Hindawi
Journal of Mathematics
Volume 2021, Article ID 6646091, 10 pages
https://doi.org/10.1155/2021/6646091

Research Article

Interior GE-Algebras

Jeong-Gon Lee[1] **Ravikumar Bandaru**[2] **Kul Hur,**[1] **and Young Bae Jun**[3]

[1]*Division of Applied Mathematics, Wonkwang University, 460 Iksan-daero, Iksan-Si, Jeonbuk 54538, Republic of Korea*
[2]*Department of Mathematics, GITAM (Deemed to be University), Hyderabad Campus, Telangana 502329, India*
[3]*Department of Mathematics Education, Gyeongsang National University, Jinju 52828, Republic of Korea*

Correspondence should be addressed to Jeong-Gon Lee; jukolee@wku.ac.kr

Received 11 November 2020; Revised 26 November 2020; Accepted 21 December 2020; Published 8 February 2021

Academic Editor: Hee S. Kim

The concepts of (commutative, transitive, left exchangeable, belligerent, antisymmetric) interior GE-algebras and bordered interior GE-algebras are introduced, and their relations and properties are investigated. Many examples are given to support these concepts. A semigroup is formed using the set of interior GE-algebras. An example is given that the set of interior GE-algebras is not a GE-algebra. It is clear that if X is a transitive (resp., commutative, belligerent, and left exchangeable) GE-algebra, then the interior GE-algebra (X, f) is transitive (resp., commutative, belligerent, and left exchangeable), but examples are given to show that the converse is not true in general. An interior GE-algebra is constructed using a bordered interior GE-algebra with certain conditions, and an example is given to explain this.

1. Introduction

In 1966, Imai and Iséki introduced BCK-algebras (see [1]) as the algebraic semantics for a nonclassical logic possessing only implication. Since then, the generalized concepts of BCK-algebras have been studied by various scholars. Kim and Kim introduced the notion of a BE-algebra as a generalization of a dual BCK-algebra (see [2]). Hilbert algebras were introduced by Henkin and Skolem in the fifties for investigations in intuitionistic and other nonclassical logics. Diego proved that Hilbert algebras form a variety which is locally finite (see [3]). Rezaei et al. discussed relations between Hilbert algebras and BE-algebras (see [4]). The generalization process in the study of algebraic structures is also an important area of study. As a generalization of Hilbert algebras, Bandaru et al. introduced the notion of GE-algebras and investigated several properties (see [5–8]).

The notion of the interior operator was introduced by Vorster [9] in an arbitrary category, and it was used in [10] to study the notions of connectedness and disconnectedness in topology. Interior algebras are a certain type of algebraic structures that encode the idea of the topological interior of a set and are a generalization of topological spaces defined by means of topological interior operators. Rachůnek and Svoboda [11] studied interior operators on bounded residuated lattices, and Svrcek [12] studied multiplicative interior operators on GMV-algebras.

In this article, we apply the interior operator theory to GE-algebras. We introduce the concepts of (commutative, transitive, left exchangeable, belligerent, antisymmetric) interior GE-algebras and bordered interior GE-algebras, and investigate their relations and properties. We find and present many examples to illustrate these concepts. We use the set of interior GE-algebras to make up a semigroup. We give examples to show that the set of interior GE-algebras is not a GE-algebra. It is clear that if X is a transitive (resp., commutative, belligerent and left exchangeable) GE-algebra, then the interior GE-algebra (X, f) is transitive (resp., commutative, belligerent and left exchangeable), but we give examples to show that its inverse is not established. We make up the internal GE-algebra using a bordered interior GE-algebra with certain conditions and give examples describing this.

2. Preliminaries

Definition 1 (see [5]). By a GE-algebra, we mean a nonempty set X with constant 1 and binary operation $*$ satisfying the following axioms:

(GE1) $u * u = 1$,

(GE2) $1 * u = u$,

(GE3) $u * (v * w) = u * (v * (u * w))$,

for all $u, v, w \in X$.

In a GE-algebra X, a binary relation "$\leq$" is defined by

$$(\forall x, y \in X), \quad (x \leq y \Longleftrightarrow x * y = 1). \tag{1}$$

Definition 2 (see [5, 6, 8]). A GE-algebra X is said to be

(i) transitive if it satisfies

$$(\forall x, y, z \in X), \quad (x * y \leq (z * x) * (z * y)). \tag{2}$$

(ii) commutative if it satisfies

$$(\forall x, y \in X), \quad ((x * y) * y = (y * x) * x). \tag{3}$$

(iii) left exchangeable if it satisfies

$$(\forall x, y, z \in X), \quad (x * (y * z) = y * (x * z)). \tag{4}$$

(iv) belligerent if it satisfies

$$(\forall x, y, z \in X), \quad (x * (y * z) = (x * y) * (x * z)). \tag{5}$$

(v) antisymmetric if the binary relation "$\leq$" is antisymmetric.

Proposition 1 (see [5]). *Every GE-algebra X satisfies the following items:*

$$\begin{aligned}
&(\forall u \in X), \quad (u * 1 = 1),\\
&(\forall u, v \in X), \quad (u * (u * v) = u * v),\\
&(\forall u, v \in X), \quad (u \leq v * u),\\
&(\forall u, v, w \in X), \quad (u * (v * w) \leq v * (u * w)),\\
&(\forall u \in X), \quad (1 \leq u \Longrightarrow u = 1),\\
&(\forall u, v \in X), \quad (u \leq (v * u) * u),\\
&(\forall u, v \in X), \quad (u \leq (u * v) * v),\\
&(\forall u, v, w \in X), \quad (u \leq v * w \Longleftrightarrow v \leq u * w).
\end{aligned} \tag{6}$$

