Research Article

ARL Estimation of the Control Chart of Log Likelihood Ratios' Sum for Markov Sequence

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To evaluate the surveillance performance of a control chart with the charting statistic of the sum of log likelihood ratios in the statistical process control (SPC), in this paper, we give the proof procedure based on Markov chains for the asymptotic estimation of the average run length (ARL) for this kind of chart. The out-of-control ARL_1 is approximately equal to 1 for any fixed in-control ARL_0 with a negative control limit. By the equivalence between limit distribution of a sum and that of a suprema sum of Markov chain, we derive the estimation of ARL_1 with a large enough positive control limit. Numerical experiments are conducted to confirm our results.

1. Introduction

The main aim of SPC is to detect an abrupt change in the observation of time series as soon as possible after the change has happened. The study, found in [1], is the first to show the design of the control chart for quickly detecting possible changes in the underlying process. Subsequently, a great number of control charts have been proposed and come into a wider use in many fields, such as environmental control [2, 3], biostatistics [4, 5], clinical medicine [6, 7], economics and finance [8], industrial quality and process control [9, 10], process monitoring [11–16], public health [7, 14, 16], and social network [17].

Obviously, the research on constructing various control charts for surveillance has never stopped since Shewhart presented this method. The existing theoretical studies in statistical monitoring, on the whole, can be classified into three areas. The first category is the optimal economic or economic-statistical design, where, in most cases, the net sum of all costs is minimized [18–21]. The ideas of the second category are the choice of the statistical parameters of a control chart to minimize the out-of-control ARL_1 for a

given in-control ARL_0 or a probability of false alarm, and it is named as the optimal statistical design [16, 22–25]. Recently, the theoretical approximation to ARL was proposed in [26]. Detecting changes in distribution of the optimal control charts is the last category, which can be regarded as a kind of optimal stopping time [27–29]. This metric for defining optimality is based on that it has smallest out-of-control ARL_1 among all control charts with either a given probability of a false alarm no greater than a preset level, or a given false alarm rate no less than a given value. So, this more mathematical area is thus related to statistical design. In the following argument, we will focus on the third type of these bibliographies mentioned above.

Consider the Gaussian observation sequence $\{X_0, X_1, X_2, \dots\}$, whose distribution may change at time τ (≥ 1). Let $p_0(x_0, x_1, x_2, \dots, x_n)$ ($n < \tau$) and $p_\tau(x_\tau, x_{\tau+1}, \dots)$ ($n \geq \tau$) be the prechange probability density function and the postchange probability density function of $\{X_0, X_1, X_2, \dots\}$, respectively. Denote the postchange joint probability distribution, expectation, and variance by $\mathbb{P}_\tau$, $\mathbb{E}_\tau$, and $\mathbb{D}_\tau$, respectively. Especially, when the change time $\tau = \infty$, we suppose that a change never occurs. Frisén

[30] showed that there exists a positive value c such that the control chart with the charting statistic of the sum of log likelihood ratios (SLR),

$$T_{\text{SLR}}(c) = \min \left\{ n: \sum_{k=1}^n \left(X_k - \frac{\mu}{2} \right) \geq c, n \geq 1 \right\}, \quad (1)$$

is optimal in the sense,

$$T: \mathbb{P}_0(T \geq \tau) \geq 1 - \alpha \quad \mathbb{E}_1(T) = \mathbb{E}_1(T_{\text{SLR}}(c)), \quad (2)$$

when the change is the case of a shift in the mean of $\{X_0, X_1, X_2, \dots\}$, where T denotes the control chart test, $\tau = 1$ is the change-point time, and $c > 0$ is the constant control limit such that $\mathbb{P}_0(T_{\text{SLR}}(c) \geq \tau) = 1 - \alpha > 0$. Although the optimality of T_{SLR} has been proved in [30], there were few surveys and bibliographies which considered the performance of the control chart T_{SLR} for monitoring the change point in the process.

Therefore, in this paper, we regard the ARL as the criterion of optimality of a control chart. This is because the ARL is the connection among itself, control limit, and statistical properties of the observation sequence $\{X_0, X_1, X_2, \dots\}$. In Section 2, we prove that the out-of-control ARL_1 is approximately equal to 1 for a given ARL_0 as the control limit c is negative. In Section 3, we apply the equivalence between limit distribution of a sum and that of a suprema sum of Markov chain to obtain the estimation of ARL_1 when the control limit is some large enough positive constant. Finally, we conduct some numerical experiments to verify our theoretical analysis in Section 4.

2. Estimation of ARL_1 with a Negative Control Limit

In this section, we consider a constant control limit $\exp(c)$ to construct a control chart $T_Z(c)$ and estimate the corresponding in-control ARL_0 and out-of-control ARL_1 . Let the observations $\{X(t_0), X(t_1), X(t_2), \dots\}$ be a time-homogeneous Markov chain with the discrete state space $S = \{s_1, s_2, \dots\}$. Here, we consider only detecting the change at the initial time $\tau = 1$, and the Markov chain $\{X(t_k), k \geq 0\}$ is positive recurrence with the prechange transition probability $p_0(y|x)$ and the postchange transition probability $p_1(y|x)$. For convenience, we use X_k instead of $X(t_k)$ for $k \geq 0$. Similar to the notations defined in Section 1, we use $\mathbb{P}_0, \mathbb{E}_0$, and $\mathbb{D}_0$ to denote, respectively, the prechange joint probability distribution, expectation, and variance of $\{X_0, X_1, X_2, \dots\}$.

