

The proofs of (2) and (3) are similar to the proof of (1).

Theorem 3. *Let L be a left α-ideal and R a right β-ideal of a Γ-semigroup S. Then, $L\gamma R$ is an (α, β)-ideal of S for all $\gamma \in \Gamma$.*

Proof. Let L and R be a left α-ideal and a right β-ideal of S, respectively, and let $\gamma \in \Gamma$. Clear that $L\gamma R \neq \emptyset$. We have $S\alpha(L\gamma R) = (S\alpha L)\gamma R \subseteq L\gamma R$ and $(L\gamma R)\beta S = L\gamma(R\beta S) \subseteq L\gamma R$. Therefore, $L\gamma R$ is an (α, β)-ideal of S.

Theorem 4. *Let L_1 and L_2 be two left α-ideals of a Γ-semigroup S. The following statements are true:*

(1) $L_1 \cup L_2$ is a left α-ideal of S.
(2) $L_1 \cap L_2$ is a left α-ideal of S, where $L_1 \cap L_2 \neq \emptyset$.

Proof

(1) Let L_1 and L_2 be two left α-ideals of S. Clear that $L_1 \cup L_2 \neq \emptyset$. Then, $S\alpha(L_1 \cup L_2) \subseteq S\alpha L_1 \cup S\alpha L_2 \subseteq L_1 \cup L_2$. Hence, $L_1 \cup L_2$ is a left α-ideal of S.

(2) Since $L_1 \cap L_2 \neq \emptyset$, we have $S\alpha(L_1 \cap L_2) \subseteq S\alpha L_1 \cap S\alpha L_2 \subseteq L_1 \cap L_2$. Hence, $L_1 \cap L_2$ is a left α-ideals of S.

Theorem 5. *Let R_1 and R_2 be two right β-ideals of a Γ-semigroup S. Then,*

(1) $R_1 \cup R_2$ is a right β-ideal of S.
(2) $R_1 \cap R_2$ is a right β-ideal of S, where $R_1 \cap R_2 \neq \emptyset$.

Proof. It is similar to Theorem 4.

Theorem 6. *Let I_1 and I_2 be two (α, β)-ideals of a Γ-semigroup S. Then,*

(1) $I_1 \cup I_2$ is an (α, β)-ideal of S.
(2) $I_1 \cap I_2$ is an (α, β)-ideal of S, where $I_1 \cap I_2 \neq \emptyset$.

Proof. It follows by Theorems 4 and 5.

2.2. (α, β)-Quasi-Ideals. We define (α, β)-quasi-ideals of Γ-semigroups as follows.

Definition 4. Let S be a Γ-semigroup. A nonempty subset Q of S is called

(1) An (α, β)-quasi-ideal of S if $S\alpha Q \cap Q\beta S \subseteq Q$.
(2) An α-quasi-ideal of S if it is an (α, α)-quasi-ideal of S.

Theorem 7. *Let S be a Γ-semigroup and Q_i an (α, β)-quasi-ideal of S for all $i \in I$. If $\cap_{i\in I} Q_i \neq \emptyset$, then $\cap_{i\in I} Q_i$ is an (α, β)-quasi-ideal of S.*

Proof. Let S be a Γ-semigroup and Q_i an (α, β)-quasi-ideal of S for all $i \in I$. Then, $(S\alpha \cap_{i\in I} Q_i) \cap (\cap_{i\in I} Q_i\beta S) \subseteq S\alpha Q_i \cap$

$Q_i\beta S\subseteq Q_i$ for all $i\in I$, so $(S\alpha\cap_{i\in I}Q_i)\cap(\cap_{i\in I}Q_i\beta S)\subseteq\cap_{i\in I}Q_i$. Therefore, $\cap_{i\in I}Q_i$ is an (α,β)-quasi-ideal of S.

Let A be a nonempty subset of a Γ-semigroup S. Let $(A)_{q(\alpha,\beta)}$ be the (α,β)-quasi-ideal of S generated by A.

Theorem 8. *Let A be a nonempty subset of a Γ-semigroup S. Then,*

$$(A)_{q(\alpha,\beta)}=A\cup(S\alpha A\cap A\beta S). \tag{16}$$

Proof. Let A be a nonempty subset of a Γ-semigroup S. Let $Q=A\cup(S\alpha A\cap A\beta S)$. Clearly, $A\subseteq Q$. We have $S\alpha Q\cap Q\beta S=S\alpha[A\cup(S\alpha A\cap A\beta S)]\cap[A\cup(S\alpha A\cap A\beta S)]\beta S\subseteq Q$. Therefore, Q is an (α,β)-quasi-ideal of S. □

Let C be any (α,β)-quasi-ideal of S containing A. Since C is an (α,β)-quasi-ideal of S and $A\subseteq C$, $S\alpha A\cap A\beta S\subseteq C$. Therefore, $Q=A\cup(S\alpha A\cap A\beta S)\subseteq C$.

Hence, Q is the smallest (α,β)-quasi-ideal of S containing A. Therefore, $(A)_{q(\alpha,\beta)}=Q=A\cup(S\alpha A\cap A\beta S)$, as required.

Theorem 9. *Let S be a Γ-semigroup. Let L and R be a left α-ideal and right β-ideal of S, respectively. If* $L\cap R\neq\varnothing$, *then* $L\cap R$ *is an* (α,β)*-quasi-ideal of S.*

Proof. Let L and R be a left α-ideal and a right β-ideal of a Γ-semigroup S, respectively. Then, $S\alpha(L\cap R)\cap(L\cap R)\beta S\subseteq S\alpha L\cap R\beta S\subseteq L\cap R$. Hence, $L\cap R$ is an (α,β)-quasi-ideal of S.

Corollary 1. *Let S be a Γ-semigroup. Let L and R be a left α-ideal and right α-ideal of S, respectively. Then,* $L\cap R$ *is an α-quasi-ideal of S.*

Proof. We have $R\alpha L\subseteq L\cap R$; this implies $L\cap R\neq\varnothing$. By Theorem 9, $L\cap R$ is an α-quasi-ideal of S.

Theorem 10. *Every* (α,β)*-quasi-ideal Q of a Γ-semigroup S is the intersection of a left α-ideal and a right β-ideal of S.*

Proof. Let Q be an (α,β)-quasi-ideal of a Γ-semigroup S. Let $L=Q\cup S\alpha Q$ and $R=Q\cup Q\beta S$. Then, $S\alpha L=S\alpha(Q\cup S\alpha Q)=S\alpha Q\cup S\alpha S\alpha Q\subseteq S\alpha Q\subseteq L$, and also, $R\beta S\subseteq R$. We have that $L\cap R=(Q\cup S\alpha Q)\cap(Q\cup Q\beta S)=Q\cup(S\alpha Q\cap Q\beta S)=Q$. Therefore, $Q=L\cap R$.

Definition 5. A Γ-semigroup S is called (α,β)-quasi-simple if S does not contain proper (α,β)-quasi-ideals.

A Γ-semigroup S is called α-quasi-simple if S is (α,α)-quasi-simple.

Theorem 11. *Let S be a Γ-semigroup. Then, S is α-quasi-simple if and only if* $S\alpha s\cap s\alpha S=S$ *for all* $s\in S$.

Proof. Assume that S is α-quasi-simple. Let $s\in S$; we claim that $S\alpha s\cap s\alpha S$ is an α-quasi-ideal of S. We have $s\alpha s\in S\alpha s\cap s\alpha S$; this implies $S\alpha s\cap s\alpha S\neq\varnothing$. Moreover, $S\alpha(S\alpha s\cap s\alpha S)\cap(S\alpha s\cap s\alpha S)\alpha S\subseteq S\alpha(S\alpha S)\cap S\alpha(s\alpha S)\cap(S\alpha S)\alpha S\cap(s\alpha S)\alpha S\subseteq S\alpha(S\alpha s)\cap(s\alpha S)\alpha S=(S\alpha S)\alpha s\cap s\alpha(S\alpha S)\subseteq S\alpha s\cap s\alpha S$. Therefore, $S\alpha s\cap s\alpha S$ is an α-quasi-ideal of S. Since S is α-quasi-simple, we have $S=S\alpha s\cap s\alpha S$.

Conversely, assume that $S\alpha s\cap s\alpha S=S$ for all $s\in S$. Let Q be an α-quasi-ideal of S and $q\in Q$. By assumption, $S=S\alpha q\cap q\alpha S$. Since Q is an α-quasi-ideal of S, $S=S\alpha q\cap q\alpha S\subseteq S\alpha Q\cap Q\alpha S\subseteq Q$. Then, $Q=S$. Therefore, S is α-quasi-simple.

Definition 6. An (α,β)-quasi-ideal Q of a Γ-semigroup S is called minimal if for all (α,β)-quasi-ideal C of S, if $C\subseteq Q$, then $C=Q$.

Theorem 12. *Let S be a Γ-semigroup and Q an* (α,β)*-quasi-ideal of S. If Q is* (α,β)*-quasi-simple, then Q is a minimal* (α,β)*-quasi-ideal of S.*

Proof. Assume that S is a Γ-semigroup and Q an (α,β)-quasi-ideal of S. Let Q be (α,β)-quasi-simple. Let C be an (α,β)-quasi-ideal of S such that $C\subseteq Q$. Then, $Q\alpha C\cap C\beta Q\subseteq S\alpha C\cap C\beta S\subseteq C$. Therefore, C is an (α,β)-quasi-ideal of Q. Since Q is (α,β)-quasi-simple, $C=Q$. Hence, Q is a minimal (α,β)-quasi-ideal of S.

2.3. (α,β)*-Bi-Ideals.* We will define (α,β)-bi-ideals of Γ-semigroups as follows.