If X is transitive, then

$$(\forall u, v, w \in X), \quad (u \leq v \Longrightarrow w * u \leq w * v, \; v * w \leq u * w), \tag{7}$$

$$(\forall u, v, w \in X), \quad (u * v \leq (v * w) * (u * w)). \tag{8}$$

Lemma 1 (see [5]). *In a GE-algebra X, the following facts are equivalent to each other:*

$$\begin{aligned}
&(\forall x, y, z \in X), \quad (x * y \leq (z * x) * (z * y)),\\
&(\forall x, y, z \in X), \quad (x * y \leq (y * z) * (x * z)).
\end{aligned} \tag{9}$$

3. Interior GE-Algebras

Definition 3. By an interior GE-algebra, we mean a pair (X, f) in which X is a GE-algebra and $f\colon X \longrightarrow X$ is a mapping such that

$$(\forall x \in X), \quad (x \leq f(x)), \tag{10}$$

$$(\forall x \in X), \quad ((f \circ f)(x) = f(x)), \tag{11}$$

$$(\forall x, y \in X), \quad (x \leq y \Longrightarrow f(x) \leq f(y)). \tag{12}$$

Example 1. Consider a GE-algebra $X = \{1, a, b, c, d\}$ with the binary operation $*$ which is given in the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	c	c
b	1	1	1	d	d
c	1	a	b	1	1
d	1	a	a	1	1

(13)

Then, it is routine to verify that (X, f) is an interior GE-algebra, where

$$f\colon X \longrightarrow X, x \longmapsto \begin{cases} 1, & \text{if } x \in \{1, c, d\},\\ a, & \text{if } x \in \{a, b\}. \end{cases} \tag{14}$$

It is clear that if X is a GE-algebra, then (X, id_X) and $(X, \widetilde{1})$ are interior GE-algebras, where $id_X\colon X \longrightarrow X, x \longmapsto x$ and $\widetilde{1}\colon X \longrightarrow X, x \longmapsto 1$.

In the following example, we know that there is a constant map $\widetilde{c}\colon X \longrightarrow X, x \longmapsto c$, where $c (\neq 1) \in X$, on a GE-algebra X such that $(X, \widetilde{c})$ is not an interior GE-algebra.

Example 2. Consider a GE-algebra $(X, *, 1)$, where $X = \{1, a, b, c, d\}$ and $*$ is a binary operation on X, which is given in the following table:

$$\begin{array}{c|ccccc} * & 1 & a & b & c & d \\ \hline 1 & 1 & a & b & c & d \\ a & 1 & 1 & b & 1 & 1 \\ b & 1 & d & 1 & 1 & d \\ c & 1 & a & 1 & 1 & a \\ d & 1 & 1 & b & c & 1 \end{array}. \quad (15)$$

If we take a constant mapping $\tilde{c}$: $X \longrightarrow X, x \longmapsto c$, then $d \nleq c = \tilde{c}(d)$. Hence, $(X, \tilde{c})$ is not an interior GE-algebra.

Let $\mathrm{Int}(X)$ be the set of all interior GE-algebras. For every $(X, f), (X, g) \in \mathrm{Int}(X)$, we define

$$\begin{aligned} &(X, f) = (X, g) \Longleftrightarrow f = g, \quad i.e, f(x) = g(x), \text{ for all } x \in X, \\ &(X, f) \circ (X, g) := (X, f \circ g), \\ &(X, f) \circledast (X, g) := (X, f \circledast g), \end{aligned} \quad (16)$$

where $f \circ g$: $X \longrightarrow X, x \longmapsto f(g(x))$ and $f \circledast g$: $X \longrightarrow X$, $x \longmapsto f(x) * g(x)$.

Let $(X, f), (X, g) \in \mathrm{Int}(X)$. The following example shows that the composition $(X, f) \circ (X, g) = (X, f \circ g)$ of (X, f) and (X, g) may not be an interior GE-algebra, and $(X, f \circ g) \neq (X, g \circ f)$.

Example 3. Let $X = \{1, a, b, c, d\}$ be a set with the binary operation $*$ given in the following table:

$$\begin{array}{c|ccccc} * & 1 & a & b & c & d \\ \hline 1 & 1 & a & b & c & d \\ a & 1 & 1 & 1 & c & c \\ b & 1 & 1 & 1 & d & d \\ c & 1 & 1 & b & 1 & 1 \\ d & 1 & 1 & 1 & 1 & 1 \end{array}. \quad (17)$$

Then, X is a GE-algebra. Define two mappings:

$$\begin{aligned} &f: X \longrightarrow X, x \longmapsto \begin{cases} 1, & \text{if } x = 1, \\ a, & \text{if } x \in \{a, b\}, \\ d, & \text{if } x \in \{c, d\}, \end{cases} \\ &g: X \longrightarrow X, x \longmapsto \begin{cases} 1, & \text{if } x = 1, \\ a, & \text{if } x \in \{a, c\}, \\ b, & \text{if } x \in \{b, d\}. \end{cases} \end{aligned} \quad (18)$$

Then, (X, f) and (X, g) are interior GE-algebras, and the composition $g \circ f$ of f and g is calculated as follows:

$$g \circ f: X \longrightarrow X, x \longmapsto \begin{cases} 1, & \text{if } x = 1, \\ a, & \text{if } x \in \{a, b\}, \\ b, & \text{if } x \in \{c, d\}. \end{cases} \quad (19)$$

Since $c * (g \circ f)(c) = c * g(f(c)) = c * g(d) = c * b = b \neq 1$, the composition $(X, g) \circ (X, f)$ of (X, f) and (X, g) is not an interior GE-algebra. Also, $(X, f \circ g) \neq (X, g \circ f)$ since

$$\begin{aligned} (f \circ g)(c) &= f(g(c)) = f(a) = a \neq b = g(d) = g(f(c)) \\ &= (g \circ f)(c). \end{aligned} \quad (20)$$

We consider the following condition:

$$(X, f \circ g) = (X, g \circ f), \quad (21)$$

for $(X, f), (X, g) \in \mathrm{Int}(X)$.