For ease of the subsequent analysis processing, we choose a large enough positive number N to cut out the first $N + 1$ terms of the sequence $\{X_0, X_1, X_2, \dots\}$. In other words, the sequence $\{X_0, X_1, X_2, \dots, X_N\}$ tends to $\{X_0, X_1, X_2, \dots\}$ when N goes to infinity. The optimality of the control chart $T_Z(c)$ for the Markov chain

$\{X_0, X_1, X_2, \dots, X_N\}$ appeared in [31]. Now, we define the charting statistics as

$$Y_k = \prod_{j=1}^k \frac{p_1(X_j|X_{j-1})}{p_0(X_j|X_{j-1})}, \quad 1 \leq k \leq N. \quad (3)$$

Let $Z_j = \ln\{p_1(X_j|X_{j-1})/p_0(X_j|X_{j-1})\}$ for $1 \leq j \leq N$; then, $\ln Y_k = \sum_{j=1}^k Z_j$ for $1 \leq k \leq N$. For any state s_i , let τ_k^i be the time of k th visit of state s_i and N_n^i be the number of times for visiting s_i from time 0 to time n . The cumulative sums of Z_k and $Z_k - \mathbb{E}_1 Z_k$ on the k th block of time are, respectively, defined by

$$\eta_k^i = \sum_{l=\tau_{k-1}^i+1}^{\tau_k^i} Z_l, \quad \tilde{\eta}_k^i = \sum_{l=\tau_{k-1}^i+1}^{\tau_k^i} (Z_l - \mathbb{E}_1 Z_l), \quad (4)$$

for $k = 1, 2, \dots, N_n^i$ and $\tau_0^i = 0$. Then, both $\eta_1, \dots, \eta_{N_n^i}$ and $\tilde{\eta}_1, \dots, \tilde{\eta}_{N_n^i}$ are sequences of independent and identically distributed (i.i.d.) random variables.

Throughout this paper, we assume that

- (i) $\mathbb{P}_0(s_j|s_i) = 0$ if and only if $\mathbb{P}_1(s_j|s_i) = 0$, as well as $\mathbb{P}_1(s_j|s_i)/\mathbb{P}_0(s_j|s_i)$ has no atom with respect to $\mathbb{P}_0$ for $s_i, s_j \in S$. Here, we define $0/0 = 1$.
- (ii) $\sup_{s_i \in S} \mathbb{E}_0(Z_1^2|X_0 = s_i) < +\infty$.

The sum of $Z_1, \dots, Z_N$ is written by $S_N = \sum_{k=1}^N Z_k$. Set $\mu_{i,j} = \mathbb{E}_j(\eta_1^i)/\mathbb{E}\tau_1^i$, $\bar{\mu}_{i,j} = \mathbb{E}_j|\tilde{\eta}_1^i|/\mathbb{E}\tau_1^i$, and $\sigma_{i,j} = \mathbb{E}_j(\eta_1^i - \mathbb{E}_j\eta_1^i)^2/\mathbb{E}\tau_1^i$ and $\bar{\sigma}_{i,j} = \mathbb{E}_j(|\tilde{\eta}_1^i| - \mathbb{E}_j|\tilde{\eta}_1^i|)^2/\mathbb{E}\tau_1^i$. Moreover, for $j = 0, 1$, let

$$\begin{aligned} \underline{\mu}_j &= \min_{i \in \{1, 2, \dots\}} \mu_{i,j}, \\ \bar{\mu}_j &= \max_{i \in \{1, 2, \dots\}} \mu_{i,j}, \end{aligned} \quad (5)$$

$$\begin{aligned} \underline{\sigma}_j &= \min_{i \in \{1, 2, \dots\}} \sigma_{i,j}, \\ \bar{\sigma}_j &= \max_{i \in \{1, 2, \dots\}} \sigma_{i,j}, \end{aligned}$$

$$\bar{\mu}_1 = \max_{i \in \{1, 2, \dots\}} \bar{\mu}_{i,1}, \quad \bar{\sigma}_1 = \max_{i \in \{1, 2, \dots\}} \bar{\sigma}_{i,1}. \quad (6)$$

Then, for large N , we have

$$\begin{aligned} \mathbb{E}_j \left(\sum_{k=1}^{N_n^i} \eta_k^i \right) &= n\mu_{i,j} (1 + o(1)), \\ \mathbb{D}_j \left(\sum_{k=1}^{N_n^i} \eta_k^i \right) &= n\sigma_{i,j}^2 (1 + o(1)). \end{aligned} \quad (7)$$

Now, we define a control chart $T_Z(c)$ in the following with the charting statistics of the sum of log likelihood ratios for detecting the change in distribution of the Markov sequence $X_0, X_1, \dots, X_N$:

$$T_Z(c) = \min\{n: Y_n \geq \exp(c), 1 \leq n \leq N\} = \min\left\{n: \sum_{k=1}^n Z_k \geq c, 1 \leq n \leq N\right\}, \quad (8)$$

for some constant c .

