Research Article

Extending of Edge Even Graceful Labeling of Graphs to Strong r -Edge Even Graceful Labeling

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Edge even graceful labeling of a graph G with p vertices and q edges is a bijective f from the set of edge $E(G)$ to the set of positive integers $\{2, 4, \dots, 2q\}$ such that all the vertex labels $f^*[V(G)]$, given by $f^*(u) = (\sum_{uv \in E(G)} f(uv)) \bmod (2k)$, where $k = \max(p, q)$, are pairwise distinct. There are many graphs that do not have edge even graceful labeling, so in this paper, we have extended the definition of edge even graceful labeling to r -edge even graceful labeling and strong r -edge even graceful labeling. We have obtained the necessary conditions for more path-related graphs and cycle-related graphs to be an r -edge even graceful graph. Furthermore, the minimum number r for which the graphs: tortoise graph, double star graph, ladder and diagonal ladder graphs, helm graph, crown graph, sunflower graph, and sunflower planar graph, have an r -edge even graceful labeling was found. Finally, we proved that the even cycle C_{2n} has a strong 2-edge even graceful labeling when n is even.

1. Introduction and Preliminaries

The field of graph theory plays vital role in various mathematical fields which are used in structural models. This structural arrangement of various objects or technologies lead to new inventions and modifications in the existing environment for improvement in these fields. Nowadays, graph labeling has become an important and rich branch of different research studies in graph theory because of its huge applications in many disciplines, e.g., coding theory, X-ray, crystallography, radar, communication networks, astronomy, communication network addressing, circuit design, and graph decomposition problems. For more interesting applications of graph labeling, see [1–3].

Throughout this paper, a graph $G = (V(G), E(G))$ always means a finite and a simple connected graph (without loops and parallel edges), where $q = |E(G)|$ is the cardinality of $E(G)$ and $p = |V(G)|$ is that of $V(G)$.

There are two main types of graceful labeling in graph theory. The oldest one is the vertex graceful labeling and the

recent one is the edge graceful labeling. Here we are dealing with the recent one.

A function f is called an edge graceful labeling of a graph G if $f: E(G) \rightarrow \{1, 2, \dots, q\}$ is bijective and the induced function $f^*: V(G) \rightarrow \{0, 1, 2, \dots, p-1\}$, defined as $f^*(u) = (\sum_{xy \in E(G)} f(xy)) \bmod (p)$, is bijective. This type of graph labeling was first introduced by Lo in 1985 [4].

A function f is called an edge odd graceful labeling of a graph if $f: E(G) \rightarrow \{1, 3, \dots, 2q-1\}$ is bijective and the induced function $f^*: V(G) \rightarrow \{0, 1, 2, 3, \dots, 2q-1\}$, defined as $f^*(u) = (\sum_{xy \in E(G)} f(xy)) \bmod (2q)$, is bijective. This type of graph labeling was first introduced by Solairaju and Chithra in 2009 [5] (see also [6, 7]).

A function f is called an edge even graceful labeling of a graph G if $f: E(G) \rightarrow \{2, 4, \dots, 2q\}$ is bijective and the induced function $f^*: V(G) \rightarrow \{0, 2, 4, \dots, 2q\}$, defined as $f^*(x) = (\sum_{xy \in E(G)} f(xy)) \bmod (2k)$, is an injective function, where $k = \max(p, q)$. This type of graph labeling was first introduced by Elsonbaty and Daoud in 2017 [8, 9]. They introduced some path- and cycle-related graphs which are edge even graceful. Following this, Daoud [10] investigated

the edge even graceful labeling of polar grid graphs and Zeen El Deen et al. [11–13] studied more graphs having an edge even graceful labeling. The graph that admits an edge even graceful labeling is called an edge even graceful graph.

A summary of the results on these three types of graceful labels as well as all known labels so far can be found in [14].

Because there are many graphs which do not have edge even graceful labeling, we introduce the extension of the definition of edge even graceful labeling to r -edge even graceful labeling.

Definition 1. A function f is called r -edge even graceful labeling of the graph G if $f: E(G) \rightarrow \{2, 4, \dots, 2q + 2r - 2\}$ is an injective function such that the induced mapping $f^*: V(G) \rightarrow \{0, 2, 4, \dots, 2q + 2r - 2\}$, given by: $f^*(x) = (\sum_{xy \in E(G)} f(xy)) \bmod (2k + 2r - 2)$, is an injective function, where $k = \max(p, q)$. We will call the graph that admits r -edge even graceful labeling an r -edge even graceful graph.

Note that the standard definition of edge even graceful labeling corresponds to a 1-edge even graceful labeling.

Definition 2. A function f is called strong r -edge even graceful labeling of the graph G if $f: E(G) \rightarrow \{2, 4, \dots, 2q + 2r - 2\}$ is an injective function such that the induced mapping $f^*: V(G) \rightarrow \{2r - 2, 2r, 2r + 2, \dots, 2q + 2r - 2\}$, given by $f^*(x) = (\sum_{xy \in E(G)} f(xy)) \bmod (2k + 2r - 2)$, is an injective function, where $k = \max(p, q)$.

Illustration. A 2-edge even and a strong 2-edge even labeling of the graph K_4 is shown in Figure 1.

2. The r -Edge Even Graceful Labeling for Some Path-Related Graphs

2.1. The r -Edge Even Graceful Labeling of the Path Graph P_n . The path graph P_n has edge even graceful labeling only when n is an odd number [8].

Theorem 1. When n is an even number, the path graph P_n has 3-edge even graceful labeling.

Proof. The path graph P_n has $p = n$ vertices and $q = n - 1$ edges. Let e_i denote the edge $v_i v_{i+1}$ in P_n (see Figure 2).

Let $r = 3$; then, $2p + 2r - 2 = 2n + 4$, and there are two cases.

(i) *Case (1).* If $n \equiv 0 \pmod{4}$, we define the labeling function $f: E(P_n) \rightarrow \{2, 4, \dots, 2n + 2\}$ as follows:

$$f(e_i) = \begin{cases} i + 1, & \text{if } i = 1, 3, 5, \dots, n - 1, \\ n + i, & \text{if } i = 2, 4, \dots, \frac{n}{2}, \\ n + i + 4, & \text{if } i = \frac{n}{2} + 2, \frac{n}{2} + 4, \dots, n - 2. \end{cases} \quad (1)$$

Now, the labeling of the vertices is

$$\begin{aligned} f^*(v_1) &= f(e_1) = 2, & f^*(v_n) &= [f(e_{n-1})] \bmod (2n + 4) = n, \\ f^*(v_i) &= [f(e_i) + f(e_{i-1})] \bmod (2n + 4) = (n + 2i) \bmod (2n + 4), & i &= 2, 3, \dots, \frac{n}{2} + 1, \\ f^*(v_i) &= [f(e_i) + f(e_{i-1})] \bmod (2n + 4) = (n + 2i + 4) \bmod (2n + 4), & i &= \frac{n}{2} + 2, \frac{n}{2} + 3, \dots, n - 1. \end{aligned} \quad (2)$$

So, the labels of the vertices $v_2, v_3, \dots, v_{(n/2)}, v_{(n/2)+1}$ will be $n + 4, n + 6, \dots, 2n, 2n + 2$, respectively, and the labels of the vertices $v_{(n/2)+2}, v_{(n/2)+3}, \dots, v_{n-2}, v_{n-1}$ will be $4, 6, \dots, n - 4, n - 2$, respectively.