Definition 7. Let S be a Γ-semigroup and $\alpha,\beta\in\Gamma$. A nonempty subset B of S is called

(1) An (α,β)-bi-ideal of S if $B\alpha S\beta B\subseteq B$.
(2) An α-bi-ideal of S if it is an (α,α)-bi-ideal of S.

Theorem 13. *Every* (α,β)*-quasi-ideal of a Γ-semigroup S is a* (β,α)*-bi-ideal of S.*

Proof. Let Q be an (α,β)-quasi-ideal of S. Then,

$$Q\beta S\alpha Q\subseteq Q\beta S\cap S\alpha Q\subseteq Q. \tag{17}$$

Hence, Q is a (β,α)-bi-ideal of S.

Theorem 14. *Let S be a Γ-semigroup and* B_i *an* (α,β)*-bi-ideal of S for all* $i\in I$. *If* $\cap_{i\in I}B_i\neq\varnothing$, *then* $\cap_{i\in I}B_i$ *is an* (α,β)*-bi-ideal of S.*

Proof. Let S be a Γ-semigroup and B_i an (α,β)-bi-ideal of S for all $i\in I$. Then, $(\cap_{i\in I}B_i)\alpha S\beta(\cap_{i\in I}B_i)\subseteq B_i\alpha S\beta B_i\subseteq B_i$ for all $i\in I$, so $(\cap_{i\in I}B_i)\alpha S\beta(\cap_{i\in I}B_i)\subseteq\cap_{i\in I}B_i$. Therefore, $\cap_{i\in I}B_i$ is an (α,β)-bi-ideal of S.

Let A be a nonempty subset of a Γ-semigroup S, and let $(A)_{b(\alpha,\beta)}$ denote the (α,β)-bi-ideal of S generated by A.

Theorem 15. *Let A be a nonempty subset of a Γ-semigroup S and* $\alpha,\beta\in\Gamma$. *Then,*

$$(A)_{b(\alpha,\beta)}=A\cup(A\alpha S\beta A). \tag{18}$$

Proof. Let A be a nonempty subset of a Γ-semigroup S. Let $B = A \cup (A\alpha S\beta A)$. Clearly, $A \subseteq B$. We have that $B\alpha S\beta B = [A \cup (A\alpha S\beta A)]\alpha S\beta [A \cup (A\alpha S\beta A)] \subseteq B$. Therefore, B is an (α, β)-bi-ideal of S.

Let C be any (α, β)-bi-ideal of S containing A. Since C is an (α, β)-bi-ideal of S and $A \subseteq C$, $A\alpha S\beta A \subseteq C$. Therefore, $B = A \cup (A\alpha S\beta A) \subseteq C$.

Hence, B is the smallest (α, β)-bi-ideal of S containing A. Therefore, $(A)_{b(\alpha,\beta)} = B = A \cup (A\alpha S\beta A)$.

Theorem 16. *Let S be a Γ-semigroup, A a nonempty subset of S, and B an (α, β)-bi-ideal of S. The following statements are true:*

(1) $B\alpha A$ is an (α, β)-bi-ideal of S.

(2) $A\beta B$ is an (α, β)-bi-ideal of S.

Proof. We have that

$$(B\alpha A)\alpha S\beta (B\alpha A) = B\alpha (A\alpha S)\beta (B\alpha A) \subseteq (B\alpha S\beta B)\alpha A \subseteq B\alpha A. \tag{19}$$

Then, $B\alpha A$ is an (α, β)-bi-ideal of S. Similarly, $A\beta B$ is an (α, β)-bi-ideal of S.

Theorem 17. *Let S be a Γ-semigroup. Assume that B_1, B_2, and B_3 are (α, β)-bi-ideals of S. Then, $B_1\alpha B_2\beta B_3$ is an (α, β)-bi-ideal of S.*

Proof. Since B_1, B_2, and B_3 are (α, β)-bi-ideals of S, we have

$$\begin{aligned} &(B_1\alpha B_2\beta B_3)\alpha (B_1\alpha B_2\beta B_3) \subseteq B_1\alpha B_2\beta B_3, \\ &(B_1\alpha B_2\beta B_3)\beta (B_1\alpha B_2\beta B_3) \subseteq B_1\alpha B_2\beta B_3. \end{aligned} \tag{20}$$

Then,

$$\begin{aligned} &(B_1\alpha B_2\beta B_3)\alpha S\beta (B_1\alpha B_2\beta B_3) \\ &= (B_1\alpha B_2\beta B_3\alpha S\beta B_1)\alpha B_2\beta B_3 \\ &\subseteq (B_1\alpha S\beta B_1)\alpha B_2\beta B_3 \\ &\subseteq B_1\alpha B_2\beta B_3. \end{aligned} \tag{21}$$

Therefore, $B_1\alpha B_2\beta B_3$ is an (α, β)-bi-ideal of S.

Definition 8. A Γ-semigroup S is called (α, β)-bi-simple if S does not contain proper (α, β)-bi-ideals.

Theorem 18. *Let S be a Γ-semigroup. Then, S is (α, β)-bi-simple if and only if $s\alpha S\beta s = S$ for all $s \in S$.*

Proof. Assume that S is (α, β)-bi-simple. Let $s \in S$. We claim that $s\alpha S\beta s$ is an (α, β)-bi-ideal of S. We have $s\alpha s\beta s \in s\alpha S\beta s$; this implies that $s\alpha S\beta s \neq \varnothing$. Moreover, $(s\alpha S\beta s)\alpha S\beta (s\alpha S\beta s) \subseteq (s\alpha S\beta S\alpha S\beta s) \subseteq s\alpha S\beta s$. Therefore, $s\alpha S\beta s$ is an (α, β)-bi-ideal of S. Since S is (α, β)-bi-simple, $S = s\alpha S\beta s$.

Conversely, assume that $s\alpha S\beta s = S$ for all $s \in S$. Let B be an (α, β)-bi-ideal of S and $b \in B$. By assumption, $S = b\alpha S\beta b$. Since B is an (α, β)-bi-ideal of S, $b\alpha S\beta b \subseteq B\alpha S\beta B \subseteq B$. Then, $B = S$. Therefore, S is (α, β)-bi-simple.

Definition 9. An (α, β)-bi-ideal B of a Γ-semigroup S is called minimal if for all (α, β)-bi-ideal C of S, if $C \subseteq B$, then $C = B$.

Theorem 19. *Let S be a Γ-semigroup and B an (α, β)-bi-ideal of S. If B is (α, β)-bi-simple, then B is a minimal (α, β)-bi-ideal of S.*

Proof. Assume that S is a Γ-semigroup and B an (α, β)-bi-ideal of S. Let B be (α, β)-bi-simple. Let C be an (α, β)-bi-ideal of S such that $C \subseteq B$. Then, $C\alpha B\beta C \subseteq C\alpha S\beta C \subseteq C$. Therefore, C is an (α, β)-bi-ideal of B. Since B is (α, β)-bi-simple, $C = B$. Then, B is a minimal (α, β)-bi-ideal of S.

3. New Types of Fuzzy Ideals

3.1. Fuzzy (α, β)-Ideals. We will define fuzzy (α, β)-ideals of Γ-semigroups as follows.

Definition 10. Let $\alpha, \beta \in \Gamma$ and f be a fuzzy subset of a Γ-semigroup S. Then, f is called

(1) A fuzzy left α-ideal of S if $f(x\alpha y) \geq f(y)$ for all $x, y \in S$.

(2) A fuzzy right β-ideal of S if $f(x\beta y) \geq f(x)$ for all $x, y \in S$.

(3) A fuzzy (α, β)-ideal of S if it is both a fuzzy left α-ideal and a fuzzy right β-ideal of S.

The following theorems show the relationship between (α, β)-ideals and fuzzy (α, β)-ideals.

Theorem 20. *Let A be a nonempty subset of a Γ-semigroup S. Then, the following statements are true:*

(1) A is a left α-ideal of S if and only if C_A is a fuzzy left α-ideal of S.

(2) A is a right β-ideal of S if and only if C_A is a fuzzy right β-ideal of S.

(3) A is an (α, β)-ideal of S if and only if C_A is a fuzzy (α, β)-ideal of S.

Proof

(1) Suppose that A is a left α-ideal of S. Let $x, y \in S$.

If $y \in A$, then $x\alpha y \in A$. Thus, $C_A(x\alpha y) = 1$ so that $C_A(x\alpha y) \geq C_A(y)$.

If $y \notin A$, then $C_A(y) = 0 \leq C_A(x\alpha y)$.

Therefore, C_A is a fuzzy left α-ideal of S.

Conversely, assume that C_A is a fuzzy left α-ideal of S. Let $x \in S$ and $y \in A$. Then, $C_A(y) = 1$. Since $C_A(x\alpha y) \geq C_A(y)$, we have $x\alpha y \in A$. Hence, A is a left α-ideal of S.

(2) is similar to (1).

(3) follows by (1) and (2).

Theorem 21. *Let f be a fuzzy subset of a Γ-semigroup S. Then, the following properties hold:*

(1) f is a fuzzy left α-ideal of S if and only if $S \circ_\alpha f \subseteq f$.

(2) f is a fuzzy right β-ideal of S if and only if $f \circ_\beta S \subseteq f$.

(3) f *is a fuzzy* (α,β)*-ideal of* S *if and only if* $S\circ_{\alpha} f\subseteq f$ *and* $f\circ_{\beta} S\subseteq f$.

Proof

(1) Assume that f is a fuzzy left α-ideal of a Γ-semigroup S. Let $x\in S$ and $\alpha\in\Gamma$.

If $x\notin S\alpha S$, then $(S\circ_{\alpha} f)(x)=0$. So, $(S\circ_{\alpha} f)(x)\leq f(x)$.