Denote by $c\,\mathrm{Int}(X)$ the set of all interior GE-algebras satisfying condition (21).

Theorem 1. *If X is a GE-algebra, then $(c\,\mathrm{Int}(X), \circ)$ is a semigroup.*

Proof. It is sufficient to show that $c\,\mathrm{Int}(X)$ is closed under $\circ$. Let $(X, f), (X, g) \in c\,\mathrm{Int}(X)$. Using (10), we have $x \leq g(x) \leq f(g(x)) = (f \circ g)(x)$ for all $x \in X$, and so, $f \circ g$ satisfies condition (10). Also,

$$\begin{aligned} ((f \circ g) \circ (f \circ g))(x) &= ((f \circ f) \circ (g \circ g))(x) \\ &= (f \circ f)((g \circ g)(x)) \\ &= (f \circ f)(g(x)) = f(g(x)) \\ &= (f \circ g)(x), \end{aligned} \quad (22)$$

for all $x \in X$, which shows that $f \circ g$ satisfies condition (11). For every $x, y \in X$, if $x \leq y$, then $g(x) \leq g(y)$, and so, $(f \circ g)(x) = f(g(x)) \leq f(g(y)) = (f \circ g)(y)$. This shows that $(X, f) \circ (X, g) = (X, f \circ g)$ is an interior GE-algebra, that is, $c\,\mathrm{Int}(X)$ is closed under $\circ$. Therefore, $(c\,\mathrm{Int}(X), \circ)$ is a semigroup. □

The following example describes Theorem 1.

Example 4

(1) Consider a GE-algebra $(X, {}^*, 1)$, where $X = \{1, a, b\}$ and $*$ is a binary operation on X, which is given in the following Cayley table:

$$\begin{array}{c|ccc} * & 1 & a & b \\ \hline 1 & 1 & a & b \\ a & 1 & 1 & b \\ b & 1 & a & 1 \end{array}. \quad (23)$$

The set of all interior GE-algebras is $\mathrm{Int}(X) = \{(X, \tilde{1}), (X, id_X), (X, f_1), (X, f_2)\}$, where the self-maps f_1 and f_2 are given by Table 1.

We can check $c\mathrm{Int}(X) = \mathrm{Int}(X)$ by the following Cayley table:

Table 1: Self-maps f_i, $i = 1, 2$.

x	1	a	b
$f_1(x)$	1	1	b
$f_2(x)$	1	a	1

$\circ$	$\tilde{1}$	id_X	f_1	f_2
$\tilde{1}$	$\tilde{1}$	$\tilde{1}$	$\tilde{1}$	$\tilde{1}$
id_X	$\tilde{1}$	id_X	f_1	f_2
f_1	$\tilde{1}$	f_1	f_1	$\tilde{1}$
f_2	$\tilde{1}$	f_2	$\tilde{1}$	f_2

(24)

And $(c\,\mathrm{Int}(X), \circ)$ is a semigroup.

(2) Consider a GE-algebra $X = \{1, a, b, c, d\}$ with the binary operation which is given in the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	d	d
b	1	1	1	c	c
c	1	1	b	1	1
d	1	1	b	1	1

(25)

Table 2: Self-maps f_i, $i = 1, 2, \ldots, 10$.

x	1	a	b	c	d
$f_1(x)$	1	a	a	a	a
$f_2(x)$	1	a	a	c	c
$f_3(x)$	1	a	a	c	d
$f_4(x)$	1	a	a	d	d
$f_5(x)$	1	a	b	a	a
$f_6(x)$	1	1	1	c	c
$f_7(x)$	1	1	1	c	d
$f_8(x)$	1	1	1	d	d
$f_9(x)$	1	a	b	c	c
$f_{10}(x)$	1	a	b	d	d

The set of all interior GE-algebras $\mathrm{Int}(X)$ consists of

$$\mathrm{Int}(X) = \{(X, \tilde{1}), (X, \mathrm{id}_X), (X, f_i)|, \quad i = 1, 2, \ldots, 10\}, \tag{26}$$

in which each f_i, $i = 1, 2, \ldots, 10$, is given in Table 2. The operation "$\circ$" in $\mathrm{Int}(X)$ is calculated as follows:

$\circ$	$\tilde{1}$	id_X	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}
$\tilde{1}$	$\tilde{1}$	$\tilde{1}$	$\tilde{1}$	$\tilde{1}$	$\tilde{1}$	$\tilde{1}$	$\tilde{1}$	$\tilde{1}$	$\tilde{1}$	$\tilde{1}$	$\tilde{1}$	$\tilde{1}$
id_X	$\tilde{1}$	id_X	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}
f_1	$\tilde{1}$	f_1	f_1	f_1	f_1	f_1	f_1	f_1	f_1	f_1	f_1	f_1
f_2	$\tilde{1}$	f_2	f_1	f_2	f_2	f_4	f_1	f_6	f_6	f_6	f_2	f_2
f_3	$\tilde{1}$	f_3	f_1	f_2	f_3	f_4	f_1	f_6	f_7	f_8	f_2	f_4
f_4	$\tilde{1}$	f_4	f_1	f_4	f_4	f_4	f_1	f_8	f_8	f_8	f_4	f_4
f_5	$\tilde{1}$	f_5	f_1	f_1	f_1	f_1	f_5	f_{11}	f_{11}	f_{11}	f_5	f_5
f_6	$\tilde{1}$	f_6	$\tilde{1}$	f_6	f_6	f_6	$\tilde{1}$	f_6	f_6	f_6	f_6	f_6
f_7	$\tilde{1}$	f_7	$\tilde{1}$	f_6	f_7	f_8	$\tilde{1}$	f_6	f_7	f_8	f_6	f_8
f_8	$\tilde{1}$	f_8	$\tilde{1}$	f_8	f_8	f_8	$\tilde{1}$	f_8	f_8	f_8	f_8	f_8
f_9	$\tilde{1}$	f_9	f_1	f_2	f_2	f_2	f_5	f_6	f_6	f_6	f_9	f_9
f_{10}	$\tilde{1}$	f_{10}	f_1	f_4	f_4	f_4	f_5	f_8	f_8	f_8	f_{10}	f_{10}

(27)

in which (X, f_{11}), which is not contained in $\mathrm{Int}(X)$, is given as follows:

x	1	a	b	c	d
$f_{11}(x)$	1	1	1	a	a

We know that $\mathrm{int}(X)$ is not closed under the operation "$\circ$," and $(c\,\mathrm{Int}(X), \circ)$ is a semigroup, where $c\,\mathrm{Int}(X) = \{\tilde{1}, id_X, f_1, f_2, f_3, f_4, f_5\}$.

