Research Article

Generalized Hadamard Fractional Integral Inequalities for Strongly (s, m) -Convex Functions

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This article deals with Hadamard inequalities for strongly (s, m) -convex functions using generalized Riemann–Liouville fractional integrals. Several generalized fractional versions of the Hadamard inequality are presented; we also provide refinements of many known results which have been published in recent years.

1. Introduction

Fractional calculus is related to the integrals and derivatives of any arbitrary real or complex order. Its history starts from the end of the seventeenth century, but now it has many applications in almost every field of mathematics, science, and engineering such as electromagnetic, viscoelasticity, fluid mechanics, and signal processing. Fractional integral and derivative operators are of great importance in fractional calculus. The Riemann–Liouville fractional integrals are playing key role in its development. Sarikaya et al. [1, 2] studied Hadamard inequality through Riemann–Liouville fractional integrals of convex functions. This study has encouraged a number of researchers to work further in the field of mathematical inequalities by using fractional integral operators. As a consequence, Hadamard's inequality is generalized and extended by fractional integral operators in many ways (see [3–9] and the references therein). The following inequality is the well-known Hadamard inequality for convex functions which is stated in [10].

Let $f: I \rightarrow \mathbb{R}$ be a convex function defined on an interval $I \subset \mathbb{R}$ and $x, y \in I$ where $x < y$. Then, the following inequality holds:

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x} \int_x^y f(v) dv \leq \frac{f(x) + f(y)}{2}. \quad (1)$$

For the history of this inequality, we refer the readers to [11, 12]. Use of convex functions in the fields of statistics [13], economics [14], and optimization [15] is of prime importance because they play an important role in development of new concepts and notions. Various scholars extended the research on integral inequalities to fractal sets [16]. In this paper, the Hadamard inequality is studied for generalized Riemann–Liouville fractional integrals of strongly (s, m) -convex functions; also, by using two integral identities, some error bounds of already established fractional inequalities are studied. Bracamonte et al. [17] defined the strongly (s, m) -convex function as follows.

Definition 1. A function $f: [0, +\infty) \rightarrow \mathbb{R}$ is said to be strongly (s, m) -convex function with modulus $c \geq 0$ in second sense, where $(s, m) \in (0, 1]^2$, if

$$f(xt + m(1-t)y) \leq t^s f(x) + m(1-t)^s f(y) - cmt(1-t)|y-x|^2, \quad (2)$$

holds for all $x, y \in [0, +\infty)$ and $t \in [0, 1]$.

The well-known definition of Riemann–Liouville fractional integral is given as follows.

Definition 2 (see [18]) (see also [19]). Let $f \in L[a, b]$. Then, left-sided and right-sided Riemann–Liouville fractional

integrals of a function f of order μ where $\Re(\mu) > 0$ are given by

$$I_{a^+}^\mu f(x) = \frac{1}{\Gamma(\mu)} \int_a^x (x-t)^{\mu-1} f(t) dt, \quad x > a, \quad (3)$$

$$I_{b^-}^\mu f(x) = \frac{1}{\Gamma(\mu)} \int_x^b (t-x)^{\mu-1} f(t) dt, \quad x < b, \quad (4)$$

where $\Re(\mu)$ is real part of μ and $\Gamma(\mu) = \int_0^\infty e^{-z} z^{\mu-1} dz$. The following theorems are the fractional versions of Hadamard inequality by Riemann–Liouville fractional integrals.

Theorem 1 (see [1]). Let $f: [a, b] \rightarrow \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L[a, b]$. If f is a convex function on $[a, b]$, then the following fractional integral inequality holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\mu+1)}{2(b-a)^\mu} [I_{a^+}^\mu f(b) + I_{b^-}^\mu f(a)] \leq \frac{f(a) + f(b)}{2}, \quad (5)$$

with $\mu > 0$.

Theorem 2 (see [2]). Under the assumptions of Theorem 1, the following fractional integral inequality holds:

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{2^{\mu-1} \Gamma(\mu+1)}{(b-a)^\mu} [I_{((a+b)/2)^+}^\mu f(b) + I_{((a+b)/2)^-}^\mu f(a)] \\ &\leq \frac{f(a) + f(b)}{2}, \end{aligned} \quad (6)$$

with $\mu > 0$.

By establishing an integral identity, the following error estimation of inequality (6) is proved.

Theorem 3 (see [1]). Let $f: [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following fractional integral inequality holds:

$$\begin{aligned} &\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\mu+1)}{2(b-a)^\mu} [I_{a^+}^\mu f(b) + I_{b^-}^\mu f(a)] \right| \\ &\leq \frac{b-a}{2(\mu+1)} \left(1 - \frac{1}{2^\mu}\right) [|f'(a)| + |f'(b)|]. \end{aligned} \quad (7)$$

A k -analogue of Riemann–Liouville integral is defined as follows.

Definition 3 (see [20]). Let $f \in L[a, b]$. Then, k -fractional Riemann–Liouville integrals of order μ where $\Re(\mu) > 0$, $k > 0$, are defined as

$${}_k I_{a^+}^\mu f(x) = \frac{1}{k \Gamma_k(\mu)} \int_a^x (x-t)^{(\mu/k)-1} f(t) dt, \quad x > a, \quad (8)$$

$${}_k I_{b^-}^\mu f(x) = \frac{1}{k \Gamma_k(\mu)} \int_x^b (t-x)^{(\mu/k)-1} f(t) dt, \quad x < b, \quad (9)$$

where $\Gamma_k(\cdot)$ is defined by [21]

$$\Gamma_k(\mu) = \int_0^\infty t^{\mu-1} e^{-(t^k/k)} dt, \quad \Re(\mu) > 0. \quad (10)$$

If $k = 1$, (8) and (9) coincide with (3) and (4).

Two k -fractional versions of Hadamard inequality for k -fractional Riemann–Liouville integrals are given in the next two theorems.

Theorem 4 (see [22]). Let $f: [a, b] \rightarrow \mathbb{R}$ be a positive function with $0 \leq a < b$. If f is a convex function on $[a, b]$, then the following inequality for k -fractional integrals holds:

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{\Gamma_k(\mu+k)}{2(b-a)^{(\mu/k)}} [{}_k I_{a^+}^\mu f(b) + {}_k I_{b^-}^\mu f(a)] \\ &\leq \frac{f(a) + f(b)}{2}. \end{aligned} \quad (11)$$

Theorem 5 (see [23]). Under the assumption of Theorem 4, the following inequality for k -fractional integrals holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{2^{(\mu/k)-1} \Gamma_k(\mu+k)}{(b-a)^{(\mu/k)}} [{}_k I_{((a+b)/2)^+}^\mu f(b) + {}_k I_{((a+b)/2)^-}^\mu f(a)] \leq \frac{f(a) + f(b)}{2}. \quad (12)$$

By establishing an integral identity, in the following theorem, the error estimation of Theorem 4 is proved.

Theorem 6 (see [22]). Let $f: [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) with $0 \leq a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality for k -fractional integrals holds:

$$\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma_k(\mu + k)}{2(b-a)^{\mu/k}} [{}_k I_{a^+}^{\mu} f(b) + {}_k I_{b^-}^{\mu} f(a)] \right| \leq \frac{b-a}{2((\mu/k) + 1)} \left(1 - \frac{1}{2^{(\mu/k)}} \right) [|f'(a)| + |f'(b)|]. \quad (13)$$

In the following, we recall the definition of generalized Riemann–Liouville fractional integrals by a monotonically increasing function.

Definition 4 (see [24]). Let $f: [a, b] \rightarrow \mathbb{R}$ be an integrable function. Also, let ψ be an increasing and positive function on

$(a, b]$, having a continuous derivative ψ' on (a, b) . The left-sided and right-sided fractional integrals of a function f with respect to another function ψ on $[a, b]$ of order μ where $\Re(\mu) > 0$ are given by

$$I_{a^+}^{\mu, \psi} f(x) = \frac{1}{\Gamma(\mu)} \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\mu-1} f(t) dt, \quad x > a, \quad (14)$$

$$I_{b^-}^{\mu, \psi} f(x) = \frac{1}{\Gamma(\mu)} \int_x^b \psi'(t) (\psi(t) - \psi(x))^{\mu-1} f(t) dt, \quad x < b. \quad (15)$$

If ψ is identity function, then (14) and (15) coincide with (3) and (4).

The k -analogue of generalized Riemann–Liouville fractional integral is defined as follows.

