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Research Article

Monotone Iterative Method for Two Types of Integral Boundary Value Problems of a Nonlinear Fractional Differential System with Deviating Arguments

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This paper concerns on two types of integral boundary value problems of a nonlinear fractional differential system, *i.e.*, nonlocal strip integral boundary value problems and coupled integral boundary value problems. With the aid of the monotone iterative method combined with the upper and lower solutions, the existence of extremal system of solutions for the above two types of differential systems is investigated. In addition, a new comparison theorem for fractional differential system is also established, which is crucial for the proof of the main theorem of this paper. At the end, an example explaining how our studies can be used is also given.

1. Introduction

Differential equations with integral boundary conditions have been applied in many fields such as thermoelasticity, blood flow phenomena, and groundwater systems. For specific details, readers interested in this topic can see papers [1–6] and the references therein. In addition, the advantages of fractional derivatives make fractional differential equations a hot topic. At present, it exhibits great vitality and splendor in a number of applications of interest such as biophysics, hemodynamics, complex media circuit analysis and simulation, control optimization theory, and earthquake prediction models. For more details, refer to books in [7–12]. For the latest developments and trends, refer to [13–24]. Fractional differential system, as an important branch of differential system, is attracting more and more scholars research interest, which comes from its good practical application background (see [25–32]).

In [25], by applying the monotone iterative method, Wang, Agarwal, and Cabada investigated the existence of extremal solutions for a nonlinear Riemann–Liouville fractional differential system:

$$\begin{cases} \mathscr{D}^{\alpha}\varphi(\varepsilon) = \mathscr{X}(\varepsilon,\varphi(\varepsilon),\psi(\varepsilon)), & \varepsilon \in (0,\mathscr{B}], \\ \mathscr{D}^{\alpha}\psi(\varepsilon) = \mathscr{Y}(\varepsilon,\psi(\varepsilon),\varphi(\varepsilon)), & \varepsilon \in (0,\mathscr{B}], \\ \varepsilon^{1-\alpha}\varphi(\varepsilon)|_{\varepsilon=0} = x_0, & \varepsilon^{1-\alpha}\psi(\varepsilon)|_{\varepsilon=0} = y_0, \end{cases} \tag{1}$$

where $0<\mathscr{B}<\infty, \mathscr{X}, \mathscr{Y} \in C([0,\mathscr{B}]\times\mathbb{R}\times\mathbb{R},\mathbb{R}), x_0, y_0 \in \mathbb{R}$ and $x_0 \le y_0$.

In [32], Ahmad and Nieto studied a three point-coupled nonlinear Riemann–Liouville fractional differential system given by

$$\begin{cases} \mathscr{D}^{\alpha}\varphi(\varepsilon) = \mathscr{X}(\varepsilon,\psi(\varepsilon),\mathscr{D}^{\delta}\psi(\varepsilon)), & \varepsilon \in (0,1), \\ \mathscr{D}^{\beta}\psi(\varepsilon) = \mathscr{Y}(\varepsilon,\varphi(\varepsilon),\mathscr{D}^{\sigma}\varphi(\varepsilon)), & \varepsilon \in (0,1), \\ \varphi(0)=0, \varphi(1)=\gamma\varphi(\eta), & \psi(0)=0, \psi(1)=\gamma\psi(\eta), \end{cases} \tag{2}$$

where $1<\alpha,\beta<2,\delta,\sigma,\gamma>0,0<\eta<1,\alpha-\sigma\geq 1,\beta-\delta\geq 1,$ $\gamma\eta^{\alpha-1}<1,\gamma\eta^{\beta-1}<1,\mathscr{X},\quad \mathscr{Y}\in C([0,1]\times\mathbb{R}\times\mathbb{R},\mathbb{R}),x_0,y_0\in\mathbb{R}.$ By using the Schauder fixed-point theorem, the authors successfully obtained the existence of solution of the system.

