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Research Article

Common Fixed Point Theorems via Inverse C_k- Class Functions in Metric Spaces

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In this paper, we firstly introduce a new notion of inverse C_k-class functions which extends the notion of inverse $C-$class functions introduced by Saleem et al., 2018. Secondly, some common fixed point theorems are stated under some compatible conditions such as weak semicompatible of type (A), weak semicompatibility, and conditional semicompatibility in metric spaces. Moreover, we introduce a new kind of compatibility called $S_\tau-$compatibility which is weaker than $(E.A.)$ property and also present a common fixed point theorem in metric spaces via inverse C_k-class functions. Some examples are provided to support our results.

1. Introduction and Preliminaries

As a follow-up work of A.H. Ansari's research on fixed point (or common fixed point) theory via auxiliary C-class functions, very recently, Saleem et al. [1] introduced the new concept of inverse C-class functions and obtained some corresponding fixed point theorems under certain weak compatibility assumption via inverse C-class functions. In 1976, Jungck [2] defined the concept of commutative maps and initiated the study of the existence of a common fixed point of such maps in metric spaces. After which, Sessa [3] introduced the weak version of commuting maps called weak commuting maps. Next, Jungck [4, 5] provided some generalizations of weak commuting maps by providing the notions of compatible maps and compatible maps of type (A). Minor relaxations of compatible of type (A) are introduced by Pathak and Khan [6], which are well known as $g-$compatible and $f-$compatible (see [6], for more details).

Singh et al. [7] proposed the notion of compatibility of type (E) by making a minor modification of compatibility of type (A). By splitting the concept of compatibility of type (E), Singh et al. [7] also gave some relaxations of compatibility type of (E) which are known as $g-$compatibility of type (E) and $f-$compatibility of type (E).

Definition 1 (see [7]). Two self-maps f and g of a metric space (X, d) are said to be $f-$compatible of type (E), if $\lim_{n\longrightarrow+\infty} ffx_n = \lim_{n\longrightarrow+\infty} fgx_n = gt$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\longrightarrow+\infty} fx_n = \lim_{n\longrightarrow+\infty} gx_n = t$, for some $t \in X$. Similarly, two self-maps f and g of a metric space (X, d) are said to be $g-$compatible of type (E), if $\lim_{n\longrightarrow+\infty} ggx_n = \lim_{n\longrightarrow+\infty} gfx_n = ft$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\longrightarrow+\infty} fx_n = \lim_{n\longrightarrow+\infty} gx_n = t$, for some $t \in X$.

It is easy to see that compatibility of type (E) implies both $g-$compatibility of type (E) and $f-$compatibility of

type (E); however, g–compatibility or f–compatibility of type (E) does not imply compatibility of type (E).

In 1994, Pant [8] introduced the following definition.

Definition 2 (see [8]). Two self-maps f and g defined on a metric space (X, d) are said to be R–weakly commuting, if there exists a real number $R > 0$ such that $d(fgx, gfx) \leq R\, d(fx, gx)$, for all $x \in X$.

Note that $R = 1$; then, f and g are weakly commuting.

In 1997, Pathak et al. [9] introduced the notions of R–weak commuting of type (A_g) and R–weak commuting of type (A_f) as follows.

Definition 3 (see [9]). Two self-maps f and g of a metric space (X, d) are said to be R–weakly commuting of type (A_g), if there exists a real number $R > 0$ such that $d(gfx, ffx) \leq R\, d(fx, gx)$, for all $x \in X$.

Definition 4 (see [9]). Two self-maps f and g of a metric space (X, d) are said to be R–weakly commuting of type (A_f), if there exists a real number $R > 0$ such that $d(fgx, ggx) \leq R\, d(fx, gx)$, for all $x \in X$.

It is noted that compatible maps f and g are also R–weakly commuting of type (A_g) and R–weakly commuting of type (A_f). Moreover, we can find suitable examples which show that R–weakly commuting mappings and R–weakly commuting of type (A_f) (or (A_g)) are independent concepts (see examples of [9, 10]).

In 2008, Gopal et al. [10] introduced the notions of g–absorbing and f–absorbing stated as follows.

Definition 5 (see [10]). Let f and $g\,(f \neq g)$ be two self-maps of a metric space (X, d); then, f is said to be g-absorbing, if there exists a real number $R > 0$ such that $d(gx, gfx) \leq R\, d(fx, gx)$ for all $x \in X$. Similarly, let f and $g\,(f \neq g)$ be two self-maps of a metric space (X, d); then, g is said to be f-absorbing, if there exists a real number $R > 0$ such that $d(fx, fgx) \leq R\, d(fx, gx)$ for all $x \in X$.

Jungck and Rhoades [11], in 1998, introduced the concept of weak compatibility which is weaker than the concept of compatibility.

Another generalization of compatible maps called semicompatible maps was firstly introduced by Cho et al. [12] under the setting of d–topological spaces in which a pair (S, T) of self-maps are called to be semicompatible if condition $(i) Sy = Ty$ implies that $STy = TSy$; (ii) for sequence $\{x_n\}$ in X and $x \in X$, whenever $\{Sx_n\} \longrightarrow x$, $\{Tx_n\} \longrightarrow x$, and then $STx_n \longrightarrow Tx$, as $n \longrightarrow +\infty$, hold. However, Singh and Jain [13] redefined this concept by using condition (ii) only stated as follows.

Definition 6 (see [13]). A pair (f, g) of self-maps of a metric space (X, d) is said to be semicompatible, if $\lim_{n\longrightarrow+\infty} fgx_n = gt$ holds whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\longrightarrow+\infty} fx_n = \lim_{n\longrightarrow+\infty} gx_n = t$ for some $t \in X$.

It follows that if (f, g) is semicompatible and $fx = gx$, then $fgx = gfx$. It is also noted that if the pair (f, g) is semicompatible, then it is weak compatible; however, the converse is not true. Further, the semicompatibility of the pair (f, g) does not imply the semicompatibility of the pair (g, f) (see Example 3.2 in [13]).

Now, we make a minor modification of semicompatibility to introduce the notion of semicompatible of type (A) as follows.

Definition 7. A pair (f, g) of self-maps of a metric space (X, d) is said to be semicompatible of type (A), if $\lim_{n\longrightarrow+\infty} fgx_n = gt$ and $\lim_{n\longrightarrow+\infty} gfx_n = ft$ hold whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\longrightarrow+\infty} fx_n = \lim_{n\longrightarrow+\infty} gx_n = t$ for some $t \in X$.

It is obvious that semicompatibility of type (A) implies semicompatibility; however, the converse is not true.

Recently, Saluja et al. [14, 15] introduced the weak semicompatible maps and conditional semicompatible maps and obtained corresponding fixed point theorems (see [14–16], for more details).

Definition 8 (see [14]). A pair (f, g) of self-maps of a metric space (X, d) is said to be weakly semicompatible, if $\lim_{n\longrightarrow+\infty} fgx_n = gt$ or $\lim_{n\longrightarrow+\infty} gfx_n = ft$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\longrightarrow+\infty} fx_n = \lim_{n\longrightarrow+\infty} gx_n = t$ for some $t \in X$.

Definition 9 (see [15]). A pair (f, g) of self-maps of a metric space (X, d) is said to be conditionally semicompatible; if whenever the set of sequences $\{x_n\}$ satisfying $\lim_{n\longrightarrow+\infty} fx_n = \lim_{n\longrightarrow+\infty} gx_n$ is nonempty, then there exists at least a sequence $\{y_n\}$ satisfying $\lim_{n\longrightarrow+\infty} fy_n = \lim_{n\longrightarrow+\infty} gy_n = t$ such that $\lim_{n\longrightarrow+\infty} fgy_n = gt$ and $\lim_{n\longrightarrow+\infty} gfy_n = ft$.

It is obvious that semicompatibility of type (A) implies weak semicompatibility. From the definition itself, it is clear that if a pair (f, g) of self-maps is semicompatible of type (A), then it is necessarily conditionally semicompatible; however, the conditionally semicompatible maps are not necessarily semicompatible of type (A).

Example 1. Let $X = [1, +\infty)$ and d be the usual metric on X. Define f, g: $X \mapsto X$ as follows:

$$fx = \begin{cases} 1, & \text{if } x = 1, \\ \dfrac{x+7}{2}, & \text{if } 1 < x < 3, \\ 3, & \text{if } x \geq 3, \end{cases} \qquad (1)$$

$$gx = \begin{cases} 1, & \text{if } x = 1, \\ x+3, & \text{if } 1 < x < 3, \\ x, & \text{if } x \geq 3. \end{cases}$$

Let us consider the sequence $x_n = 1 + (1/n)$; we have

$$\begin{aligned}
\lim_{n\longrightarrow+\infty} fx_n &= \lim_{n\longrightarrow+\infty} f\left(1+\frac{1}{n}\right) = 4,\\
\lim_{n\longrightarrow+\infty} gx_n &= \lim_{n\longrightarrow+\infty} g\left(1+\frac{1}{n}\right) = 4,\\
\lim_{n\longrightarrow+\infty} fgx_n &= \lim_{n\longrightarrow+\infty} f\left(4+\frac{1}{n}\right) = 3 \neq g(4),\\
\lim_{n\longrightarrow+\infty} gfx_n &= \lim_{n\longrightarrow+\infty} g\left(4+\frac{1}{2n}\right) = 4 \neq f(4).
\end{aligned} \tag{2}$$

However, if we take $y_n = 3 + (1/n)$, we have that

$$\begin{aligned}
\lim_{n\longrightarrow+\infty} fy_n &= \lim_{n\longrightarrow+\infty} f\left(3+\frac{1}{n}\right) = 3,\\
\lim_{n\longrightarrow+\infty} gy_n &= \lim_{n\longrightarrow+\infty} g\left(3+\frac{1}{n}\right) = 3,\\
\lim_{n\longrightarrow+\infty} fgx_n &= \lim_{n\longrightarrow+\infty} f\left(3+\frac{1}{n}\right) = 3 = g(3),\\
\lim_{n\longrightarrow+\infty} gfx_n &= \lim_{n\longrightarrow+\infty} g(3) = 3 = f(3).
\end{aligned} \tag{3}$$

Thus, the pair (f, g) is conditional semicompatible.

Finally, we introduce a new kind of compatibility of a pair (f, g) of self-maps called S_τ−compatible firstly proposed by Jain et al. [17] as follows.

Definition 10. Let τ be a self-map defined on X satisfying $\lim_{n\longrightarrow+\infty}\tau x_n = t$ for some sequence $\{x_n\} \in X$ and $t \in X$. Then, a pair (f, g) of self-maps defined on X is called S_τ−compatible if $\lim_{n\longrightarrow+\infty} f\tau x_n = \lim_{n\longrightarrow+\infty} g\tau x_n = \tau t$.

Example 2. Let $X = \mathbb{R}$, $fx = 2x$, $gx = 4 - 2x$, and $\tau x = 1 + x$. Take $\{x_n\} = 1/n$. Since $\lim_{n\longrightarrow+\infty}\tau x_n = 1$ with $\lim_{n\longrightarrow+\infty} f\tau x_n = 2 = \tau(1)$ and $\lim_{n\longrightarrow+\infty} g\tau x_n = 2 = \tau(1)$, then pair (f, g) is S_τ−compatible. However, $\lim_{n\longrightarrow+\infty}\tau x_n \neq \lim_{n\longrightarrow+\infty} fx_n$ and $\lim_{n\longrightarrow+\infty}\tau x_n \neq \lim_{n\longrightarrow+\infty} gx_n$.

It is obvious that S_τ−compatibility of a pair (f, g) self-maps implies *E.A.* property of a pair (f, g) of self-maps by taking self-map τ as an identity map.

Let $X = \mathbb{R}$, $fx = x$, $gx = x^2$, and $\tau = I_x$(identity function on X). Take $\{x_n\} = 1/n$. Here, $\lim_{n\longrightarrow+\infty}\tau x_n = \lim_{n\longrightarrow+\infty} g\tau x_n = \lim_{n\longrightarrow+\infty} f\tau x_n = \lim_{n\longrightarrow+\infty} x_n = 0$. Hence, pair self-maps (f, g) satisfy $(E.A.)$ property.

Now, we introduce one more example of S_τ−compatibility including four maps as follows.

Example 3. If $X = [1, +\infty)$ with the usual metric. Define f, g, B, T: $X \mapsto X$ by

$$fx = \begin{cases} x+1, & \text{if } x \in [1,2),\\ 3, & \text{if } x \in [2,+\infty),\end{cases}$$

$$gx = \begin{cases} 2, & \text{if } x \in [1,2),\\ \dfrac{x+4}{2}, & \text{if } x \in [2,+\infty),\end{cases}$$

$$Bx = \begin{cases} 2x, & \text{if } x \in [1,2),\\ x+1, & \text{if } x \in [2,+\infty),\end{cases} \tag{4}$$

$$Tx = \begin{cases} 3x-1, & \text{if } x \in [1,2),\\ 2x-1, & \text{if } x \in [2,+\infty).\end{cases}$$

Choose $x_n = 1 + \varepsilon_n$, where $\varepsilon_n \longrightarrow 0$ when $n \longrightarrow +\infty$, then $\lim_{n\longrightarrow+\infty} fx_n = 2$, $\lim_{n\longrightarrow+\infty} gx_n = 2$, $\lim_{n\longrightarrow+\infty} Bx_n = 2$, and $\lim_{n\longrightarrow+\infty} Tx_n = 2$.