Definition 5 (see [25]). Let $f: [a, b] \rightarrow \mathbb{R}$ be an integrable function. Also, let ψ be an increasing and positive function on $(a, b]$, having a continuous derivative ψ' on (a, b) . The left-sided and right-sided fractional integrals of a function f with respect to another function ψ on $[a, b]$ of order μ where $\Re(\mu) > 0$, $k > 0$, are defined by

$${}_k I_{a^+}^{\mu, \psi} f(x) = \frac{1}{k\Gamma_k(\mu)} \int_a^x \psi'(t) (\psi(x) - \psi(t))^{(\mu/k)-1} f(t) dt, \quad x > a, \quad (16)$$

$${}_k I_{b^-}^{\mu, \psi} f(x) = \frac{1}{k\Gamma_k(\mu)} \int_x^b \psi'(t) (\psi(t) - \psi(x))^{(\mu/k)-1} f(t) dt, \quad x < b. \quad (17)$$

For further study of fractional integrals, see [26, 27]. We will utilize the following well-known hypergeometric, beta, and incomplete beta functions in our results [28].

$${}_2F_1[a, b; c; z] = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} dt, \quad c > b > 0, |z| < 1,$$

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad (18)$$

$$B(x, y; z) = \int_0^z t^{x-1} (1-t)^{y-1} dt.$$

The rest of the paper is organized as follows. In Section 2, we obtain Hadamard inequalities for generalized Riemann-Liouville fractional integrals of strongly (s, m) -convex functions. Many specific cases are given as outcomes of these inequalities; they are related to the results which have been published in different papers. In Section 3, by using two integral identities for generalized fractional integrals, the error bounds of fractional Hadamard inequalities are established for differentiable strongly (s, m) -convex functions. This paper reproduces the results which are explicitly given in [1, 2, 22, 23, 29–37].

2. Main Results

This section is dedicated to the Hadamard inequality for strongly (s, m) -convex functions via generalized Riemann-Liouville fractional integrals. We will give two versions of this inequality. First one is stated and proved in the following theorem.

Theorem 7. Let $f: [a, b] \rightarrow \mathbb{R}$, $[a, b] \subset [0, +\infty)$ be a positive function and $f \in L_1[a, b]$. Also, let f be strongly (s, m) -convex function on $[a, b]$ with modulus c , such that $(a/m), (a/m^2), mb \in [a, b]$. Then, for $k > 0$ and $(s, m) \in (0, 1]^2$, the following k -fractional integral inequalities hold for operators given in (16) and (17):

$$\begin{aligned} & f\left(\frac{a+mb}{2}\right) + \frac{cm}{4(\mu+k)(\mu+2k)} \left[\mu(\mu+k)(b-a)^2 + 2k^2\left(\frac{a}{m} - mb\right)^2 + 2k\mu(b-a)\left(\frac{a}{m} - mb\right) \right] \\ & \leq \frac{\Gamma_k(\mu+k)}{2^s(mb-a)^{\mu/k}} \left[{}_k I_{\psi^{-1}(a)^+}^{\mu, \psi} (f \circ \psi)(\psi^{-1}(mb)) + m^{(\mu/k)+1} {}_k I_{\psi^{-1}(b)^-}^{\mu, \psi} (f \circ \psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right) \right] \\ & \leq \frac{\mu[f(a) + mf(b)]}{2^s(\mu+sk)} + \frac{m\mu B((\mu/k), s+1)[f(b) + mf(a/m^2)]}{k2^s} - \frac{cmk\mu \left[(b-a)^2 + m(b - (a/m^2))^2 \right]}{2^s(\mu+k)(\mu+2k)}, \end{aligned} \quad (19)$$

with $\mu > 0$.

Proof. The following inequality holds for strongly (s, m) -convex functions.

$$f\left(\frac{x+my}{2}\right) \leq \frac{f(x) + mf(y)}{2^s} - \frac{cm|y-x|^2}{4}. \quad (20)$$

By setting $x = at + m(1-t)b$, $y = (a/m)(1-t) + tb$, $t \in [0, 1]$, in (20), multiplying resulting inequality with $t^{(\mu/k)-1}$, and then integrating with respect to t , we get

$$\begin{aligned} \frac{k}{\mu} f\left(\frac{a+mb}{2}\right) & \leq \frac{1}{2^s} \left[\int_0^1 f(at + m(1-t)b) t^{(\mu/k)-1} dt + m \int_0^1 f\left(\frac{a}{m}(1-t) + bt\right) t^{(\mu/k)-1} dt \right] \\ & \quad - \frac{cm}{4} \left[\frac{k(b-a)^2}{\mu+2k} + \frac{2k^3((a/m) - mb)^2}{\mu(\mu+k)(\mu+2k)} + \frac{2k^2(b-a)((a/m) - mb)}{(\mu+k)(\mu+2k)} \right]. \end{aligned} \quad (21)$$

By setting $\psi(u) = at + m(1-t)b$ and $\psi(v) = (a/m)(1-t) + bt$ in (21) and by applying Definition 5, we get the following inequality:

$$\begin{aligned} f\left(\frac{a+mb}{2}\right) & \leq \frac{\Gamma_k(\mu+k)}{2^s(mb-a)^{\mu/k}} \left[{}_k I_{\psi^{-1}(a)^+}^{\mu, \psi} (f \circ \psi)(\psi^{-1}(mb)) + m^{(\mu/k)+1} {}_k I_{\psi^{-1}(b)^-}^{\mu, \psi} (f \circ \psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right) \right] \\ & \quad - \frac{cm\mu}{4} \left[\frac{(b-a)^2}{\mu+2k} + \frac{2k^2((a/m) - mb)^2}{\mu(\mu+k)(\mu+2k)} + \frac{2k(b-a)((a/m) - mb)}{(\mu+k)(\mu+2k)} \right]. \end{aligned} \quad (22)$$

The above inequality leads to the first inequality of (19). On the other hand, f is strongly (s, m) -convex function with modulus c ; for $t \in [0, 1]$, we have the following inequality:

$$\begin{aligned} f(ta + m(1-t)b) + mf\left(\frac{a}{m}(1-t) + tb\right) &\leq t^s[f(a) + mf(b)] \\ &+ m(1-t)^s\left[f(b) + mf\left(\frac{a}{m^2}\right)\right] - cmt(1-t)\left[(b-a)^2 + m\left(b - \frac{a}{m^2}\right)^2\right]. \end{aligned} \quad (23)$$

By integrating (23) over $[0, 1]$ after multiplying with $t^{(\mu/k)-1}$, the following inequality holds:

$$\begin{aligned} &\int_0^1 t^{(\mu/k)-1} f(ta + m(1-t)b) dt + m \int_0^1 t^{(\mu/k)-1} f\left(\frac{a}{m}(1-t) + tb\right) dt \\ &\leq \frac{[f(a) + mf(b)]k}{\mu + sk} + m\left[f(b) + mf\left(\frac{a}{m^2}\right)\right] B\left(\frac{\mu}{k}, s+1\right) \\ &\quad - \frac{cmk^2[(b-a)^2 + m(b - (a/m^2))^2]}{(\mu + k)(\mu + 2k)}. \end{aligned} \quad (24)$$

Again using substitutions as considered in (21), we get

$$\begin{aligned} &\frac{k\Gamma_k(\mu)}{(mb-a)^{(\mu/k)}} \left[{}_kI_{\psi^{-1}(a)^+}^{\mu, \psi} (f \circ \psi)(\psi^{-1}(mb)) + m^{(\mu/k)+1} {}_kI_{\psi^{-1}(b)^-}^{\mu, \psi} (f \circ \psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right) \right] \\ &\leq \frac{[f(a) + mf(b)]k}{\mu + sk} + m\left[f(b) + mf\left(\frac{a}{m^2}\right)\right] B\left(\frac{\mu}{k}, s+1\right) \\ &\quad - \frac{cmk^2[(b-a)^2 + m(b - (a/m^2))^2]}{(\mu + k)(\mu + 2k)}. \end{aligned} \quad (25)$$

This leads to the second inequality of (19). $\square$

Remark 1. Under the assumption of Theorem 7, the following outcomes are noted.

- (i) If $s = 1$, $m = 1$, then the inequality stated in [[32], Theorem 9] is obtained.
- (ii) If $c = 0$, $s = 1$, $m = 1$, and ψ is the identity function in (19), then Theorem 4 is obtained.
- (iii) If $c = 0$, $s = 1$, $m = 1$, $k = 1$, and ψ is the identity function in (19), then Theorem 1 is obtained.
- (iv) If $k = 1$, $s = 1$, $m = 1$, and ψ is the identity function in (19), then refinement of Theorem 1 is obtained.

- (v) If $\mu = 1$, $k = 1$, $s = 1$, $m = 1$, $c = 0$, and ψ is the identity function in (19), then Hadamard inequality is obtained.
- (vi) If $m = 1$, $s = 1$, and $c = 0$ in (19), then the inequality [[34], Theorem 1] is obtained.
- (vii) If $c = 0$, $k = 1$, $m = 1$, and $s = 1$ in (19), then the inequality stated in [[33], Theorem 2.1] is obtained.
- (viii) If $s = 1$, $k = 1$, and ψ is the identity function in (19), then the inequality stated in [[31], Theorem 6] is obtained.
- (ix) If $k = 1$, $m = 1$, $s = 1$, $\mu = 1$, and ψ is the identity function in (19), then the inequality stated in [[35], Theorem 6] is obtained.

(x) If $\alpha = 1$, $k = 1$, $c = 0$, and ψ is the identity function in (19), then the inequality stated in [[29], Theorem 2.1] is obtained.