Inspired by these papers, we concern on the following nonlinear Riemann–Liouville fractional differential system of order $0<\alpha\leq 1$:

$$\begin{cases} \mathscr{D}^{\alpha}\varphi(\varepsilon)=\mathscr{F}(\varepsilon,\varphi(\varepsilon),\psi(\varepsilon),\varphi(\theta(\varepsilon))),\\ \mathscr{D}^{\alpha}\psi(\varepsilon)=\mathscr{G}(\varepsilon,\psi(\varepsilon),\varphi(\varepsilon),\psi(\theta(\varepsilon))),\end{cases} \tag{3}$$

where $\varepsilon\in(0,L](0<L<\infty)$, $\mathscr{J}=[0,L]$, $\mathscr{F},\mathscr{G}\in C(\mathscr{J}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, $\theta\in C(\mathscr{J},\mathscr{J})$. Notice that our system contains the unknown functions $\varphi(\varepsilon),\psi(\varepsilon)$ and deviating arguments $\varphi(\theta(\varepsilon)),\psi(\theta(\varepsilon))$.

In order to approximate the solution of the nonlinear Riemann–Liouville fractional differential system mentioned above, we firstly give a new comparison result for fractional differential system. Also, we develop the monotone iterative technique for the system. The advantage of the technique needs no special emphasis [33–40]. It is worth to point that, in this paper, only half pair of upper and lower solutions is assumed to the system, which is weaker than a pair of upper and lower solutions. It is believed that this is also an attempt to apply the monotone iterative method to solve nonlinear Riemann–Liouville fractional differential systems with deviating arguments and families of nonlocal coupled and strip integral boundary conditions.

To this end, we study the following two types of integral boundary conditions:

(i) Nonlocal coupled integral boundary conditions of the form:

$$\begin{cases} \varepsilon^{1-\alpha}\varphi(\varepsilon)|_{\varepsilon=0}=\varepsilon^{1-\alpha}\displaystyle\int_0^{\tau}W(s)\psi(s)\mathrm{d}s+x_0,\\ \varepsilon^{1-\alpha}\psi(\varepsilon)|_{\varepsilon=0}=\varepsilon^{1-\alpha}\displaystyle\int_0^{\tau}W(s)\varphi(s)\mathrm{d}s+y_0,\end{cases} \tag{4}$$

where $0<\tau<L,W(s)\in C(\mathscr{J},(-\infty,0])$, $x_0,y_0\in\mathbb{R}$ and $x_0\leq y_0$.

In the present study, nonlocal type of integral boundary condition with limits of integration involving the parameters $0<\tau<L$ has been introduced. It is worth mentioning that, in practical situations, such nonlocal integral boundary conditions may be regarded as a continuous distribution of arbitrary finite length; for instance, refer to [41].

(ii) Nonlocal strip condition of the form:

$$\begin{cases} \varepsilon^{1-\alpha}\varphi(\varepsilon)|_{\varepsilon=0}=\varepsilon^{1-\alpha}\displaystyle\int_{\nu}^{\tau}V(s)\varphi(s)\mathrm{d}s+x_0,\\ \varepsilon^{1-\alpha}\psi(\varepsilon)|_{\varepsilon=0}=\varepsilon^{1-\alpha}\displaystyle\int_{\nu}^{\tau}V(s)\psi(s)\mathrm{d}s+y_0,\end{cases} \tag{5}$$

where $0<\nu<\tau<L,V(s)\in C(\mathscr{J},[0,+\infty)),x_0,y_0\in\mathbb{R}$ and $x_0\leq y_0$.

In fact, nonlocal strip condition is used to describe a continuous distribution of the values of the unknown function on an arbitrary finite segment of the interval. If $\nu\longrightarrow 0,\tau\longrightarrow L$, the condition is degenerated to a classic integral boundary condition (see [42] for details).

2. Comparison Theorem: The Unique Solution of Linear System

Let

$$C_{1-\alpha}(\mathscr{J})=\left\{\varphi\in C(0,L];\varepsilon^{1-\alpha}\varphi\in C(\mathscr{J})\right\}, \tag{6}$$

with the norm

$$\|\varphi\|_{C_{1-\alpha}}=\left\|\varepsilon^{1-\alpha}\varphi\right\|_C. \tag{7}$$

Next, we provide a comparison result from Wang's paper [43]. Notice that the comparison result is valid for $I(\varepsilon)$ which is a nonnegative bounded integrable function.

Lemma 1. *Let $\xi\in C_{1-\alpha}(\mathscr{J})$be locally Hölder continuous, and ξ satisfies*

$$\begin{cases} \mathscr{D}^{\alpha}\xi(\varepsilon)\geq -I(\varepsilon)\xi(\varepsilon),\\ \varepsilon^{1-\alpha}\xi(\varepsilon)|_{\varepsilon=0}\geq 0,\end{cases} \tag{8}$$

where $I(\varepsilon)$ is a nonnegative bounded integrable function and satisfies $\sup_{\varepsilon\in J}I(\varepsilon)L^{\alpha}\Gamma(1-\alpha)<1$.
Then, $\xi(\varepsilon)\geq 0,\quad$ for all $\varepsilon\in(0,L]$.