Since $\lim_{n\longrightarrow+\infty} fBx_n = 3 = B(2)$ and $\lim_{n\longrightarrow+\infty} gBx_n = 3 = B(2)$, then the pair (f, g) is S_B−compatible. Next, since $\lim_{n\longrightarrow+\infty} fTx_n = 3 = T(2)$ and $\lim_{n\longrightarrow+\infty} gTx_n = 3 = T(2)$, then the pair (f, g) is S_T−compatible. Further, since $\lim_{n\longrightarrow+\infty} Bgx_n = 3 = g(2)$ and $\lim_{n\longrightarrow+\infty} Tgx_n = 3 = g(2)$, then the pair (B, T) is S_g−compatible. Finally, since $\lim_{n\longrightarrow+\infty} Bfx_n = 3 = f(2)$ and $\lim_{n\longrightarrow+\infty} Tfx_n = 3 = f(2)$, then the pair (B, T) is S_f−compatible.

Ansari, in 2014, firstly [18], introduced the concept of C−class functions and proved some fixed point theorems via C−class functions (see [19, 20] for more details).

Definition 11 (see [18]). A mapping F: $[0, +\infty)^2 \mapsto \mathbb{R}$ is called a C−class function if it is continuous and the following axioms hold:

(1) $F(s, t) \leq s$ for all $s, t \in [0, +\infty)$

(2) $F(s, t) = s$ implies that $s = 0$ or $t = 0$

Denote the family of C−class functions by $\mathscr{C}$.

Example 4 (see [18]). The following functions F: $[0, +\infty)^2 \mapsto \mathbb{R}$ are elements of $\mathscr{C}$, for all $s, t \in [0, +\infty)$:

(1) $F(s, t) = s - t$, $F(s, t) = s$ implies $t = 0$

(2) $F(s, t) = ms$, for some $m \in (0, 1)$, $F(s, t) = s$ implies $s = 0$

(3) $F(s, t) = (s/(1 + t)^r)$, for some $r \in (0, +\infty)$, $F(s, t) = s$ implies $s = 0$ or $t = 0$

(4) $F(s, t) = \log_a[(t + a^s)/(1 + t)]$, for some $a > 1$, $F(s, t) = s$ implies $s = 0$ or $t = 0$

(5) $F(s, t) = \log_a[(1 + a^s/2)]$, for $a > e$, $F(s, 1) = s$ implies $s = 0$

(6) $F(s,t) = (s+l)^{(1/(1+t)^r)} - l$, $l > 1$, for $r \in (0,+\infty)$, $F(s,t) = s$ implies $t = 0$

(7) $F(s,t) = s\log_{t+a} a$, for $a > 1$, $F(s,t) = s$ implies $s = 0$ or $t = 0$

(8) $F(s,t) = s - (1 + s/2 + s)(t/1 + t)$, $F(s,t) = s$ implies $t = 0$

(9) $F(s,t) = s\beta(s)$, where β: $[0,+\infty)\mapsto[0,1)$ is continuous, $F(s,t) = s$ implies $s = 0$

(10) $F(s,t) = s - (t/k + t)$, $F(s,t) = s$ implies $t = 0$

(11) $F(s,t) = s - \varphi(t)$, $F(s,t) = s$ implies $s = 0$, where φ: $[0,+\infty) \longmapsto [0,+\infty)$ is a continuous function such that $\varphi(t) = 0$ if and only if $t = 0$

(12) $F(s,t) = sh(s,t)$, $F(s,t) = s$ implies $s = 0$, where h: $[0,+\infty)\times[0,+\infty)\mapsto[0,+\infty)$ is a continuous function such that $h(t,s) < 1$ for all $t,s > 0$

(13) $F(s,t) = s - (2 + t/1 + t)t$, $F(s,t) = s$ implies $t = 0$

(14) $F(s,t) = \sqrt[n]{\ln(1+s^n)}$, $F(s,t) = s$ implies $s = 0$

(15) $F(s,t) = \phi(s)$, $F(s,t) = s$ implies $s = 0$, where ϕ: $[0,+\infty) \longmapsto [0,+\infty)$ is a continuous function such that $\phi(0) = 0$ and $\phi(t) < t$, for $t > 0$

Afterward, by the motivation of C−class functions, Saleem et al. [1, 21] introduced a new notion of inverse C−class functions as follows.

Definition 12 (see [1]). A mapping F: $[0,+\infty)^2\mapsto\mathbb{R}$ is called an inverse C−class function if it is continuous and the following axioms hold:

(1) $F(s,t) \geq s$ for all $s,t \in [0,+\infty)$

(2) $F(s,t) = s$ implies that $s = 0$ or $t = 0$

Denote the family of inverse C−class functions by $\mathscr{C}_{\text{inv}}$.

Example 5 (see [1]). The following functions F: $[0,+\infty)^2\mapsto\mathbb{R}$ are elements of $\mathscr{C}_{\text{inv}}$, for all $s,t \in [0,+\infty)$:

(1) $F(s,t) = s + t, F(s,t) = s$ implies $t = 0$

(2) $F(s,t) = ms$, for some $m \in (1,+\infty), F(s,t) = s$ implies $s = 0$

(3) $F(s,t) = s(1+t)^r$, for some $r \in (0,+\infty)$, $F(s,t) = s$ implies $s = 0$ or $t = 0$

(4) $F(s,t) = \log_a[(t + a^s)(1+t)]$, for some $a > 1$, $F(s,t) = s$ implies $t = 0$

(5) $F(s,t) = \phi(s)$, $F(s,t) = s$ implies $s = 0$, where ϕ: $[0,+\infty)\mapsto[0,+\infty)$ is an upper semicontinuous function such that $\phi(0) = 0$ and $\phi(t) > t$, for $t > 0$

Motivated by the above definition, we now define inverse C_k−class functions as follows.

Definition 13. A mapping F: $[0,+\infty)^2\mapsto\mathbb{R}$ is called an inverse C_k−class function if it is continuous and the following axioms hold:

(1) $F(s,t) \geq ks$ for all $s,t \in [0,+\infty)$ and some $k \geq 1$

(2) $F(s,t) = ks$ implies that $s = 0$ or $t = 0$

Denote the family of inverse C_k−class functions by $\mathscr{C}_{\text{inv}-k}$. Every inverse C−class function and inverse C_k−class function are equivalent when $k = 1$; however, an inverse C_k−class function may not be an inverse C−class function.

Example 6. A mapping F: $[0,+\infty)\mapsto\mathbb{R}$ is defined by $F(s,t) = 2s + t$ for all $s,t \in [0,+\infty)$. Then, clearly, F is an inverse C_k−class function for $k = 2$, but it is not an inverse C−class function.

Example 7. The following functions F: $[0,+\infty)^2\mapsto\mathbb{R}$ are elements of $\mathscr{C}_{\text{inv}-k}$, for all $s,t \in [0,+\infty)$:

(1) $F(s,t) = ks + lt, F(s,t) = ks$ implies $t = 0$ for some $k \geq 1$ and $l > 0$

(2) $F(s,t) = kms, F(s,t) = ks$ implies $s = 0$ for $m \in (1,+\infty)$ and some $k \geq 1$

(3) $F(s,t) = ks(1+kt)^r, F(s,t) = ks$ implies $s = 0$ or $t = 0$ for $r \in (0,+\infty)$ and some $k \geq 1$

(4) $F(s,t) = \log_a[(kt + a^{ks})(1+kt)], F(s,t) = ks$ implies $t = 0$ for some $a > 1$ and some $k \geq 1$

(5) $F(s,t) = k\phi(s)$, $F(s,t) = ks$ implies $s = 0$, where ϕ: $[0,+\infty)\mapsto[0,+\infty)$ is an upper semicontinuous function such that $\phi(0) = 0$ and $\phi(t) > t$, for $t > 0$ and some $k \geq 1$

Definition 14. Let Ψ denote the class of functions ψ: $[0,+\infty)\mapsto[0,+\infty)$ which satisfy the following conditions:

(a) ψ is continuous and increasing with $\psi(t) \geq t$

(b) $\psi(t) = 0 \Leftrightarrow t = 0$

Definition 15 (see [22]). A function ϕ: $[0,+\infty)\mapsto[0,+\infty)$ is said to be ultra-altering distance function if $(a)\phi$ is nondecreasing and continuous; $(b)\phi(t) > 0$, for all $t > 0$ and $\phi(t) = 0 \Longrightarrow t = 0$. Denote the class of ultra-altering distance functions by Φ.

Lemma 1. *Every sequence $\{x_n\}$ in metric space (X,d) will be Cauchy if there exists $\lambda \in [0,1)$ such that $d(x_{n+1},x_n) \leq \lambda\, d(x_n,x_{n-1})$, for all $n \in \mathbb{N}$.*

The aim of this presented paper is to provide some common fixed point theorems under several compatible conditions mentioned above via inverse C_k−class functions, which extend, generalize, and improve the existing results in the literature. Some examples are provided to illustrate the validity of our results.

2. Main Results

Theorem 1. *Let (X,d) be a complete metric space and a let pair (g,f) of self-maps be semicompatible, satisfying the following assumptions:*

$(A1)$ $f(X)\subseteq g(X)$

$(A2)$ $k\psi(d(fx,fy)) \geq F[\psi(M(x,y)),\phi(M(x,y))] + LN(x,y)$

Here, $M(x,y) = a_1 d(fx,gx) + a_2 d(fy,gy) + a_3 d(fx,gy) + a_4 d(fy,gx) + a_5 d(gx,gy)$ *and* $N(x,y) = \min\{d(fx,gx), d(fy,gx)\}$, *for all* $x, y \in X$. *Moreover,* $a_i > 0$, $(i = 1, 2, \ldots, 5)$, *with* $1 \geq a_1 - a_3$, $a_1 + a_2 + a_5 > 1$, $a_3 + a_4 > a_1 + a_2$, $L \in \mathbb{R}$, $F \in \mathscr{C}_{inv-k}$, *for some* $k \geq 1$ *and* $\psi \in \Psi$, $\phi \in \Phi$. *If the pair* (g, f) *is* $f-$*compatible of type* (E), *then* f *and* g *have a unique common fixed point* t *in* X.

Proof. Let x_0 be any point in X. Since $f(X) \subseteq g(X)$, there exists $x_1 \in X$ such that

$$fx_1 = gx_0 = y_0. \tag{5}$$

Continuing this way, we can construct a sequence $\{x_n\}$ in X satisfying

$$fx_{n+1} = gx_n = y_n. \tag{6}$$

By the assumption of $(A2)$, we have

$$\begin{aligned}
k\psi(d(fx_n, fx_{n+1})) &\geq F[\psi(M(x_n, x_{n+1})), \phi(M(x_n, x_{n+1})] + LN(x_n, x_{n+1}) \\
&\geq k\psi(M(x_n, x_{n+1})) + LN(x_n, x_{n+1}) \\
&= k\psi(a_1 d(fx_n, gx_n) + a_2 d(fx_{n+1}, gx_{n+1}) + a_3 d(fx_n, gx_{n+1}) + a_4 d(fx_{n+1}, gx_n) + a_5 d(gx_n, gx_{n+1})) \\
&\quad + L\min\{d(fx_n, gx_n), d(fx_{n+1}, gx_n)\} \\
&= k\psi(a_1 d(y_{n-1}, y_n) + a_2 d(y_n, y_{n+1}) + a_3 d(y_{n-1}, y_{n+1}) + a_4 d(y_n, y_n) + a_5 d(y_n, y_{n+1})) \\
&\quad + L\min\{d(y_{n-1}, y_n), d(y_n, y_n)\} \\
&= k\psi(a_1 d(y_{n-1}, y_n) + a_2 d(y_n, y_{n+1}) + a_3 d(y_{n-1}, y_{n+1}) + a_5 d(y_n, y_{n+1})).
\end{aligned} \tag{7}$$

By the monotonicity of ψ, we have

$$d(y_{n-1}, y_n) \geq a_1 d(y_{n-1}, y_n) + a_2 d(y_n, y_{n+1}) + a_3 d(y_{n-1}, y_{n+1}) + a_5 d(y_n, y_{n+1}). \tag{8}$$

Again, from the triangle inequality, that is, $d(y_n, y_{n+1}) \leq d(y_n, y_{n-1}) + d(y_{n-1}, y_{n+1})$, we have

$$\begin{aligned}
d(y_{n-1}, y_n) &\geq a_1 d(y_{n-1}, y_n) + a_2 d(y_n, y_{n+1}) + a_3 (d(y_n, y_{n+1}) - d(y_n, y_{n-1})) + a_5 d(y_n, y_{n+1}) \\
&= (a_1 - a_3) d(y_{n-1}, y_n) + (a_2 + a_3 + a_5) d(y_n, y_{n+1}),
\end{aligned} \tag{9}$$

which further yields that

$$d(y_n, y_{n+1}) \leq \frac{1 - a_1 + a_3}{a_2 + a_3 + a_5} d(y_{n-1}, y_n). \tag{10}$$

Since $1 \geq a_1 - a_3$, $a_1 + a_2 + a_5 > 1$, then $0 \leq (1 - a_1 + a_3)/(a_2 + a_3 + a_5) < 1$; from Lemma 1, it follows that sequence $\{y_n\}$ is a Cauchy sequence. Since X is complete, there exists a point $t \in X$ such that

$$\lim_{n \longrightarrow +\infty} y_n = \lim_{n \longrightarrow +\infty} fx_n = \lim_{n \longrightarrow +\infty} gx_n = t. \tag{11}$$

Now, we will show that t is a common fixed point of f and g.