Corollary 1. Under the assumption of Theorem 7 with $c = 0$ in (19), the following inequality holds:

$$\begin{aligned} f\left(\frac{a+mb}{2}\right) &\leq \frac{\Gamma_k(\mu+k)}{2^s(mb-a)^{(\mu/k)} k I_{\psi^{-1}(a)^+}^{\mu,\psi}} \left[(f \circ \psi)(\psi^{-1}(mb)) + m^{(\mu/k)+1} k I_{\psi^{-1}(b)^-}^{\mu,\psi} (f \circ \psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right) \right] \\ &\leq \frac{\mu[f(a) + mf(b)]}{2^s(\mu + sk)} \\ &\quad + \frac{m\mu B((\mu/k), s+1)[f(b) + mf(a/m^2)]}{k2^s}. \end{aligned} \quad (26)$$

Theorem 8. Under the assumption of Theorem 7, the following k -fractional integral inequality holds:

$$\begin{aligned} f\left(\frac{a+mb}{2}\right) &+ \frac{cm}{16(\mu+k)(\mu+2k)} \left[\mu(b-a)^2 + \left(\frac{a}{m} - mb\right)^2 (\mu^2 + 5k\mu + 8k^2) + 2\mu(b-a)\left(\frac{a}{m} - mb\right)(\mu + 3k) \right] \\ &\leq \frac{2^{(\mu/k)-s} \Gamma_k(\mu+k)}{(mb-a)^{(\mu/k)}} \left[k I_{\psi^{-1}((a+mb)/2)^+}^{\mu,\psi} (f \circ \psi)(\psi^{-1}(mb)) + m^{(\mu/k)+1} k I_{\psi^{-1}((a+mb)/2m)^-}^{\mu,\psi} (f \circ \psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right) \right] \\ &\leq \frac{\mu[f(a) + mf(b)]}{2^{2s}(\mu + sk)} + \frac{m_2 F_1(-s, (\mu/k), ((\mu+k)/k); (1/2)) [f(b) + mf(a/m^2)]}{2^s} - \frac{cm(\mu + 3k)\mu[(b-a)^2 + m(b - (a/m^2))]}{2^{s+2}(\mu + k)(\mu + 2k)}, \end{aligned} \quad (27)$$

with $\mu > 0$.

inequality with $t^{(\mu/k)-1}$, and then integrating with respect to t , we get

Proof. By setting $x = (at/2) + m((2-t)/2)b$, $y = (a/m)((2-t)/2) + (bt/2)$, $t \in [0, 1]$ in (20), multiplying resulting

$$\begin{aligned} \frac{k}{\mu} f\left(\frac{a+mb}{2}\right) &\leq \frac{1}{2^s} \left[\int_0^1 f\left(\frac{at}{2} + m\left(\frac{2-t}{2}\right)b\right) t^{(\mu/k)-1} dt + m \int_0^1 f\left(\frac{a}{m}\left(\frac{2-t}{2}\right) + \frac{bt}{2}\right) t^{(\mu/k)-1} dt \right] \\ &\quad - \frac{cm}{4} \left[\frac{(b-a)^2 k}{4(\mu+2k)} + \frac{k((a/m) - mb)^2 (\mu^2 + 5k\mu + 8k^2)}{4\mu(\mu+k)(\mu+2k)} + \frac{(b-a)((a/m) - mb)(\mu + 3k)k}{2(\mu+k)(\mu+2k)} \right]. \end{aligned} \quad (28)$$

By setting $\psi(u) = (at/2) + bm((2-t)/2)$ and $\psi(v) = (a/m)((2-t)/2) + (bt/2)$ and by applying Definition 5, we get the following inequality:

$$\begin{aligned} \frac{k}{\mu} f\left(\frac{a+mb}{2}\right) &\leq \frac{2^{\mu/k} k \Gamma_k(\mu)}{2^s (mb-a)^{\mu/k}} \left[{}_k I_{\psi^{-1}((a+mb)/2)^+}^{\mu, \psi} (f \circ \psi)(\psi^{-1}(mb)) + m^{(\mu/k)+1} {}_k I_{\psi^{-1}((a+mb)/2m)^-}^{\mu, \psi} (f \circ \psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right) \right] \\ &\quad - \frac{cm}{4} \left[\frac{k(b-a)^2}{4(\mu+2k)} + \frac{k((a/m)-mb)^2(\mu^2+5k\mu+8k^2)}{4\mu(\mu+k)(\mu+2k)} + \frac{k(b-a)((a/m)-mb)(\mu+3k)}{2(\mu+k)(\mu+2k)} \right]. \end{aligned} \quad (29)$$

The above inequality leads to the first inequality of (27). On the other hand, f is strongly (s, m) -convex function with modulus c ; for $t \in [0, 1]$, we have the following inequality:

$$\begin{aligned} f\left(\frac{at}{2} + m\left(\frac{2-t}{2}\right)b\right) + mf\left(\frac{a}{m}\left(\frac{2-t}{2}\right) + \frac{bt}{2}\right) &\leq \left(\frac{t}{2}\right)^s [f(a) + mf(b)] \\ &\quad + m\left(\frac{2-t}{2}\right)^s \left[f(b) + mf\left(\frac{a}{m^2}\right) \right] - \frac{cmt(2-t) \left[(b-a)^2 + m(b-(a/m^2))^2 \right]}{4}. \end{aligned} \quad (30)$$

By integrating (30) over $[0, 1]$ after multiplying with $t^{(\mu/k)-1}$, the following inequality holds:

$$\begin{aligned} &\int_0^1 f\left(\frac{at}{2} + m\left(\frac{2-t}{2}\right)b\right) t^{(\mu/k)-1} dt + m \int_0^1 f\left(\frac{a}{m}\left(\frac{2-t}{2}\right) + \frac{bt}{2}\right) t^{(\mu/k)-1} dt \\ &\leq \frac{k[f(a) + mf(b)]}{2^s (sk + \mu)} + \frac{mk[f(b) + f(a/m^2)] {}_2F_1(-s, (1 + \mu/k), (2 + (\mu + k)/k); (1/2))}{\mu} \\ &\quad - \frac{cmk(\mu + 3k) \left[(b-a)^2 + m(b-(a/m^2))^2 \right]}{4(\mu + k)(\mu + 2k)}. \end{aligned} \quad (31)$$

Again using substitutions as considered in (28), we get

$$\begin{aligned}
& \frac{2^{\mu/k} k \Gamma_k(\mu)}{(mb-a)^{\mu/k}} \left[{}_k I_{\psi^{-1}((a+mb)/2)^+}^{\mu, \psi} (f \circ \psi)(\psi^{-1}(mb)) + m^{(\mu/k)+1} {}_k I_{\psi^{-1}((a+mb)/2m)^-}^{\mu, \psi} (f \circ \psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right) \right] \\
& \leq \frac{k[f(a) + mf(b)]}{2^s(sk + \mu)} + \frac{mk[f(b) + mf(a/m^2)] {}_2F_1(-s, (\mu/k), ((\mu+k)/k); (1/2))}{\mu} \\
& \quad - \frac{cmk(\mu + 3k) \left[(b-a)^2 + m(b - (a/m^2))^2 \right]}{4(\mu + k)(\mu + 2k)}.
\end{aligned} \tag{32}$$

This leads to the second inequality of (27). $\square$

Remark 2. Under the assumption of Theorem 8, the following outcomes are noted.

- (i) If $s = 1$ and $m = 1$ in (27), then the inequality stated in [[32], Theorem 10] is obtained.
- (ii) If $s = 1$, $m = 1$, $k = 1$, $c = 0$, and ψ is the identity function in (27), then Theorem 2 is obtained.
- (iii) If $s = 1$, $m = 1$, $k = 1$, and ψ is the identity function in (27), then refinement of Theorem 2 is obtained.
- (iv) If $s = 1$, $m = 1$, $k = 1$, $\mu = 1$, $c = 0$, and ψ is the identity function in (27), then Hadamard inequality is obtained.

- (v) If $s = 1$, $m = 1$, $c = 0$, and ψ is the identity function in inequality (27), then Theorem 5 is obtained.
- (vi) If $s = 1$, $m = 1$, and $c = 0$ in (27), then the inequality stated in [[32], Corollary 5] is obtained.
- (vii) If $s = 1$, $k = 1$, and ψ is the identity function in inequality (27), then the inequality stated in [[31], Theorem 7] is obtained.

Corollary 3. Under the assumption of Theorem 8 with $c = 0$ in (27), the following inequality holds:

$$\begin{aligned}
f\left(\frac{a+mb}{2}\right) & \leq \frac{2^{(\mu/k)-s} \Gamma_k(\mu+k)}{(mb-a)^{(\mu/k)}} \left[{}_k I_{\psi^{-1}((a+mb)/2)^+}^{\mu, \psi} (f \circ \psi)(\psi^{-1}(mb)) + m^{(\mu/k)+1} {}_k I_{\psi^{-1}((a+mb)/2m)^-}^{\mu, \psi} (f \circ \psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right) \right] \\
& \leq \frac{\mu[f(a) + mf(b)]}{2^{2s}(\mu + sk)} \\
& \quad + \frac{m[f(b) + mf(a/m^2)] {}_2F_1(-s, (\mu/k), ((\mu+k)/k); (1/2))}{2^s}.
\end{aligned} \tag{33}$$

3. Error Estimations of Hadamard Inequalities via Strongly (s, m) -Convex Functions

In this section, we will study error estimations of Hadamard inequalities for generalized Riemann–Liouville fractional integrals of strongly (s, m) -convex functions. The estimations obtained here provide refinements of many well-known results. The Mathematica program is used for integration. We recall the well-known Hölder's integral inequality.