If $x\in S\alpha S$, then there exist $y,z\in S$ such that $x=y\alpha z$. Since f is a fuzzy left α-ideal of S, we have

$$\begin{aligned}(S\circ_{\alpha} f)(x) &= \sup_{x=y\alpha z}\{\min\{S(y),f(z)\}\}\\ &\leq \min\{S(y),f(z)\}\\ &= f(z)\\ &\leq f(y\alpha z)\\ &= f(x).\end{aligned} \tag{22}$$

We conclude that $S\circ_{\alpha} f\subseteq f$.

Conversely, assume that $S\circ_{\alpha} f\subseteq f$. Let $x,y,z\in S$ be such that $x=y\alpha z$. Then,

$$\begin{aligned}f(y\alpha z) &= f(x)\\ &\geq (S\circ_{\alpha} f)(x)\\ &= \sup_{x=y\alpha z}\{\min\{S(y),f(z)\}\}\\ &\geq \min\{S(y),f(z)\}\\ &= \min\{1,f(z)\}\\ &= f(z).\end{aligned} \tag{23}$$

Hence, f is a fuzzy left α-ideal of S.

(2) and (3) can be seen in a similar fashion.

Theorem 22. *Let* f *and* g *be fuzzy left* α*-ideals of a* Γ*-semigroup* S*. Then,*

(1) $f\cap g$ *is a fuzzy left* α*-ideal of* S.

(2) $f\cup g$ *is a fuzzy left* α*-ideal of* S.

Proof. Let $x,y\in S$. Then,

$$\begin{aligned}(f\cap g)(x\alpha y) &= \min\{f(x\alpha y),g(x\alpha y)\}\\ &\geq \min\{f(y),g(y)\}\\ &= (f\cap g)(y).\end{aligned} \tag{24}$$

Hence, $f\cap g$ is a fuzzy left α-ideal of S. Next,

$$\begin{aligned}(f\cup g)(x\alpha y) &= \max\{f(x\alpha y),g(x\alpha y)\}\\ &\geq \max\{f(y),g(y)\}\\ &= (f\cup g)(y).\end{aligned} \tag{25}$$

Hence, $f\cup g$ is a fuzzy left α-ideal of S.

Theorem 23. *Let* f *and* g *be fuzzy right* β*-ideals of a* Γ*-semigroup* S*. Then,*

(1) $f\cap g$ *is a fuzzy right* β*-ideal of* S.

(2) $f\cup g$ *is a fuzzy right* β*-ideal of* S.

Proof. It is similar to Theorem 22.

Theorem 24. *Let* f *and* g *be fuzzy* (α,β)*-ideals of a* Γ*-semigroup* S*. Then,*

(1) $f\cap g$ *is a fuzzy* (α,β)*-ideal of* S.

(2) $f\cup g$ *is a fuzzy* (α,β)*-ideal of* S.

Proof. It follows by Theorems 22 and 23.

Theorem 25. *Let* f *be a nonzero fuzzy subset of a* Γ*-semigroup* S *and* $f_t=\{x\in S\mid f(x)\geq t\}$*. The following statements are true:*

(1) f *is a fuzzy left* α*-ideal of* S *if and only if for all* $t\in(0,1]$*, if* f_t *is a nonempty set, then* f_t *is a left* α*-ideal of* S.

(2) f *is a fuzzy right* β*-ideal of* S *if and only if for all* $t\in(0,1]$*, if* f_t *is a nonempty set, then* f_t *is a right* β*-ideal of* S.

(3) f *is a fuzzy* (α,β)*-ideal of* S *if and only if for all* $t\in(0,1]$*, if* f_t *is a nonempty set, then* f_t *is an* (α,β)*-ideal of* S.

Proof

(1) Suppose that f is a fuzzy left α-ideal of S. Then, $f(x\alpha y)\geq f(y)$ for all $x,y\in S$. Let $t\in(0,1]$ be such that $f_t\neq\varnothing$. Let $x\in f_t$ and $s\in S$. Since $f(x)\geq t$, $f(s\alpha x)\geq f(x)\geq t$. Thus, $s\alpha x\in f_t$. Hence, f_t is a left α-ideal of S.

Conversely, assume that f_t is a left α-ideal of S if $t\in(0,1]$ and $f_t\neq\varnothing$. Let $x,y\in S$ and $t=f(y)$. Since $f(y)\geq t$, we have $y\in f_t$ so that $f_t\neq\varnothing$. Thus, f_t is a left α-ideal of S. Since $y\in f_t$ and $x\in S$, we have $x\alpha y\in f_t$. Then, $f(x\alpha y)\geq t=f(y)$. Hence, f is a fuzzy left α-ideal of S.

(2) is similar to (1).

(3) follows by (1) and (2).

3.2. Fuzzy (α,β)*-Quasi-Ideals.* We will define fuzzy (α,β)-quasi-ideals of Γ-semigroups as follows.

Definition 11. Let $\alpha,\beta\in\Gamma$ and f be a fuzzy subset of a Γ-semigroup S. Then, f is called a fuzzy (α,β)-quasi-ideal of S if $(S\circ_{\alpha} f)\cap(f\circ_{\beta} S)\subseteq f$.

A fuzzy subset f of S is called a fuzzy α-quasi-ideal of S if f is a fuzzy (α,β)-quasi-ideal of S.

Theorem 26. *Let* S *be a* Γ*-semigroup. Let* f *and* g *be a fuzzy left* α*-ideal and a fuzzy right* α*-ideal of* S*, respectively. Then,* $f\cap g$ *is a fuzzy* α*-quasi-ideal of* S.

Proof. Let f and g be a fuzzy left α-ideal and a fuzzy right α-ideal of a Γ-semigroup S, respectively. We have $g\circ_{\alpha} f\subseteq f\cap g$; this implies $f\cap g\neq\varnothing$. Then, $S\circ_{\alpha}(f\cap g)\cap(f\cap g)\circ_{\alpha} S\subseteq S\circ_{\alpha} f\cap f\circ_{\alpha} S\subseteq f\cap g$. Hence, $f\cap g$ is a fuzzy α-quasi-ideal of S.

Theorem 27. *Every fuzzy (α,β)-quasi-ideal of a Γ-semigroup S is the intersection of a fuzzy left α-ideal and a fuzzy right β-ideal of S.*

Proof. Let f be a fuzzy (α,β)-quasi-ideal of a Γ-semigroup S. Let $g = f \cup (S \circ_\alpha f)$ and $h = f \cup (f \circ_\beta S)$. Then, $S \circ_\alpha g = S \circ_\alpha (f \cup (S \circ_\alpha f)) = (S \circ_\alpha f) \cup (S \circ_\alpha (S \circ_\alpha f)) \subseteq S \circ_\alpha f \subseteq g$, and also, $h \circ_\beta S \subseteq h$. Thus, g and h are a fuzzy left α-ideal and a fuzzy right β-ideal of S, respectively. We claim that $f = g \cap h$. That is, $f \subseteq (f \cup (S \circ_\alpha f)) \cap (f \cup (f \circ_\beta S)) \subseteq g \cap h$, and conversely, $g \cap h = (f \cup (S \circ_\alpha f)) \cap (f \cup (f \circ_\beta S)) \subseteq f \cup ((S \circ_\alpha f) \cap (f \circ_\beta S)) \subseteq f$. Therefore, $f = g \cap h$.

Theorem 28. *Let Q be a nonempty subset of a Γ-semigroup S. Then, Q is an (α,β)-quasi-ideal of S if and only if C_Q is a fuzzy (α,β)-quasi-ideal of S.*

Proof. Assume that Q is an (α,β)-quasi-ideal of a Γ-semigroup S. If $y \notin (S\alpha Q) \cap (Q\beta S)$, $(S \circ_\alpha C_Q) \cap (C_Q \circ_\beta S)(y) = 0 \leq C_Q(y)$. Let $y \in (S\alpha Q) \cap (Q\beta S)$. Then, $y \in Q$. So $C_Q(y) = 1$. Hence, $(S \circ_\alpha C_Q) \cap (C_Q \circ_\beta S) \subseteq C_Q$. Therefore, C_Q is a fuzzy (α,β)-quasi-ideal of S.

Conversely, assume that C_Q is a fuzzy (α,β)-quasi-ideal of S. Then, $(S \circ_\alpha C_Q) \cap (C_Q \circ_\beta S) \subseteq C_Q$. Let $x \in (S\alpha Q) \cap (Q\beta S)$. Then, $[(S \circ_\alpha C_Q) \cap (C_Q \circ_\beta S)](x) = 1$, and this implies that $C_Q(x) = 1$. So, $(s\alpha Q) \cap (Q\beta s) \subseteq Q$. Consequently, Q is an (α,β)-quasi-ideal of S.

3.3. Fuzzy (α,β)-Bi-Ideals

Definition 12. Let $\alpha,\beta \in \Gamma$ and f be a fuzzy subset of a Γ-semigroup S. Then, f is called a fuzzy (α,β)-bi-ideal of S if $f \circ_\alpha S \circ_\beta f \subseteq f$.

Theorem 29. *Let S be a Γ-semigroup, g a fuzzy subset of S, and f a fuzzy (α,β)-bi-ideal of S. Then, the following statements are true:*

(1) $f \circ_\alpha g$ is a fuzzy (α,β)-bi-ideal of S.

(2) $g \circ_\beta f$ is a fuzzy (α,β)-bi-ideal of S.

Proof

(1) Let x_t be a fuzzy point of S. Then,

$$\begin{aligned}(f \circ_\alpha g) \circ_\alpha S \circ_\beta (f \circ_\alpha g) &= f \circ_\alpha (g \circ_\alpha x_t) \circ_\beta (f \circ_\alpha g)\\ &\subseteq \left(f \circ_\alpha x_t \circ_\beta f\right) \circ_\alpha g \subseteq f \circ_\alpha g.\end{aligned} \tag{26}$$

Hence, $f \circ_\alpha g$ is a fuzzy (α,β)-bi-ideal of S.

(2) is similar to (1).