Now, we are in a position to prove the following new comparison result for fractional differential system.

Lemma 2 (comparison theorem). *Let $\varphi,\psi\in C_{1-\alpha}(\mathscr{J})$be locally Hölder continuous and satisfy*

$$\begin{cases} \mathscr{D}^{\alpha}\varphi(\varepsilon)\geq -I(\varepsilon)\varphi(\varepsilon)+J(\varepsilon)\psi(\varepsilon), & \varepsilon\in(0,L],\\ \mathscr{D}^{\alpha}\psi(\varepsilon)\geq -I(\varepsilon)\psi(\varepsilon)+J(\varepsilon)\varphi(\varepsilon), & \varepsilon\in(0,L],\\ \varepsilon^{1-\alpha}\varphi(\varepsilon)|_{\varepsilon=0}\geq 0,\\ \varepsilon^{1-\alpha}\psi(\varepsilon)|_{\varepsilon=0}\geq 0,\end{cases} \tag{9}$$

where $I(\varepsilon),J(\varepsilon)$ are nonnegative bounded integrable functions and $I(\varepsilon)\geq J(\varepsilon)\geq 0,\quad$ for all $\varepsilon\in\mathscr{J}$.

If

$$\sup_{\varepsilon\in\mathscr{J}}(I+J)(\varepsilon)L^{\alpha}\Gamma(1-\alpha)<1, \tag{10}$$

then $\varphi(\varepsilon)\geq 0,\psi(\varepsilon)\geq 0,\quad$ for all $\varepsilon\in(0,L]$.

Proof. Put $\xi(\varepsilon)=\varphi(\varepsilon)+\psi(\varepsilon),\quad$ for all $\varepsilon\in(0,L]$. Then, by (9), we have

$$\begin{cases} \mathscr{D}^{\alpha}\xi(\varepsilon)\geq -(I-J)(\varepsilon)\xi(\varepsilon), & \varepsilon\in(0,L],\\ \varepsilon^{1-\alpha}\xi(\varepsilon)|_{\varepsilon=0}\geq 0.\end{cases} \tag{11}$$

Thus, by (11) and Lemma 1, we have that

$$\xi(\varepsilon)\geq 0,\ \text{for}\ \varepsilon\in(0,L],\ i.e.\ \varphi(\varepsilon)+\psi(\varepsilon)\geq 0,\quad \text{for}\ \varepsilon\in(0,L]. \tag{12}$$

Next, we show that $\varphi(\varepsilon)\geq 0, \psi(\varepsilon)\geq 0, \quad$ for $\varepsilon\in(0,L]$. In fact, by (9) and (12), we have that

$$\begin{cases} \mathscr{D}^{\alpha}\varphi(\varepsilon)\geq -(I+J)(\varepsilon)\varphi(\varepsilon), & \varepsilon\in(0,L],\\ \varepsilon^{1-\alpha}\varphi(\varepsilon)|_{\varepsilon=0}\geq 0. \end{cases} \tag{13}$$

By (13) and Lemma 1, we have that $\varphi(\varepsilon)\geq 0, \quad$ for $\varepsilon\in(0,L]$. Similarly, we can show that $\nu(\varepsilon)\geq 0, \quad$ for $\varepsilon\in(0,L]$.

Finally, we consider the linear system:

$$\begin{cases} \mathscr{D}^{\alpha}\varphi(\varepsilon)=\sigma_1(\varepsilon)-I(\varepsilon)\varphi(\varepsilon)-J(\varepsilon)\psi(\varepsilon), & \varepsilon\in(0,L],\\ \mathscr{D}^{\alpha}\psi(\varepsilon)=\sigma_2(\varepsilon)-I(\varepsilon)\psi(\varepsilon)-J(\varepsilon)\varphi(\varepsilon), & \varepsilon\in(0,L],\\ \varepsilon^{1-\alpha}\varphi(\varepsilon)|_{\varepsilon=0}=r_1,\varepsilon^{1-\alpha}\psi(\varepsilon)|_{\varepsilon=0}=r_2, \end{cases} \tag{14}$$

where $I(\varepsilon), J(\varepsilon)$ are nonnegative bounded integrable functions $\sigma_1,\sigma_2\in C(\mathscr{J},\mathbb{R}), r_1, r_2\in\mathbb{R}$.