Theorem 9 (see [38]). Let $p > 1$ and $(1/p) + (1/q) = 1$. If f and g are real functions defined on $[a, b]$ and if $|f|^p$ and $|g|^q$ are integrable functions on $[a, b]$, then

$$\int_a^b |f(x)g(x)| dx \leq \left(\int_a^b |f(x)|^p dx \right)^{1/p} \left(\int_a^b |g(x)|^q dx \right)^{1/q}, \tag{34}$$

with equality holding iff $A|f(x)|^p = B|g(x)|^q$ almost everywhere, where A and B are constants.

In order to prove the next result, the following lemma is useful.

Lemma 1 (see [34]). Let $a < b$ and $f: [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) . Also, suppose that $f' \in L[a, b]$, $\psi(x)$ is an increasing and positive monotone function on (a, b) , having a continuous derivative $\psi'(x)$ on (a, b) , and $\alpha \in (0, 1)$. Then, for $k > 0$, the following identity holds:

$$\begin{aligned}
& \frac{f(a) + f(b)}{2} - \frac{\Gamma_k(\mu + k)}{2(b-a)^{(\mu/k)}} \left[{}_k I_{\psi^{-1}(a)^+}^{\mu, \psi} (f \circ \psi)(\psi^{-1}(b)) + {}_k I_{\psi^{-1}(b)^-}^{\mu, \psi} (f \circ \psi)(\psi^{-1}(a)) \right] \\
&= \frac{b-a}{2} \int_0^1 \left[(1-t)^{(\mu/k)} - t^{(\mu/k)} \right] f'(ta + (1-t)b) dt.
\end{aligned} \tag{35}$$

Theorem 10. Let $f: [a, b] \longrightarrow \mathbb{R}$, $[a, b] \subset [0, +\infty)$ be a differentiable mapping on (a, b) such that $f' \in L_1[a, b]$. Also, suppose that $|f'|$ is strongly (s, m) -convex on $[a, b]$ with

modulus c . Then, for $k > 0$ and $(s, m) \in (0, 1]^2$, the following k -fractional integral inequality holds for operators given in (16) and (17):

$$\begin{aligned}
& \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma_k(\mu + k)}{2(b-a)^{(\mu/k)}} \left[{}_k I_{\psi^{-1}(a)^+}^{\mu, \psi} (f \circ \psi)(\psi^{-1}(b)) + {}_k I_{\psi^{-1}(b)^-}^{\mu, \psi} (f \circ \psi)(\psi^{-1}(a)) \right] \right| \\
& \leq \frac{b-a}{2} \left[|f'(a)| \left(2B\left(\frac{1}{2}; s+1, \frac{\mu}{k}+1\right) + \frac{1-(1/2)^{s+(\mu/k)}}{s+(\mu/k)+1} - B\left(1+s, 1+\frac{\mu}{k}\right) \right) \right. \\
& \quad + m \left| f'\left(\frac{b}{m}\right) \right| \left(\frac{1-(1/2)^{s+(\mu/k)+1}}{s+(\mu/k)+1} - \frac{{}_2F_1(-s, 1+(\mu/k), 2+(\mu/k); (1/2))}{2^{1+(\mu/k)}((\mu/k)+1)} \right) \\
& \quad + B(1+s, 1+(\mu/k)) - \frac{k(k+\mu+2^s(k+ks+\mu)){}_2F_1(-s, 1+(\mu/k), 2+(\mu/k); (1/2))}{2^{s+(\mu/k)+1}(sk+\mu+k)(\mu+k)} \Bigg] \\
& \quad - \frac{c((b/m)-a)^2}{((\mu/k)+2)((\mu/k)+3)} \left(1 - \frac{(\mu/k)+4}{2^{(\mu/k)+2}} \right),
\end{aligned} \tag{36}$$

with $\mu > 0$ and ${}_2F_1(-s, 1+(\mu/k), 2+(\mu/k); (1/2))$ being the hypergeometric function.

Proof. By Lemma 1, it follows that

$$\begin{aligned}
& \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma_k(\mu + k)}{2(b-a)^{\mu/k}} \left[{}_k I_{\psi^{-1}(a)^+}^{\mu, \psi} (f \circ \psi)(\psi^{-1}(b)) + {}_k I_{\psi^{-1}(b)^-}^{\mu, \psi} (f \circ \psi)(\psi^{-1}(a)) \right] \right| \\
& \leq \frac{b-a}{2} \int_0^1 \left| (1-t)^{\mu/k} - t^{\mu/k} \right| |f'(ta + (1-t)b)| dt.
\end{aligned} \tag{37}$$

Since $|f'|$ is strongly (s, m) -convex function on $[a, b]$, for $t \in [0, 1]$, we have

$$|f'(ta + (1-t)b)| \leq t^s |f'(a)| + m(1-t)^s \left| f'\left(\frac{b}{m}\right) \right| - cmt(1-t) \left(\frac{b}{m} - a \right)^2. \tag{38}$$

Now using (38) in (37), we have

$$\begin{aligned}
& \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma_k(\mu + k)}{2(b-a)^{\mu/k}} \left[{}_k I_{\psi^{-1}(a)^+}^{\mu, \psi} (f \circ \psi)(\psi^{-1}(b)) + {}_k I_{\psi^{-1}(b)^-}^{\mu, \psi} (f \circ \psi)(\psi^{-1}(b)) \right] \right| \\
& \leq \frac{b-a}{2} \int_0^1 \left| (1-t)^{\mu/k} - t^{\mu/k} \right| \left(t^s |f'(a)| + m(1-t)^s \left| f' \left(\frac{b}{m} \right) \right| - cmt(1-t) \left(\frac{b}{m} - a \right)^2 \right) dt. \\
& \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma_k(\mu + k)}{2(b-a)^{\mu/k}} \left[{}_k I_{\psi^{-1}(a)^+}^{\mu, \psi} (f \circ \psi)(\psi^{-1}(b)) + {}_k I_{\psi^{-1}(b)^-}^{\mu, \psi} (f \circ \psi)(\psi^{-1}(b)) \right] \right| \\
& \leq \frac{b-a}{2} \int_0^{1/2} (1-t)^{\mu/k} - t^{\mu/k} \left(t^s |f'(a)| + m(1-t)^s \left| f' \left(\frac{b}{m} \right) \right| - cmt(1-t) \left(\frac{b}{m} - a \right)^2 \right) dt \\
& \quad + \int_{1/2}^1 (t^{\mu/k} - (1-t)^{\mu/k}) \left(t^s |f'(a)| + m(1-t)^s \left| f' \left(\frac{b}{m} \right) \right| - cmt(1-t) \left(\frac{b}{m} - a \right)^2 \right) dt \\
& \leq \frac{b-a}{2} \left[|f'(a)| \left(\int_0^{1/2} t^s ((1-t)^{\mu/k} - t^{\mu/k}) dt + \int_{1/2}^1 t^s (t^{\mu/k} - (1-t)^{\mu/k}) dt \right) \right. \\
& \quad + m \left| f' \left(\frac{b}{m} \right) \right| \left(\int_0^{1/2} (1-t)^s ((1-t)^{\mu/k} - t^{\mu/k}) dt + \int_{1/2}^1 (1-t)^s (t^{\mu/k} - (1-t)^{\mu/k}) dt \right) \\
& \quad \left. - cm \left(\frac{b}{m} - a \right)^2 \left(\int_0^{1/2} t(1-t) ((1-t)^{\mu/k} - t^{\mu/k}) dt + \int_{1/2}^1 t(1-t) (t^{\mu/k} - (1-t)^{\mu/k}) dt \right) \right].
\end{aligned} \tag{39}$$

We now evaluate integrals that appear on the right side of the above inequality:

$$\begin{aligned}
& \int_0^{1/2} t^s ((1-t)^{\mu/k} - t^{\mu/k}) dt + \int_{1/2}^1 t^s (t^{\mu/k} - (1-t)^{\mu/k}) dt \\
& = 2B\left(\frac{1}{2}; s+1, \frac{\mu}{k}+1\right) + \frac{1 - (1/2)^{s+(\mu/k)}}{s + (\mu/k) + 1} - B\left(1+s, 1 + \frac{\mu}{k}\right),
\end{aligned} \tag{40}$$

$$\begin{aligned}
& \int_0^{1/2} (1-t)^s ((1-t)^{\mu/k} - t^{\mu/k}) dt + \int_{1/2}^1 (1-t)^s (t^{\mu/k} - (1-t)^{\mu/k}) dt \\
& = \frac{1 - (1/2)^{s+(\mu/k)+1}}{s + (\mu/k) + 1} - \frac{(1/2)^{1+(\mu/k)} {}_2F_1(-s, 1 + (\mu/k), 2 + (\mu/k); (1/2))}{(\mu/k) + 1} + B(1+s, 1 + (\mu/k)) \\
& \quad - \frac{(1/2)^{s+(\mu/k)+1} k(k+\mu+2^s(k+ks+\mu)) {}_2F_1(-s, 1 + (\mu/k), 2 + (\mu/k); (1/2))}{(sk + \mu + k)(\mu + k)},
\end{aligned} \tag{41}$$

$$\int_0^{1/2} t(1-t) ((1-t)^{\mu/k} - t^{\mu/k}) dt + \int_{1/2}^1 t(1-t) (t^{\mu/k} - (1-t)^{\mu/k}) dt = \frac{1 - ((\mu/k) + 4)/2^{(\mu/k)+2}}{((\mu/k) + 2)((\mu/k) + 3)}. \tag{42}$$

Using (40)–(42) in (39), we get (36). $\square$

Remark 3. Under the assumption of Theorem 10, the following outcomes are noted.