Since the pair (g, f) is semicompatible, we have

$$\lim_{n \longrightarrow +\infty} gfx_n = ft. \tag{12}$$

Since f and g are $f-$compatible of type (E), it follows that

$$\lim_{n \longrightarrow +\infty} ffx_n = \lim_{n \longrightarrow +\infty} fgx_n = gt. \tag{13}$$

Now, by the definition of ψ and assumption $(A2)$, we obtain

$$\begin{aligned} k\psi(d(ffx_n, ft)) &\geq F[\psi(M(fx_n, t)), \phi(M(fx_n, t))] + LN(fx_n, t) \\ &= F[\psi(a_1 d(ffx_n, gfx_n) + a_2 d(ft, gt) + a_3 d(ffx_n, gt) + a_4 d(ft, gfx_n) + a_5 d(gfx_n, gt)), \\ &\quad \phi(a_1 d(ffx_n, gfx_n) + a_2 d(ft, gt) + a_3 d(ffx_n, gt) + a_4 d(ft, gfx_n) + a_5 d(gfx_n, gt))] \\ &\quad + L\min\{d(ffx_n, gfx_n), d(ft, gfx_n)\}. \end{aligned} \tag{14}$$

Taking the limit as $n \longrightarrow +\infty$ in above inequality, it follows that

$$\begin{aligned} k\psi(d(gt, ft)) &\geq F[\psi(a_1 d(gt, ft) + a_2 d(ft, gt) + a_3 d(gt, gt) + a_4 d(ft, ft) + a_5 d(ft, gt)), \\ &\quad \phi(a_1 d(gt, ft) + a_2 d(ft, gt) + a_3 d(gt, gt) + a_4 d(ft, ft) + a_5 d(ft, gt))] \\ &\quad + L\min\{d(gt, ft), d(ft, ft)\} \\ &= F[\psi(a_1 d(ft, gt) + a_2 d(ft, gt) + a_5 d(ft, gt)), \phi(a_1 d(ft, gt) + a_2 d(ft, gt) + a_5 d(ft, gt))] \\ &\geq k\psi((a_1 + a_2 + a_5) d(gt, ft)) \\ &\geq k\psi(d(gt, ft)), \end{aligned} \tag{15}$$

which implies that

$$\begin{aligned} d(ft, gt) &= 0 \\ \text{or } ft &= gt. \end{aligned} \tag{16}$$

Again, it follows from the definition of ψ and assumption $(A2)$ that

$$\begin{aligned} k\psi(d(fx_n, ft)) &\geq F[\psi(a_1 d(fx_n, gx_n) + a_2 d(ft, gt) + a_3 d(fx_n, gt) + a_4 d(ft, gx_n) + a_5 d(gx_n, gt)), \\ &\quad \phi(a_1 d(fx_n, gx_n) + a_2 d(ft, gt) + a_3 d(fx_n, gt) + a_4 d(ft, gx_n) + a_5 d(gx_n, gt))] \\ &\quad + L\min\{d(fx_n, gx_n), d(ft, gx_n)\}. \end{aligned} \tag{17}$$

Taking the limit as $n \longrightarrow +\infty$ in above inequality, we conclude that

$$\begin{aligned} k\psi(d(t, ft)) &\geq F[\psi(a_3 d(t, gt) + a_4 d(ft, t) + a_5 d(t, gt)), \phi(a_3 d(t, gt) + a_4 d(ft, t) + a_5 d(t, gt))] \\ &= F[\psi(a_3 d(t, ft) + a_4 d(ft, t) + a_5 d(t, ft)), \phi(a_3 d(t, ft) + a_4 d(ft, t) + a_5 d(t, ft))] \\ &\geq k\psi((a_3 + a_4 + a_5) d(t, ft)). \end{aligned} \tag{18}$$

Since $a_1 + a_2 < a_3 + a_4$ and $a_1 + a_2 + a_5 > 1$, it follows that

$$\begin{aligned} k\psi(d(t, ft)) &\geq F[\psi((a_3 + a_4 + a_5) d(t, ft)), \phi((a_3 + a_4 + a_5) d(t, ft))] \\ &\geq k\psi((a_3 + a_4 + a_5) d(t, ft)) \\ &\geq k\psi((a_1 + a_2 + a_5) d(t, ft)) \\ &\geq k\psi(d(t, ft)), \end{aligned} \tag{19}$$

which implies that

$$F[\psi((a_3+a_4+a_5)d(t,ft)),\phi((a_3+a_4+a_5)d(t,ft))]=k\psi(d(t,ft)). \tag{20}$$

From the definition of $\mathscr{C}_{\text{inv}-k}$−class functions, we have

$$\begin{aligned}&\psi((a_3+a_4+a_5)d(t,ft))=0\\ &\text{or}\ \phi((a_3+a_4+a_5)d(t,ft))=0.\end{aligned} \tag{21}$$

By the definitions of ψ and ϕ, we have that $d(ft,t)=0$, which proves that $ft=gt=t$, that is, t is a common fixed point of f and g.

Next, we will prove the uniqueness of common fixed point of f and g in X.

Suppose that u is another common fixed point of f and g, that is $fu=gu=u$.

From above argument, it may be concluded that

$$\begin{aligned}k\psi(d(fu,ft)) &\geq F[\psi(M(u,t)),\phi(M(u,t))]+LN(u,t)\\ &=F[\psi(a_1d(fu,gu)+a_2d(ft,gt)+a_3d(fu,gt)+a_4d(ft,gu)+a_5d(gu,gt)),\\ &\quad\phi(a_1d(fu,gu)+a_2d(ft,gt)+a_3d(fu,gt)+a_4d(ft,gu)+a_5d(gu,gt))]\\ &\quad+L\min\{d(fu,gu),d(ft,gu)\}.\\ &=F[\psi(a_3d(u,t)+a_4d(t,u)+a_5d(u,t)),\phi(a_3d(u,t)+a_4d(t,u)+a_5d(u,t))]\\ &\geq k\psi((a_3+a_4+a_5)d(u,t)).\end{aligned} \tag{22}$$

Since $a_1+a_2<a_3+a_4$ and $a_1+a_2+a_5>1$, it follows that

$$\begin{aligned}k\psi(d(u,t)) &\geq F[\psi((a_3+a_4+a_5)d(u,t)),\phi((a_3+a_4+a_5)d(u,t))]\\ &\geq k\psi((a_3+a_4+a_5)d(u,t))\\ &\geq k\psi((a_1+a_2+a_5)d(u,t))\\ &\geq k\psi(d(u,t)),\end{aligned} \tag{23}$$

which implies that

$$F[\psi((a_3+a_4+a_5)d(u,t)),\phi((a_3+a_4+a_5)d(u,t))]=k\psi(d(u,t)). \tag{24}$$

From the definition of $\mathscr{C}_{\text{inv}-k}$−class functions, we have

$$\begin{aligned}&\psi((a_3+a_4+a_5)d(u,t))=0\\ &\text{or}\ \phi((a_3+a_4+a_5)d(u,t))=0.\end{aligned} \tag{25}$$

By the definitions of ψ and ϕ, we have that $d(u,t)=0$, which proves that $u=t$. □

Remark 1. (i) The conclusion of Theorem 1 still holds under the assumptions of semi-compatibility of the pair (f,g) and g−compatibility of type (E).

(ii) If the inequality in assumption $(A2)$ is replaced by

$$k\psi(d(fx,gx))\geq F[\psi(M(x,y)),\phi(M(x,y))], \tag{26}$$

the conclusion still holds.

Here is an illustrated example to support the validity of Theorem 1 as follows.

Example 8. Let $X=[0,2]$ be a usual metric space. We define f and g on X as follows:

$$\begin{aligned}&fx=\frac{3}{2},\quad \forall x\in X,\\ &gx=\begin{cases}\dfrac{1}{2}, & x\in[0,1],\\[2ex] \dfrac{3}{2}, & x\in(1,2].\end{cases}\end{aligned} \tag{27}$$

To verify that the pair (g,f) is semicompatible as well as f−compatible of type (E), we take any sequence

$\{x_n\} \in (1,2]$; then, $\lim_{n\longrightarrow+\infty} fx_n = \lim_{n\longrightarrow+\infty} gx_n = 3/2$ and $\lim_{n\longrightarrow+\infty} fgx_n = 3/2 = g(3/2)$, $\lim_{n\longrightarrow+\infty} ffx_n = 3/2 = g(3/2)$, $\lim_{n\longrightarrow+\infty} gfx_n = 3/2 = g(3/2)$. Define $F \in \mathscr{C}_{\text{inv}-k}$ by $F(s,t) = 2s$, with $k = 1$ and $\psi(t) = t = \phi(t)$, for all $t \in [0,+\infty)$. Then, for all $x, y \in [0,1]$, we have

$$\begin{aligned}
&k\psi(d(fx,fy)) = 0,\\
&F[\psi(M(x,y)),\phi(M(x,y))] + LN(x,y) = 2\psi(M(x,y)) + LN(x,y)\\
&= 2[a_1 d(fx,gx) + a_2 d(fy,gy) + a_3 d(fx,gy) + a_4 d(fy,gx) + a_5 d(gx,gy)]\\
&\quad + L\min\{d(fx,gx), d(fy,gx)\}\\
&= 2\left[a_1\left|\frac{3}{2}-\frac{1}{2}\right| + a_2\left|\frac{3}{2}-\frac{1}{2}\right| + a_3\left|\frac{3}{2}-\frac{1}{2}\right| + a_4\left|\frac{3}{2}-\frac{1}{2}\right|\right]\\
&\quad + L\min\left\{\left|\frac{3}{2}-\frac{1}{2}\right|, \left|\frac{3}{2}-\frac{1}{2}\right|\right\}\\
&= 2[a_1 + a_2 + a_3 + a_4] + L.
\end{aligned} \tag{28}$$

It is verified that assumption $(A2)$ holds true provided by $a_1 = 2, a_2 = a_3 = 1, a_4 = 3$, and $a_5 > 0$ such that $1 \geq a_1 - a_3$, $a_1 + a_2 + a_5 > 1$, $a_3 + a_4 > a_1 + a_2$, and $L \leq -14$.

Also, it is obvious that assumption $(A2)$ holds true, for all $x, y \in (1,2]$.

Hence, functions f and g satisfy all conditions of Theorem 1 with 3/2 are common fixed point.

Taking f as the identity map in Theorem 1, we have the following fixed point theorem.

Corollary 1. *Let g be a self-map defined on a complete metric space (X,d) satisfying the following assumption:*

$$k\psi(d(x,y)) \geq F[\psi(M(x,y)),\phi(M(x,y))] + LN(x,y), \tag{29}$$

where $M(x,y) = a_1 d(x,gx) + a_2 d(y,gy) + a_3 d(x,gy) + a_4 d(y,gx) + a_5 d(gx,gy)$ and $N(x,y) = \min\{d(x,gx), d(y,gy)\}$, for all $x, y \in X$. Moreover, $a_i > 0$, $(i = 1,2,\ldots,5)$ with $1 \geq a_1 - a_3$, $a_1 + a_2 + a_5 > 1$, $a_1 + a_2 < a_3 + a_4$, $L \in \mathbb{R}$, $F \in \mathscr{C}_{inv-k}$ for some $k \geq 1$ and $\psi \in \Psi$, $\phi \in \Phi$. Then, g has a unique fixed point in X.

Since semicompatibility of type (A) implies semicompatibility, we can obtain the following theorem.

Theorem 2. *Let (X,d) be a complete metric space and let a pair (g,f) of self-maps be semicompatible of type (A) satisfying the following assumptions:*

$(A1)$ $f(X) \subseteq g(X)$

$(A2)$ $k\psi(d(fx,fy)) \geq F[\psi(M(x,y)),\phi(M(x,y))] + LN(x,y)$

where $M(x,y) = a_1 d(fx,gx) + a_2 d(fy,gy) + a_3 d(fx,gy) + a_4 d(fy,gx) + a_5 d(gx,gy)$ and $N(x,y) = \min\{d(fx,gx), d(fy,gx)\}$, for all $x, y \in X$. Moreover, $a_i > 0$, $(i = 1,2,\ldots,5)$ with $1 \geq a_1 - a_3$, $a_1 + a_2 + a_5 > 1$, $a_1 + a_2 < a_3 + a_4$, $L \in \mathbb{R}$, $F \in \mathscr{C}_{inv-k}$ for some $k \geq 1$ and $\psi \in \Psi$, $\phi \in \Phi$. If the pair (f,g) is either f−compatible of type (E) or g−compatible of type (E), then f and g have a unique common fixed point t in X.

Now, we will present a common fixed point theorem under weakly semicompatible condition via $\mathscr{C}_{inv-k}$-class functions as follows.

Theorem 3. *Let (X,d) be a complete metric space and let a pair (g,f) of self-maps be weakly semicompatible and R−weakly commuting type of A_f satisfying the following assumptions:*

$(A1)$ $f(X) \subseteq g(X)$

$(A2')$ $k\psi(d(fx,fy)) \geq F[\psi(M(x,y)),\phi(M(x,y))] + LN(x,y)$

Here, $M(x,y) = a_1 d(fx,gx) + a_2 d(fy,gy) + a_3 d(fx,gy) + a_4 d(fy,gx) + a_5 d(gx,gy)$ and $N(x,y) = \min\{d(fx,gx), d(fy,gx)\}$, for all $x, y \in X$. Moreover, $a_i > 0$, $(i = 1,2,\ldots,5)$, $L \in \mathbb{R}$, $F \in \mathscr{C}_{inv-k}$ for some $k \geq 1$ with $1 \geq a_1 - a_3$, $a_1 + a_2 + a_5 > 1$, $a_1 + a_2 < a_3 + a_4$, $k(a_1 + a_2 + a_3 + a_4) + L > 1$, and $\psi \in \Psi$, $\phi \in \Phi$. If the pair (f,g) is either f−compatible of type (E) or g−compatible of type (E), then f and g have a unique common fixed point t in X.