Theorem 30. *Let S be a Γ-semigroup. If f_1, f_2 and f_3 are fuzzy (α,β)-bi-ideals of S, then $f_1 \circ_\alpha f_2 \circ_\beta f_3$ is a fuzzy bi-(α,β)-ideal of S.*

Proof. Let x_t be a fuzzy point of S. Then,

$$\begin{aligned}&\left(f_1 \circ_\alpha f_2 \circ_\beta f_3\right) \circ_\alpha x_t \circ_\beta \left(f_1 \circ_\alpha f_2 \circ_\beta f_3\right)\\ &\quad\subseteq f_1 \circ_\alpha \left(f_2 \circ_\beta S \circ_\alpha f_2\right) \circ_\beta f_3\\ &\quad\subseteq f_1 \circ_\alpha f_2 \circ_\beta f_3.\end{aligned} \tag{27}$$

Hence, $f \circ_\alpha f_2 \circ_\beta f_3$ is a fuzzy (α,β)-bi-ideal of S.

Theorem 31. *Let S be a Γ-semigroup and B a nonempty subset of S. Then, B is an (α,β)-bi-ideal of S if and only if C_B is a fuzzy (α,β)-bi-ideal of S.*

Proof. Assume that B is an (α,β)-bi-ideal of S. Then, $B\alpha S\beta B \subseteq B$. If $z \notin B\alpha S\beta B$, we have $C_B \circ_\alpha S \circ_\beta C_B(z) = 0 \leq C_B(z)$. Let $z \in B\alpha S\beta B$. Then, $z \in B$. Thus, $(C_B \circ_\alpha S \circ_\beta C_B)(z) = 1$; then, $C_B(z) = 1$. This implies that $C_B \circ_\alpha S \circ_\beta C_B \subseteq C_B$. Hence, C_B is a fuzzy (α,β)-bi-ideal of S.

Conversely, assume that C_B is a fuzzy (α,β)-bi-ideal of S. Then, $C_B \circ_\alpha S \circ_\beta C_B \subseteq C_B$. Let $z \in B\alpha S\beta B$. Then, $(C_B \circ_\alpha S \circ_\beta C_B)(z) = 1$; thus, $C_B(z) = 1$. Hence, $z \in B$ so that $B\alpha S\beta B \subseteq B$. Thus, B is an (α,β)-bi-ideal of S.

Theorem 32. *Let f and g be two fuzzy (α,β)-bi-ideals of a Γ-semigroup S. If $f \cap g \neq \varnothing$, then $f \cap g$ is a fuzzy (α,β)-bi-ideal of S.*

Proof. Let f and g be fuzzy (α,β)-bi-ideals of a Γ-semigroup S. Then, $(f \cap g) \circ_\alpha S \circ_\beta (f \cap g) \subseteq f \circ_\alpha S \circ_\beta f \cap f \circ_\alpha S \circ_\beta f \cap g \subseteq f \cap g$. Hence, $f \cap g$ is a fuzzy (α,β)-bi-ideal of S.

4. New Types of Almost Ideals

4.1. Almost (α,β)-Ideals

Definition 13. Let S be a Γ-semigroup and L, R, and I be nonempty subsets of S. Let $\alpha,\beta \in \Gamma$. Then,

(1) L is called an almost left α-ideal of S if $S\alpha L \cap L \neq \varnothing$.

(2) R is called an almost right β-ideal of S if $R\beta S \cap R \neq \varnothing$.

(3) I is called an almost (α,β)-ideal of S if it is both an almost left α-ideal and an almost right β-ideal of S.

Theorem 33. *Let S be a Γ-semigroup. If L is a left α-ideal of S, then L is an almost left α-ideal of S.*

Proof. Let L be a left α-ideal of S. Then, $S\alpha L \subseteq L$ so that $S\alpha L \cap L \neq \varnothing$. Thus, L is an almost left α-ideal of S.

Theorem 34. *Let S be a Γ-semigroup. If R is a right β-ideal of S, then R is an almost right β-ideal of S.*

Proof. It is similar to Theorem 33.

Theorem 35. *Let S be a Γ-semigroup. If I is an (α,β)-ideal of S, then I is an almost (α,β)-ideal of S.*

Proof. It follows by Theorems 33 and 34.

Theorem 36. *Let S be a Γ-semigroup. If L is an almost left α-ideal of S and $L \subseteq H \subseteq S$, then H is an almost left α-ideal of S.*

Proof. Let L be an almost left α-ideal of S and $L \subseteq H \subseteq S$. Since $S\alpha L \cap L \neq \emptyset$ and $S\alpha L \cap L \subseteq S\alpha H \cap H$, we have $S\alpha H \cap H \neq \emptyset$. Therefore, H is an almost left α-ideal of S.

Theorem 37. *Let S be a Γ-semigroup. If R is an almost right β-ideal of S and $R \subseteq H \subseteq S$, then H is an almost right β-ideal of S.*

Proof. It is similar to Theorem 36.

Theorem 38. *Let S be a Γ-semigroup. If I is an almost (α, β)-ideal of S and $I \subseteq H \subseteq S$, then H is an almost (α, β)-ideal of S.*

Proof. It follows by Theorems 36 and 37.

Corollary 2. *Let S be a Γ-semigroup. If L_1 and L_2 are almost left α-ideals of S, then $L_1 \cup L_2$ is an almost left α-ideal of S.*

Corollary 3. *Let S be a Γ-semigroup. If R_1 and R_2 are almost right β-ideals of S, then $R_1 \cup R_2$ is an almost right β-ideal of S.*

Corollary 4. *Let S be a Γ-semigroup. If I_1 and I_2 are almost (α, β)-ideals of S, then $I_1 \cup I_2$ is an almost (α, β)-ideal of S.*

Example 2. Consider a Γ-semigroup $S = \{a, b, c, d, e\}$ with $\Gamma = \{\alpha, \beta\}$ and

α	a	b	c	d	e
a	a	b	c	d	e
b	b	c	d	e	a
c	c	d	e	a	b
d	d	e	a	b	c
e	e	a	b	c	d

,

β	a	b	c	d	e
a	b	c	d	e	a
b	c	d	e	a	b
c	d	e	a	b	c
d	e	a	b	c	d
e	a	b	c	d	e

.

We have $\{a, b, d\}$ and $\{a, c, d\}$ are almost left α-ideals of S. However, $\{a, b, d\} \cap \{a, c, d\} = \{a, d\}$ is not an almost left α-ideal of S.

Remark 3. The intersection of two almost left α-ideals of a Γ-semigroup S need not be an almost left α-ideal of S.

Remark 4. The intersection of two almost right β-ideals of a Γ-semigroup S need not be an almost right β-ideal of S.

Remark 5. The intersection of two almost (α, β)-ideals of a Γ-semigroup S need not be an almost (α, β)-ideal of S.

Definition 14. A Γ-semigroup S is called

(1) Almost left α-simple if S does not contain proper almost left α-ideals.

(2) Almost right β-simple if S does not contain proper almost right β-ideals.

(3) Almost (α, β)-simple if S does not contain proper almost (α, β)-ideals.

Theorem 39. *Let S be a Γ-semigroup. Then, S is almost left α-simple if and only if for each $a \in S$ there exists $s \in S$ such that $s\alpha(S \setminus \{a\}) = \{a\}$.*

Proof. Assume that S is almost left α-simple. Then, S has no proper almost left α-ideals. Let $a \in S$. Then, $S \setminus \{a\}$ is not an almost left α-ideal of S. Thus, there exists $s \in S$ such that $s\alpha(S \setminus \{a\}) \cap (S \setminus \{a\}) = \emptyset$; this implies that $s\alpha(S \setminus \{a\}) = \{a\}$.

Conversely, for each $a \in S$, there exists $s \in S$ such that $s\alpha(S \setminus \{a\}) = \{a\}$. Assume that L is a proper almost left α-ideal of S, and let $a \in S \setminus L$. Since $L \subset S \setminus \{a\}$, we have $S \setminus \{a\}$ is an almost left α-ideal of S by Theorem 36. Thus, $s\alpha(S \setminus \{a\}) \cap (S \setminus \{a\}) \neq \emptyset$; we get a contradiction. Hence, S has no proper left almost α-ideals. Therefore, S is almost left α-simple.

Theorem 40. *Let S be a Γ-semigroup. Then, S is almost right β-simple if and only if for each $a \in S$, there exists $s \in S$ such that $(S \setminus \{a\})\beta s = \{a\}$.*

Proof. It is similar to Theorem 39.

4.2. Almost (α, β)-Quasi-Ideals

Definition 15. A nonempty subset Q of a Γ-semigroup S is called an almost (α, β)-quasi-ideal of S if $s\alpha Q \cap Q\beta s \cap Q \neq \emptyset$ for all $s \in S$.

Proposition 1. *Every (α, β)-quasi-ideal of a Γ-semigroup S is either $s\alpha Q \cap Q\beta s = \emptyset$ for some $s \in S$ or an almost (α, β)-quasi-ideal of S.*

Proof. Assume that Q is an (α, β)-quasi-ideal of a Γ-semigroup S. Assume that $s\alpha Q \cap Q\beta s \neq \emptyset$ for all $s \in S$. Let $s \in S$. Then, $s\alpha Q \cap Q\beta s \subseteq S\alpha Q \cap Q\beta S \subseteq Q$. That is, $(s\alpha Q \cap Q\beta s) \cap Q \neq \emptyset$. Hence, Q is an almost (α, β)-quasi-ideal of S.

Theorem 41. *Every almost (α, β)-quasi-ideal of a Γ-semigroup S is an almost left α-ideal of S.*

Proof. Assume that Q is an almost (α, β)-quasi-ideal of a Γ-semigroup S. Let $s \in S$. Then, $\emptyset \neq (s\alpha Q \cap Q\beta s) \cap Q \subseteq s\alpha Q \cap Q$. Hence, Q is an almost left α-ideal of S.