(ii) *Case (2).* If $n \equiv 2 \pmod{4}$, we define the labeling f as follows:

$$f(e_i) = \begin{cases} i + 1, & \text{if } i = 1, 3, 5, \dots, n - 1, \\ n + i, & \text{if } i = 2, 4, \dots, \frac{n}{2} + 1, \\ n + i + 4, & \text{if } i = \frac{n}{2} + 3, \frac{n}{2} + 5, \dots, n - 2. \end{cases} \quad (3)$$

Now, the labeling of the vertices is

$$\begin{aligned} f^*(v_1) &= f(e_1) = 2, & f^*(v_n) &= [f(e_{n-1})] \bmod (2n + 4) = n, \\ f^*(v_i) &= [f(e_i) + f(e_{i-1})] \bmod (2n + 4) = (n + 2i) \bmod (2n + 4), & i &= 2, 3, \dots, \frac{n}{2} + 2, \\ f^*(v_i) &= [f(e_i) + f(e_{i-1})] \bmod (2n + 4) = (n + 2i + 4) \bmod (2n + 4), & i &= \frac{n}{2} + 3, \frac{n}{2} + 4, \dots, n - 1. \end{aligned} \quad (4)$$

FIGURE 1: (a) A 2-edge even graceful labeling for K_4 . (b) A strong 2-edge even graceful labeling for K_4 .

FIGURE 2: The path graph P_n with ordinary labeling when n is even.

So, the labels of the vertices $v_2, v_3, \dots, v_{(n/2)}$, $v_{(n/2)+1}, v_{(n/2)+2}$ are $n+4, n+6, \dots, 2n, 2n+2, 0$, respectively, and the labels of the vertices $v_{(n/2)+2}, v_{(n/2)+3}, \dots, v_{n-2}, v_{n-1}$ are $6, 8, \dots, n-4, n-2$ respectively.

Therefore, all the vertex labels are even and distinct numbers. Thus, the path graph P_n has 3-edge even graceful labeling when n is even. $\square$

Illustration. A 3-edge even graceful labeling of the graphs P_{12} and P_{14} is shown in Figure 3.

2.2. The r -Edge Even Graceful Labeling of the Star Graph $K_{1,n}$. It should be noted that the star graph $K_{1,n}$ has edge even graceful labeling only when n is an even number [8].

Theorem 2. *The star graph $K_{1,n}$ has 3-edge even graceful labeling when n is odd.*

Proof. The graph $K_{1,n}$ has $p = n+1$ vertices and $q = n$ edges obtained by connecting one vertex v_0 with each of the v_i , where $1 \leq i \leq n$. Let $r = 3$; then, $2p + 2r - 2 = 2n + 6$, and we define the labeling $f: E(K_{1,n}) \rightarrow \{2, 4, \dots, 2n+4\}$ as follows:

$$f(v_0 v_i) = \begin{cases} 2i, & \text{if } i = 1, 2, 3, \dots, \frac{n+3}{2}, \\ 2i+2, & \text{if } i = \frac{n+5}{2}, \dots, n. \end{cases} \quad (5)$$

Note that we do not take the labels $n+5$ and $2n+4$ in the label of edges. Since the vertices $v_i, i = 1, 2, 3, \dots, n$ are pendant, then $f^*(v_i)$ take their edge labels which are even and distinct and

$$\begin{aligned} f^*(v_0) &= \left[\sum_{i=1}^n f(v_0 v_i) \right] \bmod (2n+6) \\ &= \left[\sum_{i=1}^{(n+3)/2} (2i) + \sum_{i=(n+5)/2}^n (2i+2) \right] \bmod (2n+6) = 0. \end{aligned} \quad (6)$$

Hence, the labels of the vertices are all even and distinct numbers. $\square$

Illustration. A 3-edge even labeling of the graph $K_{1,11}$ is shown in Figure 4.

2.3. The r -Edge Even Graceful Labeling of the Comb Graph $P_n \odot K_1$

Theorem 3. *The comb graph [15] $P_n \odot K_1$ has 3-edge even graceful labeling for any positive integer n .*

Proof. The graph $P_n \odot K_1$ has $p = 2n$ vertices and $q = 2n-1$ edges. Let the comb graph $P_n \odot K_1$ be given as indicated in Figure 5.

Let $r = 3$; then, $2p + 2r - 2 = 4n + 4$, and we define the labeling $f: E(P_n \odot K_1) \rightarrow \{2, 4, \dots, 4n+2\}$ as follows:

$$\begin{aligned} f(a_i) &= 4n - 2i + 2, & \text{if } i = 1, 2, \dots, n, \\ f(b_i) &= 2i + 2, & \text{if } i = 1, 2, \dots, n-1. \end{aligned} \quad (7)$$

Then, the induced vertex labels are

$$\begin{aligned} f^*(v_1) &= [f(a_1) + f(b_1)] \bmod (4n+4) = 0, \\ f^*(v_n) &= [f(a_n) + f(b_{n-1})] \bmod (4n+4) = 4n+2, \\ f^*(v_i) &= [f(b_i) + f(b_{i-1}) + f(a_i)] \bmod (4n+4) = 2i, \end{aligned}$$

for $i = 2, \dots, n-1$.

(8)

So, the labels of the vertices $v_2, v_3, v_4, \dots, v_{n-1}$ are $4, 6, 8, \dots, 2n-2$, respectively. Again, each pendant vertex will take the labels of its edges; then, $f^*(u_i) = f(a_i) = 4n - 2i + 2, \forall i = 1, 2, 3, \dots, n$ which are even and distinct from all the labels of the vertices v_i . Hence, the comb graph $P_n \odot K_1$ has 3-edge even graceful labeling for any number n . $\square$

FIGURE 3: A 3-edge even graceful labeling of the path graphs P_{12} and P_{14} . FIGURE 4: A 3-edge even labeling of the graph $K_{1,11}$.

Illustration. A 3-edge even labeling of the comb graph $P_{10} \odot K_1$ is shown in Figure 6.

2.4. The r -Edge Even Graceful Labeling of the Graph $T_{n,n-2}$. The graph $T_{n,n-2}$ obtained from a path P_n by attaching exactly one pendant edge to each internal vertex of the path P_n , i.e., $T_{n,n-2}$ has vertices $\{v_i, u_j, i = 1, 2, \dots, n, j = 1, 2, \dots, n-2\}$ and edges $\{v_i v_{i+1}, u_j v_{j+1}, i = 1, 2, \dots, n-1, j = 1, 2, \dots, n-2\}$.

Theorem 4. *The graph $T_{n,n-2}$ has 3-edge even graceful labeling for all positive integer $n \geq 4$.*

Proof. The graph $T_{n,n-2}$ has $p = 2n - 2$ vertices and $q = 2n - 3$ edges (see Figure 7).

Let $r = 3$; then, $2p + 2r - 2 = 4n$. The labeling $f: E(T_{n,n-2}) \rightarrow \{2, 4, \dots, 4n - 2\}$ can be defined as follows:

$$f(u_i v_{i+1}) = f(a_i) = \begin{cases} 4n - 2i, & \text{if } i = 1, 2, \dots, n-3, \\ 2n + 2, & \text{if } i = n-2, \end{cases}$$

$$f(v_i v_{i+1}) = f(b_i) = \begin{cases} 2i, & \text{if } i = 1, 2, \dots, n-2, \\ 2n, & \text{if } i = n-1. \end{cases} \quad (9)$$

Note that we do not take the labels $2n - 2$ and $2n + 4$ in the label of edges.

Then, the induced vertex labels are

$$f^*(v_1) = f(b_1) = 2, \quad f^*(v_n) = f(b_{n-1}) = 2n - 2,$$

$$f^*(v_i) = [f(b_i) + f(b_{i-1}) + f(a_{i-1})] \bmod (4n) = 2i,$$

$$\text{for } i = 2, \dots, n-2. \quad (10)$$

So, the labels of the vertices $v_2, v_3, v_4, \dots, v_{n-2}$ are $4, 6, 8, \dots, 2n - 4$, respectively, which are all even and distinct numbers. Once again, since the vertices u_i are pendant, then $f^*(u_i), i = 1, 2, \dots, n-2$ take their edge labels. Therefore, there is no repetition of the labels, and $T_{n,n-2}$ has 3-edge even graceful labeling for all n . $\square$

Illustration. A 3-edge even labeling of the graph $T_{9,7}$ is shown in Figure 8.