Proof. Let x_0 be any point in X. Since $f(X) \subseteq g(X)$, there exists $x_1 \in X$ such that

$$fx_1 = gx_0 = y_0. \tag{30}$$

Continuing this way, we can have a sequence $\{x_n\}$ in X satisfying

$$fx_{n+1} = gx_n = y_n. \tag{31}$$

From the discussion of Theorem 1, we have that there exists a point $t \in X$ such that

$$\lim_{n\longrightarrow+\infty} y_n = \lim_{n\longrightarrow+\infty} fx_n = \lim_{n\longrightarrow+\infty} gx_n = t. \tag{32}$$

Now consider the following possibilities:

Case 1. (f and g are f−compatible of type (E)).

Since f and g are weakly semicompatible, it follows that

$$\begin{aligned} &\lim_{n\longrightarrow+\infty} fgx_n = gt \\ \text{or } &\lim_{n\longrightarrow+\infty} gfx_n = ft. \end{aligned} \tag{33}$$

Firstly, we take

$$\lim_{n\longrightarrow+\infty} gfx_n = ft. \tag{34}$$

Since f and g are f−compatible of type (E), it yields that

$$\lim_{n\longrightarrow+\infty} ffx_n = \lim_{n\longrightarrow+\infty} fgx_n = gt. \tag{35}$$

Hence, the conclusion can directly follow from the proof of Theorem 1.

Secondly, we take

$$\lim_{n\longrightarrow+\infty} fgx_n = gt. \tag{36}$$

Since f and g are f−compatible of type (E), it yields that

$$\lim_{n\longrightarrow+\infty} ffx_n = \lim_{n\longrightarrow+\infty} fgx_n = gt. \tag{37}$$

Again since f and g are R−weakly commuting of type A_f, which implies

$$d(fgx_n, ggx_n) \le R\, d(fx_n, gx_n). \tag{38}$$

Taking limit as $n \longrightarrow +\infty$ in above inequality, we have

$$\lim_{n\longrightarrow+\infty} ggx_n = gt. \tag{39}$$

From the monotonicity of ψ and assumption $(A2')$, we have

$$\begin{aligned} k\psi(d(fgx_n, fx_n)) &\ge F[\psi(M(gx_n, x_n)), \phi(M(gx_n, x_n))] + LN(gx_n, x_n) \\ &= F[\psi(a_1 d(fgx_n, ggx_n) + a_2 d(fx_n, gx_n) + a_3 d(fgx_n, gx_n) + a_4 d(fx_n, ggx_n) + a_5 d(ggx_n, gx_n)), \\ &\quad \phi(a_1 d(fgx_n, ggx_n) + a_2 d(fx_n, gx_n) + a_3 d(fgx_n, gx_n) + a_4 d(fx_n, ggx_n) + a_5 d(ggx_n, gx_n))] \\ &\quad + L\min\{d(fgx_n, ggx_n), d(fx_n, ggx_n)\}. \end{aligned} \tag{40}$$

Taking limit as $n \longrightarrow +\infty$ in above inequality, we have

$$\begin{aligned} &k\psi(d(gt, t)) \\ &\ge F[\psi(a_1 d(gt, gt) + a_2 d(t, t) + a_3 d(gt, t) + a_4 d(t, gt) + a_5 d(gt, t)), \\ &\phi(a_1 d(gt, gt) + a_2 d(t, t) + a_3 d(gt, t) + a_4 d(t, gt) + a_5 d(gt, t))] \\ &\quad + L\min\{d(gt, gt), d(t, gt)\} \\ &= F[\psi((a_3 + a_4 + a_5) d(t, gt)), \phi((a_3 + a_4 + a_5) d(t, gt))] \\ &\ge k\psi((a_3 + a_4 + a_5) d(gt, t)). \end{aligned} \tag{41}$$

Since $a_1 + a_2 < a_3 + a_4$ and $a_1 + a_2 + a_5 > 1$, it follows that

$$\begin{aligned}
&k\psi(d(gt,t))\\
&\geq F[\psi((a_3+a_4+a_5)d(gt,t)),\phi((a_3+a_4+a_5)d(gt,t))]\\
&\geq k\psi((a_3+a_4+a_5)d(gt,t))\\
&\geq k\psi((a_1+a_2+a_5)d(gt,t))\\
&\geq k\psi(d(gt,t)),
\end{aligned} \tag{42}$$

which implies that

$$F[(a_3+a_4+a_5)d(gt,t)),\phi((a_3+a_4+a_5)d(gt,t))] = k\psi(d(gt,t)). \tag{43}$$

From the definition of $\mathscr{C}_{\text{inv}-k}$–class functions, we have

$$\begin{aligned}
&\psi((a_3+a_4+a_5)d(gt,t)) = 0\\
&\text{or } \phi((a_3+a_4+a_5)d(gt,t)) = 0.
\end{aligned} \tag{44}$$

By the definitions of ψ and ϕ, we have that $d(gt,t)=0$, which proves that $gt=t$.

Again, from the definition of ψ and assumption $(A2')$, we have

$$\begin{aligned}
0 &= k\psi(d(ft,ft))\\
&\geq F[a_1d(ft,gt)+a_2d(ft,gt)+a_3d(ft,gt)+a_4d(ft,gt)+a_5d(gt,gt)),\\
&\phi(a_1d(ft,gt)+a_2d(ft,gt)+a_3d(ft,gt)+a_4d(ft,gt)+a_5d(gt,gt))]\\
&\quad + L\min\{d(ft,gt),d(ft,gt)\}\\
&= F[\psi(a_1d(ft,gt)+a_2d(ft,gt)+a_3d(ft,gt)+a_4d(ft,gt)),\phi(a_1d(ft,gt)+a_2d(ft,gt)+a_3d(ft,gt)+a_4d(ft,gt))]\\
&\quad + L\,d(ft,t) \geq k\psi((a_1+a_2+a_3+a_4)d(ft,t)) + L\,d(ft,t)\\
&\geq (k(a_1+a_2+a_3+a_4)+L)\,d(ft,t).
\end{aligned} \tag{45}$$

Together with $a_i>0, i=\{1,2,\ldots,5\}$, $k(a_1+a_2+a_3+a_4)+L>1$, from the above inequality, it follows that $d(ft,t)=0$, which proves that $ft=t=gt$. Hence, t is a common fixed point of f and g.

Case 2 (f and g are g–compatible of type (E)).

Since f and g are weakly compatible, it follows that

$$\begin{aligned}
&\lim_{n\longrightarrow+\infty} fgx_n = gt\\
&\text{or } \lim_{n\longrightarrow+\infty} gfx_n = ft.
\end{aligned} \tag{46}$$

Firstly, we take

$$\lim_{n\longrightarrow+\infty} fgx_n = gt. \tag{47}$$

Since f and g are g–compatible of type (E), it yields that

$$\lim_{n\longrightarrow+\infty} ggx_n = \lim_{n\longrightarrow+\infty} gfx_n = ft. \tag{48}$$

Therefore, the conclusion directly follows from Theorem 1 and (i) of Remark 1.

Secondly, we take

$$\lim_{n\longrightarrow+\infty} gfx_n = ft. \tag{49}$$

Since f and g are g–compatible of type (E), it yields that

$$\lim_{n\longrightarrow+\infty} ggx_n = \lim_{n\longrightarrow+\infty} gfx_n = ft. \tag{50}$$

Also f and g are R–weakly commuting of type A_f, which implies

$$d(fgx_n, ggx_n) \leq R\,d(fx_n, gx_n). \tag{51}$$

Taking limit as $n\longrightarrow+\infty$ in the above inequality, we have

$$\lim_{n\longrightarrow+\infty} fgx_n = ft. \tag{52}$$

From the monotonicity of ψ and assumption $(A2')$, we have

$$\begin{aligned} k\psi(d(fx_n, fgx_n)) &\geq F[\psi(M(x_n, gx_n)), \phi(M(x_n, gx_n))] + LN(x_n, gx_n) \\ &= F[\psi(a_1 d(fx_n, gx_n) + a_2 d(fgx_n, ggx_n) + a_3 d(fx_n, ggx_n) + a_4 d(fgx_n, gx_n) + a_5 d(gx_n, ggx_n)) \\ &\phi(a_1 d(fx_n, gx_n) + a_2 d(fgx_n, ggx_n) + a_3 d(fx_n, ggx_n) + a_4 d(fgx_n, gx_n) + a_5 (gx_n, ggx_n))] \\ &\quad + L\min\{d(fx_n, gx_n), d(fgx_n, gx_n)\}. \end{aligned} \tag{53}$$

Taking limit as $n \longrightarrow +\infty$ in the above inequality, we have

$$\begin{aligned} k\psi(d(t, ft)) &\geq F[\psi(a_1 d(t,t) + a_2 d(ft, ft) + a_3 d(t, ft) + a_4 d(ft, t) + a_5 d(t, ft)), \\ &\phi(a_1 d(t,t) + a_2 d(ft, ft) + a_3 d(t, ft) + +a_4 d(ft, t) + a_5 d(t, ft))] \\ &\quad + L\min\{d(t,t), d(ft, t)\} \\ &= F[\psi((a_3 + a_4 + a_5) d(t, ft)), \phi((a_3 + a_4 + a_5) d(t, ft))] \\ &\geq k\psi((a_3 + a_4 + a_5) d(t, ft)). \end{aligned} \tag{54}$$

Since $a_1 + a_2 < a_3 + a_4$ and $a_1 + a_2 + a_5 > 1$, it follows that

$$\begin{aligned} k\psi(d(t, ft)) &\geq F[\psi((a_3 + a_4 + a_5) d(t, ft)), \phi((a_3 + a_4 + a_5) d(t, ft))] \\ &\geq k\psi((a_3 + a_4 + a_5) d(t, ft)) \\ &\geq k\psi((a_1 + a_2 + a_5) d(t, ft)) \\ &\geq k\psi(d(t, ft)), \end{aligned} \tag{55}$$

which implies that

$$F[\psi((a_3 + a_4 + a_5) d(t, ft)), \phi((a_3 + a_4 + a_5) d(t, ft))] = k\psi(d(t, ft)). \tag{56}$$

From the definition of $\mathscr{C}_{\text{inv}-k}$–class functions, we have

$$\begin{aligned} &\psi((a_3 + a_4 + a_5) d(t, ft)) = 0 \\ &\text{or } \phi((a_3 + a_4 + a_5) d(t, ft)) = 0. \end{aligned} \tag{57}$$

By the definitions of ψ and ϕ, we have that $d(t, ft) = 0$, which proves that $ft = t$.

Again, from the monotonicity of ψ and assumption $(A2')$, we have

$$\begin{aligned} k\psi(d(fgx_n, ft)) &\geq F[\psi(M(gx_n, t)), \phi(M(gx_n, t))] + LN(gx_n, t) \\ &= F[\psi(a_1 d(fgx_n, ggx_n) + a_2 d(ft, gt) + a_3 d(fgx_n, gt) + a_4 d(ft, ggx_n) + a_5 d(ggx_n, gt)), \\ &\phi(a_1 d(fgx_n, ggx_n) + a_2 d(ft, gt) + a_3 d(fgx_n, gt) + a_4 d(ft, ggx_n) + a_5 d(ggx_n, gt))] \\ &\quad + L\min\{d(fgx_n, ggx_n), d(ft, ggx_n)\}. \end{aligned} \tag{58}$$

Taking limit as $n \longrightarrow +\infty$ in above inequality and by the definition of ψ, we have

$$\begin{aligned}
0 &= k\psi(d(ft, ft)) \\
&\geq F[\psi(a_1 d(ft, ft) + a_2 d(ft, gt) + a_3 d(ft, gt) + a_4 d(ft, ft) + a_5 d(ft, gt)), \\
&\phi(a_1 d(ft, ft) + a_2 d(ft, gt) + a_3 d(ft, gt) + a_4 d(ft, ft) + a_5 d(ft, gt))] \\
&+ L\min\{d(ft, ft), d(ft, ft)\} \\
&= F[\psi((a_2 + a_3 + a_5)d(ft, gt)), \phi((a_2 + a_3 + a_5)d(ft, gt))] \\
&\geq k\psi((a_2 + a_3 + a_5)d(ft, gt)).
\end{aligned} \tag{59}$$

Since $a_i > 0, i = \{1, 2, \ldots, 5\}$, it yields that $d(ft, gt) = 0$ or $ft = gt$, which further implies $ft = gt = t$; hence, t is a common fixed point of f and g.

Finally, the uniqueness of the common fixed point of f and g can be directly obtained from the second half proof of Theorem 1. □

Remark 2. If f and g are assumed to be R−weakly commuting of type A_g, the conclusion of Theorem 3 still holds true.

Now, we provide an example to verify the validity of Theorem 3 as follows.

Example 9. Let $X = [1, 2]$ be a usual metric space and $x, y \in X$. We define f and g on X as follows: $f(x) = 1$ for all $x \in X$ and $g(x) = x$ for all $x \in X$. It is easy to check that $f(X) \subseteq g(X)$.