Now, we are ready to estimate ARL_1 . Based on Section 3 presented in [30], ARL_0 is the average run length until an alarm is signaled, so ARL_0 and $\mathbb{E}_0(T_Z(c))$ are equivalent. Consequently, we use the estimator of $\mathbb{E}_0(T_Z(c))$ to substitute ARL_0 in the subsequent analysis. The main result is proposed in the following theorem.

Theorem 1. *Let $\{X_k, k \geq 0\}$ be a time-homogeneous Markov chain satisfying conditions (i) and (ii). Then, for a given $ARL_0 = L$ and $1 < ARL_0 < N + 1$, there exists a negative number $c^* = c^*(N, L)$ such that*

$$ARL_0(T_Z(c^*)) = \mathbb{E}_0(T_Z(c^*)) = L, \quad (9)$$

$$ARL_1(T_Z(c^*)) = \mathbb{E}_1(T_Z(c^*)) = 1 + o(1), \quad (10)$$

as $N \rightarrow \infty$.

Proof. Choose the control limit $c_1 = c_1(N) = \underline{\mu}_0 N$ and note that

$$\begin{aligned} \mathbb{E}_0(\eta_1^i | X_0 = s_i) &= \mathbb{E}_0\left(\sum_{n=1}^N \left(I_{\{\tau_1=n\}} \sum_{k=1}^N Z_k\right) | X_0 = s_i\right) \\ &= \mathbb{E}_0\left(\sum_{k=1}^{\infty} \left(I_{\{\tau_1 \geq k\}} Z_k\right) | X_0 = s_i\right) \\ &= \sum_{k=1}^{\infty} \mathbb{E}_0[\zeta(X_0, \dots, X_{k-1}) Z_k | X_0 = s_i], \end{aligned} \quad (11)$$

where $I_{\{\tau_1 \geq k\}} = \zeta(X_0, \dots, X_{k-1}) \in \mathcal{F}_{k-1}$. Let $\mathbb{E}_{0,s_i} = \mathbb{E}_0[\zeta(X_0, \dots, X_{k-1}) Z_k | X_0 = s_i]$. Using the law of total expectation and Markov property, we obtain

$$\begin{aligned} \mathbb{E}_{0,s_i}[\zeta(X_0, \dots, X_{k-1}) Z_k] &= \mathbb{E}_{0,s_i}[\mathbb{E}_{0,s_i}(\zeta(X_0, \dots, X_{k-1}) Z_k | \mathcal{F}_{k-1})] \\ &= \mathbb{E}_{0,s_i}[\zeta(X_0, \dots, X_{k-1}) \mathbb{E}_{0,s_i}(Z_k | X_{k-1})]. \end{aligned} \quad (12)$$

Since $\ln x$ is a concave function, by Jensen's inequality, we have

$$\begin{aligned} \mathbb{E}_{0,s_i}(Z_k | X_{k-1}) &= \mathbb{E}_{0,s_i}\left(\ln \frac{p_1(X_k | X_{k-1})}{p_0(X_k | X_{k-1})} | X_{k-1}\right) \\ &\leq \ln \mathbb{E}_{0,s_i}\left(\frac{p_1(X_k | X_{k-1})}{p_0(X_k | X_{k-1})} | X_{k-1}\right) = 0. \end{aligned} \quad (13)$$

Substituting (13) into (12) and combining (11), we can get that $\mu_{i,0} < 0$ for any i . Similarly, we can prove $\mu_{i,1} > 0$ for any i . Hence, $\underline{\mu}_0, \bar{\mu}_0 < 0$ and $\underline{\mu}_1, \bar{\mu}_1 > 0$; then, $\lim_{N \rightarrow \infty} c_1(N) = -\infty$. Let $\tilde{m} = [\varepsilon_1 N]$ for some $\varepsilon_1 < 1$, where $[x]$ denotes the integer part of x . Then,

$$\begin{aligned} \mathbb{E}_0(T_Z(c_1)) &= \mathbb{P}_0(T_Z(c_1) > 0) + \sum_{m=1}^N \mathbb{P}_0(T_Z(c_1) > m) \\ &= 1 + \sum_{m=1}^{\tilde{m}-1} \mathbb{P}_0(T_Z(c_1) > m) + \sum_{m=\tilde{m}}^N \mathbb{P}_0(T_Z(c_1) > m) \leq 1 + \sum_{m=1}^{\tilde{m}-1} \mathbb{P}_0(T_Z(c_1) > m) + (N - \tilde{m} + 1) \mathbb{P}_0(T_Z(c_1) > \tilde{m}). \end{aligned} \quad (14)$$

The total probability formula tells us that

$$\begin{aligned}\mathbb{P}_0(T_Z(c_1) > \tilde{m}) &= \mathbb{P}_0\left(\max_{1 \leq k \leq \tilde{m}} S_k \leq c_1\right) \leq \mathbb{P}_0(S_{\tilde{m}} \leq c_1) \\ &= \sum_{i=1}^{\infty} \mathbb{P}_0(X_0 = s_i) \mathbb{P}_0(s_{\tilde{m}} \leq c_1 | X_0 = s_i).\end{aligned}\tag{15}$$