2.5. The r -Edge Even Graceful Labeling of the Double Star $B_{n,m}$. The double star is the graph obtained by joining the center of two stars $K_{1,n}$ and $K_{1,m}$ with an edge, denoted by $B_{n,m}$. The number of vertices is $p = n + m + 2$, and the number of edges is $q = n + m + 1$. The double star $B_{n,m}$ has edge even graceful labeling when one of $(m \text{ or } n)$ is an odd number and the other is an even number [11].

Theorem 5

- (1) *The double star $B_{n,m}$ is 2-edge even graceful graph when both m and n are odd numbers.*
- (2) *The double star $B_{n,m}$ is 3-edge even graceful graph when both m and n are even numbers.*

Proof. Let the vertex and edge symbols be given as in Figure 9.

Case (1). If m and n are odd numbers. Let $r = 2$; then, $2q + 2r - 2 = 2m + 2n + 4$.

We define the labeling function $f: E(G) \rightarrow \{2, 4, \dots, 2m + 2n + 4\}$ as follows:

$$f(uv) = 2,$$

$$f(e_i) = 2i + 2, \quad \text{for } i = 1, 2, \dots, \frac{m-1}{2},$$

$$f(E_i) = 2m + 2n + 6 - 2i, \quad \text{for } i = 1, 2, \dots, \frac{m+1}{2},$$

$$f(a_i) = m + 1 + 2i, \quad \text{for } i = 1, 2, \dots, \frac{n+1}{2},$$

$$f(b_i) = (m + 2n + 5) - 2i, \quad \text{for } i = 1, 2, \dots, \frac{n-1}{2}. \quad (11)$$

FIGURE 5: The comb graph $P_n \odot K_1$ with ordinary labeling. FIGURE 6: A 3-edge even labeling of the comb graph $P_{10} \odot K_1$. FIGURE 7: The graph $T_{n,n-2}$ with ordinary labeling. FIGURE 8: A 3-edge even labeling of the graph $T_{9,7}$. FIGURE 9: (a) $B_{m,n}$ when m and n are odd numbers. (b) $B_{m,n}$ when m and n are even numbers.

In this labeling, we can note that

- (i) $[f(e_i) + f(E_{i+1})] \bmod (2m + 2n + 6) \equiv 0 \bmod (2m + 2n + 6)$, for $i = 1, 2, \dots, (m - 1/2)$.
- (ii) $[f(a_i) + f(b_i)] \bmod (2m + 2n + 6) \equiv 0 \bmod (2m + 2n + 6)$, for $i = 1, 2, \dots, (n - 1/2)$.

- (iii) $[f(uv) + f(E_1)] \bmod (2m + 2n + 6) \equiv 0 \bmod (2m + 2n + 6)$.

So, the vertex labels will be

$$f^*(u) = \left[\sum_{i=1}^{(m-1)/2} f(e_i) + \sum_{i=1}^{(m+1)/2} f(E_i) + f(uv) \right] \bmod (2m + 2n + 6) = 0, \quad (12)$$

$$f^*(v) = \left[\sum_{i=1}^{(n-1)/2} f(b_i) + \sum_{i=1}^{(n+1)/2} f(a_i) + f(a_{(n+1)/2}) + f(uv) \right] \bmod (2m + 2n + 6) = m + n + 4.$$

Each pendant vertex takes the labels of its incident edge which are different from $f^*(u)$ and $f^*(v)$.

Case (2). If m and n are even numbers. Let $r = 3$; then, $2q + 2r - 2 = 2m + 2n + 6$. Consider the vertex and edge symbols given in Figure 9(b). We define the mapping $f: (G) \longrightarrow \{2, 4, \dots, 2m + 2n + 6\}$ as follows:

$$f(uv) = 2, f(e_i) = 2i + 2 \quad \text{for } i = 1, 2, \dots, (m/2), \\ f(E_i) = 2p - 2(i - 1) \text{ for } i = 1, 2, \dots, (m/2)$$

$$f(a_i) = \begin{cases} m + 2 + 2i, & \text{if } i = 1, 2, \dots, \frac{n}{2} + 1, \\ (m + 4) + 2i, & \text{if } i = \frac{n}{2} + 2, \dots, n. \end{cases} \quad (13)$$

$$f^*(u) = \left[\sum_{i=1}^{(m/2)} f(e_i) + \sum_{i=1}^{(m/2)} f(E_i) + f(uv) \right] \bmod (2m + 2n + 8) = f(uv) = 2, \\ f^*(v) = \left[\sum_{i=1}^{(n/2)-1} f(a_{n-i+1}) + \sum_{i=1}^{(n/2)-1} f(a_i) + f(a_{(n/2)}) + f(a_{(n/2)+1}) + f(uv) \right] \bmod (2m + 2n + 8) = 0. \quad (14)$$

Also, each pendant vertex takes the labels of its incident edge. Thus, we get a 3-edge even graceful graph. $\square$

Illustration. A 2-edge even labeling of the star graph $B_{9,7}$ and 3-edge even labeling of the star graph $B_{10,8}$ are shown in Figure 10.

2.6. The r -Edge Even Graceful Labeling of the Ladder Graph $L_n = P_2 \times P_n$

Theorem 6. The ladder graph $L_n = P_2 \times P_n$ has 2-edge even graceful labeling.

In this labeling, we can see that

- (i) $[f(e_i) + f(E_i)] \bmod (2m + 2n + 8) \equiv 0 \bmod (2m + 2n + 8)$, for $i = 1, 2, \dots, (m/2)$.
- (ii) $[f(a_i) + f(a_{n-i+1})] \bmod (2m + 2n + 8) \equiv 0 \bmod (2m + 2n + 8)$, for $i = 1, 2, \dots, (n/2) - 1$.

So, the vertex labels will be

Proof. The graph L_n has $p = 2n$ vertices and $q = 3n - 2$ edges. Let $r = 2$; then, $2q + 2r - 2 = 6n - 2$.

Consider the ladder graph $L_n = P_2 \times P_n$ shown in Figure 11.

We define the labeling function $f: E(L_n) \longrightarrow \{2, 4, \dots, 6n - 2\}$ as follows:

$$f(a_i) = 2i, \quad i = 1, 2, \dots, n, \\ f(c_i) = 6n - 2i - 2, \quad i = 1, 2, \dots, n - 1, \\ f(b_i) = \begin{cases} 4n - 2i - 2, & \text{if } i = 1, 2, \dots, n - 2, \\ 6n - 2, & \text{if } i = n - 1. \end{cases} \quad (15)$$

Then, the induced vertex labels are

$$f^*(v_1) = [f(a_1) + f(c_1)] \bmod (6n - 2) = 0, \\ f^*(v_n) = [f(a_n) + f(c_{n-1})] \bmod (6n - 2) = 2, \\ f^*(v_i) = [f(c_i) + f(c_{i-1}) + f(a_i)] \bmod (6n - 2) = 6n - 2i, \quad i = 2, \dots, n - 1, \\ f^*(u_1) = [f(a_1) + f(b_1)] \bmod (6n - 2) = 4n - 2, \\ f^*(u_{n-1}) = [f(a_{n-1}) + f(b_{n-1}) + f(b_{n-2})] \bmod (6n - 2) = 4n, \\ f^*(u_n) = [f(a_n) + f(b_{n-1})] \bmod (6n - 2) = 2n, \\ f^*(u_i) = [f(b_i) + f(b_{i-1}) + f(a_i)] \bmod (6n - 2) = 2n - 2i, \quad i = 2, \dots, n - 2. \quad (16)$$

So, the labels of the vertices $v_2, v_3, \dots, v_{n-2}, v_{n-1}$ will be $6n - 4, 6n - 6, \dots, 4n + 4, 4n + 2$, respectively, and the labels of the vertices $u_2, u_3, \dots, u_{n-3}, u_{n-2}$ will be $2n - 4, 2n - 6, \dots, 6, 4$, respectively. Clearly, all the labels of the vertices are even and there is no repetition on the labels. Hence, the ladder graph L_n has 2-edge even graceful labeling. $\square$

Illustration. A 2-edge even labeling of the ladder graph L_{10} is shown in Figure 12.