Let $(X, f), (X, g) \in \mathrm{Int}(X)$. The following example shows that $(X, f) \circledast (X, g)$ may not be an interior GE-algebra, and $(X, f \circledast g) \neq (X, g \circledast f)$.

Example 5. Consider $\mathrm{Int}(X)$ in Example 4 (2). Then, the operation "$\circledast$" in $\mathrm{Int}(X)$ is calculated as follows:

$$\begin{array}{c|cccccccccccc}
\circledast & \tilde{1} & id_X & f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & f_8 & f_9 & f_{10} \\
\tilde{1} & \tilde{1} & id_X & f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & f_8 & f_9 & f_{10} \\
id_X & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} \\
f_1 & \tilde{1} & f_8 & \tilde{1} & f_8 & f_8 & f_8 & \tilde{1} & f_8 & f_8 & f_8 & f_8 & f_8 \\
f_2 & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} \\
f_3 & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} \\
f_4 & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} \\
f_5 & \tilde{1} & f_8 & \tilde{1} & f_8 & f_8 & f_8 & \tilde{1} & f_{f_8} & f_8 & f_8 & f_8 & f_8 \\
f_6 & \tilde{1} & f_{6id} & f_{12} & f_{12} & f_{12} & f_{12} & f_{6id} & \tilde{1} & \tilde{1} & \tilde{1} & f_{6id} & f_{6id} \\
f_7 & \tilde{1} & f_{7id} & f_{12} & f_{12} & f_{12} & f_{12} & f_{7id} & \tilde{1} & \tilde{1} & \tilde{1} & f_{7id} & f_{7id} \\
f_8 & \tilde{1} & f_{8id} & f_{12} & f_{12} & f_{12} & f_{12} & f_{8id} & \tilde{1} & \tilde{1} & \tilde{1} & f_{8id} & f_{8id} \\
f_9 & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} \\
f_{10} & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1}
\end{array}, \tag{28}$$

in which $f_{6\mathrm{id}}, f_{7\mathrm{id}}, f_{8\mathrm{id}}, f_{12} \in \mathrm{Int}(X)$, and they are calculated as follows:

x	1	a	b	c	d
$f_{6id}(x)$	1	a	b	1	1
$f_{7id}(x)$	1	a	b	1	1
$f_{8id}(x)$	1	a	b	1	1
$f_{12}(x)$	1	a	a	1	1

We also know that $(X, f \circledast g) \neq (X, g \circledast f)$; for example, $f_1 \circledast f_7 = f_8 \neq f_{12} = f_7 \circledast f_1$.

Example 5 generally shows that $(\mathrm{Int}(X), \circledast, \tilde{1})$ cannot be a GE-algebra. However, the following example shows that $(\mathrm{Int}(X), \circledast, \tilde{1})$ becomes a GE-algebra sometimes.

Example 6. Consider $\mathrm{Int}(X) = \{(X, \tilde{1}), (X, id_X), (X, f_1), (X, f_2)\}$ in Example 4 (1). Then, the operation "$\circledast$" in $\mathrm{Int}(X)$ is calculated as follows:

$$\begin{array}{c|cccc}
\circledast & \tilde{1} & id_X & f_1 & f_2 \\
\tilde{1} & \tilde{1} & id_X & f_1 & f_2 \\
id_X & \tilde{1} & \tilde{1} & \tilde{1} & \tilde{1} \\
f_1 & \tilde{1} & f_2 & \tilde{1} & f_2 \\
f_2 & \tilde{1} & f_1 & f_1 & \tilde{1}
\end{array}. \tag{29}$$

It is routine to verify that $(\mathrm{Int}(X), \circledast, \tilde{1})$ is a GE-algebra.

For every $(X, f), (X, g) \in \mathrm{Int}(X)$, we define

$$(X, f) \ll (X, g) \Longleftrightarrow (\forall x \in X), \quad (g(x) \leq f(x)). \tag{30}$$

For every $(X, f) \in \mathrm{Int}(X)$, the sets

$$\begin{aligned} \mathcal{I}(f) &:= \{x \in X \mid f(x) = x\}, \\ \ker(f) &:= \{x \in X \mid f(x) = 1\} \end{aligned} \tag{31}$$

are called the identity part and the kernel of f, respectively.

Proposition 2. *Let X be an antisymmetric and transitive GE-algebra. For every $(X, f), (X, g) \in Int(X)$, we have*

(i) $(X, f) \ll (X, g) \Longleftrightarrow (X, f) \circ (X, g) = (X, f)$

(ii) $(X, f) = (X, g) \Longleftrightarrow \mathcal{I}(f) = \mathcal{I}(g)$

Proof

(i) If $(X, f) \ll (X, g)$, then $g(x) \leq f(x)$ for all $x \in X$, and so, $(f \circ g)(x) \leq (f \circ f)(x) = f(x)$ for all $x \in X$. Also, $x \leq g(x) = (g \circ g)(x) \leq (f \circ g)(x)$ which implies that $f(x) \leq (f \circ (f \circ g))(x) = (f \circ f)(g(x)) = (f \circ g)(x)$ for all $x \in X$. Hence, $(f \circ g)(x) = f(x)$ for all $x \in X$, and therefore, $(X, f \circ g) = (X, f)$.