Equations (4) and (7) yield that

$$\begin{aligned}\mathbb{P}_0(s_{\tilde{m}} \leq c_1 | X_0 = s_i) &= \mathbb{P}_0\left(\sum_{k=1}^{N_{\tilde{m}}^i} \eta_k^i \leq c_1 - \sum_{k=r_{N_{\tilde{m}}^i}^i+1}^{N_{\tilde{m}}^i} Z_k | X_0 = s_i\right) \\ &= \mathbb{P}_0\left(\frac{\sum_{k=1}^{N_{\tilde{m}}^i} \eta_k^i - \mathbb{E}_0\left(\sum_{k=1}^{N_{\tilde{m}}^i} \eta_k^i\right)}{\mathbb{D}_0\left(\sum_{k=1}^{N_{\tilde{m}}^i} \eta_k^i\right)} \geq \frac{-c_1 + \tilde{m}\mu_{i,0}(1+o(1))}{\sigma_{i,0}\sqrt{\tilde{m}}} + \frac{\sum_{k=r_{N_{\tilde{m}}^i}^i+1}^{N_{\tilde{m}}^i} Z_k}{\sigma_{i,0}\sqrt{\tilde{m}}} | X_0 = s_i\right).\end{aligned}\tag{16}$$

Note that $\tilde{m} = [\varepsilon_1 N]$ and $\sum_{k=r_{N_{\tilde{m}}^i}^i+1}^{N_{\tilde{m}}^i} Z_k / \sigma_{i,0}\sqrt{\tilde{m}}$ almost everywhere converges to 0 as $N \xrightarrow{s_m^i} \infty$; therefore,

$$\frac{-c_1 + \tilde{m}\mu_{i,0}(1+o(1))}{\sigma_{i,0}\sqrt{\tilde{m}}} + \frac{\sum_{k=r_{N_{\tilde{m}}^i}^i+1}^{N_{\tilde{m}}^i} Z_k}{\sigma_{i,0}\sqrt{\tilde{m}}} = \frac{-\underline{\mu}_0 + \varepsilon_1\mu_{i,0}}{\sigma_{i,0}\sqrt{\varepsilon_1}} \sqrt{N} + o(1).\tag{17}$$

Notice that, for a large x , the relationship between probability distribution function $\Phi(x)$ and density function $p(x)$ of standard normal distribution is

$$1 - \Phi(x) = \frac{p(x)}{x} (1 + o(1)).\tag{18}$$

Combining (16)–(18) and the central limit theorem of Markov chain (see [32]), we can get that

$$\begin{aligned}\mathbb{P}_0(S_{\tilde{m}} \leq c_1 | X_0 = s_i) &= 1 - \Phi\left(\frac{-\underline{\mu}_0 + \varepsilon_1\mu_{i,0}}{\sigma_{i,0}\sqrt{\varepsilon_1}} \sqrt{N} + o(1)\right) (1 + o(1)) \\ &= \frac{\sigma_{i,0}\sqrt{\varepsilon_1}(1+o(1))}{\sqrt{2\pi N}(-\underline{\mu}_0 + \varepsilon_1\mu_{i,0})} \exp\left[-\frac{(-\underline{\mu}_0 + \varepsilon_1\mu_{i,0})^2}{2\sigma_{i,0}^2\varepsilon_1} N\right] \\ &\leq \frac{\bar{\sigma}_0\sqrt{\varepsilon_1}(1+o(1))}{\sqrt{2\pi N}\underline{\mu}_0(\varepsilon_1-1)} \exp\left[-\frac{\underline{\mu}_0^2(\varepsilon_1-1)^2}{2\bar{\sigma}_0^2\varepsilon_1} N\right].\end{aligned}\tag{19}$$

Put (19) into (15) and combine $\sum_{i=1}^{\infty} \mathbb{P}_0(X_0 = s_i) = 1$ to yield that

$$\mathbb{P}_0(T_Z(c_1) > \tilde{m}) \leq \frac{\bar{\sigma}_0 \sqrt{\varepsilon_1} (1 + o(1))}{\sqrt{2\pi} N \underline{\mu}_0 (\varepsilon_1 - 1)} \exp \left[-\frac{\underline{\mu}_0^2 (\varepsilon_1 - 1)^2}{2\bar{\sigma}_0^2 \varepsilon_1} N \right]. \quad (20)$$

For $1 \leq m \leq \tilde{m} - 1$, by the Markov inequality and $\tilde{m} = \lfloor \varepsilon_1 N \rfloor$, we have

$$\begin{aligned} \sum_{m=1}^{\tilde{m}-1} \mathbb{P}_0(T_Z(c_1) > m) &= \sum_{m=1}^{\tilde{m}-1} \mathbb{P}_0 \left(Z_1 \leq c_1, \dots, \sum_{k=1}^m Z_k \leq c_1 \right) \\ &\leq (\tilde{m} - 1) \mathbb{P}_0(Z_1 \leq c_1) \\ &\leq \tilde{m} \mathbb{P}_0(|Z_1| \geq -\mu_0 N) \\ &\leq \frac{\tilde{m} \mathbb{E}_{0,s_i} |Z_1|^2}{\mu_0^2 N^2} = \frac{\varepsilon_1 \mathbb{E}_{0,s_i} |Z_1|^2}{\mu_0^2 N}. \end{aligned} \quad (21)$$

Inequality (21) implies that the value of $\sum_{m=1}^{\tilde{m}-1} \mathbb{P}_0(T_Z(c_1) > m)$ tends to 0 as $N \rightarrow \infty$. Put (20) and (21) into (14); then, there exists some constant L such that

$$\mathbb{E}_0(T_Z(c_1)) \leq 1 + \frac{(N - \tilde{m} + 1) \bar{\sigma}_0 \sqrt{\varepsilon_1} (1 + o(1))}{\sqrt{2\pi} N \underline{\mu}_0 (\varepsilon_1 - 1)} \exp \left[-\frac{\underline{\mu}_0^2 (\varepsilon_1 - 1)^2}{2\bar{\sigma}_0^2 \varepsilon_1} N \right] < L, \quad (22)$$

for a large N .