2.7. The r -Edge Even Graceful Labeling of the Diagonal Ladder DL_n

Diagonal ladder graph is a ladder with additional edges $v_i u_{i+1}$ and $u_i v_{i+1}$, $i = 1, 2, \dots, n - 1$, denoted by DL_n , where n

FIGURE 10: (a) A 2-edge even labeling of $B_{9,7}$. (b) A 3-edge even labeling of $B_{10,8}$. FIGURE 11: The ladder graph $L_n = P_2 \times P_n$ with ordinary labeling. FIGURE 12: A 2-edge even labeling of the ladder graph $L_{10} = P_2 \times P_{10}$.

is half its vertices and the number of vertices $|V(DL_n)| = 2n$ and edges $|E(DL_n)| = 5n - 4$.

Theorem 7. The diagonal ladder DL_n has 2-edge even graceful labeling.

Proof. Consider the diagonal ladder DL_n shown in Figure 13. Let $r = 2$; then, $2q + 2r - 2 = 10n - 6$.

The labeling $f: E(DL_n) \rightarrow \{2, 4, \dots, 10n - 6\}$ can be defined as follows:

$$\begin{aligned} f(v_i u_{i+1}) &= 2i, \quad i = 1, 2, \dots, n-1, \\ f(v_{i+1} u_i) &= 2n + 2i, \quad i = 1, 2, \dots, n-1, \\ f(v_i v_{i+1}) &= 10n - 6 - 2i, \quad i = 1, 2, \dots, n-1, \\ f(u_i u_{i+1}) &= 8n - 4 - 2i, \quad i = 1, 2, \dots, n-1, \\ f(v_i u_i) &= \begin{cases} 10n - 6, & \text{if } i = 1, \\ 6n - 2i, & \text{if } i = 2, 3, \dots, n. \end{cases} \end{aligned} \quad (17)$$

Then, the induced vertex labels are

$$\begin{aligned} f^*(v_1) &= [f(v_1 v_2) + f(v_1 u_1) + f(v_1 u_2)] \bmod (10n - 6) = 0, \\ f^*(u_1) &= [f(u_1 u_2) + f(v_1 u_1) + f(u_1 v_2)] \bmod (10n - 6) = 2. \end{aligned} \quad (18)$$

For $i = 2, 3, \dots, n-1$,

$$\begin{aligned} f^*(v_i) &= [f(v_i v_{i-1}) + f(v_i v_{i+1}) + f(v_i u_i) + f(v_i u_{i+1}) + f(v_i u_{i-1})] \bmod (10n - 6) = 8n - 2i, \\ f^*(u_i) &= [f(u_i u_{i-1}) + f(u_i u_{i+1}) + f(v_i u_i) + f(u_i v_{i+1}) + f(u_i v_{i-1})] \bmod (10n - 6) = 4n + 4 - 2i. \end{aligned} \quad (19)$$

FIGURE 13: The diagonal ladder DL_n with ordinary labeling.

So, the labels of the vertices $v_2, v_3, \dots, v_{n-2}, v_{n-1}$ are $8n-4, 8n-6, \dots, 6n+4, 6n+2$, respectively, and the labels

of the vertices $u_2, u_3, \dots, u_{n-2}, u_{n-1}$ are $4n, 4n-2, \dots, 2n+8, 2n+6$, respectively, and

$$\begin{aligned} f^*(v_n) &= [f(v_n v_{n-1}) + f(v_n u_n) + f(v_n u_{n-1})] \bmod (10n-6) = 6n, \\ f^*(u_n) &= [f(u_n u_{n-1}) + f(v_n u_n) + f(u_n v_{n-1})] \bmod (10n-6) = 2n+2. \end{aligned} \quad (20)$$

Clearly, all the labels of the vertices are even and distinct, and thus we get the result. $\square$

Illustration. A 2-edge even labeling of the diagonal ladder DL_8 is shown in Figure 14.

2.8. The r -Edge Even Graceful Labeling of the Tortoise Graph T_n . The tortoise graph T_n is the graph obtained from a path P_n , where n is odd, by attaching an edge between v_i and v_{n-i+1} for $1 \leq i \leq [n/2]$, so $V(T_n) = V(P_n)$ and $E(T_n) = E(P_n) \cup \{e_i = v_i v_{n-i+1}, i = n, n-1, \dots, (n+3/2)\}$.

Theorem 8. *The tortoise graph T_n has 2-edge even graceful labeling.*

Proof. The graph T_n has n vertices and $q = 3(n-1)/2$ edges. Consider the tortoise graph T_n shown in Figure 15.

Let $r = 2$; then, $2q + 2r - 2 = 3n - 1$, and we define the labeling $f: E(T_n) \rightarrow \{2, 4, \dots, 3n-1\}$ as follows:

$$f(e_i) = \begin{cases} 2i, & \text{if } i = 1, 2, 3, \dots, \frac{n+1}{2}, \\ 3n-2i+1, & \text{if } i = \frac{n+3}{2}, \dots, n-1, \\ 3n-1, & \text{if } i = n, \\ 5n-2i-1, & \text{if } i = n+1, n+2, \dots, \frac{3(n-1)}{2}. \end{cases} \quad (21)$$

In view of the above labeling pattern, we have

$$\begin{aligned} f^*(v_1) &= [f(e_1) + f(e_n)] \bmod (3n-1) = 2, \\ f^*(v_n) &= [f(e_n) + f(e_{n-1})] \bmod (3n-1) = n+3, \\ f^*(v_i) &= [f(e_i) + f(e_{i-1}) + f(e_{i+n-1})] \bmod (3n-1) = 2i, \quad i = 2, 3, \dots, \frac{n-1}{2}, \\ f^*(v_{(n+1)/2}) &= [f(e_{(n+1)/2}) + f(e_{(n-1)/2})] \bmod (3n-1) = 2n, \\ f^*(v_{(n+3)/2}) &= [f(e_{(n+3)/2}) + f(e_{(n+1)/2}) + f(e_{3(n-1)/2})] \bmod (3n-1) = 2n+2, \\ f^*(v_i) &= [f(e_i) + f(e_{i-1}) + f(e_{2n-i})] \bmod (3n-1) = n-2i+5, \quad i = \frac{n+5}{2}, \dots, n-1. \end{aligned} \quad (22)$$

Clearly all the label of the vertices are even and distinct, changed inside the set $\{0, 2, 4, \dots, 2q + 2r - 2\} = \{0, 2, \dots, 3n-1\}$. Hence the tortoise graph T_n has a 2-edge even labeling. $\square$

Illustration. A 2-edge even labeling of the tortoise graph T_{11} is shown in Figure 16.

3. The r -Edge Even Graceful Labeling of Some Cycle-Related Graphs

3.1. The r -Edge Even Graceful Labeling of the Even Cycle Graph C_m . The cycle graph C_m has an edge even graceful labeling only when m is an odd number [8], so we will study the case when m is an even number, i.e., $m = 2n$.

FIGURE 14: A 2-edge even graceful for the diagonal ladder DL_8 . FIGURE 15: The tortoise graph T_n with ordinary labeling. FIGURE 16: A 2-edge even graceful labeling for the tortoise graph T_{11} .

Theorem 9. The cycle graph C_{2n} has 2-edge even graceful labeling. If n is even, the label is strong.

Proof. In the cycle graph C_{2n} , $p = q = 2n$. If we assume that $r = 2$, then $2q + 2r - 2 = 4n + 2$. We will try to define an injective mapping $f: E(G) \rightarrow \{2, 4, \dots, 4n + 2\}$ such that $f^*(v) = (\sum_{u \in V(G)} f(uv)) \bmod (4n + 2)$ is an injective. Now there are two cases.