Conversely, assume that $(X, f \circ g) = (X, f)$. Then, $g(x) \leq (f \circ g)(x) = f(x)$ for all $x \in X$. Thus, $(X, f) \ll (X, g)$.

(ii) It is clear that if $(X, f) = (X, g)$, then $\mathcal{I}(f) = \mathcal{I}(g)$. Suppose that $\mathcal{I}(f) = \mathcal{I}(g)$. Condition (11) induces $f(x) \in \mathcal{I}(f) = \mathcal{I}(g)$, and so, $g(f(x)) = f(x)$ for all $x \in X$. Hence, $g \circ f = f$. Similarly, $f \circ g = g$. Using (10) and (12), we have $g(x) \leq (g \circ f)(x) = f(x)$ and $f(x) \leq (f \circ g)(x) = g(x)$. Thus, $f(x) = g(x)$ for all $x \in X$, and therefore, $(X, f) = (X, g)$. □

Lemma 2 (see [5]). *Every GE-algebra X satisfies*

(i) $x \leq y * x$,

(ii) $x \leq (x * y) * y$,

(iii) $x \leq (y * x) * x$,

(iv) $x \leq (x * y) * x$,

(v) $x \leq y * (y * x)$,

(vi) $x * (y * z) \leq y * (x * z)$,

for all $x, y, z \in X$.

Proposition 3. *If* (X, f) *is an interior GE-algebra, then*

(i) $f(1) = 1$

(ii) $(\forall x, y \in X), (f(x) \leq f(y * x))$

(iii) $(\forall x, y \in X), (f(x) \leq f((x * y) * y))$

(iv) $(\forall x, y \in X), (f(x) \leq f((y * x) * x)$

(v) $(\forall x, y \in X), (f(x) \leq f((x * y) * x)$

(vi) $(\forall x, y \in X), (f(x) \leq f(y * (y * x))$

(vii) $(\forall x, y, z \in X), (f(x * (y * z)) \leq f(y * (x * z))$

Proof. (i) is straightforward, and (ii)–(vii) follow from (12) and Lemma 2. □

Question 1. If (X, f) is an interior GE-algebra, will the next items be established?

$$(\forall x, y, z \in X), \quad (f(x * y) \leq f((z * x) * (z * y))), \tag{32}$$

$$(\forall x, y \in X), \quad (f((x * y) * y) = f((y * x) * x)), \tag{33}$$

$$(\forall x, y, z \in X), \quad (f(x * (y * z)) = f((x * y) * (x * z))), \tag{34}$$

$$(\forall x, y, z \in X), \quad (f(x * (y * z)) = (f(y * (x * z))). \tag{35}$$

The following example shows that the answer to the above question is negative.

Example 7

(1) Consider a GE-algebra $X = \{1, a, b, c, d\}$ with the binary operation $*$ given in the following Cayley table:

*	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	1	1
b	1	a	1	1	1
c	1	a	b	1	1
d	1	1	1	c	1

(36)

Define $f: X \longrightarrow X$ by

$$f(x) = \begin{cases} 1, & \text{if } x = 1, \\ b, & \text{if } x = b, \\ d, & \text{if } x \in \{a, c, d\}. \end{cases} \tag{37}$$

Then, (X, f) is an interior GE-algebra, but (X, f) does not satisfy (32) and (33) since

$$\begin{aligned} f(b * c) * f((d * b) * (d * c)) = f(1) * f(1 * c) = 1 * f(c) = 1 * d = d \neq 1, \\ f((b * c) * c) = f(1 * c) = f(c) = d \neq 1 = f(1) = f(b * b) = f((c * b) * b). \end{aligned} \tag{38}$$

(2) Consider a GE-algebra $X = \{1, a, b, c, d\}$ with the binary operation $*$ given in the following Cayley table:

*	1	a	b	c	d
1	1	a	b	c	d
a	1	1	c	c	1
b	1	d	1	1	d
c	1	1	1	1	1
d	1	1	c	c	1

(39)

Define $f: X \longrightarrow X$ by

$$f(x) = \begin{cases} 1, & \text{if } x = 1, \\ c, & \text{if } x \in \{b, c\}, \\ d, & \text{if } x \in \{a, d\}. \end{cases} \tag{40}$$

Then, (X, f) is an interior GE-algebra, but (X, f) does not satisfy (34) since

$$\begin{aligned} f(b * (c * d)) = f(b * 1) = f(1) = 1 \neq d = f(d) \\ = f(1 * d) = f((b * c) * (b * d)). \end{aligned} \tag{41}$$

(3) Consider a GE-algebra $X = \{1, a, b, c, d\}$ with the binary operation $*$ given in the following Cayley table:

$$\begin{array}{cccccc} * & 1 & a & b & c & d \\ 1 & 1 & a & b & c & d \\ a & 1 & 1 & 1 & c & c \\ b & 1 & 1 & 1 & d & d \\ c & 1 & a & a & 1 & 1 \\ d & 1 & a & a & 1 & 1 \end{array}. \quad (42)$$

Define $f\colon X \longrightarrow X$ by

$$f(x) = \begin{cases} 1, & \text{if } x = 1, \\ a, & \text{if } x \in \{a, b\}, \\ c, & \text{if } x = c, \\ d, & \text{if } x = d. \end{cases} \quad (43)$$

Then, (X, f) is an interior GE-algebra, but (X, f) does not satisfy (35) since

$$\begin{aligned} f(a * (b * c)) &= f(a * d) = f(c) = c \neq d = f(d) \\ &= f(b * c) = f(b * (a * c)). \end{aligned} \quad (44)$$

Definition 4. An interior GE-algebra (X, f) is said to be transitive (resp., commutative, belligerent, and left exchangeable) if it satisfies (32) (resp., (33), (34), and (35)).