Lemma 3. *If 10 holds, then the problem 14has a unique system of solutions in* $C_{1-\alpha}([0,L])\times C_{1-\alpha}([0,L])$.

Proof. Let

$$\begin{aligned} \xi_1(\varepsilon)&=\frac{\varphi(\varepsilon)+\psi(\varepsilon)}{2},\\ \xi_2(\varepsilon)&=\frac{\varphi(\varepsilon)-\psi(\varepsilon)}{2}, \quad \varepsilon\in(0,L], \end{aligned} \tag{15}$$

where ξ_1 and ξ_2 solve the problems

$$\begin{cases} \mathscr{D}^{\alpha}\xi_1(\varepsilon)=(\sigma_1+\sigma_2)(\varepsilon)-(I+J)(\varepsilon)\xi_1(\varepsilon), \varepsilon\in(0,L],\\ \varepsilon^{1-\alpha}\xi_1(\varepsilon)|_{\varepsilon=0}=r_1+r_2, \end{cases} \tag{16}$$

and

$$\begin{cases} \mathscr{D}^{\alpha}\xi_2(\varepsilon)=(\sigma_1-\sigma_2)(\varepsilon)-(I-J)(\varepsilon)\xi_2(\varepsilon), & \varepsilon\in(0,L],\\ \varepsilon^{1-\alpha}\xi_2(\varepsilon)|_{\varepsilon=0}=r_1-r_2. \end{cases} \tag{17}$$

It is obvious that the problems (16) and (17) have the unique solution $\xi^*,\xi^{**}\in C_{1-\alpha}([0,L])$, respectively. Since ξ^*,ξ^{**} are unique, then by (14) and (15), we can show that the problem (14) has a unique system of solutions in $C_{1-\alpha}([0,L])\times C_{1-\alpha}([0,L])$. □

3. Extremal Solutions of Nonlinear System

Theorem 1. *Assume that the following holds:*

(H_1) *There exist two locally Hölder continuous functions* $\varphi_0,\psi_0\in C_{1-\alpha}([0,L])$ *satisfying* $\varphi_0(\varepsilon)\leq\psi_0(\varepsilon)$ *such that*

$$\begin{cases} \mathscr{D}^{\alpha}\varphi_0(\varepsilon)\leq\mathscr{F}(\varepsilon,\varphi_0(\varepsilon),\psi_0(\varepsilon),\varphi_0(\theta(\varepsilon))), & \varepsilon\in(0,L],\\ \varepsilon^{1-\alpha}\varphi_0(\varepsilon)|_{\varepsilon=0}\leq\varepsilon^{1-\alpha}\displaystyle\int_0^{\tau}W(s)\psi_0(s)ds+x_0,\\ \mathscr{D}^{\alpha}\psi_0(\varepsilon)\geq\mathscr{G}(\varepsilon,\psi_0(\varepsilon),\varphi_0(\varepsilon),\psi_0(\theta(\varepsilon))), & \varepsilon\in(0,L],\\ \varepsilon^{1-\alpha}\psi_0(\varepsilon)|_{\varepsilon=0}\geq\varepsilon^{1-\alpha}\displaystyle\int_0^{\tau}W(s)\varphi_0(s)ds+y_0. \end{cases} \tag{18}$$

(H_2) *There exist nonnegative bounded integrable functions* $I(\varepsilon), J(\varepsilon)$ *which satisfy* (10), *and* $I(\varepsilon)\geq J(\varepsilon)$, *such that*

$$\begin{cases} \mathscr{F}(\varepsilon,\varphi(\varepsilon),\psi(\varepsilon),\varphi(\theta(\varepsilon)))-\mathscr{F}(\varepsilon,\mu(\varepsilon),\nu(\varepsilon),\mu(\theta(\varepsilon)))\geq -I(\varepsilon)(\varphi-\mu)(\varepsilon)-J(\varepsilon)(\psi-\nu)(\varepsilon),\\ \mathscr{G}(\varepsilon,\varphi(\varepsilon),\psi(\varepsilon),\varphi(\theta(\varepsilon)))-\mathscr{G}(\varepsilon,\mu(\varepsilon),\nu(\varepsilon),\mu(\theta(\varepsilon)))\geq -I(\varepsilon)(\varphi-\mu)(\varepsilon)-J(\varepsilon)(\psi-\nu)(\varepsilon),\\ \mathscr{G}(\varepsilon,\varphi(\varepsilon),\psi(\varepsilon),\varphi(\theta(\varepsilon)))-\mathscr{F}(\varepsilon,\mu(\varepsilon),\nu(\varepsilon),\mu(\theta(\varepsilon)))\geq -I(\varepsilon)(\varphi-\mu)(\varepsilon)-J(\varepsilon)(\psi-\nu)(\varepsilon), \end{cases} \tag{19}$$