Using Theorem 5.1.7 in [33], there exists a nonpositive number $c_2 = c_2(N) > (c_1(N))$ such that

$$\mathbb{E}_0(T_Z(c_2)) > L \quad (23)$$

holds for a large enough N . Thus, we can choose a negative number $c^* = c^*(N)$ satisfying $c_1 < c^* < c_2$ and $\mathbb{E}_0(T_Z(c^*)) = L$.

Next, we prove (10). For any $x > 0$, let $c_3 = -(|\underline{\mu}_0| + x)N$. Similar to (20), we can get that

$$\begin{aligned} 1 &\leq \mathbb{E}_1(T_Z(c_3)) = 1 + \mathbb{P}_1(T_Z(c_3) > m) \\ &\leq 1 + \sum_{m=1}^{\tilde{m}-1} \mathbb{P}_1(T_Z(c_3) > m) + (N - \tilde{m} + 1) \sum_{i=1}^{\infty} \mathbb{P}_1(X_0 = s_i) \mathbb{P}_1(S_m \leq c_3 | X_0 = s_i) \\ &= 1 + o(1) + (1 + o(1))(N - \tilde{m} + 1) \Phi \left(-\frac{(|\underline{\mu}_0| + x)N + \tilde{m}\mu_{i,1}}{\sigma_{i,0} \sqrt{\tilde{m}}} \right) \\ &= 1 + o(1) + \frac{(1 + o(1))(N - \tilde{m} + 1) \sigma_{i,1} \sqrt{\tilde{m}}}{(|\underline{\mu}_0| + x)N + \tilde{m}\mu_{i,1}} \exp \left[-\frac{((|\underline{\mu}_0| + x)N + \tilde{m}\mu_{i,1})^2}{2\sigma_{i,1}^2 \tilde{m}} \right] \\ &= 1 + o(1) + \frac{(1 + o(1))(1 - \varepsilon_1) \bar{\sigma}_1 \sqrt{\varepsilon_1} N}{[(|\underline{\mu}_0| + x) + \varepsilon_1 \underline{\mu}_1] \sqrt{2\pi}} \exp \left[-\frac{((|\underline{\mu}_0| + x) + \varepsilon_1 \underline{\mu}_1)^2}{2\bar{\sigma}_1^2} N \right]. \end{aligned} \quad (24)$$

Note that the second and third terms on the right-hand side of the last inequality in (24) tends to 0 as N goes to infinity.

This completes the proof. $\square$

3. Estimation of ARL_1 with a Positive Control Limit

In the subsequent discussion, we use the equivalence between limit distribution of a sum and that of a suprema sum

of Markov chain to estimate the out-of-control ARL_1 when the control limit c is a sufficiently large positive constant. The main result is presented in the following theorem.

Theorem 2. Assume that $\{X_k, k \geq 0\}$ is a homogeneous Markov chain. For any state $s_i, i = 1, 2, \dots$ if

$$\mathbb{E}_1(\eta_1^i)^2 \leq \bar{\mu}_1^2 + \bar{\sigma}_1^2 < \infty, \quad (25)$$

then

$$\begin{aligned} & \frac{c - \sqrt{c \ln c}}{\bar{\mu}_1} + \frac{(1 + o(1))\bar{\sigma}_1}{\varepsilon_2 \sqrt{2\pi \bar{\mu}_1 \ln c}} \exp\left(-\frac{\bar{\mu}_1(c - \sqrt{c \ln c})}{2\bar{\sigma}_1^2}\right) \leq \mathbb{E}_1(T_Z(c)) \\ & \leq \frac{c + \sqrt{c \ln c}}{\bar{\mu}_1} + \frac{\bar{\sigma}_1(1 + o(1))\sqrt{c + \sqrt{c \ln c}}}{\varepsilon_2 \sqrt{2\pi \bar{\mu}_1 c \ln c}} \exp\left(-\frac{\bar{\mu}_1(\sqrt{c \ln c})}{2\bar{\sigma}_1^2}\right), \end{aligned} \quad (26)$$

for large N and c , where $\bar{\mu}_1, \bar{\sigma}_1, \bar{\sigma}_1$ and $\bar{\mu}_1, \bar{\sigma}_1$ are defined in (5) and (6), respectively. The sign $o(\cdot)$ denotes the infinitesimal of higher order and $\varepsilon_2 = \exp[\bar{\mu}_1^2/2\bar{\sigma}_1^2] - 1$.