Example 8

(1) Consider a GE-algebra $X = \{1, a, b, c, d\}$ with the binary operation $*$ given in the following Cayley table:

$$\begin{array}{cccccc} * & 1 & a & b & c & d \\ 1 & 1 & a & b & c & d \\ a & 1 & 1 & 1 & c & d \\ b & 1 & a & 1 & d & d \\ c & 1 & 1 & b & 1 & 1 \\ d & 1 & 1 & b & 1 & 1 \end{array}. \quad (45)$$

Define $f\colon X \longrightarrow X$ by

$$f(x) = \begin{cases} 1, & \text{if } x \in \{1, a, b\}, \\ c, & \text{if } x = c, d. \end{cases} \quad (46)$$

Then, (X, f) is a transitive interior GE-algebra.

(2) Consider a GE-algebra $X = \{1, a, b, c, d\}$ with the binary operation $*$ given in the following Cayley table:

$$\begin{array}{cccccc} * & 1 & a & b & c & d \\ 1 & 1 & a & b & c & d \\ a & 1 & 1 & a & b & c \\ b & 1 & a & 1 & c & 1 \\ c & 1 & a & b & 1 & 1 \\ d & 1 & 1 & 1 & c & 1 \end{array}. \quad (47)$$

Define $f\colon X \longrightarrow X$ by

$$f(x) = \begin{cases} 1, & \text{if } x \in \{1, a, b, d\}, \\ c, & \text{if } x = c. \end{cases} \quad (48)$$

Then, (X, f) is a commutative interior GE-algebra.

(3) Consider a GE-algebra $X = \{1, a, b, c, d\}$ with the binary operation $*$ given in the following Cayley table:

$$\begin{array}{cccccc} * & 1 & a & b & c & d \\ 1 & 1 & a & b & c & d \\ a & 1 & 1 & b & c & d \\ b & 1 & 1 & 1 & c & d \\ c & 1 & 1 & 1 & 1 & d \\ d & 1 & a & b & c & 1 \end{array}. \quad (49)$$

Define $f\colon X \longrightarrow X$ by

$$f(x) = \begin{cases} 1, & \text{if } x = 1, \\ a, & \text{if } x = a, \\ b, & \text{if } x \in \{b, c\}, \\ d, & \text{if } x = d. \end{cases} \quad (50)$$

Then, (X, f) is a belligerent interior GE-algebra.

(4) Consider a GE-algebra $X = \{1, a, b, c, d\}$ with the binary operation $*$ given in the following Cayley table:

$$\begin{array}{cccccc} * & 1 & a & b & c & d \\ 1 & 1 & a & b & c & d \\ a & 1 & 1 & 1 & 1 & d \\ b & 1 & c & 1 & c & 1 \\ c & 1 & b & b & 1 & 1 \\ d & 1 & b & b & 1 & 1 \end{array}. \quad (51)$$

Define $f\colon X \longrightarrow X$ by

$$f(x) = \begin{cases} 1, & \text{if } x = 1, \\ b, & \text{if } x \in \{a, b\}, \\ d, & \text{if } x \in \{c, d\}. \end{cases} \quad (52)$$

Then, (X, f) is a left exchangeable interior GE-algebra.

It is clear that if X is a transitive (resp., commutative, belligerent, and left exchangeable) GE-algebra, then the interior GE-algebra (X, f) is transitive (resp., commutative, belligerent, and left exchangeable), but the converse is not true in general as seen in the following example.

Example 9

(1) Consider a GE-algebra $X = \{1, a, b, c, d\}$ with the binary operation $*$ given in the following Cayley table:

$$\begin{array}{c|ccccc} * & 1 & a & b & c & d \\ \hline 1 & 1 & a & b & c & d \\ a & 1 & 1 & 1 & c & d \\ b & 1 & 1 & 1 & 1 & d \\ c & 1 & a & b & 1 & d \\ d & 1 & a & a & c & 1 \end{array}. \quad (53)$$

Define $f: X \longrightarrow X$ by

$$f(x) = \begin{cases} 1, & \text{if } x \in \{1, c, d\}, \\ a, & \text{if } x \in \{a, b\}. \end{cases} \quad (54)$$

Then, (X, f) is a transitive interior GE-algebra, but X is not transitive GE-algebra since

$$(b * c) * ((d * b) * (d * c)) = 1 * (a * c) = a * c = c \neq 1. \quad (55)$$

(2) Consider a GE-algebra $X = \{1, a, b, c, d\}$ with the binary operation $*$ given in the following Cayley table:

$$\begin{array}{c|ccccc} * & 1 & a & b & c & d \\ \hline 1 & 1 & a & b & c & d \\ a & 1 & 1 & 1 & c & d \\ b & 1 & 1 & 1 & c & d \\ c & 1 & b & b & 1 & d \\ d & 1 & b & b & 1 & 1 \end{array}. \quad (56)$$

Define $f: X \longrightarrow X$ by

$$f(x) = \begin{cases} 1, & \text{if } x \in \{1, c\}, \\ b, & \text{if } x \in \{a, b\}, \\ d, & \text{if } x = d. \end{cases} \quad (57)$$

Then, (X, f) is a commutative interior GE-algebra, but X is not commutative GE-algebra since

$$(a * b) * b = 1 * b = b \neq a = 1 * a = (b * a) * a. \quad (58)$$

(3) Consider a GE-algebra $X = \{1, a, b, c, d\}$ with the binary operation $*$ given in the following Cayley table:

$$\begin{array}{c|ccccc} * & 1 & a & b & c & d \\ \hline 1 & 1 & a & b & c & d \\ a & 1 & 1 & c & c & 1 \\ b & 1 & a & 1 & 1 & d \\ c & 1 & a & 1 & 1 & a \\ d & 1 & 1 & c & c & 1 \end{array}. \quad (59)$$