where $\varphi_0(\varepsilon)\leq\mu\leq\varphi\leq\psi_0(\varepsilon), \varphi_0(\varepsilon)\leq\psi\leq\nu\leq\psi_0(\varepsilon)$.

Then, (3) and (4) *have extremal systems of solutions* $(\varphi^*,\psi^*)\in[\varphi_0,\psi_0]\times[\varphi_0,\psi_0]$. *Moreover, there exist monotone iterative sequences* $\{\varphi_n\},\{\psi_n\}\subset[\varphi_0,\psi_0]$ *such that* $\varphi_n\longrightarrow\varphi^*, \psi_n\longrightarrow\psi^*\ (n\longrightarrow\infty)$ *uniformly on compact subsets of* $(0,L]$ *and*

$$\varphi_0\leq\varphi_1\leq\cdots\leq\varphi_n\leq\cdots\leq\varphi^*\leq\psi^*\leq\cdots\leq\psi_n\leq\cdots\leq\psi_1\leq\psi_0. \tag{20}$$

Proof. For any $\varphi_{n-1},\psi_{n-1}\in C_{1-\alpha}([0,L])$, $n\geq 1$, considering (14) with

$$\begin{cases} \sigma_1^n(\varepsilon) = \mathscr{F}(\varepsilon, \varphi_{n-1}(\varepsilon), \psi_{n-1}(\varepsilon), \varphi_{n-1}((\theta(\varepsilon))) + I(\varepsilon)\varphi_{n-1}(\varepsilon) + J(\varepsilon)\psi_{n-1}(\varepsilon), \\ \sigma_2^n(\varepsilon) = \mathscr{G}(\varepsilon, \psi_{n-1}(\varepsilon), \varphi_{n-1}(\varepsilon), \psi_{n-1}((\theta(\varepsilon))) + I(\varepsilon)\psi_{n-1}(\varepsilon) + J(\varepsilon)\varphi_{n-1}(\varepsilon), \\ r_1^n = \varepsilon^{1-\alpha}\int_0^\tau W(s)\psi_{n-1}(s)\mathrm{d}s + x_0, r_2^n = \varepsilon^{1-\alpha}\int_0^\tau W(s)\varphi_{n-1}(s)\mathrm{d}s + y_0, \end{cases} \quad (21)$$

we have

$$\begin{cases} \mathscr{D}^\alpha\varphi_n(\varepsilon) = \sigma_1^n(\varepsilon) - I(\varepsilon)\varphi_n(\varepsilon) - J(\varepsilon)\psi_n(\varepsilon), & \varepsilon \in (0, L], \\ \mathscr{D}^\alpha\psi_n(\varepsilon) = \sigma_2^n(\varepsilon) - I(\varepsilon)\psi_n(\varepsilon) - J(\varepsilon)\varphi_n(\varepsilon), & \varepsilon \in (0, L], \\ \varepsilon^{1-\alpha}\varphi_n(\varepsilon)|_{\varepsilon=0} = r_1^n, \varepsilon^{1-\alpha}\psi_n(\varepsilon)|_{\varepsilon=0} = r_2^n. \end{cases} \quad (22)$$

By Lemma 3, we know that (22) has a unique system of solutions in $C_{1-\alpha}([0,L]) \times C_{1-\alpha}([0,L])$.