Similarly, every almost (α, β)-quasi-ideal of a Γ-semigroup S is an almost right β-ideal of S, but the converse is not true.

Theorem 42. *If Q is an almost (α, β)-quasi-ideal of a Γ-semigroup S and $Q \subseteq H \subseteq S$, then H is an almost (α, β)-quasi-ideal of S.*

Proof. Assume that Q is an almost (α,β)-quasi-ideal of a Γ-semigroup S and $Q\subseteq H\subseteq S$. Let $s\in S$. Then, $\varnothing\neq(s\alpha Q\cap Q\beta s)\cap Q\subseteq(s\alpha H\cap H\beta s)\cap H$. Therefore, H is an almost (α,β)-quasi-ideal of S.

Corollary 5. *The union of two almost (α,β)-quasi-ideals of a Γ-semigroup S is an almost (α,β)-quasi-ideal of S.*

Example 3. Consider a Γ-semigroup $S=\{a,b,c,d,e\}$ with $\Gamma=\{\alpha,\beta\}$ and

α	a	b	c	d	e
a	a	b	c	d	e
b	b	c	d	e	a
c	c	d	e	a	b
d	d	e	a	b	c
e	e	a	b	c	d

,

β	a	b	c	d	e
a	b	c	d	e	a
b	c	d	e	a	b
c	d	e	a	b	c
d	e	a	b	c	d
e	a	b	c	d	e

.

We have $\{b,c,e\}$ and $\{b,d,e\}$ are almost (α,β)-quasi-ideals of S.

Remark 6. The intersection of two almost (α,β)-quasi-ideals of a Γ-semigroup S need not be an almost (α,β)-quasi-ideal of S.

Definition 16. A Γ-semigroup S is called almost (α,β)-quasi-simple if S does not contain proper almost (α,β)-quasi-ideals.

Theorem 43. *A Γ-semigroup S is almost (α,β)-quasi-simple if and only if for any $a\in S$, there exists s_a such that $s_a\alpha(S\backslash\{a\})\cap(S\backslash\{a\})\beta s_a\subseteq\{a\}$.*

Proof. Assume that a Γ-semigroup S is almost (α,β)-quasi-simple and $S\backslash\{a\}$ is not an almost (α,β)-quasi-ideal of S. Then, there exists $s_a\in S$ such that $[s_a\alpha(S\backslash\{a\})\cap(S\backslash\{a\})\beta s_a]\cap(S\backslash\{a\})=\varnothing$. Therefore, $s_a\alpha(S\backslash\{a\})\cap(S\backslash\{a\})\beta s_a\subseteq\{a\}$.

Conversely, assume that, for any $a\in S$, there exists s_a such that $s_a\beta(S\backslash\{a\})\cap(S\backslash\{a\})\beta s_a\subseteq\{a\}$. Then, $[s_a\alpha(S\backslash\{a\})\cap(S\backslash\{a\})\beta s_a]\cap(S\backslash\{a\})=\varnothing$. Hence, $S\backslash\{a\}$ is not an almost (α,β)-quasi-ideal of S. Let A be a proper almost (α,β)-quasi-ideal of S. Then, $A\subseteq S\backslash\{a\}\subseteq S$ for some $a\in S$; this is a contradiction. Therefore, S has no proper almost (α,β)-quasi-ideals. Hence, S is almost (α,β)-quasi-simple.

4.3. Almost (α,β)-Bi-Ideals

Definition 17. A nonempty subset B of a Γ-semigroup S is called an almost (α,β)-bi-ideal of S if $B\alpha s\beta B\cap B\neq\varnothing$ for all $s\in S$.

Theorem 44. *If B is an almost (α,β)-bi-ideal of a Γ-semigroup S and $B\subseteq C\subseteq S$, then C is an almost (α,β)-bi-ideal of S.*

Proof. Let B be an almost (α,β)-bi-ideal of a Γ-semigroup S and $B\subseteq C\subseteq S$. Since $B\alpha s\beta B\cap B\neq\varnothing$ for all $s\in S$ and $B\alpha s\beta B\cap B\subseteq C\alpha s\beta C\cap C$, we have $C\alpha s\beta C\cap C\neq\varnothing$. Therefore, C is an almost (α,β)-bi-ideal of S.

Corollary 6. *The union of two almost (α,β)-bi-ideals of a Γ-semigroup S is also an almost (α,β)-bi-ideal of S.*

Example 4. Consider a Γ-semigroup $S=\{a,b,c,d,e\}$ with $\Gamma=\{\alpha,\beta\}$ and

α	a	b	c	d	e
a	a	b	c	d	e
b	b	c	d	e	a
c	c	d	e	a	b
d	d	e	a	b	c
e	e	a	b	c	d

,

β	a	b	c	d	e
a	b	c	d	e	a
b	c	d	e	a	b
c	d	e	a	b	c
d	e	a	b	c	d
e	a	b	c	d	e

.

We have $\{b,c,e\}$ and $\{b,d,e\}$ are almost (α,β)-bi-ideals of S.

Remark 7. The intersection of two almost (α,β)-bi-ideals of a Γ-semigroup S need not be an almost (α,β)-bi-ideal of S.

Theorem 45. *A Γ-semigroup S has a proper almost (α,β)-bi-ideal if and only if there exists an element $a\in S$ such that $(S\backslash\{a\})\alpha s\beta(S\backslash\{a\})\cap(S\backslash\{a\})\neq\varnothing$ for all $a\in S$.*

Proof Assume that a Γ-semigroup S contains a proper almost (α,β)-bi-ideal B and $a\notin B$. Then, $B\subset S\backslash\{a\}\subset S$. By Theorem 44, $S\backslash\{a\}$ is a proper almost (α,β)-bi-ideal of S, that is, $(S\backslash\{a\})\alpha s\beta(S\backslash\{a\})\cap(S\backslash\{a\})\neq\varnothing$ for all $s\in S$.

Conversely, let $a\in S$ such that $(S\backslash\{a\})\alpha s\beta(S\backslash\{a\})\cap(S\backslash\{a\})\neq\varnothing$ for all $s\in S$. Since $S\backslash\{a\}\subset S$, we have $S\backslash\{a\}$ is a proper almost (α,β)-bi-ideal of S.

Definition 18. A Γ-semigroup S is called almost (α,β)-bi-simple if S does not contain proper almost (α,β)-bi-ideals.

Theorem 46. *A Γ-semigroup S is almost (α,β)-bi-simple if and only if for all $a\in S$, there exists $s\in S$ such that $(S\setminus\{a\})\alpha s\beta(S\setminus\{a\})=\{a\}$.*

Proof. Assume that a Γ-semigroup S is almost (α,β)-bi-simple. Then, S has no proper almost (α,β)-bi-ideals, and let $a\in S$. Then, $S\setminus\{a\}$ is not an almost (α,β)-bi-ideal of S. Thus, there exists $s\in S$ such that $(S\setminus\{a\})\alpha s\beta(S\setminus\{a\})\cap(S\setminus\{a\})\neq\varnothing$. This implies that $(S\setminus\{a\})\alpha s\beta(S\setminus\{a\})=\{a\}$.

Conversely, suppose that S has a proper almost (α,β)-bi-ideal B, that is, $B\subset S$. Since $B\subseteq S\setminus\{a\}\subset S$, we have $S\setminus\{a\}$ is an almost (α,β)-bi-ideal of S by Theorem 44; this is a contradiction. Hence, S has no proper almost (α,β)-bi-ideals. Therefore, S is almost (α,β)-bi-simple.

5. New Types of Fuzzy Almost Ideals

5.1. Fuzzy Almost (α, β)-Ideals

Definition 19. Let $\alpha, \beta \in \Gamma$. Let f be a fuzzy subset of a Γ-semigroup S. Then, f is called

(1) A fuzzy almost left α-ideal of S if $(x_t \circ_\alpha f) \cap f \neq 0$ for all fuzzy point x_t of S.

(2) A fuzzy almost right β-ideal of S if $(f \circ_\beta x_t) \cap f \neq 0$ for all fuzzy point x_t of S.

(3) A fuzzy almost (α, β)-ideal of S if it is both a fuzzy almost left α-ideal and a fuzzy almost right β-ideal of S.

Theorem 47. *Let f be a fuzzy almost left α-ideal of a Γ-semigroup S and g be a fuzzy subset of S such that $f \subseteq g$. Then, g is a fuzzy almost left α-ideal of S.*

Proof. Assume that f is a fuzzy almost left α-ideal of a Γ-semigroup S and g is a fuzzy subset of S such that $f \subseteq g$. Then, for each fuzzy point x_t, $(x_t \circ_\alpha f) \cap f \neq 0$. We have $(x_t \circ_\alpha f) \cap f \subseteq (x_t \circ_\alpha g) \cap g$; this implies $(x_t \circ_\alpha g) \cap g \neq 0$. Therefore, g is a fuzzy almost left α-ideal of S.

Theorem 48. *Let f be a fuzzy almost right β-ideal of a Γ-semigroup S and g be a fuzzy subset of S such that $f \subseteq g$. Then, g is a fuzzy almost right β-ideal of S.*

Proof. It is similar to Theorem 47.

Theorem 49. *Let f be a fuzzy almost (α, β)-ideal of a Γ-semigroup S and g be a fuzzy subset of S such that $f \subseteq g$. Then, g is a fuzzy almost (α, β)-ideal of S.*

Proof. It follows by Theorem 47 and Theorem 48.

Corollary 7. *The union of two fuzzy almost left α-ideals of a Γ-semigroup S is a fuzzy almost left α-ideal of S.*

Corollary 8. *The union of two fuzzy almost right β-ideals of a Γ-semigroup S is a fuzzy almost right β-ideal of S.*

Corollary 9. *The union of two fuzzy almost (α, β)-ideals of a Γ-semigroup S is a fuzzy almost (α, β)-ideal of S.*

Theorem 50. *Let A be a nonempty subset of a Γ-semigroup S. Then,*

(1) A is an almost left α-ideal of S if and only if C_A is a fuzzy almost left α-ideal of S.