- (i) If $s = 1$ and $m = 1$ in (45), then the inequality stated in [[32], Theorem 11] is obtained.
- (ii) If $s = 1$, $m = 1$, and $c = 0$ in (45), then the inequality stated in [[32], Corollary 10] is obtained.

- (iii) If $s = 1$, $m = 1$, $k = 1$, and ψ is the identity function in (45), then a refined error estimation of the fractional Hadamard inequality is obtained.
- (iv) If $m = 1$ and $c = 0$ in (45), then the inequality stated in [[34], Theorem 2] is obtained.
- (v) If $m = 1$, $s = 1$, $c = 0$, and ψ is the identity function in (45), then Theorem 6 is obtained.

- (vi) If $k = 1$, $m = 1$, $s = 1$, $c = 0$, and ψ is the identity function in (45), then Theorem 3 is obtained.
- (vii) If $k = 1$, $s = 1$, and ψ is the identity function in (45), then the inequality stated in [[31], Theorem 8] is obtained.

Corollary 5. Under the assumption of Theorem 10 with $c = 0$ in (3.10), the following inequality holds:

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma_k(\mu + k)}{2(b-a)^{\mu/k}} \left[{}_k I_{\psi^{-1}(a)^+}^{\mu, \psi} (f \circ \psi)(\psi^{-1}(b)) + {}_k I_{\psi^{-1}(b)^-}^{\mu, \psi} (f \circ \psi)(\psi^{-1}(a)) \right] \right| \\ & \leq \frac{b-a}{2} \left[|f'(a)| \left(2B\left(\frac{1}{2}; s+1, \frac{\mu}{k}+1\right) + \frac{1 - (1/2)^{s+(\mu/k)}}{s + (\mu/k) + 1} - B\left(1+s, 1 + \frac{\mu}{k}\right) \right) \right. \\ & \quad + m \left| f'\left(\frac{b}{m}\right) \right| \left(\frac{1 - (1/2)^{s+(\mu/k)+1}}{s + (\mu/k) + 1} - \frac{{}_2F_1(-s, 1 + (\mu/k), 2 + (\mu/k); (1/2))}{2^{1+(\mu/k)}((\mu/k) + 1)} \right) \\ & \quad \left. + B\left(1+s, 1 + \frac{\mu}{k}\right) - \frac{k(k + \mu + 2^s(k + ks + \mu)_2 F_1(-s, 1 + (\mu/k), 2 + (\mu/k); (1/2)))}{2^{s+(\mu/k)+1}(sk + \mu + k)(\mu + k)} \right). \end{aligned} \quad (43)$$

Corollary 6. Under the assumption of Theorem 10 with $k = 1$, $\mu = 1$, $s = 1$, $m = 1$, and ψ as the identity function in (3.10), the following inequality holds:

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{(b-a)} - \int_a^b f(v) dv \right| \leq \frac{b-a}{8} [|f'(a)| + |f'(b)|] - \frac{c(b-a)^3}{32}. \quad (44)$$

Lemma 2. Let $f: [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) such that $f' \in L[a, b]$. Then, for $k > 0$ and

$m \in (0, 1]$, the following identity holds for operators given in (16) and (17):

$$\begin{aligned} & \frac{2^{(\mu/k)-1} \Gamma_k(\mu + k)}{(mb-a)^{\mu/k}} \left[{}_k I_{\psi^{-1}((a+mb)/2)^+}^{\mu, \psi} (f \circ \psi)(\psi^{-1}(mb)) + m^{(\mu/k)+1} {}_k I_{\psi^{-1}((a+mb)/2m)^-}^{\mu, \psi} (f \circ \psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right) \right] \\ & - \frac{1}{2} \left[f\left(\frac{a+mb}{2}\right) + m f\left(\frac{a+mb}{2m}\right) \right] \\ & = \frac{mb-a}{4} \left[\int_0^1 t^{\mu/k} f'\left(\frac{at}{2} + m\left(\frac{2-t}{2}\right)b\right) dt - \int_0^1 t^{\mu/k} f'\left(\frac{a}{m}\left(\frac{2-t}{2}\right) + \frac{bt}{2}\right) dt \right]. \end{aligned} \quad (45)$$

Proof. Let

$$\begin{aligned} I_1 &= \frac{2^{(\mu/k)-1} \Gamma_k(\mu + k)}{(mb-a)^{(\mu/k)}} \left[{}_k I_{\psi^{-1}((a+mb)/2)^+}^{\mu, \psi} (f \circ \psi)(\psi^{-1}(mb)) \right], \\ I_2 &= \frac{m^{(\mu/k)+1} 2^{(\mu/k)-1} \Gamma_k(\mu + k)}{(mb-a)^{(\mu/k)}} \left[{}_k I_{\psi^{-1}((a+mb)/2m)^-}^{\mu, \psi} (f \circ \psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right) \right]. \end{aligned} \quad (46)$$

First, we evaluate I_1 :

$$\begin{aligned} I_1 &= \frac{2^{(\mu/k)-1} \mu}{k(mb-a)^{(\mu/k)}} \left[\int_{\psi^{-1}((a+mb)/2)}^{\psi^{-1}(mb)} \psi'(u) (mb - \psi(u))^{(\mu/k)-1} (f \circ \psi)(u) du \right] \\ &= \frac{-2^{(\mu/k)-1}}{(mb-a)^{(\mu/k)}} \left[\int_{\psi^{-1}((a+mb)/2)}^{\psi^{-1}(mb)} (d(mb - \psi(u))^{(\mu/k)}) (f(\psi(u))) \right]. \end{aligned} \quad (47)$$

Now integrating by parts, we have

$$I_1 = \frac{1}{2} f\left(\frac{a+mb}{2}\right) + \frac{1}{2} \int_{\psi^{-1}((a+mb)/2)}^{\psi^{-1}(mb)} \left(\frac{2(mb - \psi(u))}{mb-a}\right)^{\mu/k} \psi'(u) f'(\psi(u)) du. \quad (48)$$

Substituting $t = 2(mb - \psi(u))/(mb - a)$, so that $\psi(u) = (at/2) + m((2-t)/2)b$ in (48), we get the following inequality:

$$I_1 = \frac{1}{2} f\left(\frac{a+mb}{2}\right) + \frac{mb-a}{4} \int_0^1 t^{\mu/k} f'\left(\frac{at}{2} + m\left(\frac{2-t}{2}\right)b\right) dt. \quad (49)$$

Now, we evaluate I_2 :

$$\begin{aligned} I_2 &= \frac{m^{(\mu/k)+1} 2^{(\mu/k)-1} \Gamma_k(\mu+k)}{(mb-a)^{(\mu/k)}} \left[{}_k I_{\psi^{-1}((a+mb)/2m)-}^{\mu, \psi} (f \circ \psi) \left(\psi^{-1}\left(\frac{a}{m}\right) \right) \right] \\ &= \frac{m^{(\mu/k)+1} \mu 2^{(\mu/k)-1}}{k(mb-a)^{(\mu/k)}} \left[\int_{\psi^{-1}(a/m)}^{\psi^{-1}((a+mb)/2m)} \psi'(v) \left(\psi(v) - \frac{a}{m} \right)^{(\mu/k)-1} (f \circ \psi)(v) dv \right] \\ &= \frac{m^{(\mu/k)+1} 2^{(\mu/k)-1}}{(mb-a)^{(\mu/k)}} \left[\int_{\psi^{-1}(a/m)}^{\psi^{-1}((a+mb)/2m)} \left(f(\psi(v)) \left(d\left(\psi(v) - \frac{a}{m} \right)^{(\mu/k)} \right) \right) \right]. \end{aligned} \quad (50)$$

Integrating by parts, we get

$$I_2 = \frac{m}{2} f\left(\frac{a+mb}{2m}\right) - \frac{m}{2} \int_{\psi^{-1}(a/m)}^{\psi^{-1}((a+mb)/2m)} \left(\frac{2m(\psi(v)) - (a/m)}{mb-a} \right)^{\mu/k} \psi'(v) (f'(\psi(v))) dv. \quad (51)$$

Substituting $s = 2m((\psi(v)) - (a/m))/(mb - a)$, so that $\psi(v) = (a/m)((2-t)/2) + (bt/2)$ in (51), we get the following inequality:

$$I_2 = \frac{m}{2} f\left(\frac{a+mb}{2m}\right) - \frac{(mb-a)}{4} \int_0^1 s^{\mu/k} f'\left(\frac{a}{m}\left(\frac{2-s}{2}\right) + \frac{bs}{2}\right) ds. \quad (52)$$

Adding (49) and (52), (45) is obtained. $\square$

Remark 4. Under the assumption of Lemma 2, the following outcomes are noted.