Now, we show that $\{\varphi_n(\varepsilon)\}, \{\psi_n(\varepsilon)\}$ satisfy

$$\varphi_{n-1} \le \varphi_n \le \psi_n \le \psi_{n-1}, \quad n = 1, 2, \ldots. \quad (23)$$

Let $\xi = \varphi_1 - \varphi_0, \zeta = \psi_0 - \psi_1$. By condition (H_1) and (22), we have that

$$\begin{cases} \mathscr{D}^\alpha\xi(\varepsilon) \ge -I(\varepsilon)\xi(\varepsilon) + J(\varepsilon)\zeta(\varepsilon), & \mathscr{D}^\alpha\zeta(\varepsilon) \ge -I(\varepsilon)\zeta(\varepsilon) + J(\varepsilon)\xi(\varepsilon), \\ \varepsilon^{1-\alpha}\xi(\varepsilon)|_{\varepsilon=0} \ge 0, & \varepsilon^{1-\alpha}\zeta(\varepsilon)|_{\varepsilon=0} \ge 0. \end{cases} \quad (24)$$

Thus, by Lemma 2, we have that $\xi(\varepsilon) \ge 0, \zeta(\varepsilon) \ge 0, \quad \text{forall}\, \varepsilon \in (0, L]$.

Let $w = \psi_1 - \varphi_1$, by condition (H_2) and (22), we get

$$\begin{cases} \mathscr{D}^\alpha w(\varepsilon) & = \mathscr{D}^\alpha\psi_1(\varepsilon) - \mathscr{D}^\alpha\varphi_1(\varepsilon) \\ & = \mathscr{G}(\varepsilon, \psi_0(\varepsilon), \varphi_0(\varepsilon), \psi_0(\theta(\varepsilon))) + I(\varepsilon)\psi_0(\varepsilon) + J(\varepsilon)\varphi_0(\varepsilon) - I(\varepsilon)\psi_1(\varepsilon) - J(\varepsilon)\varphi_1(\varepsilon) \\ & -\mathscr{F}(\varepsilon, \varphi_0(\varepsilon), \psi_0(\varepsilon), \varphi_0(\theta(\varepsilon))) - I(\varepsilon)\varphi_0(\varepsilon) - J(\varepsilon)\psi_0(\varepsilon) + I(\varepsilon)\varphi_1(\varepsilon) + J(\varepsilon)\psi_1(\varepsilon) \\ & \ge -I(\varepsilon)(\psi_0 - \varphi_0)(\varepsilon) - J(\varepsilon)(\varphi_0 - \psi_0)(\varepsilon) + I(\varepsilon)\psi_0(\varepsilon) + J(\varepsilon)\varphi_0(\varepsilon) - I(\varepsilon)\psi_1(\varepsilon) \\ & -J(\varepsilon)\varphi_1(\varepsilon) - I(\varepsilon)\varphi_0(\varepsilon) - J(\varepsilon)\psi_0(\varepsilon) + I(\varepsilon)\varphi_1(\varepsilon) + J(\varepsilon)\psi_1(\varepsilon) \\ & = -(I - J)(\varepsilon)w(\varepsilon). \end{cases} \quad (25)$$

Besides,

$$\varepsilon^{1-\alpha}w(\varepsilon)|_{\varepsilon=0} = \varepsilon^{1-\alpha}\int_0^\tau W(s)(\varphi_0 - \psi_0)(s)\mathrm{d}s + y_0 - x_0 \ge 0. \quad (26)$$

By Lemma 1, we can get $w(\varepsilon) \ge 0, \quad \text{for}\, \varepsilon \in (0, L]$. Therefore, we have the relation $\varphi_0 \le \varphi_1 \le \psi_1 \le \psi_0$.

Assume that $\varphi_{k-1} \le \varphi_k \le \psi_k \le \psi_{k-1}$, for some $k \ge 1$. Then, using the same way as above, by Lemmas 1 and 2 again, we can obtain $\varphi_k \le \varphi_{k+1} \le \psi_{k+1} \le \psi_k$. By induction, it is not difficult to show that

$$\varphi_0 \le \varphi_1 \le \cdots \le \varphi_n \le \cdots \le \psi_n \le \cdots \le \psi_1 \le \psi_0. \quad (27)$$

Employing the standard arguments, we have

$$\lim_{n\longrightarrow\infty}\varphi_n(\varepsilon) = \varphi^*(\varepsilon), \lim_{n\longrightarrow\infty}\psi_n(\varepsilon) = \psi^*(\varepsilon) \quad (28)$$

uniformly on compact subsets of $(0, L]$, and $\varphi^*, \psi^* \in [\varphi_0, \psi_0]$ satisfy (3) and (4). Letting $n \longrightarrow \infty$ in (3.15), we get that (φ^*, ψ^*) is solutions of (3) and (4) in $[\varphi_0, \psi_0] \times [\varphi_0, \psi_0]$ and (20) holds.