(2) A is an almost right β-ideal of S if and only if C_A is a fuzzy almost right β-ideal of S.

(3) A is an almost (α, β)-ideal of S if and only if C_A is a fuzzy almost (α, β)-ideal of S.

Proof

(1) Assume that A is an almost left α-ideal of a Γ-semigroup S. Then, $x\alpha A \cap A \neq \emptyset$ for all $x \in S$. Thus, there exists $y \in x\alpha A$, and $y \in A$. So, $(x_t \circ_\alpha C_A)(y) = 1$ and $C_A(y) = 1$. Hence, $(x_t \circ_\alpha C_A) \cap C_A \neq 0$. Therefore, C_A is a fuzzy almost left α -ideal of S.

Conversely, assume that C_A is a fuzzy almost left α -ideal of S. Let $x \in S$. Then, $(x_t \circ_\alpha C_A) \cap C_A \neq 0$. Then, there exists $a \in S$ such that $[(x_t \circ_\alpha C_A) \cap C_A](a) \neq 0$. Hence, $a \in x\alpha A \cap A$. So, $x\alpha A \cap A \neq \emptyset$. Consequently, A is an almost left α -ideal of S.

The proofs of (2) and (3) are similar to the proof of (1).

Theorem 51. *Let f be a fuzzy subset of a α-semigroup S. Then,*

(1) f is a fuzzy almost left α -ideal of S if and only if $supp(f)$ is an almost left α -ideal of S.

(2) f is a fuzzy almost right β -ideal of S if and only if $supp(f)$ is an almost right β-ideal of S.

(3) f is a fuzzy almost (α, β)-ideal of S if and only if $supp(f)$ is an almost (α, β)-ideal of S.

Proof

(1) Assume that f is a fuzzy almost left α -ideal of a Γ -semigroup S. Let $x \in S$. Then, $(x_t \circ_\alpha f) \cap f \neq 0$. Hence, there exists $a \in S$ such that $[(x_t \circ_\alpha f) \cap f](a) \neq 0$. So, there exists $b \in S$ such that $a = x\alpha b, f(a) \neq 0, f(b) \neq 0$. That is, $a, b \in \text{supp}(f)$. Thus, $(x_t \circ_\alpha C_{\text{supp}(f)})(a) \neq 0$ and $C_{\text{supp}(f)}(a) \neq 0$. Therefore, $(C_x \circ_\alpha C_{\text{supp}(f)}) \cap C_{\text{supp}(f)} \neq 0$. Hence, $C_{\text{supp}(f)}$ is a fuzzy almost left α -ideal of S. By Theorem 50, $\text{supp}(f)$ is an almost left α-ideal of S.

Conversely, assume that $\text{supp}(f)$ is an almost left α -ideal of S. By Theorem 50, $C_{\text{supp}(f)}$ is a fuzzy almost left α -ideal of S. Then, $(x_t \circ_\alpha C_{\text{supp}(f)}) \cap C_{\text{supp}(f)} \neq 0$ for all $x \in S$. Then, there exists $a \in S$ such that $[(x_t \circ_\alpha C_{\text{supp}(f)}) \cap C_{\text{supp}(f)}](a) \neq 0$. Hence, $(x_t \circ_\alpha C_{\text{supp}(f)})(a) \neq 0$ and $C_{\text{supp}(f)}(a) \neq 0$. Then, there exists $y \in S$ such that $a = x\alpha y, f(a) \neq 0, f(y) \neq 0$. This means $(x_t \circ_\alpha f) \cap f \neq 0$. Therefore, f is a fuzzy almost left α ideal of S.

The proofs of (2) and (3) are similar to the proof of (1).

Definition 20. A fuzzy almost left α -ideal f of a Γ -semigroup S is minimal if for all fuzzy almost left α -ideal g of S such that $g \subseteq f$, we obtain $\text{supp}(g) = \text{supp}(f)$.

Theorem 52. *Let A be a nonempty subset of a Γ-semigroup S. Then,*

(1) A is a minimal almost left α-ideal of S if and only if C_A is a minimal fuzzy almost left α-ideal of S.

(2) A is a minimal almost right β-ideal of S if and only if C_A is a minimal fuzzy almost right β-ideal of S.

(3) A is a minimal almost (α, β)-ideal of S if and only if C_A is a minimal fuzzy almost (α, β)-ideal of S.

Proof

(1) Assume that A is a minimal almost left α-ideal of a Γ-semigroup S. By Theorem 50 (1), C_A is a fuzzy almost left α-ideal of S. Let g be a fuzzy almost left α-ideal of S such that

$g \subseteq C_A$. By Theorem 51 (1), $\text{supp}(g)$ is an almost left α-ideal of S. Then, $\text{supp}(g) \subseteq \text{supp}(C_A) = A$. Since A is minimal, $\text{supp}(g) = A = \text{supp}(C_A)$. Therefore, C_A is minimal.

Conversely, assume that C_A is a minimal fuzzy almost left α-ideal of S. By Theorem 50 (1), A is an almost left α-ideal of S. Let L be an almost left α-ideal of S such that $L \subseteq A$. By Theorem 50 (1), C_L is a fuzzy almost left α-ideal of S such that $C_L \subseteq C_A$. Hence, $L = \text{supp}(C_L) = \text{supp}(C_A) = A$. Therefore, A is minimal.

(2) and (3) can be proved similarly.

Corollary 10. *Let A be a sub-Γ-semigroup of a Γ-semigroup S. Then,*

(1) A is almost left α-simple if and only if for each fuzzy almost left α-ideal f of S, $\text{supp}(f) = A$.

(2) A is almost right β-simple if and only if for each fuzzy almost right β-ideal f of S, $\text{supp}(f) = A$.

(3) A is almost (α, β)-simple if and only if for each fuzzy almost (α, β)-ideal of S, $\text{supp}(f) = A$.

5.2. Fuzzy Almost (α, β)-Quasi-Ideals

Definition 21. Let $\alpha, \beta \in \Gamma$ and f be a fuzzy subset of a Γ-semigroup S. Then, f is called a fuzzy almost (α, β)-quasi-ideal of S if $[(f \circ_\alpha x_t) \cap (x_t \circ_\beta f)] \cap f \neq 0$ for all fuzzy points x_t of S.

Theorem 53. *Let f be a fuzzy almost (α, β)-quasi-ideal of a Γ-semigroup S and g a fuzzy subset of S such that $f \subseteq g$. Then, g is a fuzzy almost (α, β)-quasi-ideal of S.*

Proof. Assume that f is a fuzzy almost (α, β)-quasi-ideal of a Γ-semigroup S and g is a fuzzy subset of S such that $f \subseteq g$. Then, for all fuzzy point x_t of S, $[(f \circ_\alpha x_t) \cap (x_t \circ_\beta f)] \cap f \neq 0$. We have that $[(f \circ_\alpha x_t) \cap (x_t \circ_\beta f)] \cap f \subseteq [(g \circ_\alpha x_t) \cap (x_t \circ_\beta g)] \cap g$. This implies that $[(g \circ_\alpha x_t) \cap (x_t \circ_\beta g)] \cap g \neq 0$. Therefore, g is a fuzzy almost (α, β)-quasi-ideal of S.

Corollary 11. *Let f and g be fuzzy almost (α, β)-quasi-ideals of a Γ-semigroup S. Then, $f \cup g$ is a fuzzy almost (α, β)-quasi-ideal of S.*

Proof. Since $f \subseteq f \cup g$, by Theorem 53, $f \cup g$ is a fuzzy almost (α, β)-quasi-ideal of S.

Example 5. Consider the Γ-semigroup $\mathbb{Z}_5$ where $\Gamma = \{\overline{0}, \overline{2}, \overline{4}\}$ and $\overline{a}\gamma\overline{b} = \overline{a} + \gamma + \overline{b}$, where $\overline{a}, \overline{b} \in \mathbb{Z}_5$ and $\gamma \in \Gamma$. Let $f: \mathbb{Z}_5 \longrightarrow [0,1]$ defined by

$$\begin{aligned} f(\overline{0}) &= 0, \\ f(\overline{1}) &= 0.6, \\ f(\overline{2}) &= 0, \\ f(\overline{3}) &= 0.4, \\ f(\overline{4}) &= 0.4, \end{aligned} \tag{28}$$

and $g: \mathbb{Z}_5 \longrightarrow [0,1]$ defined by

$$\begin{aligned} g(\overline{0}) &= 0, \\ g(\overline{1}) &= 0.3, \\ g(\overline{2}) &= 0.6, \\ g(\overline{3}) &= 0, \\ g(\overline{4}) &= 0.8. \end{aligned} \tag{29}$$

We have f and g are fuzzy almost $(\overline{0}, \overline{0})$-quasi-ideals of $\mathbb{Z}_5$, but $f \cap g$ is not a fuzzy almost $(\overline{0}, \overline{0})$-quasi-ideal of $\mathbb{Z}_5$.

Theorem 54. *Let Q be a nonempty subset of a Γ-semigroup S. Then, Q is an almost (α, β)-quasi-ideal of S if and only if C_Q is a fuzzy almost (α, β)-quasi-ideal of S.*

Proof. Assume that Q is an almost (α, β)-quasi-ideal of a Γ-semigroup S, and let x_t be a fuzzy point of S. Then, $[(Q\alpha x) \cap (x\beta Q)] \cap Q \neq \varnothing$. Thus, there exists $y \in (Q\alpha x) \cap (x\beta Q)$ and $y \in Q$. So, $[(C_Q \circ_\alpha x_t) \cap (x_t \circ_\beta C_Q)](y) \neq 0$ and $C_Q(y) = 1$. Hence, $[(C_Q \circ_\alpha x_t) \cap (x_t \circ_\beta C_Q)] \cap C_Q \neq 0$. Therefore, C_Q is a fuzzy almost (α, β)-quasi-ideal of S.