Case (1). When n is odd. Let the cyclic graph C_{2n} be given as shown in Figure 17(a). The cyclic edges

$E(C_{2n}) = \{e_1, e_2, \dots, e_n, e_{n+1}, e_{n+2}, \dots, e_{2n-1}, e_{2n}\}$ will be labeled as $\{e_1, e_2, \dots, e_n, a_1, a_2, \dots, a_{n-1}, a_n\}$, where $\{a_i = e_{n+i}, i = 1, \dots, n\}$ such that a_i is put opposite to e_i .

We define the labeling function $f: E(G) \rightarrow \{2, 4, \dots, 4n + 2\}$ as follows:

$$\begin{aligned} f(e_i) &= 2i, \quad i = 1, 2, \dots, n, \\ f(a_i) &= 4n + 4 - 2i, \quad i = 1, 2, \dots, n, \end{aligned} \quad (23)$$

we see that

FIGURE 17: (a) C_{2n} with ordinary labeling when n is odd. (b) C_{2n} with ordinary labeling when n is even.

$$\begin{aligned}
 f^*(v_2) &= [f(e_2) + f(e_1)] \bmod (4n+2) = 6, \\
 f^*(v_n) &= [f(e_n) + f(a_{n-1})] \bmod (4n+2) = (4n+6) \bmod (4n+2) = 4, \\
 f^*(v_i) &= \begin{cases} \{f(e_i) + f(e_{i-2})\} \bmod (4n+2) = 4i-4, & \text{if } i = 4, 6, \dots, n-1, \\ \{f(e_i) + f(e_{i+2})\} \bmod (4n+2) = 4i+4, & \text{if } i = 1, 3, \dots, n-2, \end{cases} \\
 f^*(u_1) &= [f(a_3) + f(a_1)] \bmod (4n+2) = f(a_3) = 4n-2, \\
 f^*(u_2) &= [f(a_2) + f(a_1)] \bmod (4n+2) = f(a_2) = 4n, \\
 f^*(u_n) &= [f(e_{n-1}) + f(a_n)] \bmod (4n+2) = 0, \\
 f^*(u_i) &= \begin{cases} \{f(a_i) + f(a_{i-2})\} \bmod (4n+2) = 4n+10-4i, & \text{if } i = 4, 6, \dots, n-1, \\ \{f(a_i) + f(a_{i+2})\} \bmod (4n+2) = 4n+2-4i, & \text{if } i = 3, 5, \dots, n-2. \end{cases}
 \end{aligned} \tag{24}$$

Hence, the labels of the vertices $v_1, v_3, v_5, \dots, v_{n-2}$ are $8, 16, 24, \dots, 4n-4$, respectively, and the labels of the vertices $v_4, v_6, v_8, \dots, v_{n-1}$ are $12, 20, 28, \dots, 4n-8$, respectively. The labels of the vertices $u_3, u_5, \dots, u_{n-2}$ are $4n-10, 4n-18, \dots, 10$ and the labels of the vertices $u_4, u_6, \dots, v_{n-1}$ will be $4n-6, 4n-14, \dots, 14$, respectively. Therefore, the vertex labels are even and distinct. So, the graph has 2-edge even graceful labeling.

Case (2). When n is even. Let C_{2n} be given as shown in Figure 17(b), and we define the labeling function $f: E(G) \rightarrow \{2, 4, \dots, 4n+2\}$ as follows:

$$\begin{aligned}
 f(e_i) &= 2i, \quad i = 1, 2, \dots, n, \\
 f(a_i) &= (4n+2) - f(e_{2i}) = 4n+2-4i, \quad i = 1, 2, \dots, \frac{n}{2}, \\
 f(b_i) &= (4n+2) - f(e_{2i-1}) = 4n+4-4i, \quad i = 1, 2, \dots, \frac{n}{2}.
 \end{aligned} \tag{25}$$

Thus, the vertex labels are

$$\begin{aligned}
 f^*(v_i) &= [f(e_i) + f(e_{i+1})] \bmod (4n+2) = 4i+2; \quad i = 1, 2, \dots, n-1, \\
 f^*(v_n) &= (f(e_n) + f(b_{n/2})) \bmod (4n+2) = (4n+4) \bmod (4n+2) = 2, \\
 f^*(v_{2n}) &= [f(e_1) + f(a_1)] \bmod (4n+2) = 4n, \\
 f^*(v_{2n-i}) &= \begin{cases} \{f(a_{((i+1)/2})} + f(b_{((i+1)/2)})\} \bmod (4n+2) = 4n-4i, & \text{if } i = 1, 3, 5, \dots, n+1, \\ \{f(a_{(i/2)+1}) + f(b_{(i/2)})\} \bmod (4n+2) = 4n-4i, & \text{if } i = 2, 4, 6, \dots, n+2. \end{cases}
 \end{aligned} \tag{26}$$

Hence, the labels of the vertices $v_1, v_2, v_3, v_4, \dots, v_{n-1}$ will be $6, 10, 14, 18, \dots, 4n - 2$, respectively, and the labels of the vertices $v_{n+1}, v_{n+2}, v_{n+3}, v_{n+4}, \dots, v_{2n-1}$ will be $4, 8, 12, 16, \dots, 4n - 4$. Therefore the vertex labels are even and distinct and changed inside the set $\{2r - 2, 2r, \dots, 2q + 2r - 2\} = \{2, -4, \dots, 4n + 2\}$. So, the graph C_{2n} has strong 2-edge even graceful labeling when n is even. $\square$

Illustration. The graphs C_{18} and C_{16} labeled according to Theorem 9 are presented in Figure 18.

3.2. The r -Edge Even Graceful Labeling of the Crown R_n and $C_n \odot \overline{K_{2m-1}}$. A crown R_n is formed by adding n more pendent vertices $\{u_1, u_2, u_3, \dots, u_n\}$ and n more edges $\{(v_i u_i), i = 1, 2, \dots, n\}$ for $n \geq 3$ to the n vertices $\{v_1, v_2, v_3, \dots, v_n\}$ of a cycle C_n .

Proposition 1. *The crown graph R_n has 2-edge even graceful labeling.*

Proof. The crown graph R_n has $p = 2n$ vertices and $q = 2n$ edges. Consider the crown graph R_n shown in Figure 19, where $a_i = (v_i u_i)$ and $e_i = (v_i v_{i+1}), 1 \leq i \leq n$.

We can define the labeling $f: E(R_n) \rightarrow \{2, 4, \dots, 4n + 2\}$ as follows:

$$\begin{aligned} f(e_i) &= 2i, \quad i = 1, 2, \dots, n, \\ f(a_i) &= 4n - 2i + 4, \quad i = 1, 2, \dots, n. \end{aligned} \quad (27)$$

Now, the induced labels of the vertices are

$$\begin{aligned} f^*(v_1) &= [f(e_1) + f(e_n) + f(a_1)] \bmod (4n + 2) \\ &= [6n + 4] \bmod (4n + 2) = 2n + 2, \\ f^*(v_i) &= [f(e_i) + f(e_{i-1}) + f(a_i)] \bmod (4n + 2) \\ &= [4n + 2 + 2i] \bmod (4n + 2) = 2i, \quad i = 2, 3, \dots, n. \end{aligned} \quad (28)$$

Also, the vertices u_i take the labels of the pendant edges which are $\{0, 4n, 4n - 2, \dots, 2n + 4\}$, so overall all the vertex labels are even and there is no repetition. Thus, the crown graph R_n has 2-edge even graceful labeling. $\square$

The generalization of the previous result is presented in the following theory.

Theorem 10. *The graph $C_n \odot \overline{K_{2m-1}}$ has 2-edge even graceful labeling.*

Proof. In the graph $C_n \odot \overline{K_{2m-1}}$, the cardinality of the vertex set and the edge set is $p = 4mn$ and $q = 4mn$, respectively. Let the vertex and edge symbols be given as in Figure 20.