- (i) If $k = 1$ and ψ is the identity function in (45), then the identity stated in [[30], Lemma 2.3] is obtained.
- (ii) If $m = 1$ in (45), then the identity stated in [[32], Lemma 2] is obtained.
- (iii) If $m = 1, k = 1$, and ψ is the identity function in (45), then the identity stated in [[2], Lemma 3] is obtained.

(iv) If $m = 1, k = 1, \mu = 1$, and ψ is the identity function in (45), then the identity stated in [[2], Corollary 1] is obtained.

(v) If $m = 1$ and ψ is the identity function in (45), then the identity stated in [[23], Lemma 3.1] is obtained.

Theorem 11. Let $f: [a, b] \rightarrow \mathbb{R}$, $[a, b] \subset [0, +\infty)$ be a differentiable mapping on (a, b) such that $f' \in L[a, b]$. Also, suppose that $|f'|^q$ is strongly (s, m) -convex function on $[a, b]$ for $q \geq 1$. Then, for $k > 0$ and $(s, m) \in (0, 1]^2$, the following k -fractional integral inequality holds for operators given in (16) and (17):

$$\begin{aligned} & \left| \frac{2^{(\mu/k)-1} \Gamma_k(\mu+k)}{(mb-a)^{(\mu/k)}} \left[{}_k I_{\psi^{-1}((a+mb)/2)^+}^{\mu, \psi} (f \circ \psi)(\psi^{-1}(mb)) + m^{(\mu/k)+1} {}_k I_{\psi^{-1}((a+mb)/2m)^-}^{\mu, \psi} (f \circ \psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right) \right] \right. \\ & \quad \left. - \frac{1}{2} \left[f\left(\frac{a+mb}{2}\right) + m f\left(\frac{a+mb}{2m}\right) \right] \right| \\ & \leq \frac{mb-a}{4((\mu/k)+1)} \left(\frac{1}{2((\mu/k)+2)} \right)^{1/q} \left[\left(\frac{2((\mu/k)+1)((\mu/k)+2)}{2^s((\mu/k)+s+1)} |f'(a)|^q + 2m |f'(b)|^q \left(\frac{\mu}{k}+2\right) \right) \right. \\ & \quad \times {}_2 F_1\left(-s, 1+\frac{\mu}{k}, 2+\frac{\mu}{k}; \frac{1}{2}\right) - \frac{cm(b-a)^2((\mu/k)+1)((\mu/k)+4)}{2((\mu/k)+3)} \Bigg]^{1/q} \\ & \quad + \left(2m \left(\frac{\mu}{k}+2\right) \left| f'\left(\frac{a}{m^2}\right) \right|^q {}_2 F_1\left(-s, 1+\frac{\mu}{k}, 2+\frac{\mu}{k}; \frac{1}{2}\right) \right. \\ & \quad \left. + \frac{2((\mu/k)+1)((\mu/k)+2)|f'(b)|^q}{2^s((\mu/k)+s+1)} - \frac{cm(b-(a/m^2))^2((\mu/k)+1)((\mu/k)+4)}{2((\mu/k)+3)} \right)^{1/q} \Bigg], \end{aligned} \quad (53)$$

with $\mu > 0$ and $(1/p) + (1/q) = 1$.

Proof. We divide the proof in two cases. $\square$

Case 1. Fix $q = 1$. Applying Lemma 2 and strongly (s, m) -convexity of $|f'|$, we have

$$\begin{aligned} & \left| \frac{2^{(\mu/k)-1} \Gamma_k(\mu+k)}{(mb-a)^{(\mu/k)}} \left[{}_k I_{\psi^{-1}((a+mb)/2)^+}^{\mu, \psi} (f \circ \psi)(\psi^{-1}(mb)) + m^{(\mu/k)+1} {}_k I_{\psi^{-1}((a+mb)/2m)^-}^{\mu, \psi} (f \circ \psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right) \right] \right. \\ & \quad \left. - \frac{1}{2} \left[f\left(\frac{a+mb}{2}\right) + m f\left(\frac{a+mb}{2m}\right) \right] \right| \leq \frac{mb-a}{4} \left[\int_0^1 t^{(\mu/k)} f'\left(\frac{at}{2} + m\left(\frac{2-t}{2}\right)b\right) dt + \int_0^1 t^{\mu/k} f'\left(\frac{a}{m}\left(\frac{2-t}{2}\right) + \frac{bt}{2}\right) dt \right] \\ & \leq \frac{mb-a}{4} \left[\left(\frac{|f'(a)| + m|f'(b)|}{2^s} \right) \int_0^1 t^{(\mu/k)+s} dt + \frac{m(|f'(b)| + m|f'(a/m^2)|)}{2^s} \int_0^1 (2-t)^s t^{\mu/k} dt \right. \\ & \quad \left. - \frac{cm\left((b-a)^2 + m(b-(a/m^2))^2\right)}{4} \int_0^1 t^{(\mu/k)+1} (2-t) dt \right] \leq \frac{mb-a}{4} \left[\frac{k(|f'(a)| + m|f'(b)|)}{2^s(\mu+sk+k)} \right. \\ & \quad \left. + \frac{mk {}_2 F_1(-s, 1+(\mu/k), 2+(\mu/k), (1/2))(|f'(b)| + m|f'(a/m^2)|)}{k+\mu} - \frac{cm\left((b-a)^2 + m(b-(a/m^2))^2\right)((\mu/k)+4)}{4((\mu/k)+2)((\mu/k)+3)} \right]. \end{aligned} \quad (54)$$

Case 2. For $q > 1$. From Lemma 2 and using power mean inequality, we get

$$\begin{aligned}
& \left| \frac{2^{(\mu/k)-1} \Gamma_k(\mu+k)}{(mb-a)^{(\mu/k)}} \left[{}_k I_{\psi^{-1}((a+mb)/2)^+}^{\mu, \psi} (f \circ \psi) \left(\psi^{-1}(mb) \right) \right. \right. \\
& \quad \left. \left. + m^{(\mu/k)+1} {}_k I_{\psi^{-1}((a+mb)/2m)^-}^{\mu, \psi} (f \circ \psi) \left(\psi^{-1}\left(\frac{a}{m}\right) \right) \right] - \frac{1}{2} \left[f\left(\frac{a+mb}{2}\right) + mf\left(\frac{a+mb}{2m}\right) \right] \right| \\
& \leq \frac{mb-a}{4} \left(\int_0^1 t^{(\mu/k)} dt \right)^{1-(1/q)} \left[\left(\int_0^1 t^{(\mu/k)} \left| f' \left(\frac{at}{2} + m \left(\frac{2-t}{2} \right) b \right) \right|^q dt \right)^{(1/q)} \right. \\
& \quad \left. + \left(\int_0^1 mt^{(\mu/k)} \left| f' \left(\frac{a}{m} \left(\frac{2-t}{2} \right) + \frac{bt}{2} \right) \right|^q dt \right)^{(1/q)} \right] \leq \frac{mb-a}{4((\mu/k)+1)^{1/p}} \left[\left(\frac{|f'(a)|^q}{2^s} \int_0^1 t^{s+(\mu/k)} dt \right. \right. \\
& \quad \left. \left. + \frac{m|f'(b)|^q}{2^s} \int_0^1 (2-t)^s t^{(\mu/k)} dt - \frac{cm(b-a)^2}{4} \int_0^1 (2-t)t^{(\mu/k)+1} dt \right)^{(1/q)} + \left(\frac{mf'(a/m^2)}{2^s} \right. \right. \\
& \quad \left. \left. \times \int_0^1 (2-t)^s t^{(\mu/k)} dt + \frac{|f'(b)|^q}{2^s} \int_0^1 t^{s+(\mu/k)} dt - \frac{cm(b-(a/m^2))^2}{4} \int_0^1 (2-t)t^{(\mu/k)+1} dt \right)^{(1/q)} \right] \\
& \leq \frac{mb-a}{4((\mu/k)+1)^{1/p}} \left[\left(\frac{k|f'(a)|^q}{2^s(sk+\mu+k)} + \frac{mk|f'(b)|^q {}_2F_1(-s, 1+(\mu/k), 2+(\mu/k), (1/2))}{k+\mu} \right. \right. \\
& \quad \left. \left. - \frac{cm((\mu/k)+4)(b-a)^2}{4((\mu/k)+2)((\mu/k)+3)} \right)^{(1/q)} + \left(\frac{mk|f'(a/m^2)|^q {}_2F_1(-s, 1+(\mu/k), 2+(\mu/k), (1/2))}{k+\mu} \right. \right. \\
& \quad \left. \left. + \frac{k|f'(b)|^q}{2^s(sk+\mu+k)} - \frac{cm((\mu/k)+4)(b-(a/m^2))^2}{4((\mu/k)+2)((\mu/k)+3)} \right)^{(1/q)} \right] \\
& \leq \frac{mb-a}{4((\mu/k)+1)^{1/p}} \left(\frac{1}{2((\mu/k)+1)((\mu/k)+2)} \right)^{1/q} \left[\left(\frac{2k|f'(a)|^q((\mu/k)+1)((\mu/k)+2)}{2^s(sk+\mu+k)} \right. \right. \\
& \quad \left. \left. + 2m \left(\frac{\mu}{k} + 2 \right) |f'(b)|^q {}_2F_1 \left(-s, 1 + \frac{\mu}{k}, 2 + \frac{\mu}{k}, \frac{1}{2} \right) - \frac{cm((\mu/k)+4)((\mu/k)+1)(b-a)^2}{2((\mu/k)+3)} \right)^{1/q} \\
& \quad \left. \left. + \left(2m \left| f' \left(\frac{a}{m^2} \right) \right|^q \left(\frac{\mu}{k} + 2 \right) {}_2F_1 \left(-s, 1 + \frac{\mu}{k}, 2 + \frac{\mu}{k}, \frac{1}{2} \right) \right. \right. \right. \\
& \quad \left. \left. + \frac{2k((\mu/k)+1)((\mu/k)+2)|f'(b)|^q}{2^s(sk+\mu+k)} - \frac{cm((\mu/k)+1)((\mu/k)+4)(b-(a/m^2))^2}{2((\mu/k)+3)} \right)^{1/q} \right].
\end{aligned} \tag{55}$$