To show that the pair of mappings (f, g) is weak semicompatible and f−compatible of type (E), we take any sequence $\{x_n\} \in (1, 2)$, then $\lim_{n\longrightarrow+\infty} fx_n = \lim_{n\longrightarrow+\infty} gx_n = 1$. Hence, $\lim_{n\longrightarrow+\infty} fgx_n = 1 = g(1)$ and $\lim_{n\longrightarrow+\infty} ffx_n = 1 = g(1)$. It also can be observed that pair (f, g) satisfy R−weak commuting of type A_f for $R = 1$. To satisfy condition $(A2')$, we choose $F(s, t) = 2s$ with $k = 1$ and $\psi(t) = t/20$, for all $t \geq 0$. Then, for all $x, y \in [1, 2]$, we have

$$\begin{aligned}
&k\psi(d(fx, fy)) = 0, \\
&F[\psi(M(x, y)), \phi(M(x, y))] + LN(x, y) \\
&= \frac{2}{20}[a_1 d(fx, gx) + a_2 d(fy, gy) + a_3 d(fx, gy) + a_4 d(fy, gx) + a_5 d(gx, gy)] \\
&\quad + L\min\{d(fx, gx), d(fy, gx)\} \\
&= \frac{1}{10}[a_1|1 - x| + a_2|1 - y| + a_3|1 - y| + a_4|1 - x| + a_5|x - y|] + L\min\{|1 - x|, |1 - x|\} \\
&= \frac{1}{10}[a_1(x - 1) + a_2(y - 1) + a_3(y - 1) + a_4(x - 1) + a_5|x - y|] + L(x - 1).
\end{aligned} \tag{60}$$

Here are three possible cases as follows:

Case (1): if $x > y$, taking $a_1 = 2, a_2 = 2, a_3 = 3, a_4 = 3, a_5 = 5$, and $-9 < L \leq -1$ satisfying $a_1 - a_3 \leq 1, a_1 + a_2 < a_3 + a_4, a_1 + a_2 + a_5 > 1$, and $k(a_1 + a_2 + a_3 + a_4) + L > 1$, then we have

$$\begin{aligned}
0 &\geq F[\psi(M(x, y)), \phi(M(x, y))] + LN(x, y) \\
&= \frac{1}{10}[a_1(x - 1) + a_2(y - 1) + a_3(y - 1) + a_4(x - 1) + a_5(x - y)] + L(x - 1) \\
&= (x - 1) + L(x - 1),
\end{aligned} \tag{61}$$

which proves that condition $(A2')$ holds true.

Case (2): if $x < y$, taking $a_1 = 2, a_2 = 2, a_3 = 3, a_4 = 3, a_5 = 5$, and $-9 < L \leq -1$ satisfying $a_1 - a_3 \leq 1$, $a_1 + a_2 < a_3 + a_4, a_1 + a_2 + a_5 > 1$, and $k(a_1 + a_2 + a_3 + a_4) + L > 1$, then we have

$$\begin{aligned} 0 &\geq F[\psi(M(x,y)), \phi(M(x,y))] + LN(x,y) \\ &= \frac{1}{10}\left[a_1(x-1) + a_2(y-1) + a_3(y-1) + a_4(x-1) + a_5(y-x)\right] + L(x-1) \\ &= (y-1) + L(x-1), \end{aligned} \tag{62}$$

which shows that condition $(A2')$ holds true.

Case (3): if $x = y$, taking $a_1 = 2, a_2 = 2, a_3 = 3, a_4 = 3, a_5 = 5$, and $-9 < L \leq -1$ satisfying $a_1 - a_3 \leq 1$, $a_1 + a_2 < a_3 + a_4, a_1 + a_2 + a_5 > 1$, and $k(a_1 + a_2 + a_3 + a_4) + L > 1$, then we have

$$\begin{aligned} 0 &\geq F[\psi(M(x,y)), \phi(M(x,y))] + LN(x,y) \\ &= \frac{1}{10}(a_1 + a_2 + a_3 + a_4)(x-1) + L(x-1) \\ &= (x-1) + L(x-1), \end{aligned} \tag{63}$$

which shows that condition $(A2')$ holds true.

Hence, the pair (f, g) satisfies all conditions of Theorems 3, with 1 being the unique common fixed point.

Next, we will present some common fixed point theorems under conditional semicompatibility as follows.

Theorem 4. *Let (X, d) be a complete metric space and a pair (g, f) of self-maps be conditional semicompatible satisfying the following assumptions:*

$(A1)$ $f(X) \subseteq g(X)$

$(A2')$ $k\psi(d(fx, fy)) \geq F[\psi(M(x,y)), \phi(M(x,y))] + LN(x,y)$

Here, $M(x,y) = a_1 d(fx, gx) + a_2 d(fy, gy) + a_3 d(fx, gy) + a_4 d(fy, gx) + a_5 d(gx, gy)$ and $N(x,y) = \min\{d(fx, gx), d(fy, gx)\}$, for all $x, y \in X$. Moreover, $a_i > 0$, $(i = 1, 2, \ldots, 5)$, $L \in \mathbb{R}$, $F \in \mathscr{C}_{inv-k}$ for some $k \geq 1$ with $1 \geq a_1 - a_3$, $a_1 + a_2 + a_5 > 1$, $a_1 + a_2 < a_3 + a_4$, $k(a_1 + a_2 + a_3 + a_4) + L > 1$, and $\psi \in \Psi$, $\phi \in \Phi$. If f is g–absorbing or g is f–absorbing, then f and g have a unique common fixed point t in X.

Proof. Similar to the first part of the proof in Theorems 1 or 3, we can construct a sequence $\{x_n\}$ in X such that

$$\lim_{n \longrightarrow +\infty} fx_n = \lim_{n \longrightarrow +\infty} gx_n = t, \quad t \in X. \tag{64}$$

Again since f and g are conditional semicompatible and $\lim_{n \longrightarrow +\infty} fx_n = \lim_{n \longrightarrow +\infty} gx_n = t$ (nonempty), then there exists a sequence $\{y_n\}$ satisfying $\lim_{n \longrightarrow +\infty} fy_n = \lim_{n \longrightarrow +\infty} gy_n = u$ such that $\lim_{n \longrightarrow +\infty} fgy_n = gu$ and $\lim_{n \longrightarrow +\infty} gfy_n = fu$.

Case 1: f is g–absorbing.

Since f is g–absorbing, this yields that

$$d(gy_n, gfy_n) \leq R\, d(fy_n, gy_n). \tag{65}$$

Taking limit as $n \longrightarrow +\infty$ in the above inequality, it follows that

$$\begin{aligned} \lim_{n \longrightarrow +\infty} gy_n = \lim_{n \longrightarrow +\infty} gfy_n = u, \\ fu = u. \end{aligned} \tag{66}$$

Since $f(X) \subseteq g(X)$, then there exists v in X such that $fu = gv$. Next, we will show that $fv = u$.

By the definition of ψ and assumption $(A2')$, we have

$$\begin{aligned} 0 &= k\psi(d(fv, fv)) \\ &\geq F[\psi(M(v,v)), \phi(M(v,v))] + LN(v,v) \\ &= F\big[\psi\big(a_1 d(fv, gv) + a_2 d(fv, gv) + a_3 d(fv, gv) + a_4 d(fv, gv) + a_5 d(gv, gv)\big), \\ &\quad \phi\big(a_1 d(fv, gv) + a_2 d(fv, gv) + a_3 d(fv, gv) + a_4 d(fv, gv) + a_5 d(gv, gv)\big)\big] \\ &\quad + L \min\{d(fv, gv), d(fv, gv)\} \\ &= F\big[\psi\big(a_1 d(fv, gv) + a_2 d(fv, gv) + a_3 d(fv, gv) + a_4 d(fv, gv)\big), \\ &\quad \phi\big(a_1 d(fv, gv) + a_2 d(fv, gv) + a_3 d(fv, gv) + a_4 d(fv, gv)\big)\big] + L\, d(fv, gv) \\ &\geq k\psi\big((a_1 + a_2 + a_3 + a_4) d(fv, u)\big) + L\, d(fv, u) \\ &\geq \big(k(a_1 + a_2 + a_3 + a_4) + L\big) d(fv, u). \end{aligned} \tag{67}$$

Together with $a_i > 0, i = \{1, 2, \ldots, 5\}$, $k(a_1 + a_2 + a_3 + a_4) + L > 1$, from above inequality, it follows that $d(fv, u) = 0$ or $fv = u = gv$.

Again, since f is g−absorbing, it yields

$$d(gv, gfv) \leq R\, d(fv, gv), \tag{68}$$

which implies that $gv = gfv$ or $ggv = gv$. Now, we will show $fgv = gv$. By the definition of ψ, $s > 1$ and assumption $(A2')$, we have

$$\begin{aligned}
0 &= k\psi(d(fgv, fgv)) \\
&\geq F[\psi(M(gv, gv)), \phi(M(gv, gv))] + LN(gv, gv) \\
&= F[\psi(a_1 d(fgv, ggv) + a_2 d(fgv, ggv) + a_3 d(fgv, ggv) + a_4 d(fgv, ggv) + a_5 d(ggv, ggv)), \\
&\phi(a_1 d(fgv, ggv) + a_2 d(fgv, ggv) + a_3 d(fgv, ggv) + a_4 d(fgv, ggv) + a_5 d(ggv, ggv))] \\
&\quad + L \min\{d(fgv, ggv), d(fgv, ggv)\} \\
&= F[\psi(a_1 d(fgv, gv) + a_2 d(fgv, gv) + a_3 d(fgv, gv) + a_4 d(fgv, gv)), \\
&\phi(a_1 d(fv, gv) + a_2 d(fgv, gv) + a_3 d(fgv, gv) + a_4 d(fgv, gv))] + L\, d(fgv, ggv) \\
&\geq k\psi((a_1 + a_2 + a_3 + a_4) d(fgv, gv)) + L\, d(fgv, gv) \\
&\geq (k(a_1 + a_2 + a_3 + a_4) + L) d(fgv, gv).
\end{aligned} \tag{69}$$

Together with $a_i > 0, i = \{1, 2, \ldots, 5\}$, $k(a_1 + a_2 + a_3 + a_4) + L > 1$, from the above inequality, it follows that $d(fgv, gv) = 0$ or $fgv = gv$.

Therefore, gv is a common fixed point of f and g.

Case 2: g is f−absorbing.

Since g is f−absorbing, this yields

$$d(fy_n, fgy_n) \leq R\, d(fy_n, gy_n). \tag{70}$$

Taking limit as $n \longrightarrow +\infty$ in the above inequality, it follows that

$$\begin{gathered}
\lim_{n \longrightarrow +\infty} gy_n = \lim_{n \longrightarrow +\infty} fgy_n = u, \\
gu = u.
\end{gathered} \tag{71}$$

Next, we will show $fu = gu$.

By the definition of ψ and assumption $(A2')$, we have

$$\begin{aligned}
0 &= k\psi(d(fu, fu)) \\
&\geq F[\psi(M(u, u)), \phi(M(u, u))] + LN(u, u) \\
&= F[\psi(a_1 d(fu, gu) + a_2 d(fu, gu) + a_3 d(fu, gu) + a_4 d(fu, gu) + a_5 d(gu, gu)), \\
&\phi(a_1 d(fu, gu) + a_2 d(fu, gu) + a_3 d(fu, gu) + a_4 d(fu, gu) + a_5 d(gu, gu))] \\
&\quad + L \min\{d(fu, gu), d(fu, gu)\} \\
&= F[a_1 d(fu, gu) + a_2 d(fu, gu) + a_3 d(fu, gu) + a_4 d(fu, gu),) \\
&\phi(a_1 d(fu, gu) + a_2 d(fu, gu) + a_3 d(fu, gu) + a_4 d(fu, gu))] + L\, d(fu, gu) \\
&\geq k\psi(a_1 d(fu, gu) + a_2 d(fu, gu) + a_3 d(fu, gu) + a_4 d(fu, gu)) + L\, d(fu, gu) \\
&\geq (k(a_1 + a_2 + a_3 + a_4) + L) d(fu, gu).
\end{aligned} \tag{72}$$

Together with $a_i > 0, i = \{1, 2, \ldots, 5\}$, $k(a_1 + a_2 + a_3 + a_4) + L > 1$, from the above inequality, it follows that $d(fu, gu) = 0$ or $fu = gu$. Therefore, u is a common fixed point of f and g.

The uniqueness of common fixed point of f and g can be directly obtained by following the same argument as Theorem 1. □

Remark 3. The results of Theorem 4 still hold by replacing condition $(A2')$ with condition $(A2'')$ stated as follows.

$$(A2''): k\theta(d(fx, fy)) \geq F[\theta(M(x, y)), \phi(M(x, y))].$$

Here, $M(x, y) = a_1 d(fx, gx) + a_2 d(fy, gy) + a_3 d(fx, gy) + a_4 d(fy, gx) + a_5 d(gx, gy)$, for all $x, y \in X$. Moreover, $a_i > 0$, $(i = 1, 2, \ldots, 5)$ with $1 \geq a_1 - a_3$, $a_1 + a_2 + a_5 > 1$, $a_1 + a_2 < a_3 + a_4$, $F \in \mathscr{C}_{\text{inv}-k}$ for some $k \geq 1$ and $\theta \in \Theta$, $\phi \in \Phi$.

Moreover, θ is said to be an altering distance function (see [19]), which satisfies $(i)\theta$: $[0, +\infty) \mapsto [0, +\infty)$ is continuous and increasing; $(ii)\theta(t) = 0 \Leftrightarrow t = 0$. Denote the class of altering distance functions by Θ.

Now, we provide an example to verify the validity of Theorem 4 as follows.