Conversely, assume that C_Q is a fuzzy almost (α, β)-quasi-ideal of S. Let $s \in S$. Then, $[(C_Q \circ_\alpha s_t) \cap (s_t \circ_\beta C_Q)] \cap C_Q \neq 0$. Then, there exists $x \in S$ such that $[(C_Q \circ_\alpha s_t) \cap (s_t \circ_\beta C_Q) \cap C_Q](x) \neq 0$. Hence, $x \in [(Q\alpha s) \cap (s\beta Q)] \cap Q$. So, $[(Q\alpha s) \cap (s\beta Q)] \cap Q \neq \varnothing$. Consequently, Q is an almost (α, β)-quasi-ideal of S.

Theorem 55. *Let f be a fuzzy subset of a Γ-semigroup S. Then, f is a fuzzy almost (α, β)-quasi-ideal of S if and only if $\text{supp}(f)$ is an almost (α, β)-quasi-ideal of S.*

Proof. Assume that f is a fuzzy almost (α, β)-quasi-ideal of a Γ-semigroup S. Let $s \in S$ and $t \in (0,1]$. Then, $[(f \circ_\alpha s_t) \cap (s_t \circ_\beta f)] \cap f \neq 0$. Hence, there exists $x \in S$ such that

$$\left[\left(f \circ_\alpha s_t\right) \cap \left(s_t \circ_\beta f\right) \cap f\right](x) \neq 0. \tag{30}$$

So, there exist $y_1, y_2 \in S$ such that $x = y_1 \alpha s = s\beta y_2$, $f(x) \neq 0, f(y_1) \neq 0$, and $f(y_2) \neq 0$. That is, $x, y_1, y_2 \in \text{supp}(f)$. Thus, $[(C_{\text{supp}(f)} \circ_\alpha s_t) \cap (s_t \circ_\beta C_{\text{supp}(f)})](x) \neq 0$ and $C_{\text{supp}(f)}(x) \neq 0$. Therefore, $[(C_{\text{supp}(f)} \circ_\alpha s_t) \cap (s_t \circ_\beta C_{\text{supp}(f)})] \cap C_{\text{supp}(f)} \neq 0$. Hence, $C_{\text{supp}(f)}$ is a fuzzy almost (α, β)-quasi-ideal of S. By Theorem 54, $\text{supp}(f)$ is an almost (α, β)-quasi-ideal of S.

Conversely, assume that $\text{supp}(f)$ is an almost (α, β)-quasi-ideal of S. By Theorem 54, $C_{\text{supp}(f)}$ is a fuzzy almost (α, β)-quasi-ideal of S. Then, for each fuzzy point s_t of S, we have $[(C_{\text{supp}(f)} \circ_\alpha s_t) \cap (s_t \circ_\beta C_{\text{supp}(f)})] \cap C_{\text{supp}(f)} \neq 0$. Then, there exists $x \in S$ such that

$$\left[\left(C_{\text{supp}(f)} \circ_\alpha s_t\right) \cap \left(s_t \circ_\beta C_{\text{supp}(f)}\right) \cap C_{\text{supp}(f)}\right](x) \neq 0. \tag{31}$$

Hence, $[(C_{\text{supp}(f)} \circ_\alpha s_t) \cap (s_t \circ_\beta C_{\text{supp}(f)})](x) \neq 0$ and $C_{\text{supp}(f)}(x) \neq 0$. Then, there exist $y_1, y_2 \in S$ such that $x = y_1 \alpha s = s\beta y_2, f(x) \neq 0, f(y_1) \neq 0$, and $f(y_2) \neq 0$. This means that $[(f \circ_\alpha s_t) \cap (s_t \circ_\beta f)] \cap f \neq 0$. Therefore, f is a fuzzy almost (α, β)-quasi-ideal of S. □

Next, we define minimal fuzzy almost (α, β)-quasi-ideals in Γ-semigroups and give some relationship between minimal almost (α,β)-quasi-ideals and minimal fuzzy almost (α,β)-quasi-ideals of Γ-semigroups.

Definition 22. A fuzzy almost (α,β)-quasi-ideal f of a Γ-semigroup is called minimal if for each fuzzy almost (α,β)-quasi-ideal g of S such that $g\subseteq f$, we have $\mathrm{supp}(g)=\mathrm{supp}(f)$.

Theorem 56. *Let Q be a nonempty subset of a Γ-semigroup S. Then, Q is a minimal almost (α,β)-quasi-ideal of S if and only if C_Q is a minimal fuzzy almost (α,β)-quasi-ideal of S.*

Proof. Assume that Q is a minimal almost (α,β)-quasi-ideal of a Γ-semigroup S. By Theorem 54, C_Q is a fuzzy almost (α,β)-quasi-ideal of S. Let g be a fuzzy almost (α,β)-quasi-ideal of S such that $g\subseteq C_Q$. Then, $\mathrm{supp}(g)\subseteq\mathrm{supp}(C_Q)=Q$. Since $g\subseteq C_{\mathrm{supp}(g)}$ and by Theorem 53, we have $C_{\mathrm{supp}(g)}$ is a fuzzy almost (α,β)-quasi-ideal of S. By Theorem 54, $\mathrm{supp}(g)$ is an almost (α,β)-quasi-ideal of S. Since Q is minimal, $\mathrm{supp}(g)=Q=\mathrm{supp}(C_Q)$. Therefore, C_Q is minimal.

Conversely, assume that C_Q is a minimal fuzzy almost (α,β)-quasi-ideal of S. Let Q' be an almost (α,β)-quasi-ideal of S such that $Q'\subseteq Q$. By Theorem 54, $C_{Q'}$ is a fuzzy almost (α,β)-quasi-ideal of S such that $C_{Q'}\subseteq C_Q$. Since C_Q is minimal, $Q'=\mathrm{supp}(C_{Q'})=\mathrm{supp}(C_Q)=Q$. Therefore, Q is minimal.

Corollary 12. *Let Q be a sub Γ-semigroup of a Γ-semigroup S. Then, Q is almost (α,β)-quasi-simple if and only if for all fuzzy almost (α,β)-quasi-ideal f of S, $\mathrm{supp}(f)=Q$.*

5.3. Fuzzy Almost (α,β)-Bi-Ideals

Definition 23. Let $\alpha,\beta\in\Gamma$ and f be a fuzzy subset of a Γ-semigroup S. Then, f is called a fuzzy almost (α,β)-bi-ideal of S if $(f\circ_\alpha x_t\circ_\beta f)\cap f\neq 0$ for all fuzzy point x_t of S.

Theorem 57. *Let f be a fuzzy almost (α,β)-bi-ideal of a Γ-semigroup S and g be a fuzzy subset of S such that $f\subseteq g$. Then, g is a fuzzy almost (α,β)-bi-ideal of S.*

Proof. Assume that f is a fuzzy almost (α,β)-bi-ideal of a Γ-semigroup S and g is a fuzzy subset of S such that $f\subseteq g$. Then, for each fuzzy point x_t, $(f\circ_\alpha x_t\circ_\beta f)\cap f\neq 0$. We have $(f\circ_\alpha x_t\circ_\beta f)\cap f\subseteq(g\circ_\alpha x_t\circ_\beta g)\cap g$; this implies $(g\circ_\alpha x_t\circ_\beta g)\cap g\neq 0$. Therefore, g is a fuzzy almost (α,β)-bi-ideal of S.

Corollary 13. *Let f and g be fuzzy almost (α,β)-bi-ideals of a Γ-semigroup S. Then, $f\cup g$ is a fuzzy almost (α,β)-bi-ideal of S.*

Example 6. Consider the Γ-semigroup $\mathbb{Z}_5$ where $\Gamma=\{\overline{0},\overline{4}\}$ and $\overline{a}\gamma\overline{b}=\overline{a}+\gamma+\overline{b}$, where $\overline{a},\overline{b}\in\mathbb{Z}_5$ and $\gamma\in\Gamma$. Let $f\colon\mathbb{Z}_5\longrightarrow[0,1]$ defined by

$$\begin{aligned} f(\overline{0})&=0,\\ f(\overline{1})&=0.7,\\ f(\overline{2})&=0,\\ f(\overline{3})&=0.6,\\ f(\overline{4})&=0.4, \end{aligned}\tag{32}$$

and $g\colon\mathbb{Z}_5\longrightarrow[0,1]$ defined by

$$\begin{aligned} g(\overline{0})&=0,\\ g(\overline{1})&=0.1,\\ g(\overline{2})&=0.6,\\ g(\overline{3})&=0,\\ g(\overline{4})&=0.8. \end{aligned}\tag{33}$$

We have f and g are fuzzy almost $(\overline{0},\overline{4})$-bi-ideals of $\mathbb{Z}_5$.

Remark 8. The intersection of two fuzzy almost (α,β)-bi-ideals of a Γ-semigroup S need not be a fuzzy almost (α,β)-bi-ideal of S.

Theorem 58. *Let B be a nonempty subset of a Γ-semigroup S. Then, B is an almost (α,β)-bi-ideal of S if and only if C_B is a fuzzy almost (α,β)-bi-ideal of S.*

Proof. Assume that B is an almost (α,β)-bi-ideal of a Γ-semigroup S. Then, $B\alpha x\beta B\cap B\neq\emptyset$ for all $x\in S$. Thus, there exists $y\in B\alpha x\beta B$ and $y\in B$. So, $(C_B\circ_\alpha x_t\circ_\beta C_B)(y)=1$ and $C_B(y)=1$. Hence, $(C_B\circ_\alpha x_t\circ_\beta C_B)\cap C_B\neq 0$. Therefore, C_B is a fuzzy almost (α,β)-bi-ideal of S.