Proof. It follows from (8) that, for $m \geq 1$,

$$\mathbb{P}_1(T_Z(c) > m) = \mathbb{P}_1\left(Z_1 \leq c, Z_1 + Z_2 \leq c, \dots, \sum_{k=1}^m Z_k \leq c\right) = \mathbb{P}_1\left(\max_{1 \leq k \leq m} S_k \leq c\right). \quad (27)$$

The estimation of $\mathbb{P}_1(T_Z(c) > m)$ is reduced to the estimation of $\mathbb{P}_1(\max_{1 \leq k \leq m} S_k \leq c)$. According to Theorem 2 in [34], we have the following conclusion:

$$\lim_{m \rightarrow \infty} \mathbb{P}_1\left(\max_{1 \leq k \leq m} S_k - \mathbb{E}_1 S_m \leq x\sqrt{m}\right) = \lim_{m \rightarrow \infty} \mathbb{P}_1(S_m - \mathbb{E}_1 S_m \leq x\sqrt{m}). \quad (28)$$

Then, we use (28) to calculate the value of $\mathbb{P}_1(T_Z(c) > m)$. Let $\hat{c} = c - \mathbb{E}_1 S_m = x\sqrt{m}$, where $x \in \mathbb{R}$ and

$\mathbb{R}$ denotes the real number field. When m becomes sufficiently large, by (5), (27), and (28), we have

$$\begin{aligned} \mathbb{P}_1(T_Z(c) > m) &= \mathbb{P}_1\left(\max_{1 \leq k \leq m} S_k \leq c\right) = \mathbb{P}_1(S_m - \mathbb{E}_1 S_m \leq \hat{c})(1 + o(1)) \\ &= \mathbb{P}_1\left(\frac{S_m - \mathbb{E}_1 S_m}{\mathbb{D}_1 S_m} \leq \frac{c - \mathbb{E}_1 S_m}{\mathbb{D}_1 S_m}\right)(1 + o(1)) \\ &= \Phi\left(\frac{c - m\mu_{i,1}}{\sqrt{m}\sigma_{i,1}}\right)(1 + o(1)). \end{aligned} \quad (29)$$

Let $\hat{m}_1 = (c + \sqrt{c \ln c})/\mu_{i,1}$, and we have

$$\begin{aligned}
\mathbb{E}_1(T_Z(c)) &= \sum_{m=1}^{\infty} \mathbb{P}_1(T_Z(c) > m) = \sum_{m=1}^{\lceil \hat{m}_1 \rceil} \mathbb{P}_1(T_Z(c) > m) + \sum_{m=\lceil \hat{m}_1 \rceil+1}^{\infty} \mathbb{P}_1(T_Z(c) > m) \\
&= \sum_{m=1}^{\lceil \hat{m}_1 \rceil} \mathbb{P}_1(T_Z(c) > m) + (1 + o(1)) \sum_{m=\lceil \hat{m}_1 \rceil+1}^{\infty} \Phi\left(\frac{c - m\mu_{i,1}}{\sqrt{m}\sigma_{i,1}}\right) \\
&= \sum_{m=1}^{\lceil \hat{m}_1 \rceil} \mathbb{P}_1(T_Z(c) > m) + (1 + o(1)) \sum_{k=1}^{\infty} \Phi\left(\frac{c - (\hat{m}_1 + k)\mu_{i,1}}{\sigma_{i,1}\sqrt{\hat{m}_1 + k}}\right),
\end{aligned} \tag{30}$$

for the large enough m and c . When c goes to infinity, we deal with the second term on the right-hand side of (30) as follows:

$$\begin{aligned}
\Phi\left(\frac{c - (\hat{m}_1 + k)\mu_{i,1}}{\sigma_{i,1}\sqrt{\hat{m}_1 + k}}\right) &= \Phi\left(\frac{c - ((c/\mu_{i,1}) + (\sqrt{c \ln c}/\mu_{i,1}) + k)\mu_{i,1}}{(\sigma_{i,1}(\sqrt{c} + \sqrt{c \ln c})/\mu_{i,1}) + k}\right) = \Phi\left(\frac{\sqrt{\mu_{i,1}c \ln c} + \mu_{i,1}\sqrt{\mu_{i,1}k}}{\sigma_{i,1}\sqrt{c} + \sqrt{c \ln c} + \mu_{i,1}k}\right) \\
&= \frac{(1 + o(1))\sigma_{i,1}}{\sqrt{2\pi}\mu_{i,1}} \frac{\sqrt{c} + \sqrt{c \ln c} + \mu_{i,1}k}{\sqrt{c \ln c} + \mu_{i,1}k} \exp\left(-\frac{\mu_{i,1}(\sqrt{c \ln c} + \mu_{i,1}k)^2}{2\sigma_{i,1}^2(c + \sqrt{c \ln c} + \mu_{i,1}k)}\right).
\end{aligned} \tag{31}$$

Note that, for a large c , we have

$$\begin{aligned}
-\frac{\mu_{i,1}(\sqrt{c \ln c} + \mu_{i,1}k)^2}{2\sigma_{i,1}^2(c + \sqrt{c \ln c} + \mu_{i,1}k)} &= -\frac{\mu_{i,1}^2k}{2\sigma_{i,1}^2} + \frac{\mu_{i,1}^2kc - \mu_{i,1}c \ln c - \mu_{i,1}^2k\sqrt{c \ln c}}{2\sigma_{i,1}^2(c + \sqrt{c \ln c} + \mu_{i,1}k)} \\
&= -\frac{\mu_{i,1}^2k}{2\sigma_{i,1}^2} - \frac{\mu_{i,1}\sqrt{c \ln c}}{2\sigma_{i,1}^2} + \frac{\mu_{i,1}c}{2\sigma_{i,1}^2} \left(1 - \frac{c}{c + \sqrt{c \ln c} + \mu_{i,1}k}\right) \\
&= -\frac{\mu_{i,1}^2k}{2\sigma_{i,1}^2} - \frac{\mu_{i,1}\sqrt{c \ln c}}{2\sigma_{i,1}^2} + o(c).
\end{aligned} \tag{32}$$