Define $f: X \longrightarrow X$ by

$$f(x) = \begin{cases} 1, & \text{if } x = 1, \\ d, & \text{if } x \in \{a, d\}, \\ c, & \text{if } x \in \{b, c\}. \end{cases} \quad (60)$$

Then, (X, f) is a belligerent interior GE-algebra, but X is not belligerent GE-algebra since

$$b * (c * d) = b * a = a \neq d = 1 * d = (b * c) * (b * d). \quad (61)$$

(4) Consider a GE-algebra $X = \{1, a, b, c, d\}$ with the binary operation $*$ given in the following Cayley table:

$$\begin{array}{c|ccccc} * & 1 & a & b & c & d \\ \hline 1 & 1 & a & b & c & d \\ a & 1 & 1 & c & c & 1 \\ b & 1 & d & 1 & 1 & d \\ c & 1 & a & 1 & 1 & a \\ d & 1 & 1 & c & c & 1 \end{array}. \quad (62)$$

Define $f: X \longrightarrow X$ by

$$f(x) = \begin{cases} 1, & \text{if } x = 1, \\ c, & \text{if } x \in \{b, c\}, \\ d, & \text{if } x \in \{a, d\}. \end{cases} \quad (63)$$

Then, (X, f) is a left exchangeable interior GE-algebra, but X is not left exchangeable GE-algebra since

$$b * (c * d) = b * a = d \neq a = c * d = c * (b * d). \quad (64)$$

The following example shows that any interior GE-algebra (X, f) does not satisfy the following:

$$(\forall x, y \in X), \quad (f(x) * y \leq x * f(y)), \quad (65)$$

$$(\forall x, y \in X), \quad (f(x) * y \leq f(x * y)). \quad (66)$$

Example 10. Consider a GE-algebra $X = \{1, a, b, c, d\}$ with the binary operation $*$ given in the following Cayley table:

$$\begin{array}{c|ccccc} * & 1 & a & b & c & d \\ \hline 1 & 1 & a & b & c & d \\ a & 1 & 1 & b & c & 1 \\ b & 1 & 1 & 1 & c & 1 \\ c & 1 & d & b & 1 & d \\ d & 1 & 1 & b & 1 & 1 \end{array}. \quad (67)$$

Define $f: X \longrightarrow X$ by

$$f(x) = \begin{cases} 1, & \text{if } x = 1, \\ a, & \text{if } x = a, \\ c, & \text{if } x = c, \\ d, & \text{if } x \in \{b, d\}. \end{cases} \quad (68)$$

Then, (X, f) is an interior GE-algebra, but X does not satisfy (65) and (66) since

$$\begin{aligned} &(f(b) * c) * (b * f(c)) = (d * c) * (b * c) = 1 * c = c \neq 1, \\ &(f(b) * c) * (f(b * c)) = (d * c) * f(c) = 1 * c = c \neq 1. \end{aligned} \quad (69)$$

Proposition 4. *Every transitive interior GE-algebra (X, f) satisfies (65) and (66).*

Proof. Let $x, y \in X$. Using (7) and (10) induces $f(x) * y \leq f(x) * f(y) \leq x * f(y)$ which proves (65). Since $x * y \leq f(x * y)$ and $x \leq f(x)$, it follows from (7) that $f(x) * y \leq x * y \leq f(x * y)$. Hence, (66) is valid. □

Definition 5 (see [7]). If a GE-algebra X has a special element, say 0, which satisfies $0 \leq x$ for all $x \in X$, we call X the bordered GE-algebra.

Definition 6 (see [7]). By a duplex bordered element in a bordered GE-algebra X, we mean an element x of X which satisfies $x^{00} = x$.

The set of all duplex bordered elements of a bordered GE-algebra X is denoted by $0^2(X)$ and is called the duplex bordered set of X. It is clear that $0, 1 \in 0^2(X)$.

Definition 7 (see [7]). A bordered GE-algebra X is said to be duplex if every element of X is a duplex bordered element, that is, $X = 0^2(X)$.

Definition 8 By a bordered interior GE-algebra, we mean an interior GE-algebra (X, f) in which X is a bordered GE-algebra.

Example 11. Consider a bordered GE-algebra $X = \{0, 1, a, b, c\}$ with the binary operation $*$ given in the following Cayley table:

$*$	0	1	a	b	c
0	1	1	1	1	1
1	0	1	a	b	c
a	1	1	1	b	1
b	a	1	a	1	1
c	0	1	0	b	1

(70)

Define $f: X \longrightarrow X$ by

$$f(x) = \begin{cases} 0, & \text{if } x \in \{0, a\}, \\ 1, & \text{if } x = 1, \\ c, & \text{if } x \in \{b, c\}. \end{cases} \quad (71)$$

It is routine to verify that (X, f) is a bordered interior GE-algebra.

Proposition 5. *In a bordered interior GE-algebra (X, f) in which X is transitive, we have*

$$\begin{aligned} &(\forall x \in X), \quad \left(f(x)^0 \leq f\left(x^0\right)\right), \\ &(\forall x, y \in X), \quad \left(x * y \leq f\left(y^0 * x^0\right)\right). \end{aligned} \quad (72)$$

Proof. If we take $y = 0$ in (66), then $f(x)^0 = f(x) * 0 \leq f(x * 0) = f(x^0)$. Taking $u = x$, $v = y$, and $w = 0$ in (8) induces $x * y \leq y^0 * x^0$, and so, $x * y \leq f(x * y) \leq f(y^0 * x^0)$ by (10) and (12). □

Lemma 3 (see [7]). *The duplex bordered set $0^2(X)$ of a transitive and antisymmetric bordered GE-algebra X is closed under the binary operation $*$ in X, that is, it is a GE-subalgebra of X and is also bordered.*