Example 10. Let $X = [2, +\infty)$, $x, y \in X$ with $x > y$ and let d be the usual metric on X. Define f, g: $X \longrightarrow X$ as follows:

$$fx = \begin{cases} \dfrac{x+8}{2}, & 2 \le x < 5, \\ 5, & x \ge 5, \end{cases} \qquad gx = \begin{cases} x+3, & 2 \le x < 5, \\ 5, & x \ge 5. \end{cases} \tag{73}$$

It is clear that $f(X) \subseteq g(X)$.

Taking $x_n = 2 + \varepsilon_n$, where $\varepsilon_n \longrightarrow 0$ as $n \longrightarrow +\infty$, we have $\lim_{n\longrightarrow+\infty} fx_n = 5$ and $\lim_{n\longrightarrow+\infty} gx_n = 5$. Therefore, $\lim_{n\longrightarrow+\infty} fx_n = \lim_{n\longrightarrow+\infty} gx_n = 5$(nonempty). Then, we have a sequence $y_n = 5 + \varepsilon_n$, where $\varepsilon_n \longrightarrow 0$ as $n \longrightarrow +\infty$, for which $\lim_{n\longrightarrow+\infty} fy_n = \lim_{n\longrightarrow+\infty} f(5+\varepsilon_n) = 5$ and $\lim_{n\longrightarrow+\infty} gy_n = \lim_{n\longrightarrow+\infty} g(5+\varepsilon_n) = 5$. Moreover, $\lim_{n\longrightarrow+\infty} fgy_n = \lim_{n\longrightarrow+\infty} fg(5+\varepsilon_n) = \lim_{n\longrightarrow+\infty} f(5) = 5 = g(5)$ and $\lim_{n\longrightarrow+\infty} gfy_n = \lim_{n\longrightarrow+\infty} gf(5+\varepsilon_n) = \lim_{n\longrightarrow+\infty} g(5) = 5 = f(5)$. Therefore, the pair (f, g) is conditional semicompatible.

For $2 \le x < 5$, $gfx = g((x+8)/2) = (x+8)/2$, then $d(gx, gfx) = |(x+3) - (x+8)/2| = (x-2)/2$ and $d(gx, fx) = |(x+3) - (x+8)/2| = (x-2)/2$. Therefore, f and g satisfy $d(gx, gfx) \le Rd(fx, gx)$ with $R = 1$. Also, for $x \ge 5$, then $0 = d(gx, gfx) = Rd(gx, fx)$ for all real number R. Therefore, f and g satisfy $d(gx, gfx) \le Rd(fx, gx)$, for all $x \in X$ and $R = 1$; that is, f is g–absorbing with $R = 1$.

To satisfy condition $(A2')$, we choose $F(s,t) = s + t$ with $k = 1$ and $\psi(t) = \phi(t) = t/40$, for all $t \ge 0$. Then, for all $x, y \in [2, +\infty)$ we have

$$k\psi(d(fx, fy)) = \begin{cases} \dfrac{|x-2|}{2}, & 2 \le y < x < 5, \\ 0, & 5 \le y < x < +\infty, \\ \dfrac{|y-2|}{2}, & 2 \le y < 5 \le x < +\infty. \end{cases} \tag{74}$$

Here are three possible cases:

Case (1): if $2 \le y < x < 5$, we have

$$\begin{aligned} &F[\psi(M(x,y)), \phi(M(x,y))] + LN(x,y) \\ &= \frac{1}{20}\left[a_1 d(fx, gx) + a_2 d(fy, gy) + a_3 d(fx, gy) + a_4 d(fy, gx) + a_5 d(gx, gy)\right] \\ &\quad + L\min\{d(fx, gx), d(fy, gx)\} \\ &= \frac{1}{20}\left[a_1\frac{|x-2|}{2} + a_2\frac{|y-2|}{2} + a_3\frac{|x-2y+2|}{2} + a_4\frac{|y-2x+2|}{2} + a_5\frac{|y-2|}{2}\right] \\ &\quad L\min\left\{\frac{|x-2|}{2}, \frac{|y-2x+2|}{2}\right\} \\ &\le \frac{1}{20}\left[a_1\frac{|x-2|}{2} + a_2\frac{|y-2|}{2} + a_3\left(\frac{|x-y|}{2} + \frac{|y-2|}{2}\right) + a_4\left(\frac{|x-y|}{2} + \frac{|x-2|}{2}\right) + a_5\frac{|y-2|}{2}\right] L\frac{|x-2|}{2} \\ &= \left(\frac{a_1+a_4}{20} + L\right)\frac{|x-2|}{2} + \left(\frac{a_2+a_3+a_5}{20}\right)\frac{|y-2|}{2} + \left(\frac{a_3+a_4}{20}\right)\frac{|x-y|}{2} \\ &\le \left(\frac{a_1+a_3+2a_4}{20} + L\right)\frac{|x-2|}{2} + \left(\frac{a_2+2a_3+a_4+a_5}{20}\right)\frac{|y-2|}{2} \\ &\le \left(\frac{a_1+a_2+3a_3+3a_4+a_5}{20} + L\right)\frac{|x-2|}{2}. \end{aligned} \tag{75}$$

It is obvious that $k\psi(d(fx, fy)) \ge F[\psi(M(x,y)), \phi(M(x,y))] + LN(x,y)$ holds true by taking $a_1 = 1, a_2 = a_3 = a_4 = 2, a_5 = 5$, and $-6 < L \le -1$ satisfying $a_1 - a_3 \le 1$, $a_1 + a_2 < a_3 + a_4$, $a_1 + a_2 + a_5 > 1$, and $k(a_1 + a_2 + a_3 + a_4) + L > 1$.

Case (2): if $5 \le y < x < +\infty$, we have

$$F[\psi(M(x,y)), \phi(M(x,y))] + LN(x,y) = 0. \tag{76}$$

It is obvious that $k\psi(d(fx, fy)) \ge F[\psi(M(x,y)), \phi(M(x,y))] + LN(x,y)$ holds true by taking $a_i > 0$, $(i = 1, 2, \ldots, 5)$ and $L \in \mathbb{R}$.

Case (3): if $2 \le y < 5 \le x < +\infty$, we have

$$\begin{aligned}
&F[\psi(M(x,y)),\phi(M(x,y))]+LN(x,y)\\
&=\frac{1}{20}\left[a_1d(fx,gx)+a_2d(fy,gy)+a_3d(fx,gy)+a_4d(fy,gx)+a_5d(gx,gy)\right]\\
&\quad+L\min\{d(fx,gx),d(fy,gx)\}\\
&=\frac{1}{20}\left[a_2\frac{|y-2|}{2}+a_3|y-2|+a_4\frac{|y-2|}{2}+a_5\frac{|y-2|}{2}\right]\\
&=\frac{1}{20}\left(\frac{a_2}{2}+a_3+\frac{a_4}{2}+\frac{a_5}{2}\right)|y-2|.
\end{aligned}\tag{77}$$

It is obvious that $k\psi(d(fx,fy))\geq F[\psi(M(x,y)),\phi(M(x,y))]+LN(x,y)$ holds true by taking $a_1=1, a_2=a_3=a_4=2, a_5=5$, and $L\in\mathbb{R}$ satisfying $a_1-a_3\leq 1$, $a_1+a_2<a_3+a_4$, $a_1+a_2+a_5>1$, and $k(a_1+a_2+a_3+a_4)+L>1$.

To sum up, condition $(A2')$ holds true by taking $a_1=1, a_2=a_3=a_4=2, a_5=5$, and $-6<L\leq -1$ satisfying $a_1-a_3\leq 1$, $a_1+a_2<a_3+a_4$, $a_1+a_2+a_5>1$, and $k(a_1+a_2+a_3+a_4)+L>1$.

Therefore, f and g satisfy all the conditions of Theorems 4 with 5 being the unique common fixed point.

Theorem 5. *Let (X,d) be a complete metric space and let a pair (g,f) of self-maps be conditional semicompatible satisfying the following assumptions:*

$(A1)$ $f(X)\subseteq g(X)$

$(A2')$ $k\psi(d(fx,fy))\geq F[\psi(M(x,y)),\phi(M(x,y))]+LN(x,y)$

Here, $M(x,y)=a_1d(fx,gx)+a_2d(fy,gy)+a_3d(fx,gy)+a_4d(fy,gx)+a_5d(gx,gy)$ and $N(x,y)=\min\{d(fx,gx),\ d(fy,gx)\}$, for all $x,y\in X$. Moreover, $a_i>0$, $(i=1,2,\ldots,5)$, $L\in\mathbb{R}$, $F\in\mathscr{C}_{inv-k}$ for some $k\geq 1$ with $1\geq a_1-a_3$, $a_1+a_2+a_5>1$, $a_1+a_2<a_3+a_4$, $k(a_1+a_2+a_3+a_4)+L>1$, and $\psi\in\Psi$, $\phi\in\Phi$. If pair (f,g) is R–weak commuting either of type A_f or of type A_g, then f and g have a unique common fixed point t in X.

Proof. From the part of the proof in Theorems 1 or 3, we can construct a sequence $\{x_n\}$ in X such that $\lim_{n\longrightarrow+\infty}fx_n=\lim_{n\longrightarrow+\infty}gx_n=t$.

Again since f and g are conditional semicompatible and $\lim_{n\longrightarrow+\infty}fx_n=\lim_{n\longrightarrow+\infty}gx_n=t$(nonempty), then there exists a sequence $\{y_n\}$ satisfying $\lim_{n\longrightarrow+\infty}fy_n=\lim_{n\longrightarrow+\infty}gy_n=u$ such that $\lim_{n\longrightarrow+\infty}fgy_n=gu$ and $\lim_{n\longrightarrow+\infty}gfy_n=fu$.

Case 1: pair (f,g) is R–weak commuting of type A_f. Since pair (f,g) is R–weak commuting of type A_f, it yields

$$d(fgy_n,ggy_n)\leq R\,d(fy_n,gy_n).\tag{78}$$

Taking limit as $n\longrightarrow+\infty$ in the above inequality, it follows that

$$\lim_{n\longrightarrow+\infty}fgy_n=\lim_{n\longrightarrow+\infty}ggy_n=gu.\tag{79}$$

Now, we will show that $fu=gu$. By the definition of ψ and assumption $(A2')$, we have

$$\begin{aligned}
0&=k\psi(d(fu,fu))\\
&\geq F[\psi(M(u,u)),\phi(M(u,u))]+LN(u,u)\\
&=F\big[\psi\big(a_1d(fu,gu)+a_2d(fu,gu)+a_3d(fu,gu)+a_4d(fu,gu)+a_5d(gu,gu)\big),\\
&\phi\big(a_1d(fu,gu)+a_2d(fu,gu)+a_3d(fu,gu)+a_4d(fu,gu)+a_5d(gu,gu)\big)\big]\\
&\quad+L\min\{d(fu,gu),d(fu,du)\}\\
&=F\big[\psi\big(a_1d(fu,gu)+a_2d(fu,gu)+a_3d(fu,gu)+a_4d(fu,gu)\big),\\
&\phi\big(a_1d(fu,gu)+a_2d(fu,gu)+a_3d(fu,gu)+a_4d(fu,gu)\big)\big]\big]+L\,d(fu,gu)\\
&\geq k\psi\big(a_1d(fu,gu)+a_2d(fu,gu)+a_3d(fu,gu)+a_4d(fu,gu)\big)+L\,d(fu,gu)\\
&\geq\big(k(a_1+a_2+a_3+a_4)+L\big)d(fu,gu).
\end{aligned}\tag{80}$$

Together with $a_i > 0, i = \{1, 2, \ldots, 5\}$, $k(a_1 + a_2 + a_3 + a_4) + L > 1$, from the above inequality, it follows that $d(fu, gu) = 0$ or $fu = gu$.

Again, since pair (f, g) is R–weak commuting of type A_f, then we have

$$d(fgu, ggu) \leq R\, d(fu, gu), \tag{81}$$

which implies $fgu = ggu$.

Now, we will show $fgu = gu$.

By the definition of ψ and assumption $(A2')$, we have

$$\begin{aligned}
&k\psi(d(fgu, fu)) \\
&\geq F[\psi(M(gu, u)), \phi(M(gu, u))] + LN(gu, u) \\
&= F[\psi(a_1 d(fgu, ggu) + a_2 d(fu, gu) + a_3 d(fgu, gu) + a_4 d(fu, ggu) + a_5 d(ggu, gu)), \\
&\phi(a_1 d(fgu, ggu) + a_2 d(fu, gu) + a_3 d(fgu, gu) + a_4 d(fu, ggu) + a_5 d(ggu, gu))] \\
&\quad + L \min\{d(fgu, ggu), d(fu, ggu)\} \\
&= F[\phi(a_3 d(fgu, gu) + a_4 d(fgu, gu) + a_5 d(fgu, gu)), \\
&\phi(a_3 d(fgu, gu) + a_4 d(fgu, gu) + a_5 d(fgu, gu))] \\
&\geq k\psi((a_3 + a_4 + a_5) d(fgu, gu)).
\end{aligned} \tag{82}$$

Since $a_1 + a_2 + a_5 > 1$, $a_1 + a_2 < a_3 + a_4$, and $fu = gu$, it follows that

$$\begin{aligned}
&k\psi(d(fgu, gu)) \\
&= F[\psi((a_3 + a_4 + a_5) d(fgu, gu)), \phi((a_3 + a_4 + a_5) d(fgu, gu))] \\
&\geq k\psi((a_3 + a_4 + a_5) d(fgu, gu)) \\
&\geq k\psi((a_1 + a_2 + a_5) d(fgu, gu)) \\
&\geq k\psi(d(fgu, gu)),
\end{aligned} \tag{83}$$

which implies that

$$F[\psi((a_3 + a_4 + a_5) d(fgu, gu)), \phi((a_3 + a_4 + a_5) d(fgu, gu))] = k\psi(d(fgu, gu)). \tag{84}$$

From the definition of $\mathscr{C}_{\text{inv}-k}$–class functions, we have

$$\begin{aligned}
&\psi((a_3 + a_4 + a_5) d(fgu, gu)) = 0 \\
&\text{or } \phi((a_3 + a_4 + a_5) d(fgu, gu)) = 0.
\end{aligned} \tag{85}$$

By the definitions of ψ and ϕ, we have that $d(fgu, gu) = 0$, which proves that $fgu = gu = ggu$.