Hence, we get (53).

Remark 5. Under the assumption of Theorem 11, the following outcomes are noted.

- (i) If $s = 1$ and $m = 1$ in (53), then the inequality stated in [[32], Theorem 12] is obtained.

- (ii) If $s = 1$, $k = 1$, and ψ is the identity function in (53), then the inequality stated in [[31], Theorem 9] is obtained.

- (iii) If $s = 1$, $k = 1$, $c = 0$, and ψ is the identity function in (53), then the inequality stated in [[30], Theorem 2.4] is obtained.

(iv) If $c = 0$, $s = 1$, $m = 1$, and ψ is the identity function in (53), then the inequality stated in [[23], Theorem 3.1] is obtained.

(v) If $s = 1$, $m = 1$, $c = 0$, $k = 1$, and ψ is the identity function in (53), then the inequality stated in [[2], Theorem 5] is obtained.

(vi) If $q = 1$, $s = 1$, $m = 1$, $c = 0$, $k = 1$, $\mu = 1$, and ψ is the identity function in (53), then the inequality stated in [[36], Theorem 2.2] is obtained.

Corollary 7. Under the assumption of Theorem 11 with $c = 0$ in (53), the following inequality holds:

$$\begin{aligned}
 & \left| \frac{2^{(\mu/k)-1} \Gamma_k(\mu+k)}{(mb-a)^{(\mu/k)}} \left[{}_k I_{\psi^{-1}((a+mb)/2)^+}^{\mu, \psi} (f \circ \psi) (\psi^{-1}(mb)) \right. \right. \\
 & \quad \left. \left. + m^{(\mu/k)+1} {}_k I_{\psi^{-1}((a+mb)/2m)^-}^{\mu, \psi} (f \circ \psi) \left(\psi^{-1}\left(\frac{a}{m}\right) \right) - \frac{1}{2} \left[f\left(\frac{a+mb}{2}\right) + mf\left(\frac{a+mb}{2m}\right) \right] \right| \\
 & \leq \frac{mb-a}{4((\mu/k)+1)} \left(\frac{1}{2((\mu/k)+2)} \right)^{1/q} \left[\left(\frac{2((\mu/k)+1)((\mu/k)+2)|f'(a)|^q}{2^s((\mu/k)+s+1)} + 2m|f'(b)|^q((\mu/k)+2) \right. \right. \\
 & \quad \left. \left. \times {}_2F_1\left(-s, 1 + \frac{\mu}{k}, 2 + \frac{\mu}{k}; \frac{1}{2}\right) \right)^{1/q} + \left(2m\left(\frac{\mu}{k}+2\right) \left| f'\left(\frac{a}{m^2}\right) \right|^q \right. \right. \\
 & \quad \left. \left. \times {}_2F_1\left(-s, 1 + \frac{\mu}{k}, 2 + \frac{\mu}{k}; \frac{1}{2}\right) + \frac{2((\mu/k)+1)((\mu/k)+2)|f'(b)|^q}{2^s((\mu/k)+s+1)} \right)^{1/q} \right]. \tag{56}
 \end{aligned}$$

Corollary 8. Under the assumption of Theorem 11 with $k = 1$, $s = 1$, $m = 1$, $q = 1$, $\mu = 1$, and ψ as the identity function in (53), the following inequality holds:

$$\left| \frac{1}{b-a} \int_a^b f(v) dv - f\left(\frac{a+b}{2}\right) \right| \leq \frac{b-a}{8} \left[|f'(a)| + |f'(b)| - \frac{5c(b-a)^2}{12} \right]. \tag{57}$$

Lemma 3 (see [39]). Let $p \geq 1$ be a real number. For $(x_1, x_2, \dots, x_n) \in [0, \infty)$ and $n \geq 2$, the following inequality holds:

$$\sum_{i=1}^n x_i^p \leq \left(\sum_{i=1}^n x_i \right)^p. \tag{58}$$

Theorem 12. Let $f: I \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$. Also, suppose that $|f'|^q$ is strongly (s, m) -convex function on $[a, b]$ for $q > 1$. Then, for $k > 0$ and $(s, m) \in (0, 1]^2$, the following k -fractional integral inequality holds for operators given in (16) and (17):

$$\begin{aligned}
& \left| \frac{2^{(\mu/k)-1} \Gamma_k(\mu+k)}{(mb-a)^{(\mu/k)}} \left[{}_k I_{\psi^{-1}((a+mb)/2)^+}^{\mu, \psi} (f \circ \psi)(\psi^{-1}(mb)) + m^{(\mu/k)+1} {}_k I_{\psi^{-1}((a+mb)/2m)^-}^{\mu, \psi} (f \circ \psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right) \right] \right. \\
& \quad \left. - \frac{1}{2} \left[f\left(\frac{a+mb}{2}\right) + mf\left(\frac{a+mb}{2m}\right) \right] \right| \\
& \leq \frac{mb-a}{16} \left(\frac{4}{(\mu p/k) + 1} \right)^{1/p} \left[\left(\left(|f'(a)| \left(\frac{4}{2^s(s+1)} \right)^{1/q} \right. \right. \right. \\
& \quad \left. \left. + |f'(b)| \left(\frac{4m(-1+2^{s+1})}{2^s(1+s)} \right)^{1/q} \right)^q - \frac{2cm(b-a)^2}{3} \right)^{1/q} + \left(\left(m \left| f'\left(\frac{a}{m^2}\right) \right| \left(\frac{4m(-1+2^{s+1})}{2^s(1+s)} \right)^{1/q} \right. \right. \right. \\
& \quad \left. \left. + \left(\frac{4}{2^s(s+1)} \right)^{1/q} |f'(b)| \right)^q - \frac{2cm(b-a)^2}{3} \right)^{1/q} \right],
\end{aligned} \tag{59}$$

with $\mu > 0$ and $(1/p) + (1/q) = 1$.