Theorem 2. *Let (X, f) be a transitive and antisymmetric bordered interior GE-algebra. Then, $(0^2(X), \tilde{f})$ is an interior GE-algebra, where*

$$\tilde{f}: 0^2(X) \longrightarrow 0^2(X), \; x \longmapsto f(x)^{00}. \quad (73)$$

Proof. Let $x \in 0^2(X)$. Then, $x = x^{00} \leq f(x)^{00} = \tilde{f}(x)$ by (7), and

$$(\tilde{f} \circ \tilde{f})(x) = \tilde{f}\left(f(x)^{00}\right) = f(x)^{0000} = f(x)^{00} = \tilde{f}(x). \quad (74)$$

Let $x, y \in 0^2(X)$ be such that $x \leq y$. Then, $f(x) \leq f(y)$ by (12), and thus, $\tilde{f}(x) = f(x)^{00} \leq f(y)^{00} = \tilde{f}(y)$. This completes the proof. □

The following example describes Theorem 2.

Example 12. Consider a GE-algebra $X = \{0, 1, a, b, c, d, e\}$ with the binary operation $*$ given in the following Cayley table:

$*$	0	1	a	b	c	d	e
0	1	1	1	1	1	1	1
1	0	1	a	b	c	d	e
a	c	1	1	1	c	c	1
b	0	1	a	1	c	d	e
c	a	1	a	1	1	e	e
d	a	1	a	1	1	1	1
e	0	1	a	1	c	c	1

(75)

Define $f: X \longrightarrow X$ by

$$f(x)=\begin{cases}0 & \text{if } x=0\\ 1 & \text{if } x=1,\\ b & \text{if } x\in\{a,b,e\},\\ c & \text{if } x\in\{c,d\}.\end{cases} \tag{76}$$

Then, (X, f) is a transitive and antisymmetric bordered interior GE-algebra, and $(0^2(X), ^*, 1)$ is a GE-algebra, where $0^2(X)=\{0,1,a,c\}$. We know that $\tilde{f}\colon 0^2(X)\longrightarrow 0^2(X)$ is calculated as follows:

$$\tilde{f}(x)=\begin{cases}0, & \text{if } x=0,\\ 1, & \text{if } x\in\{1,a\},\\ c, & \text{if } x=c,\end{cases} \tag{77}$$

and it is routine to observe that $(0^2(X), \tilde{f})$ is an interior GE-algebra.

4. Conclusions

We have introduced the concepts of (commutative, transitive, left exchangeable, belligerent, antisymmetric) interior GE-algebras and bordered interior GE-algebras, and investigated their relations and properties. We have found and presented many examples to illustrate these concepts. We have formed a semigroup using the set of interior GE-algebras. We have provided examples to show that the set of interior GE-algebras is not a GE-algebra. It is clear that if X is a transitive (resp., commutative, belligerent, and left exchangeable) GE-algebra, then the interior GE-algebra (X, f) is transitive (resp., commutative, belligerent, and left exchangeable), but we have considered examples to show that its inverse is not established. We have provided examples of how to construct and explain interior GE-algebra using a bordered interior GE-algebra under certain conditions. In the future work, we will use the idea and results given in this paper to study other (hyper) algebraic structures, for example, (hyper) hoop, (hyper) BCH-algebra, (hyper) equality algebra, and (hyper) MV-algebra.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Y.B.J. and J.-G.L. created and conceptualized ideas. R.K.B. and Y.B.J. found examples. Y.B.J. developed the methodology. K.H. and R.K.B. reviewed and edited the article. J.-G.L. contributed to funding acquisition. All authors read and agreed to the published version of the manuscript.

Acknowledgments

This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (2018R1D1A1B07049321).

References

[1] Y. Imai and K. Iséki, "On axiom systems of propositional calculi, XIV," *Proceedings of the Japan Academy*, vol. 42, no. 1, pp. 19–22, 1966.

[2] H. S. Kim and Y. H. Kim, "On BE-algebras," *Scientiae Mathematicae Japonicae*, vol. 66, no. 1, pp. 113–116, 2007.

[3] A. Diego, "Sur algébres de Hilbert," *Collect Logique Mathématiques Series A*, vol. 21, pp. 177–198, 1967.

[4] A. Rezaei, A. Borumand Saeid, and R. A. Borzooei, "Relation between Hilbert algebras and BE–algebras," *Applications and Applied Mathematics*, vol. 8, no. 2, pp. 573–584, 2013.

[5] R. K. Bandaru, A. Borumand Saeid, and Y. B. Jun, "On GE-algebras," *Bulletin of the Section of Logic*, p. 16, 2020, In press.

[6] R. K. Bandaru, A. Borumand Saeid, and Y. B. Jun, "Belligerent GE-filter in GE-algebras," *Tbilisi Mathematical Journal*.

[7] R. K. Bandaru, M. A. Öztürk, and Y. B. Jun, "Bordered GE-algebras," *Discussiones Mathematicae-General Algebra and Applications*.

[8] A. Borumand Saeid, A. Rezaei, R. K. Bandaru, and Y. B. Jun, "Voluntary GE-filters and further results of GE-filters in GE-algebras," *Journal of Algebraic systems*.

[9] S. J. R. Vorster, "Interior operators in general categories," *Quaestiones Mathematicae*, vol. 23, no. 4, pp. 405–416, 2000.

[10] G. Castellini and J. Ramos, "Interior operators and topological connectedness," *Quaestiones Mathematicae*, vol. 33, no. 3, pp. 290–304, 2010.

[11] J. Rachůnek and Z. Svoboda, "Interior and closure operators on bounded residuated lattices," *Central European Journal of Mathematics*, vol. 12, no. 3, pp. 534–544, 2014.

[12] F. Svrcek, "Operators on GMV-algebras," *Mathematica Bohemica*, vol. 129, pp. 337–347, 2004.