Let $r = 2$, and we define the labeling map $f: E(C_n \odot \overline{K_{2m-1}}) \rightarrow \{2, 4, \dots, 4mn + 2\}$ as follows:

$$\begin{aligned} f(E_i) &= 2i, \quad \text{for } i = 1, 2, \dots, n, \\ f(e_i) &= 4mn + 2(1 - i), \quad \text{for } i = 1, 2, \dots, n, \\ f(A_{ij}) &= 2n + 4(i - 1) + 2j, \quad \text{for } j = 1, 2, \dots, m - 1, \\ f(B_{ij}) &= 4mn - 2n - 4i - 2j + 6, \quad \text{for } j = 1, 2, \dots, m - 1. \end{aligned} \quad (29)$$

We realize the following:

$$\begin{aligned} [f(A_{ij}) + f(B_{ij})] \bmod (4mn + 2) &\equiv 0 \bmod (4mn + 2), \quad \text{for } j = 1, 2, \dots, m - 1, \\ [f(E_i) + f(e_i)] \bmod (4mn + 2) &\equiv 0 \bmod (4mn + 2), \quad \text{for } i = 1, 2, 3, \dots, n. \end{aligned} \quad (30)$$

Then, we have

$$\begin{aligned} f^*(v_i) &= \left[\sum_{j=1}^{m-1} f(A_{ij}) + \sum_{j=1}^{m-1} f(B_{ij}) + f(E_i) + f(e_{i-1}) + f(e_i) \right] \bmod (4mn + 2), \quad i = 2, 3, \dots, n \\ &= f(E_{i-1}) \bmod (4mn + 2) = 2(i - 1), \quad i = 2, 3, \dots, n. \end{aligned} \quad (31)$$

FIGURE 18: (a) A 2-edge even graceful labeling of $C_{18} = C_{2,9}$. (b) A strong 2-edge even graceful labeling of $C_{16} = C_{2,8}$.

FIGURE 19: The crown graph R_n with ordinary labeling.

FIGURE 20: $C_n \odot \overline{K_{2m-1}}$ with ordinary labeling.

Similarly, $f^*(v_1) = f(E_n) = 2n$.

Hence, the labels of the vertices $v_1, v_2, \dots, v_n$ take the label of the edges of the cycles and each of the pendant vertices takes the label of its edge, so they are all even and different numbers. $\square$

3.3. The r -Edge Even Graceful Labeling of the Helm Graph H_n . The helm H_n is formed from the wheel graph $W_n = \{v_0, v_1, v_2, v_3, \dots, v_n\}$ where v_0 is the center vertex by adding n more pendent vertices $\{u_1, u_2, u_3, \dots, u_n\}$ and n more edges $\{(v_i u_i), i = 1, 2, \dots, n\}$.

Illustration. In Figure 21, we present 2-edge even graceful labeling of the crown R_9 and the graph $C_6 \odot \overline{K_5}$.

Theorem 11. For $n \geq 3$, the helm graph H_n has 2-edge even graceful labeling.

FIGURE 21: (a) 2-edge even graceful labeling of the crown R_9 . (b) 2-edge even graceful labeling of $C_6 \odot \overline{K_5}$.

Proof. The helm graph H_n has $p = 2n + 1$ vertices and $q = 3n$ edges and $K = \max(p, q) = 3n$. Let the helm graph H_n be given as indicated in Figure 22. We assume that $r = 2$; then, $2q + 2r - 2 = 6n + 2$.

To find the labeling function $f: E(H_n) \longrightarrow \{2, 4, \dots, 6n + 2\}$, we have two cases.

Case (1). When n is even, we define the labeling function $f: E(H_n) \longrightarrow \{2, 4, \dots, 6n + 2\}$ as follows:

$$f(e_i) = \begin{cases} 3n + 2i, & \text{if } i = 1, 2, \dots, \frac{n}{2}, \\ 6n + 2, & \text{if } i = \frac{n}{2} + 1, \\ 3n + 2i - 2, & \text{if } i = \frac{n}{2} + 2, \dots, n, \end{cases} \quad (32)$$

$$f(v_0 v_i) = \begin{cases} 2i, & \text{if } i = 1, 2, \dots, \frac{n}{2}, \\ 4n + 2i, & \text{if } i = \frac{n}{2} + 1, \frac{n}{2} + 2, \dots, n, \end{cases}$$

$$f(v_i u_i) = \begin{cases} n + 2, & \text{if } i = 1, \\ 3n + 4 - 2i, & \text{if } i = 2, 3, \dots, n. \end{cases}$$

In this labeling, we notice that

- (i) $[f(v_0 v_i) + f(v_0 v_{n-i+1})] \bmod (6n + 2) \equiv 0 \bmod (6n + 2)$, for $i = 1, 2, \dots, (n/2)$.
- (ii) $[f(v_i u_i) + f(e_{i-1})] \bmod (6n + 2) \equiv 0 \bmod (6n + 2)$, for $i = 2, \dots, (n/2)$.
- (iii) $[f(v_i u_i) + f(e_i)] \bmod (6n + 2) \equiv 0 \bmod (6n + 2)$, for $i = (n/2) + 2, \dots, n$.

In view of the above labeling pattern, the induced vertex labels are

$$f^*(v_0) = \left[\sum_{i=1}^n f(v_0 v_i) \right] \bmod (6n + 2) = 0,$$

$$f^*(v_1) = [f(e_1) + f(e_n) + f(v_0 v_1) + f(v_1 u_1)] \bmod (6n + 2) = 3n + 2, \quad (33)$$

$$f^*(v_{(n/2)+1}) = [f(e_{(n/2)+1}) + f(e_{(n/2)}) + f(v_0 v_{(n/2)+1}) + f(v_{(n/2)+1} u_{(n/2)+1})] \bmod (6n + 2) = 5n + 2,$$

$$f^*(v_{(n/2)+2}) = [f(e_{(n/2)+2}) + f(e_{(n/2)+1}) + f(v_0 v_{(n/2)+2}) + f(v_{(n/2)+2} u_{(n/2)+2})] \bmod (6n + 2) = 5n + 4.$$

FIGURE 22: (a) The graph H_n when n is even. (b) The graph H_n when n is odd.

Since $f^*(v_i) = [f(e_i) + f(e_{i-1}) + f(v_0v_i) + f(v_iu_i)] \bmod (6n+2)$,

$$f^*(v_i) = \begin{cases} (3n+4i) \bmod (6n+2), & \text{if } i = 2, 3, \dots, \frac{n}{2}, \\ (n-6+4i) \bmod (6n+2), & \text{if } i = \frac{n}{2}+3, \dots, n. \end{cases} \quad (34)$$

Hence, the labels of the vertices $v_2, v_3, \dots, v_{(n/2)-1}, v_{(n/2)}$ will be $3n+8, 3n+12, \dots, 5n-4, 5n$, respectively, and the labels of the vertices $v_{(n/2)+3}, v_{(n/2)+4}, \dots, v_{n-1}, v_n$ will be $3n+6, 3n+10, \dots, 5n-10, 5n-6$, respectively, which are all even and distinct numbers.

Case (2). When n is odd, by considering the helm graph H_n shown in Figure 22(b), we can define the labeling $f: E(H_n) \rightarrow \{2, 4, \dots, 6n+2\}$ as follows:

$$\begin{aligned} f(e_i) &= \begin{cases} 3n+2i-1, & \text{if } i = 1, 2, \dots, \frac{n-1}{2}, \\ 3n+2i+1, & \text{if } i = \frac{n+1}{2}, \dots, n, \end{cases} \\ f(v_0v_i) &= \begin{cases} 2i, & \text{if } i = 1, 2, \dots, \frac{n-1}{2}, \\ 4n+2i+2, & \text{if } i = \frac{n+1}{2}, \dots, n, \end{cases} \\ f(v_iu_i) &= \begin{cases} n+1, & \text{if } i = 1, \\ 3n+3-2i, & \text{if } i = 2, 3, \dots, n. \end{cases} \end{aligned} \quad (35)$$

In this labeling, we notice that

- (i) $[f(v_0v_i) + f(v_0v_{n-i})] \bmod (6n+2) \equiv 0 \bmod (6n+2)$, for $i = 1, 2, \dots, (n-1)/2$.
- (ii) $[f(v_iu_i) + f(e_i)] \bmod (6n+2) \equiv 0 \bmod (6n+2)$, for $i = 2, \dots, (n-1)/2$.
- (iii) $[f(v_iu_i) + f(e_{i-1})] \bmod (6n+2) \equiv 0 \bmod (6n+2)$, for $i = (n+3)/2, \dots, n$.