Conversely, assume that C_B is a fuzzy almost (α,β)-bi-ideal of S. Let $s\in S$. Then, $(C_B\circ_\alpha s_1\circ_\beta C_B)\cap C_B\neq 0$. Thus, there exists $x\in S$ such that $[(C_B\circ_\alpha s_1\circ_\beta C_B)\cap C_B](x)\neq 0$. Hence, $x\in B\alpha s\beta B\cap B$. So, $B\alpha s\beta B\cap B\neq\emptyset$. Consequently, B is an almost (α,β)-bi-ideal of S.

Theorem 59. *Let f be a fuzzy subset of a Γ-semigroup S. Then, f is a fuzzy almost (α,β)-bi-ideal of S if and only if $\mathrm{supp}(f)$ is an almost (α,β)-bi-ideal of S.*

Proof. Assume that f is a fuzzy almost (α,β)-bi-ideal of a Γ-semigroup S. Let $s\in S$. Then, $(f\circ_\alpha s_t\circ_\beta f)\cap f\neq 0$. Hence, there exists $x\in S$ such that $[(f_\alpha\circ s_t\circ_\beta f)\cap f](x)\neq 0$. So, there exist $y_1,y_2\in S$ such that $x=y_1\alpha s\beta y_2$, $f(x)\neq 0$, $f(y_1)\neq 0$, and $f(y_2)\neq 0$. That is, $x,y_1,y_2\in\mathrm{supp}(f)$. Thus, $(C_{\mathrm{supp}(f)}\circ_\alpha s_t\circ_\beta C_{\mathrm{supp}(f)})(x)\neq 0$ and $C_{\mathrm{supp}(f)}(x)\neq 0$. Therefore, $(C_{\mathrm{supp}(f)}\circ_\alpha s_t\circ_\beta C_{\mathrm{supp}(f)})\cap C_{\mathrm{supp}(f)}\neq 0$. Hence, $C_{\mathrm{supp}(f)}$ is a fuzzy almost (α,β)-bi-ideal of S. By Theorem 58, $\mathrm{supp}(f)$ is an almost (α,β)-bi-ideal of S.

Conversely, assume that $\mathrm{supp}(f)$ is an almost (α,β)-bi-ideal of S. By Theorem 58, $C_{\mathrm{supp}(f)}$ is a fuzzy almost (α,β)-bi-ideal of S. Then, $(C_{\mathrm{supp}(f)}\circ_\alpha s_t\circ_\beta C_{\mathrm{supp}(f)})\cap C_{\mathrm{supp}(f)}\neq 0$ for all $s\in S$. Then, there exists $x\in S$ such that

$[(C_{\mathrm{supp}(f)} \circ_\alpha s_t \circ_\beta C_{\mathrm{supp}(f)}) \cap C_{\mathrm{supp}(f)}](x) \neq 0$. Hence, $(C_{\mathrm{supp}(f)} \circ_\alpha s_t \circ_\beta C_{\mathrm{supp}(f)})(x) \neq 0$ and $C_{\mathrm{supp}(f)}(x) \neq 0$. Then, there exist $y_1, y_2 \in S$ such that $x = y_1 \alpha s \beta y_2, f(x) \neq 0$, $f(y_1) \neq 0$, and $f(y_2) \neq 0$. This means $(f \circ_\alpha s_t \circ_\beta f) \cap f \neq 0$. Therefore, f is a fuzzy almost (α, β)-bi-ideal of S.

We define minimal fuzzy almost (α, β)-bi-ideals in Γ-semigroups and give some relationship between minimal almost (α, β)-bi-ideals and minimal fuzzy almost (α, β)-bi-ideals of Γ-semigroups.

Definition 24. A fuzzy almost (α, β)-bi-ideal f of a Γ-semigroup S is called minimal if for all fuzzy almost (α, β)-bi-ideal g of S such that $g \subseteq f$, we have $\mathrm{supp}(g) = \mathrm{supp}(f)$.

Theorem 60. *Let B be a nonempty subset of a Γ-semigroup S. Then, B is a minimal almost (α, β)-bi-ideal of S if and only if C_B is a minimal fuzzy almost (α, β)-bi-ideal of S.*

Proof. Assume that B is a minimal almost (α, β)-bi-ideal of a Γ-semigroup S. By Theorem 58, C_B is a fuzzy almost (α, β)-bi-ideal of S. Let g be a fuzzy almost (α, β)-bi-ideal of S such that $g \subseteq C_B$. Then, $\mathrm{supp}(g) \subseteq \mathrm{supp}(C_B) = B$. Since $g \subseteq C_{\mathrm{supp}(g)}$ and by Theorem 57, we have $C_{\mathrm{supp}(g)}$ is a fuzzy almost bi-ideal of S. By Theorem 58, $supp(g)$ is an almost (α, β)-bi-ideal of S. Since B is minimal, $\mathrm{supp}(g) = B = \mathrm{supp}(C_B)$. Therefore, C_B is minimal.

Conversely, assume that C_B is a minimal fuzzy almost (α, β)-bi-ideal of S. Let B' be an almost (α, β)-bi-ideal of S such that $B' \subseteq B$. Then, $C_{B'}$ is a fuzzy almost (α, β)-bi-ideal of S such that $C_{B'} \subseteq C_B$. Hence, $B' = \mathrm{supp}(C_{B'}) = \mathrm{supp}(C_B) = B$. Therefore, B is minimal.

Corollary 14. *Let B be a sub-Γ-semigroup of a Γ-semigroup S. Then, B is almost (α, β)-bi-simple if and only if for all fuzzy almost (α, β)-bi-ideal f of S, $supp(f) = B$.*

Next, we give the relationship between α-prime almost (α, β)-bi-ideals and α-prime fuzzy almost (α, β)-bi-ideals.

Definition 25. Let S be a Γ-semigroup and $\gamma \in \Gamma$.

(1) An almost (α, β)-bi-ideal A of S is called γ-prime if for all $x, y \in S, x\gamma y \in A$ implies $x \in A$ or $y \in A$.

(2) A fuzzy almost (α, β)-bi-ideal f of S is called γ-prime if for all $x, y \in S, f(x\gamma y) \leq \max\{f(x), f(y)\}$.

Theorem 61. *Let A be a nonempty subset of a Γ-semigroup S. Then, A is a γ-prime almost (α, β)-bi-ideal of S if and only if C_A is a γ-prime fuzzy almost (α, β)-bi-ideal of S.*

Proof. Assume that A is a γ-prime almost (α, β)-bi-ideal of S. By Theorem 58, C_A is a fuzzy almost (α, β)-bi-ideal of S. Let $x, y \in S$. We consider two cases:

Case 1: $x\gamma y \in A$. So, $x \in A$ or $y \in A$. Then, $\max\{C_A(x), C_A(y)\} = 1 \geq C_A(x\gamma y)$.

Case 2: $x\gamma y \notin A$. Then, $C_A(x\gamma y) = 0 \leq \max\{C_A(x), C_A(y)\}$.

Thus, C_A is a γ-prime fuzzy almost (α, β)-bi-ideal of S.

Conversely, assume that C_A is a γ-prime fuzzy almost (α, β)-bi-ideal of S. By Theorem 58, A is an almost (α, β)-bi-ideal of S. Let $x, y \in S$ be such that $x\gamma y \in A$. Then, $C_A(x\gamma y) = 1$. By assumption, $C_A(x\gamma y) \leq \max\{C_A(x), C_A(y)\}$. Therefore, $\max\{C_A(x), C_A(y)\} = 1$. Hence, $x \in A$ or $y \in A$. Thus, A is a γ-prime almost (α, β)-bi-ideal of S.

In this section, we give the relationship between γ-semiprime almost (α, β)-bi-ideals and γ-semiprime fuzzy almost (α, β)-bi-ideals. □

Definition 26. Let S be a Γ-semigroup and $\alpha \in \Gamma$.

(1) An almost (α, β)-bi-ideal A of S is called a γ-semiprime if for all $x \in S, x\gamma x \in A$ implies $x \in A$.

(2) A fuzzy almost (α, β)-bi-ideal f of S is called a γ-semiprime if for all $x \in S, f(x\gamma x) \leq f(x)$.

Theorem 62. *Let A be a nonempty subset of S. Then, A is a γ-semiprime almost (α, β)-bi-ideal of S if and only if C_A is a γ-semiprime fuzzy almost (α, β)-bi-ideal of S.*

Proof. Assume that A is a γ-semiprime almost (α, β)-bi-ideal of S. By Theorem 58, C_A is a fuzzy almost (α, β)-bi-ideal of S. Let $x \in S$. We consider two cases:

Case 1: $x\gamma x \in A$. Then, $x \in A$. So, $C_A(x) = 1$. Hence, $C_A(x) \geq C_A(x\gamma x)$.

Case 2: $x\gamma x \notin A$. Then, $C_A(x\gamma x) = 0 \leq C_A(x)$.

Thus, C_A is a γ-semiprime fuzzy almost (α, β)-bi-ideal of S.

Conversely, assume that C_A is a γ-semiprime fuzzy almost (α, β)-bi-ideal of S. By Theorem 58, A is an almost (α, β)-bi-ideal of S. Let $x \in S$ be such that $x\gamma x \in A$. Then, $C_A(x\gamma x) = 1$. By assumption, $C_A(x\gamma x) \leq C_A(x)$. Since $C_A(x\gamma x) = 1$, $C_A(x) = 1$. Hence, $x \in A$. Thus, A is a γ-semiprime almost (α, β)-bi-ideal of S.

6. Discussion and Conclusion

In this paper, we define new types of ideals and fuzzy ideals by using elements in Γ. We show interesting properties of these ideals and fuzzy ideals. Moreover, we show the relationships between these ideals and their fuzzifications.

Data Availability

No data were used to support this research.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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