Therefore, gu is a common fixed point of f and g.

Case 2: pair (f, g) is R–weak commuting of type A_g.

Since pair (f, g) is R–weak commuting of type A_f, it yields

$$d(gfy_n, ffy_n) \leq R\, d(fy_n, gy_n). \tag{86}$$

Taking limit as $n \longrightarrow +\infty$ in the above inequality, it follows that

$$\lim_{n \longrightarrow +\infty} gfy_n = \lim_{n \longrightarrow +\infty} ffy_n = fu. \tag{87}$$

Now, we will show that $fu = u$.

By the definition of ψ and assumption $(A2')$, we have

$$\begin{aligned}
&k\psi(d(ffy_n, fy_n)) \\
&\geq F[\psi(M(fy_n, y_n)), \phi(M(fy_n, y_n))] + LN(fy_n, y_n) \\
&= F[\psi(a_1 d(ffy_n, gfy_n) + a_2 d(fy_n, gy_n) + a_3 d(ffy_n, gy_n) + a_4 d(fy_n, gfy_n) + a_5 d(gfy_n, gy_n)), \\
&\phi(a_1 d(ffy_n, gfy_n) + a_2 d(fy_n, gy_n) + a_3 d(ffy_n, gy_n) + a_4 d(fy_n, gfy_n) + a_5 d(gfy_n, gy_n))] \\
&+ L\min\{d(ffy_n, gfy_n), d(fy_n, gy_n)\}.
\end{aligned} \tag{88}$$

Taking limit as $n \longrightarrow +\infty$ in the above inequality, we have

$$\begin{aligned}
&k\psi(d(fu,u)) \geq F[a_1 d(fu, fu) + a_2 d(u,u) + a_3 d(fu,u) + a_4 d(u, fu) + a_5 d(fu,u),) \\
&\phi(a_1 d(fu, fu) + a_2 d(u,u) + a_3 d(fu,u) + a_4 d(u, fu) + a_5 d(fu,u))] \\
&+ L\min\{d(fu, fu), d(u,u)\} \\
&= F[\psi(a_3 d(fu,u) + a_4 d(u, fu) + a_5 d(fu,u)), \phi(a_3 d(fu,u) + a_4 d(u, fu) + a_5 d(fu,u))].
\end{aligned} \tag{89}$$

Since $a_1 + a_2 + a_5 > 1$, $a_1 + a_2 < a_3 + a_4$, it follows that

$$\begin{aligned}
&k\psi(d(fu,u)) \\
&\geq F[\psi(a_3 d(fu,u) + a_4 d(u, fu) + a_5 d(fu,u)), \phi(a_3 d(fu,u) + a_4 d(u, fu) + a_5 d(fu,u))] \\
&\geq k\psi((a_3 + a_4 + a_5) d(fu,u)) \\
&\geq k\psi((a_1 + a_2 + a_5) d(fu,u)) \\
&\geq k\psi(d(fu,u)),
\end{aligned} \tag{90}$$

which implies that

$$F[\psi((a_3 + a_4 + a_5) d(fu,u)), \phi((a_3 + a_4 + a_5) d(fu,u))] = k\psi(d(fu,u)). \tag{91}$$

From the definition of $\mathscr{C}_{\text{inv}-k}$–class functions, we have

$$\begin{aligned}
&\psi((a_3 + a_4 + a_5) d(fu,u)) = 0 \\
&\text{or } \phi((a_3 + a_4 + a_5) d(fu,u)) = 0.
\end{aligned} \tag{92}$$

By the definitions of ψ and ϕ, we have that $d(fu,u) = 0$, which proves that $fu = u$.

Since $f(X) \subseteq g(X)$, there exists $v \in X$ such that $fu = gv$. Now, we will show that $fv = u$.

By the definition of ψ, $fu = gv = u$, and assumption $(A2')$, we have

$$\begin{aligned}
0 &= k\psi(d(fv, fv)) \\
&\geq F[\psi(M(v,v)), \phi(M(v,v))] + LN(v,v) \\
&= F[\psi(a_1 d(fv,gv) + a_2 d(fv,gv) + a_3 d(fv,gv) + a_4 d(fv,gv) + a_5 d(gv,gv)), \\
&\phi(a_1 d(fv,gv) + a_2 d(fv,gv) + a_3 d(fv,gv) + a_4 d(fv,gv) + a_5 d(gv,gv))] \\
&+ L\min\{d(fv,gv), d(fv,gv)\} \\
&= F[\psi(a_1 d(fv,gv) + a_2 d(fv,gv) + a_3 d(fv,gv) + a_4 d(fv,gv), \\
&\phi(a_1 d(fv,gv) + a_2 d(fv,gv) + a_3 d(fv,gv) + a_4 d(fv,gv))] + L\, d(fv,gv) \\
&\geq k\psi(a_1 d(fv,gv) + a_2 d(fv,gv) + a_3 d(fv,gv) + a_4 d(fv,gv)) + L\, d(fv,gv)) \\
&\geq (k(a_1 + a_2 + a_3 + a_4) + L) d(fv,gv).
\end{aligned} \tag{93}$$

Together with $a_i > 0, i = \{1, 2, \ldots, 5\}$, $k(a_1 + a_2 + a_3 + a_4) + L > 1$, from above inequality, it follows that $d(fv, gv) = d(fv, u) = 0$ or $fv = gv = u$.

Since pair (f, g) is R–weak commuting of type A_f, it yields

$$d(gfv, ffv) \leq R\, d(fv, gv), \tag{94}$$

which yields $ffv = gfv$.

Next, we will show $gfv = fv$.

By the definition of ψ and assumption $(A2')$, we have

$$\begin{aligned}
0 &= k\psi(d(ffv, fv)) \\
&\geq F[\psi(M(fv,v)), \phi(M(fv,v))] + LN(fv,v) \\
&= F[\psi(a_1 d(ffv,gfv) + a_2 d(fv,gv) + a_3 d(ffv,gv) + a_4 d(fv,gfv) + a_5 d(gfv,gv)), \\
&\phi(a_1 d(ffv,gfv) + a_2 d(fv,gv) + a_3 d(ffv,gv) + a_4 d(fv,gfv) + a_5 d(gfv,gv))] \\
&+ L\min\{d(ffv,gfv), d(fv,gv)\} \\
&= F[\psi(a_3 d(ffv,gv) + a_4 d(fv,gfv) + a_5 d(gfv,gv)), \\
&\phi(a_3 d(ffv,gv) + a_4 d(fv,gfv) + a_5 d(gfv,gv))] \\
&\geq k\psi((a_3 + a_4 + a_5) d(ffv,gv)).
\end{aligned} \tag{95}$$

Since $a_1 + a_2 + a_5 > 1$, $a_1 + a_2 < a_3 + a_4$, $ffv = gfv$, and $fv = gv$, it follows that

$$\begin{aligned}
&k\psi(d(gfv, fv)) \\
&= F[\psi((a_3 + a_4 + a_5) d(gfv, fv)), \phi((a_3 + a_4 + a_5) d(gfv, fv))] \\
&\geq k\psi((a_3 + a_4 + a_5) d(gfv, fv)) \\
&\geq k\psi((a_1 + a_2 + a_5) d(gfv, fv)) \\
&\geq k\psi(d(gfv, gv)),
\end{aligned} \tag{96}$$

which implies that

$$F[\psi((a_3 + a_4 + a_5) d(gfv, fv)), \phi((a_3 + a_4 + a_5) d(gfv, fv))] = k\psi(d(gfv, fv)). \tag{97}$$

From the definition of $\mathscr{C}_{\text{inv}-k}$−class functions, we have

$$\begin{aligned} &\psi\left(\left(a_3+a_4+a_5\right)d\left(gfv,fv\right)\right)=0 \\ &\text{or } \phi\left(\left(a_3+a_4+a_5\right)d\left(gfv,fv\right)\right)=0. \end{aligned} \tag{98}$$

By the definitions of ψ and ϕ, we have that $d(gfv,fv)=0$, which proves that $gfv=fv=ffv$. Therefore, fv is a common fixed point of f and g.

The uniqueness of common fixed point of f and g is also obtained by following the same proof of Theorem 2. □

Remark 4. The conclusions of Theorem 5 still hold true by replacing condition $(A2')$ with condition $(A2'')$ stated as before.

Now, we will provide an example to verify the validity of Theorem 5 as follows.

Example 11. Let $X=[0,1]$ with the usual metric and $x,y\in X$. We define f and g on X as follows:

$$\begin{aligned} fx &= \begin{cases} \dfrac{4}{5}, & x\in\left[0,\dfrac{1}{2}\right], \\ \dfrac{3}{4}, & x\in\left(\dfrac{1}{2},1\right], \end{cases} \\ gx &= \begin{cases} \dfrac{4}{5}, & x\in\left[0,\dfrac{1}{2}\right), \\ \dfrac{1}{2}, & x=\dfrac{1}{2}, \\ \dfrac{3}{4}, & x\in\left(\dfrac{1}{2},1\right]. \end{cases} \end{aligned} \tag{99}$$

It is clear that $f(X)\subseteq g(X)$.

To show that the pair (f,g) of self-maps is conditional semicompatible, we take any sequence $\{x_n\}\in(0,1/2)$, then $\lim_{n\longrightarrow+\infty}fx_n=\lim_{n\longrightarrow+\infty}gx_n=4/5$ (nonempty). Now, we choose another sequence $\{y_n\}\in(1/2,1]$, then $\lim_{n\longrightarrow+\infty}fgy_n=3/4=g(3/2)$ and $\lim_{n\longrightarrow+\infty}gfy_n=3/4=g(3/2)$. It can be observed that the pair (f,g) satisfies R−weak commuting of type (A_f) in intervals $[0,1/2)$ and $(1/2,1]$ for all real numbers. Also, the pair (f,g) satisfies R−weak commuting of type (A_f) at $x=1/2$ with $R=1$. To verify assumption $(A2')$, we define $F(s,t)=2s\in\mathscr{C}_{\text{inv}-k}$ with $k=2$, $\psi(t)=t/10$, for all $t\geq 0$. For $x=y=1/2$, we have

$$\begin{aligned} &k\psi\left(d\left(fx,fy\right)\right)=0, \\ &F\left[\psi\left(M\left(x,y\right)\right),\phi\left(M\left(x,y\right)\right)\right] \\ &=\frac{1}{5}\left[a_1d\left(fx,gx\right)+a_2d\left(fy,gy\right)+a_3d\left(fx,gy\right)+a_4d\left(fy,gx\right)+a_5d\left(gx,gy\right)\right] \\ &+L\min\left\{d\left(fx,gx\right),d\left(fy,gx\right)\right\} \\ &=\frac{1}{5}\left[a_1\left|\frac{4}{5}-\frac{1}{2}\right|+a_2\left|\frac{4}{5}-\frac{1}{2}\right|+a_3\left|\frac{4}{5}-\frac{1}{2}\right|+a_4\left|\frac{4}{5}-\frac{1}{2}\right|\right] \\ &\quad+L\min\left\{\left|\frac{4}{5}-\frac{1}{2}\right|,\left|\frac{4}{5}-\frac{1}{2}\right|\right\} \\ &=\frac{1}{5}\left[\frac{3}{10}\left(a_1+a_2+a_3+a_4\right)\right]+\frac{3}{10}L=\frac{3}{10}\left[\frac{1}{5}\left(a_1+a_2+a_3+a_4\right)+L\right]. \end{aligned} \tag{100}$$

It is obvious that $0\geq(3/10)[(1/5)(a_1+a_2+a_3+a_4)+L]$ holds true by taking $a_1=a_2=a_3=1$, $a_4=2$, $a_5>0$, and $-9<L\leq-1$ such that $a_1-a_3\leq 1$, $a_1+a_2<a_3+a_4$, $a_1+a_2+a_5>1$, and $k(a_1+a_2+a_3+a_4)+L>1$. It is also clear that the above inequality holds true for all $x,y\in[0,1/2)$ and $x,y\in(1/2,1]$.

Therefore, f and g satisfy all the conditions of Theorem 5 with 3/4 being the unique common fixed point.

Finally, we will present a common fixed point theorem under S_τ–compatibility as follows.

Theorem 6. *Let (X,d) be a complete metric space and let f,g,B,T be self-maps satisfying the following assumptions:*

$(A1')$ $f(X)\subseteq T(X)$, $g(X)\subseteq B(X)$

$(A2'')$ $k\theta(d(fx,By))\geq F[\theta(M'(x,y)),\phi(M'(x,y))]$

Here, $M'(x,y)=a_1d(fx,gx)+a_2d(By,Ty)+a_3d(fx,Ty)+a_4d(gx,By)+a_5d(gx,Ty)$, for all $x,y\in X$. Moreover, $a_i>0$, $(i=1,2,\ldots,5)$ with $1\geq a_1-a_3$, $a_1+a_2+a_5>1$, $a_1+a_2<a_3+a_4$, $F\in\mathscr{C}_{inv-k}$ for some $k\geq 1$ and $\theta\in\Theta$, $\phi\in\Phi$. If the pair (f,g) is S_B–compatible and (B,T) is S_g–compatible, then f,g,B,T have a unique common fixed point t in X.