Proof. By applying Lemma 2 and using the property of modulus, we get

$$\begin{aligned}
& \left| \frac{2^{\mu/k-1} \Gamma_k(\mu+k)}{(mb-a)^{\mu/k}} \left[{}_k I_{\psi^{-1}((a+mb)/2)^+}^{\mu, \psi} (f \circ \psi)(\psi^{-1}(mb)) \right. \right. \\
& \quad \left. \left. + m^{(\mu/k)+1} {}_k I_{\psi^{-1}((a+mb)/2m)^-}^{\mu, \psi} (f \circ \psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right) \right] - \frac{1}{2} \left[f\left(\frac{a+mb}{2}\right) + mf\left(\frac{a+mb}{2m}\right) \right] \right| \\
& \leq \frac{mb-a}{4} \left[\int_0^1 \left| t^{(\mu/k)} f'\left(\frac{at}{2} + m\left(\frac{2-t}{2}\right)b \right) \right| dt + \int_0^1 \left| t^{(\mu/k)} f'\left(\frac{a}{m}\left(\frac{2-t}{2}\right) + \frac{bt}{2}\right) \right| dt \right].
\end{aligned} \tag{60}$$

Now applying Hölder's inequality for integrals, we get

$$\begin{aligned}
& \left| \frac{2^{(\mu/k)-1} \Gamma_k(\mu+k)}{(mb-a)^{(\mu/k)}} \left[{}_k I_{\psi^{-1}((a+mb)/2)^+}^{\mu, \psi} (f \circ \psi)(\psi^{-1}(mb)) + m^{(\mu/k)+1} {}_k I_{\psi^{-1}((a+mb)/2m)^-}^{\mu, \psi} (f \circ \psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right) \right] \right. \\
& \quad \left. - \frac{1}{2} \left[f\left(\frac{a+mb}{2}\right) + mf\left(\frac{a+mb}{2m}\right) \right] \right| \leq \frac{b-a}{4} \left(\frac{1}{(\mu p/k) + 1} \right)^{1/p} \\
& \quad \left[\left(\int_0^1 \left| f'\left(\frac{at}{2} + m\left(\frac{2-t}{2}\right)b \right) \right|^q dt \right)^{1/q} + \left(\int_0^1 \left| f'\left(\frac{a}{m}\left(\frac{2-t}{2}\right) + \frac{bt}{2}\right) \right|^q dt \right)^{1/q} \right].
\end{aligned} \tag{61}$$

Using strongly (s, m) -convexity of $|f'|^q$, we get

$$\begin{aligned}
& \left| \frac{2^{(\mu/k)-1} \Gamma_k(\mu+k)}{(mb-a)^{(\mu/k)}} \left[{}_k I_{\psi^{-1}((a+mb)/2)^+}^{\mu, \psi} (f \circ \psi)(\psi^{-1}(mb)) + m^{(\mu/k)+1} {}_k I_{\psi^{-1}((a+mb)/2m)^-}^{\mu, \psi} (f \circ \psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right) \right] \right. \\
& \quad \left. - \frac{1}{2} \left[f\left(\frac{a+mb}{2}\right) + m f\left(\frac{a+mb}{2m}\right) \right] \right| \\
& \leq \frac{mb-a}{4} \left(\frac{1}{(\mu p/k)+1} \right)^{1/p} \left[\left(\frac{|f'(a)|^q}{2^s} \int_0^1 t^s dt + \frac{m|f'(b)|^q}{2^s} \int_0^1 (2-t)^s dt \right. \right. \\
& \quad \left. \left. - \frac{cm(b-a)^2}{4} \int_0^1 t(2-t) dt \right)^{1/q} + \left(\frac{m|f'(a/m^2)|^q}{2^s} \int_0^1 (2-t)^s dt + \frac{|f'(b)|^q}{2^s} \int_0^1 t^s dt \right. \right. \\
& \quad \left. \left. - \frac{cm(b-(a/m^2))^2}{4} \int_0^1 t(2-t) dt \right)^{1/q} \right] = \frac{mb-a}{4} \left(\frac{1}{(\mu p/k)+1} \right)^{1/p} \left[\left(\frac{|f'(a)|^q}{2^s(s+1)} \right. \right. \\
& \quad \left. \left. + \frac{m|f'(b)|^q(-1+2^{s+1})}{2^s(1+s)} - \frac{cm(b-a)^2}{6} \right)^{1/q} + \left(\frac{m|f'(a/m^2)|^q(-1+2^{s+1})}{2^s(1+s)} \right. \right. \\
& \quad \left. \left. + \frac{|f'(b)|^q}{2^s(s+1)} - \frac{cm(b-(a/m^2))^2}{6} \right)^{1/q} \right] \leq \frac{mb-a}{16} \left(\frac{4}{(\mu p/k)+1} \right)^{1/p} \left[\left(\frac{4|f'(a)|^q}{2^s(s+1)} \right. \right. \\
& \quad \left. \left. + \frac{4m|f'(b)|^q(-1+2^{s+1})}{2^s(1+s)} - \frac{2cm(b-a)^2}{3} \right)^{1/q} + \left(\frac{4m|f'(a/m^2)|^q(-1+2^{s+1})}{2^s(1+s)} \right. \right. \\
& \quad \left. \left. + \frac{4|f'(b)|^q}{2^s(s+1)} - \frac{2cm(b-(a/m^2))^2}{3} \right)^{1/q} \right] \leq \frac{mb-a}{16} \left(\frac{4}{(\mu p/k)+1} \right)^{1/p} \left[\left(\left(|f'(a)| \left(\frac{4}{2^s(s+1)} \right)^{1/q} \right) \right. \right. \\
& \quad \left. \left. + |f'(b)| \left(\frac{4m(-1+2^{s+1})}{2^s(1+s)} \right)^{1/q} - \frac{2cm(b-a)^2}{3} \right)^{1/q} + \left(m \left| f' \left(\frac{a}{m^2} \right) \right| \left(\frac{4m(-1+2^{s+1})}{2^s(1+s)} \right)^{1/q} \right. \right. \\
& \quad \left. \left. + \left(\frac{4}{2^s(s+1)} \right)^{1/q} |f'(b)| - \frac{2cm(b-(a/m^2))^2}{3} \right)^{1/q} \right].
\end{aligned} \tag{62}$$

Here we have used Lemma 3. This completes the proof. $\square$

Remark 6. Under the assumption of Theorem 12, the following outcomes are noted.

- (i) If $s = 1$ and $m = 1$ in (59), then the inequality stated in [[32], Theorem 13] is obtained.
- (ii) If $s = 1$ and ψ is the identity function in (53), then the inequality stated in [[31], Theorem 10] is obtained.
- (iii) If $s = 1$, $k = 1$, $c = 0$, and ψ is the identity function in (53), then the inequality stated in [[30], Theorem 2.7] is obtained.

(iv) If $c = 0$, $s = 1$, $m = 1$, and ψ is the identity function in (59), then the inequality stated in [[23], Theorem 3.2] is obtained.

(v) If $k = 1$, $s = 1$, $m = 1$, $c = 0$, and ψ is the identity function in inequality (59), then the inequality stated in [[2], Theorem 6] is obtained.

(vi) If $k = 1$, $s = 1$, $m = 1$, $c = 0$, $\mu = 1$, and ψ is the identity function in inequality (59), then the inequality stated in [[37], Theorem 2.4] is obtained.

Corollary 9. Under the assumption of Theorem 12 with $c = 0$ in (61), the following inequality holds:

$$\begin{aligned}
& \left| \frac{2^{\mu/k-1} \Gamma_k(\mu+k)}{(mb-a)^{\mu/k}} \left[{}_k I_{\psi^{-1}((a+mb)/2)^+}^{\mu, \psi} (f \circ \psi)(\psi^{-1}(mb)) + m^{(\mu/k)+1} {}_k I_{\psi^{-1}((a+mb)/2m)^-}^{\mu, \psi} (f \circ \psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right) \right] \right. \\
& \quad \left. - \frac{1}{2} \left[f\left(\frac{a+mb}{2}\right) + m f\left(\frac{a+mb}{2m}\right) \right] \right| \\
& \leq \frac{mb-a}{16} \left(\frac{4}{(\mu p/k)+1} \right)^{1/p} \left[|f'(a)| \left(\frac{4}{2^s(s+1)} \right)^{1/q} + |f'(b)| \left(\frac{4m(-1+2^{s+1})}{2^s(1+s)} \right)^{1/q} \right. \\
& \quad \left. + \left(\left| f'\left(\frac{a}{m^2}\right) \right| \left(\frac{4m(-1+2^{s+1})}{2^s(1+s)} \right)^{1/q} + \left(\frac{4}{2^s(s+1)} \right)^{1/q} |f'(b)| \right) \right].
\end{aligned} \tag{63}$$

Corollary 10. Under the assumption of Theorem 12 with $q \rightarrow 1$ and $p \rightarrow \infty$ in (59), the following inequality holds:

$$\begin{aligned}
& \left| \frac{2^{(\mu/k)-1} \Gamma_k(\mu+k)}{(mb-a)^{(\mu/k)}} \left[{}_k I_{\psi^{-1}((a+mb)/2)^+}^{\mu, \psi} (f \circ \psi)(\psi^{-1}(mb)) + m^{(\mu/k)+1} {}_k I_{\psi^{-1}((a+mb)/2m)^-}^{\mu, \psi} (f \circ \psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right) \right] \right. \\
& \quad \left. - \frac{1}{2} \left[f\left(\frac{a+mb}{2}\right) + m f\left(\frac{a+mb}{2m}\right) \right] \right| \\
& \leq \frac{mb-a}{16} \left[\frac{4(|f'(a)| + |f'(b)|)}{2^s(s+1)} + \frac{4m(-1+2^{s+1})(|f'(b)| + |f'(a/m^2)|)}{2^s(1+s)} - \frac{4cm(b-a)^2}{3} \right].
\end{aligned} \tag{64}$$

4. Conclusion

In this article, we studied the Hadamard inequalities and their estimations for generalized Riemann–Liouville fractional integrals of strongly (s, m) -convex functions. These inequalities represent the generalizations and refinements of a number of well-known inequalities stated in [1, 2, 22, 23, 29–37]. The error estimations of Hadamard inequalities for differentiable strongly (s, m) -convex functions are better as compared to those which are obtained for convex functions, strongly convex functions, and strongly m -convex functions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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