Considering the vertex labels, the induced vertex labels are

$$\begin{aligned} f^*(v_0) &= \left[\sum_{i=1}^n f(v_0v_i) \right] \bmod (6n+2) = 0, \\ f^*(v_1) &= [f(e_1) + f(e_n) + f(v_0v_1) + f(v_1u_1)] \bmod (6n+2) = 3n+3, \\ f^*(v_{(n+1)/2}) &= [f(e_{(n+1)/2}) + f(e_{(n-1)/2}) + f(v_0v_{(n+1)/2}) + f(v_{(n+1)/2}u_{(n+1)/2})] \bmod (6n+2) = 3n+1. \end{aligned} \quad (36)$$

Since $f^*(v_i) = [f(e_i) + f(e_{i-1}) + f(v_0v_i) + f(v_iu_i)] \bmod (6n+2)$,

$$f^*(v_i) = \begin{cases} (3n+4i-3) \bmod (6n+2), & \text{if } i = 2, 3, \dots, \frac{n-1}{2}, \\ (n+1+4i) \bmod (6n+2), & \text{if } i = \frac{n+3}{2}, \dots, n. \end{cases} \quad (37)$$

Hence, the labels of the vertices $v_2, v_3, \dots, v_{(n-3)/2}, v_{(n-1)/2}$ are $3n+5, 3n+9, \dots, 5n-9, 5n-5$, respectively, and the labels of the vertices $v_{(n+3)/2}, v_{(n+5)/2}, \dots, v_{n-1}, v_n$ are

$3n + 7, 3n + 11, \dots, 5n - 3, 5n + 1$, respectively, which are all even and distinct numbers. Thus, the helm graph H_n has 2-edge even graceful labeling. $\square$

Illustration. In Figure 23, we present a 2-edge even graceful labeling of H_{11} and H_{12} .

3.4. The r -Edge Even Graceful Labeling of the Sunflower Graph sf_n . The sunflower graph, sf_n , is defined as a graph obtained by starting with an n -cycle C_n with consecutive vertices $\{v_1, v_2, \dots, v_n\}$ and creating new vertices $\{u_1, u_2, \dots, u_n\}$, with u_i connected to v_i and v_{i+1} , so

$$E(sf_n) = \{a_i = v_i u_i, b_i = u_i v_{i+1}, c_i = v_i v_{i+1}\}. \quad (38)$$

Theorem 12. *The sunflower graph, sf_n , has 2-edge even graceful labeling.*

Proof. The graph sf_n has $p = 2n$ vertices and $q = 3n$ edges and $K = \max(p, q) = 3n$ (see Figure 24). Assume that $r = 2$; then, $2q + 2r - 2 = 6n + 2$, and we consider two cases.

$$f^*(v_i) = \begin{cases} (2n + 8 - 2i) \bmod (6n + 2), & \text{if } i = 3, 5, \dots, n - 1, \\ (2n + 4 - 2i) \bmod (6n + 2), & \text{if } i = 2, 4, \dots, n. \end{cases} \quad (41)$$

Hence, the labels of the vertices $u_1, u_2, u_3, \dots, u_{n-1}, u_n$ are $0, 6n, 6n - 2, \dots, 4n + 6, 4n + 4$, respectively, and the labels of the vertices $v_3, v_5, \dots, v_{n-3}, v_{n-1}$ will be $2n + 2, 2n - 2, \dots, 14, 10$, respectively. Also, the labels of the vertices $v_2, v_4, \dots, v_{n-2}, v_n$ will be $2n, 2n - 4, \dots, 8, 4$, respectively, which are all even and distinct numbers.

Case (2). When n is odd, we define the labeling $f: E(sf_n) \rightarrow \{2, 4, \dots, 6n + 2\}$ as follows:

$$\begin{aligned} f(a_i) &= 2i, \quad \text{for } i = 1, 2, 3, \dots, n, \\ f(b_i) &= 6n + 4 - 4i, \quad \text{for } i = 1, 2, 3, \dots, n, \\ f(c_i) &= \begin{cases} 2n + 2i, & \text{if } i = 1, 3, \dots, n, \\ 6n + 2 - 2i, & \text{if } i = 2, 4, \dots, n - 1. \end{cases} \end{aligned} \quad (42)$$

The induced vertex labels are

$$\begin{aligned} f^*(u_i) &= [f(a_i) + f(b_i)] \bmod (6n + 2) \\ &= (6n + 4 - 2i) \bmod (6n + 2), \quad \text{for } i = 1, 2, \dots, n, \\ f^*(v_1) &= [f(a_1) + f(b_n) + f(c_n) + f(c_1)] \bmod (6n + 2) \\ &= 2n + 6. \end{aligned} \quad (43)$$

Since $f^*(v_i) = [f(a_i) + f(b_{i-1}) + f(c_i) + f(c_{i-1})] \bmod (6n + 2)$,

Case (1). When n is even. The labeling $f: E(sf_n) \rightarrow \{2, 4, \dots, 6n + 2\}$ can be defined as follows:

$$\begin{aligned} f(a_i) &= 2i, \quad \text{for } i = 1, 2, 3, \dots, n, \\ f(b_i) &= 6n + 4 - 4i, \quad \text{for } i = 1, 2, 3, \dots, n, \\ f(c_i) &= \begin{cases} 2n + 2i, & \text{if } i = 1, 3, \dots, n - 1, \\ 6n + 2 - 2i, & \text{if } i = 2, 4, \dots, n. \end{cases} \end{aligned} \quad (39)$$

The induced vertex labels are

$$\begin{aligned} f^*(u_i) &= [f(a_i) + f(b_i)] \bmod (6n + 2) \\ &= (6n + 4 - 2i) \bmod (6n + 2), \quad \text{for } i = 1, 2, \dots, n, \\ f^*(v_1) &= [f(a_1) + f(b_n) + f(c_n) + f(c_1)] \bmod (6n + 2) \\ &= 2n + 8. \end{aligned} \quad (40)$$

Since $f^*(v_i) = [f(a_i) + f(b_{i-1}) + f(c_i) + f(c_{i-1})] \bmod (6n + 2)$,

$$f^*(v_i) = \begin{cases} (2n + 8 - 2i) \bmod (6n + 2), & \text{if } i = 3, 5, \dots, n, \\ (2n + 4 - 2i) \bmod (6n + 2), & \text{if } i = 2, 4, \dots, n - 1. \end{cases} \quad (44)$$

Hence, the labels of the vertices $u_1, u_2, u_3, \dots, u_{n-1}, u_n$ will be $0, 6n, 6n - 2, \dots, 4n + 6, 4n + 4$, respectively, and the labels of the vertices $v_3, v_5, \dots, v_{n-3}, v_n$ will be $2n + 2, 2n - 2, \dots, 12, 8$, respectively. Also, the labels of the vertices $v_2, v_4, \dots, v_{n-2}, v_{n-1}$ will be $2n, 2n - 4, \dots, 10, 6$, respectively, which are all even and distinct numbers. Thus, the sunflower graph, sf_n , has 2-edge even graceful labeling. $\square$

Illustration. In Figure 25, we present 2-edge even graceful labeling of sf_9 and sf_{10} .

3.5. The r -Edge Even Graceful Labeling of the Sunflower Planar Graph SF_n . The sunflower planar graph, SF_n , is defined as a graph obtained by starting with an n -wheel W_n with a center vertex v_0 and consecutive vertices $v_1, v_2, \dots, v_n$ and creating new vertices $u_1, u_2, \dots, u_n$, with u_i connected to v_i and v_{i+1} , so $E(sf_n) = \{v_0 v_i, a_i = v_i u_i, b_i = u_i v_{i+1}, c_i = v_i v_{i+1}\}$.