Proof. Let $x_0\in X$. Since $f(X)\subseteq T(X)$, then there exists $x_1\in X$ such that $fx_1=Tx_0=y_0$. For $x_1\in X$ and $g(X)\subseteq B(X)$, there exists $x_2\in X$ such that $Bx_2=gx_1=y_1$. Continuing this way, we can construct the sequence $\{x_n\}$ in X such that $fx_n=Tx_{n-1}=y_{n-1}$ and $Bx_{n+1}=gx_n=y_n$.

By the assumption of $(A2'')$ and properties of θ and F, we have

$$\begin{aligned}
&k\theta(d(fx_n,Bx_{n+1}))\\
&\geq F[\theta(M'(x_n,x_{n+1})),\phi(M'(x_n,x_{n+1}))]\\
&\geq k\theta(M'(x_n,x_{n+1}))\\
&=k\theta(a_1d(fx_n,gx_n)+a_2d(Bx_{n+1},Tx_{n+1})+a_3d(fx_n,Tx_{n+1})+a_4d(gx_n,Bx_{n+1})+a_5d(gx_n,Tx_{n+1}))\\
&=k\theta(a_1d(y_{n-1},y_n)+a_2d(y_n,y_{n+1})+a_3d(y_{n-1},y_{n+1})+a_4d(y_n,y_n)+a_5d(y_n,y_{n+1}))\\
&=k\theta(a_1d(y_{n-1},y_n)+a_2d(y_n,y_{n+1})+a_3d(y_{n-1},y_{n+1})+a_5d(y_n,y_{n+1})).
\end{aligned}\tag{101}$$

By the similar procedure demonstrated in Theorem 1, we can get $d(y_n,y_{n+1})\leq(1-a_1+a_3)/(a_2+a_3+a_5)d(y_{n-1},y_n)$.

From Lemma 1, it follows that sequence $\{y_n\}$ is a Cauchy sequence. Since X is complete, there exists a point $t\in X$ such that

$$\lim_{n\longrightarrow+\infty}y_n=\lim_{n\longrightarrow+\infty}fx_n=\lim_{n\longrightarrow+\infty}gx_n=\lim_{n\longrightarrow+\infty}Bx_n=\lim_{n\longrightarrow+\infty}Tx_n=t.\tag{102}$$

Next, we will show that t is a common fixed point of f,g,B,T.

Since the pair (f,g) is S_B–compatible and the pair (B,T) is S_g–compatible and by equation (102), we have the following outcomes:

$$\lim_{n\longrightarrow+\infty}fBx_n=\lim_{n\longrightarrow+\infty}gBx_n=Bt,\tag{103}$$

$$\lim_{n\longrightarrow+\infty}Bgx_n=\lim_{n\longrightarrow+\infty}Tgx_n=gt.\tag{104}$$

By the assumption of $(A2'')$ and properties of θ and F, we have

$$\begin{aligned}
&k\theta((d(fBx_n,Bx_n))\\
&\geq F[\theta(M'(Bx_n,x_n)),\phi(M'(Bx_n,x_n))]\\
&\geq k\theta(M'(Bx_n,x_n))\\
&=k\theta(a_1d(fBx_n,gBx_n)+a_2d(Bx_n,Tx_n)+a_3d(fBx_n,Tx_n)+a_4d(gBx_n,Bx_n)+a_5d(gBx_n,Tx_n)).
\end{aligned}\tag{105}$$

Letting $n \longrightarrow +\infty$ in the above inequality, with equation (103), we have

$$\begin{aligned} &k\theta(d(Bt,t)) \\ &\geq F[\theta(a_3 d(Bt,t) + a_4 d(Bt,t) + a_5 d(Bt,t), \phi(a_3 d(Bt,t) + a_4 d(Bt,t) + a_5 d(Bt,t)] \\ &\geq k\theta(a_3 d(Bt,t) + a_4 d(Bt,t) + a_5 d(Bt,t)) \\ &= k\theta((a_3 + a_4 + a_5) d(Bt,t)). \end{aligned} \tag{106}$$

Since $a_1 + a_2 + a_5 > 1$, $a_1 + a_2 < a_3 + a_4$, we have

$$\begin{aligned} &k\theta(d(Bt,t)) \\ &\geq F[\theta(a_3 d(Bt,t) + a_4 d(Bt,t) + a_5 d(Bt,t)), \phi(a_3 d(Bt,t) + a_4 d(Bt,t) + a_5 d(Bt,t))] \\ &\geq k\theta((a_3 + a_4 + a_5) d(Bt,t)) \\ &\geq k\theta((a_1 + a_2 + a_5) d(Bt,t)) \\ &\geq k\theta(d(Bt,t)), \end{aligned} \tag{107}$$

which implies that

$$F[\theta((a_3 + a_4 + a_5) d(Bt,t)), \phi((a_3 + a_4 + a_5) d(Bt,t))] = k\theta(d(Bt,t)). \tag{108}$$

From the definition of $\mathscr{C}_{\text{inv}-k}$–class functions, we have

$$\begin{aligned} &\theta((a_3 + a_4 + a_5) d(Bt,t)) = 0 \\ &\text{or } \phi((a_3 + a_4 + a_5) d(Bt,t)) = 0. \end{aligned} \tag{109}$$

By the definitions of θ and ϕ, we have that $d(Bt,t) = 0$, which proves that $Bt = t$.

We also have

$$\begin{aligned} &k\theta(d(fx_n, Bgx_n)) \\ &\geq F[\theta(M'(x_n, gx_n)), \phi(M'(x_n, gx_n))] \\ &\geq k\theta(M'(x_n, gx_n)) \\ &= k\theta(a_1 d(fx_n, gx_n) + a_2 d(Bgx_n, Tgx_n) + a_3 d(fx_n, Tgx_n) + a_4 d(gx_n, Bgx_n) + a_5 d(gx_n, Tgx_n)). \end{aligned} \tag{110}$$

Letting $n \longrightarrow +\infty$ in the above inequality, with equation (104), we have

$$\begin{aligned} &k\theta(d(t, gt)) \\ &\geq F[\theta(a_3 d(t,gt) + a_4 d(t,gt) + a_5 d(t,gt)), \phi(a_3 d(t,gt) + a_4 d(t,gt) + a_5 d(t,gt))] \\ &\geq k\theta(a_3 d(t,gt) + a_4 d(t,gt) + a_5 d(t,gt)) \\ &= k\theta((a_3 + a_4 + a_5) d(t,gt)). \end{aligned} \tag{111}$$

Since $a_1 + a_2 + a_5 > 1$, $a_1 + a_2 < a_3 + a_4$, we have

$$\begin{aligned} & k\theta(d(t, gt)) \\ & \geq F\left[\theta\left(a_3 d(t, gt) + a_4 d(t, gt) + a_5 d(t, gt)\right), \phi\left(a_3 d(t, gt) + a_4 d(t, gt) + a_5 d(t, gt)\right)\right] \\ & \geq k\theta\left((a_3 + a_4 + a_5) d(t, gt)\right) \\ & \geq k\theta\left((a_1 + a_2 + a_5) d(t, gt)\right) \\ & \geq k\theta(d(t, gt)), \end{aligned} \tag{112}$$

which implies that

$$F\left[\theta\left((a_3 + a_4 + a_5) d(t, gt)\right), \phi\left((a_3 + a_4 + a_5) d(Bt, t)\right)\right] = k\theta(d(t, gt)). \tag{113}$$

From the definition of $\mathscr{C}_{\text{inv}-k}$–class functions, we have

$$\begin{aligned} & \theta\left((a_3 + a_4 + a_5) d(t, gt)\right) = 0 \\ & \text{or } \phi\left((a_3 + a_4 + a_5) d(t, gt)\right) = 0. \end{aligned} \tag{114}$$

By the definitions of θ and ϕ, we have that $d(t, gt) = 0$, which proves that $t = gt$.

Now, we will prove $Tt = t$. If not, by the assumption of $(A2'')$ and properties of θ and F, we have

$$\begin{aligned} k\theta\left(d(fx_n, Bt)\right) & \geq F\left[\theta\left(M'(x_n, t)\right), \phi\left(M'(x_n, t)\right)\right] \\ & \geq k\theta\left(M'(x_n, t)\right) = k\theta\left(a_1 d(fx_n, gx_n) + a_2 d(Bt, Tt) + a_3 d(fx_n, Tt) + a_4 d(gx_n, Bt) + a_5 d(gx_n, Tt)\right). \end{aligned} \tag{115}$$

Letting $n \longrightarrow +\infty$ in the above inequality, we have

$$\begin{aligned} 0 = k\theta(d(t, t)) & \geq F\left[\theta\left(a_1 d(t, t) + a_2 d(t, Tt) a_3 d(t, Tt) + a_4 d(t, t) + a_5 d(t, Tt)\right),\right. \\ & \left.\phi\left(a_1 d(t, t) + a_2 d(t, Tt) a_3 d(t, Tt) + a_4 d(t, t) + a_5 d(t, Tt)\right)\right] \\ & \geq k\theta\left(a_1 d(t, t) + a_2 d(t, Tt) a_3 d(t, Tt) + a_4 d(t, t) + a_5 d(t, Tt)\right) \\ & = k\theta\left((a_2 + a_2 + a_5) d(t, Tt)\right). \end{aligned} \tag{116}$$

which implies $(a_2 + a_2 + a_5) d(t, Tt) = 0$; that is, $t = Tt$.

In the same manner, we can also get $t = ft$.

Hence, t is a common fixed point of f, g, B, T.

The uniqueness of t can also be obtained by the similar way stated in Theorem 1.

Now, we present the following example to support the validity of Theorem 6. □

Example 12. Let $X = [0, 10]$ with the usual metric. We define f, g, B, T on X as follows:

$$\begin{aligned} & f(x) = g(x) = \frac{x}{2}, \quad \text{for } \forall x \in X, \\ & B(x) = T(x) = x, \quad \text{for } \forall x \in X. \end{aligned} \tag{117}$$

It is clear that $f(X) \subseteq T(X)$ and $g(X) \subseteq B(X)$.

To show that pair (f, g) is S_B–compatible and pair (B, T) is S_g–compatible, we take $x_n = 1/n$; then, $\lim_{n \longrightarrow +\infty} fx_n = 0$, $\lim_{n \longrightarrow +\infty} gx_n = 0$, $\lim_{n \longrightarrow +\infty} Bx_n = 0$, $\lim_{n \longrightarrow +\infty} Tx_n = 0$ with $\lim_{n \longrightarrow +\infty} fBx_n = \lim_{n \longrightarrow +\infty} gBx_n = B(0)$, and $\lim_{n \longrightarrow +\infty} Bgx_n = \lim_{n \longrightarrow +\infty} Tgx_n = g(0)$.

Hence, the pair (f, g) is S_B–compatible and pair (B, T) is S_g–compatible.

To verify assumption $(A2'')$, we define $F(s,t) = ks \in \mathscr{C}_{\text{inv}-k}$ with $k \geq 1$, where $0 < \beta = a_3 + a_4 + a_5 < 1$ and $a_i > 0$, $(i = 1, 2, \ldots, 5)$ satisfying $1 \geq a_1 - a_3$, $a_1 + a_2 + a_5 > 1$ and $a_1 + a_2 < a_3 + a_4$. Define $\theta(t) = t/(2\beta^2)$, for all $t \geq 0$. Then, for all $x, y \in X$, we have

$$\begin{aligned} k\theta(d(fx, By)) &= k\frac{d(fx, By)}{2\beta^2} \\ &= \frac{k}{2\beta^2}\left|\frac{x}{2} - y\right|, \\ F[\theta(M'(x,y)), \phi(M'(x,y))] & \\ &= \frac{k}{2\beta^2}\left[a_1 d(fx, gx) + a_2 d(By, Ty) + a_3 d(fx, Ty) + a_4 d(gx, By) + a_5 d(gx, Ty)\right] \\ &= \frac{k}{2\beta}\left|\frac{x}{2} - y\right|. \end{aligned} \tag{118}$$

It is easy to verify that $k\theta(d(fx, By)) \geq F[\theta(M'(x,y)), \phi(M'(x,y))]$ for all $x, y \in X$ satisfying $a_i > 0$, $(i = 1, 2, \ldots, 5)$ with $0 < \beta = a_3 + a_4 + a_5 \leq 1$, $1 \geq a_1 - a_3$, $a_1 + a_2 + a_5 > 1$, and $a_1 + a_2 < a_3 + a_4$; that is, assumption $(A2'')$ holds true.

Then, f, g, B, and T satisfy all the conditions of Theorem 6; moreover, 0 is the unique common fixed point of f, g, B, and T.

3. Conclusion

In this paper, concepts of semicompatibility of type (A), S_τ–compatibility are introduced, in which S_τ–compatibility is weaker than (E.A) property. We also give a brief discussion on the relation between new notions and other existing types of compatibility. Motivated by the notion of inverse C-class functions, a distinct concept of inverse C_k class functions is introduced which extends the notion of inverse C-class functions introduced by Saleem et al. [1, 21]. Moreover, some common fixed point theorems are stated under some compatible conditions such as semicompatibility, semicompatibility of type (A), weak semicompatibility, conditional semicompatibility, and S_τ–compatibility in metric spaces via inverse C_k class functions which are a valuable supplement to the common fixed point theory.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All the authors contributed equally and significantly to the writing of this article. All the authors read and approved the manuscript.

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