We obtain the right-hand side of (30) as follows:

$$\begin{aligned}
\sum_{k=1}^{\infty} \Phi\left(\frac{c - (\hat{m}_1 + k)\mu_{i,1}}{\sigma_{i,1}\sqrt{\hat{m}_1 + k}}\right) &\leq \frac{\sigma_{i,1}(1 + o(1))\sqrt{c} + \sqrt{c \ln c}}{\sqrt{2\pi}\mu_{i,1}c \ln c} \exp\left(-\frac{\mu_{i,1}\sqrt{c \ln c}}{2\sigma_{i,1}^2}\right) \sum_{k=1}^{\infty} \exp\left(-\frac{\mu_{i,1}^2k}{2\sigma_{i,1}^2}\right) \\
&\leq \frac{\sigma_{i,1}(1 + o(1))\sqrt{c} + \sqrt{c \ln c}}{\varepsilon_2\sqrt{2\pi}\mu_{i,1}c \ln c} \exp\left(-\frac{\mu_{i,1}\sqrt{c \ln c}}{2\sigma_{i,1}^2}\right),
\end{aligned} \tag{33}$$

as c tends to infinity. Combining (30), (33), and $\mathbb{P}_1(T_Z(c) > m) \leq 1$, we obtain

$$\begin{aligned} \mathbb{E}_1(T_Z(c)) &= \sum_{m=1}^{\lfloor \hat{m}_1 \rfloor} \mathbb{P}_1(T_Z(c) > m) + \frac{\sigma_{i,1}(1+o(1))\sqrt{c+\sqrt{c \ln c}}}{\varepsilon_2 \sqrt{2\pi\mu_{i,1}c \ln c}} \exp\left(-\frac{\mu_{i,1}\sqrt{c \ln c}}{2\sigma_{i,1}^2}\right) \\ &\leq \hat{m}_1 + \frac{\bar{\sigma}_1(1+o(1))\sqrt{c+\sqrt{c \ln c}}}{\varepsilon_2 \sqrt{2\pi\mu_{i,1}c \ln c}} \exp\left(-\frac{\mu_{i,1}\sqrt{c \ln c}}{2\bar{\sigma}_1^2}\right) \\ &\leq \frac{c+\sqrt{c \ln c}}{\mu_{i,1}} + \frac{\bar{\sigma}_1(1+o(1))\sqrt{c+\sqrt{c \ln c}}}{\varepsilon_2 \sqrt{2\pi\mu_{i,1}c \ln c}} \exp\left(-\frac{\mu_{i,1}\sqrt{c \ln c}}{2\bar{\sigma}_1^2}\right). \end{aligned} \quad (34)$$

To prove the inequality on the left-hand side of (26), let $\hat{m}_2 = (c - \sqrt{c \ln c})/\mu_{i,1}$, and c goes to infinity; then, we use (27) to yield that

$$\begin{aligned} \mathbb{E}_1(T_Z(c)) &= \sum_{m=1}^{\lfloor \hat{m}_2 \rfloor} \mathbb{P}_1(T_Z(c) > m) + (1+o(1)) \sum_{k=1}^{\infty} \Phi\left(\frac{c - (\hat{m}_2 + k)\mu_{i,1}}{\sigma_{i,1}\sqrt{\hat{m}_2 + k}}\right) \\ &= \sum_{m=1}^{\lfloor \hat{m}_2 \rfloor} \mathbb{P}_1\left(\frac{\max_{1 \leq k \leq m} S_k - m\mu_{i,1}}{\sigma_{i,1}\sqrt{m}} \leq \frac{c - m\mu_{i,1}}{\sigma_{i,1}\sqrt{m}}\right) + (1+o(1)) \sum_{k=1}^{\infty} \Phi\left(\frac{c - (\hat{m}_2 + k)\mu_{i,1}}{\sigma_{i,1}\sqrt{\hat{m}_2 + k}}\right) \\ &= \sum_{m=1}^{\lfloor \hat{m}_2 \rfloor} \mathbb{G}_1\left(\frac{c - m\mu_{i,1}}{\sigma_{i,1}\sqrt{m}}\right) + (1+o(1)) \sum_{k=1}^{\infty} \Phi\left(\frac{c - (\hat{m}_2 + k)\mu_{i,1}}{\sigma_{i,1}\sqrt{\hat{m}_2 + k}}\right), \end{aligned} \quad (35)$$

where $\mathbb{G}_1(x) := \mathbb{P}_1((\max_{1 \leq k \leq m} S_k - m\mu_{i,1})/\sigma_{i,1}\sqrt{m} \leq x)$ is the distribution function and $m = 1, 2, \dots, \lfloor \hat{m}_2 \rfloor$. Similar to the derivation of (33), let c tend to infinity; then, we obtain

the estimation of the second term on the right-hand side of (35) as

$$\sum_{k=1}^{\infty} \Phi\left(\frac{c - (\hat{m}_2 + k)\mu_{i,1}}{\sigma_{i,1}\sqrt{\hat{m}_2 + k}}\right) \geq \frac{(1+o(1))\underline{\sigma}_1}{\varepsilon_2 \sqrt{2\pi\mu_{i,1}c \ln c}} \exp\left(-\frac{\bar{\mu}_1(c - \sqrt{c \ln c})}{2\underline{\sigma}_1^2}\right). \quad (36)$$