Theorem 13. *The sunflower planar graph, SF_n , has 2-edge even graceful labeling.*

Proof. The graph SF_n always has an odd number of vertices $p = 2n + 1$ and a number of edges $q = 4n$ and

FIGURE 23: (a) 2-edge even graceful labeling of H_{11} . (b) 2-edge even graceful labeling of H_{12} .

FIGURE 24: The graph sf_n with ordinary labeling.

FIGURE 25: (a) A 2-edge even graceful labeling of sf_9 . (b) A 2-edge even graceful labeling of sf_{10} .

$K = \max(p, q) = 4n$. Consider the graph SF_n shown in Figure 26. Assume that $r = 2$; then, $2q + 2r - 2 = 8n + 2$.

To find the label map $f: E(SF_n) \longrightarrow \{2, 4, \dots, 8n + 2\}$, there are two cases.

Case (1). When n is even. The labeling f can be defined as follows:

$$\begin{aligned} f(a_i) &= n + 2i, \quad \forall i = 1, 2, 3, \dots, n, \\ f(b_i) &= 3n + 2i, \quad \forall i = 1, 2, 3, \dots, n, \\ f(v_i v_{i+1}) &= 7n + 2 - 2i, \quad \forall i = 1, 2, 3, \dots, n, \\ f(v_0 v_i) &= \begin{cases} 2i, & \text{if } i = 1, 2, \dots, \frac{n}{2}, \\ 6n + 2i, & \text{if } i = \frac{n}{2} + 1, \frac{n}{2} + 2, \dots, n. \end{cases} \end{aligned} \quad (45)$$

In this labeling, we can check that

- (i) $[f(v_0 v_i) + f(v_0 v_{n-i+1})] \bmod (8n + 2) \equiv 0 \bmod (8n + 2)$, for $i = 1, 2, \dots, (n/2)$.
- (ii) $[f(v_i v_{i+1}) + f(a_i)] \bmod (8n + 2) \equiv 0 \bmod (8n + 2)$, for $i = 1, 2, \dots, n$.

In view of the above labeling pattern, the induced vertex labels are

$$\begin{aligned} f^*(v_0) &= \left[\sum_{i=1}^n f(v_0 v_i) \right] \bmod (8n + 2) = 0, \\ f^*(u_i) &= [f(a_i) + f(b_i)] \bmod (8n + 2) \\ &= 4n + 4i \bmod (8n + 2), \quad \text{for } i = 1, 2, \dots, n. \end{aligned} \quad (46)$$

Since

$$\begin{aligned} f^*(v_i) &= [f(a_i) + f(b_{i-1}) + f(v_0 v_i) + f(v_i v_{i+1}) + f(v_i v_{i-1})] \bmod (8n + 2), \\ f^*(v_i) &= \begin{cases} (2n + 2i) \bmod (8n + 2), & \text{if } i = 1, 2, 3, \dots, \frac{n}{2}, \\ (2i) \bmod (8n + 2), & \text{if } i = \frac{n}{2} + 1, \frac{n}{2} + 2, \dots, n. \end{cases} \end{aligned} \quad (47)$$

Hence, the labels of the vertices $u_1, u_2, u_3, \dots, u_{n-1}, u_n$ are $4n + 4, 4n + 8, 4n + 12, \dots, 8n - 4, 8n$, respectively, and the labels of the vertices $v_1, v_2, \dots, v_{(n/2)-1}, v_{(n/2)}$ are $2n + 2, 2n + 4, \dots, 3n - 2, 3n$, respectively. Also, the labels of the vertices $v_{(n/2)+1}, v_{(n/2)+2}, \dots, v_{n-1}, v_n$ will be $n, 3n + 2, \dots, 2n - 4, 2n - 2$, respectively, which are all even and distinct numbers.

Case (2). When n is odd. Define the labeling $f: E(SF_n) \longrightarrow \{2, 4, \dots, 8n + 2\}$ as follows:

$$\begin{aligned} f(a_i) &= n - 1 + 2i, \quad \forall i = 1, 2, 3, \dots, n, \\ f(b_i) &= 3n - 1 + 2i, \quad \forall i = 1, 2, 3, \dots, n, \\ f(v_i v_{i+1}) &= 7n + 3 - 2i, \quad \forall i = 1, 2, 3, \dots, n, \\ f(v_0 v_i) &= \begin{cases} 2i, & \text{if } i = 1, 2, \dots, \frac{n-1}{2}, \\ 6n + 2i + 2, & \text{if } i = \frac{n+1}{2}, \frac{n+3}{2}, \dots, n. \end{cases} \end{aligned} \quad (48)$$

We can check that

- (i) $[f(v_0 v_i) + f(v_0 v_{n-i})] \bmod (8n + 2) \equiv 0 \bmod (8n + 2)$, for $i = 1, 2, \dots, (n/2)$.
- (ii) $[f(v_i v_{i+1}) + f(a_i)] \bmod (8n + 2) \equiv 0 \bmod (8n + 2)$, for $i = 1, 2, \dots, n$.

In this labeling, the induced vertex labels are

$$\begin{aligned} f^*(v_0) &= \left[\sum_{i=1}^n f(v_0 v_i) \right] \bmod (8n + 2) = 0, \\ f^*(u_i) &= [f(a_i) + f(b_i)] \bmod (8n + 2) = 4n + 4i - 2 \bmod (8n + 2), \quad \text{for } i = 1, 2, \dots, n. \end{aligned} \quad (49)$$

FIGURE 26: The graph SF_n with ordinary labeling. FIGURE 27: (a) 2-edge even graceful labeling of SF_9 . (b) 2-edge even graceful labeling of SF_{10} .

Since

$$f^*(v_i) = [f(a_i) + f(b_{i-1}) + f(v_0 v_i) + f(v_i v_{i+1}) + f(v_i v_{i-1})] \bmod (8n+2),$$

$$f^*(v_i) = \begin{cases} (2n+2i) \bmod (8n+2), & \text{if } i = 1, 2, 3, \dots, \frac{n-1}{2}, \\ (2i) \bmod (8n+2), & \text{if } i = \frac{n+1}{2}, \frac{n+3}{2}, \dots, n. \end{cases} \quad (50)$$

Hence, the labels of the vertices $u_1, u_2, u_3, \dots, u_{n-1}, u_n$ are $4n+2, 4n+6, 4n+10, \dots, 8n-6, 8n-2$, respectively, and the labels of the vertices $v_1, v_2, \dots, v_{(n-3)/2}, v_{(n-1)/2}$ are $2n+2, 2n+4, \dots, 3n-3, 3n-1$, respectively. Also, the labels of the vertices $v_{(n+1)/2}, v_{(n+3)/2}, \dots, v_{n-1}, v_n$ are $n+1, n+3, \dots, 2n-2, 2n$, respectively, and these are all

even and distinct numbers. Thus, the sunflower planar graph, SF_n , has 2-edge even graceful labeling. $\square$

Illustration. In Figure 27, we present 2-edge even graceful labeling of SF_9 and SF_{10} .

4. Conclusions

Graph labeling has become a large and rich branch of various research studies in graph theory. It has enormous applications in many disciplines, like coding theory, X-rays, radar, communication networks, astronomy, and communication network. In this paper, we have extended the definition of edge even graceful labeling to r -edge even graceful labeling and strong r -edge even graceful labeling. We have obtained the necessary conditions for path-related graphs such as path graph P_n when n is odd, double star graph, ladder and diagonal ladder graphs, and tortoise graph and cycle-related graphs such as helm graph, crown graph, sunflower graph, and sunflower planar graph to be an r -edge even graceful graph. The minimum number r for which these graphs have an r -edge even graceful labeling was found.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors read and approved the final manuscript.

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