For the first term on the right-hand side of (35), because the function $(c - m\mu_{i,1})/\sigma_{i,1}\sqrt{m}$ is monotonically decreasing in the interval $[1, \lfloor \hat{m}_2 \rfloor]$ with respect to m and the

distribution function $\mathbb{G}_1(x)$ is monotonically nondecreasing in $\mathbb{R}$, we have

$$\mathbb{G}_1\left(\frac{c - m\mu_{i,1}}{\sigma_{i,1}\sqrt{m}}\right) \geq \mathbb{G}_1\left(\frac{c - \hat{m}_2\mu_{i,1}}{\sigma_{i,1}\sqrt{\hat{m}_2}}\right) = \mathbb{G}_1\left(\frac{\sqrt{\mu_{i,1}c \ln c}}{\sigma_{i,1}\sqrt{c - \sqrt{c \ln c}}}\right) = \mathbb{G}_1\left(\frac{\sqrt{\mu_{i,1} \ln c}}{\sigma_{i,1}\sqrt{1 - (\ln c/c)}}\right) \approx 1, \quad (37)$$

TABLE 1: Simulation values of ARL_1 for $N = 1000$.

ARL_0	250	370	500
ARL_0	251	371	501.1
c	-1.2760×10^6	-1.8869×10^6	-2.5488×10^6
ARL_1	1	1	1

TABLE 2: Ratios of $\mathbb{E}_1(T_Z(c))$ and c with different choices of N .

$N/100$	1	5	10	50	100	500	1000
$\mathbb{E}_1(T_Z(c))/c$	2.3801	2.4482	2.3827	2.3662	2.4006	2.3853	2.3880

for $c \rightarrow \infty$ and $m \in [1, [\hat{m}_2]]$. By (35)–(37), we obtain

$$\mathbb{E}_1(T_Z(c)) \geq \hat{m}_2 + \frac{(1 + o(1))\underline{\sigma}_1}{\varepsilon_2 \sqrt{2\pi\bar{\mu}_1 c \ln c}} \exp\left(-\frac{\bar{\mu}_1(c - \sqrt{c \ln c})}{2\underline{\sigma}_1^2}\right), \quad (38)$$

for a large enough c . Combine (34) and (38) to yield (26). This completes the proof. $\square$

4. Numerical Experiment

In this section, we perform two numerical experiments to verify our theoretical results. In our first numerical experiment, let $\{X_k, k \geq 0\}$ be a sequence of i.i.d. Gaussian random variables with the prechange and the postchange probability densities $p_0(x) = 1/\sqrt{2\pi} \exp(-(x - \mu_0)^2/2)$ and $p_1(x) = 1/\sqrt{2\pi} \exp(-(x - \mu_1)^2/2)$. It is assumed that the standard case of a shift in the mean of $\{X_k, k \geq 0\}$ from μ_0 to μ_1 is considered.

According to the proof of Theorem 1, the initial data is chosen as $\mu_0 = -1, \mu_1 = 100$, and $N = 1000$. Given different values of $ARL_0 = 250, 370, 500$, we obtain the corresponding values of control limit c and ARL_1 , which are listed in Table 1.

These results suggest that the value of $\mathbb{E}_1(T_Z(c))$ is equal to 1 with a negative control limit c and a given ARL_0 , which is predicted in Theorem 1.

In the second experiment, we choose $c = ((\exp(1) - 1)/\sqrt{\exp(1)} + 1)N$ and $N/100 = 1, 5, 10, 50, 100, 500, 1000$. Let $\{X_k, k \geq 0\}$ be a homogeneous Poisson process with the prechange and the postchange parameters λ_0 and λ_1 ; then, the corresponding prechange and postchange transition probabilities are defined as

$$\mathbb{P}_r(X_k = j | X_{k-1} = i) = \begin{cases} \frac{(\lambda_r \Delta t_k)^{j-i}}{(j-i)!} \exp(-\lambda_r t_k), & 0 < i \leq j, \\ 0, & 0 \leq j < i, \end{cases} \quad (39)$$

where $r = 0, 1$ and i and j are arbitrary states from the state space S and $\Delta t_k = t_k - t_{k-1}$ for $k \geq 1$.

The results are obtained and presented in Table 2, which show the ratio of $\mathbb{E}_1(T_Z(c))$ and c equals approximately 2.38 as $N \rightarrow \infty$. It implies that the value of $\mathbb{E}_1(T_Z(c))$ and the control limit c are the infinite of the same order, which is predicted by Theorem 2.

Data Availability

The data used to support the findings of the study are generated by Matlab.

Conflicts of Interest

The authors declare no conflict of interest.

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