Finally, we prove that (φ^*, ψ^*) is extremal solutions of (3) and (4) in $[\varphi_0, \psi_0] \times [\varphi_0, \psi_0]$. If $(\varphi, \psi) \in [\varphi_0, \psi_0] \times [\varphi_0, \psi_0]$ is any solutions of (3) and (4), which means

$$\begin{cases} \mathscr{D}^\alpha\varphi(\varepsilon) = \mathscr{F}(\varepsilon, \varphi(\varepsilon), \psi(\varepsilon), \varphi(\theta(\varepsilon))), & \varepsilon \in (0, L], \\ \mathscr{D}^\alpha\psi(\varepsilon) = \mathscr{G}(\varepsilon, \psi(\varepsilon), \varphi(\varepsilon), \psi(\theta(\varepsilon))), & \varepsilon \in (0, L], \\ \varepsilon^{1-\alpha}\varphi(\varepsilon)|_{\varepsilon=0} = \varepsilon^{1-\alpha}\int_0^\tau W(s)\psi(s)\mathrm{d}s + x_0, \\ \varepsilon^{1-\alpha}\psi(\varepsilon)|_{\varepsilon=0} = \varepsilon^{1-\alpha}\int_0^\tau W(s)\varphi(s)\mathrm{d}s + y_0. \end{cases} \quad (29)$$

By (22), (29), (H_2), and Lemma 2, it is easy to prove that

$$\varphi_n \le \varphi, \psi \le \psi_n, \quad n = 1, 2, \ldots. \tag{30}$$

Taking the limits in (30), we get $\varphi^* \le \varphi, \psi \le \psi^*$, which implies (φ^*, ψ^*) is extremal solutions of (3) and (4) in $[\varphi_0, \psi_0] \times [\varphi_0, \psi_0]$.

This completes the proof. □

We give the following assumption for convenience.

(H_1') There exist two locally Hölder continuous functions $\varphi_0, \psi_0 \in C_{1-\alpha}([0, L])$ satisfying $\varphi_0(\varepsilon) \le \psi_0(\varepsilon)$ such that

$$\begin{cases} \mathscr{D}^\alpha \varphi_0(\varepsilon) \le \mathscr{F}(\varepsilon, \varphi_0(\varepsilon), \psi_0(\varepsilon), \varphi_0(\theta(\varepsilon))), & \varepsilon \in (0, L], \\ \varepsilon^{1-\alpha}\varphi_0(\varepsilon)|_{\varepsilon=0} \le \varepsilon^{1-\alpha}\int_\nu^\tau V(s)\varphi_0(s)\mathrm{d}s + x_0, \\ \mathscr{D}^\alpha \psi_0(\varepsilon) \ge \mathscr{G}(\varepsilon, \psi_0(\varepsilon), \varphi_0(\varepsilon), \psi_0(\theta(\varepsilon))), & \varepsilon \in (0, L], \\ \varepsilon^{1-\alpha}\psi_0(\varepsilon)|_{\varepsilon=0} \ge \varepsilon^{1-\alpha}\int_\nu^\tau V(s)\psi_0(s)\mathrm{d}s + y_0. \end{cases} \tag{31}$$

By a proof similar to Theorem 1, we have the following.

Theorem 2. *Suppose that conditions (H_1') and (H_2) hold. Then, (3) and (5) have extremal system of solutions $(\varphi^*, \psi^*) \in [\varphi_0, \psi_0] \times [\varphi_0, \psi_0]$. Moreover, there exist monotone iterative sequences $\{\varphi_n\}, \{\psi_n\} \subset [\varphi_0, \psi_0]$ such that $\varphi_n \longrightarrow \varphi^*, \psi_n \longrightarrow \psi^*\ (n \longrightarrow \infty)$ uniformly on compact subsets of $(0, L]$ and*

$$\varphi_0 \le \varphi_1 \le \cdots \le \varphi_n \le \cdots \le \varphi^* \le \psi^* \le \cdots \le \psi_n \le \cdots \le \psi_1 \le \psi_0. \tag{32}$$

4. Example

Consider the following problem:

$$\begin{cases} \mathscr{D}^\alpha \varphi(\varepsilon) = \dfrac{1}{15}\varepsilon^3[\varepsilon - \varphi(\varepsilon)] - \dfrac{1}{20}\varepsilon^4\psi^2(\varepsilon) + \dfrac{1}{15}\varepsilon^4\tan\left(\dfrac{\pi}{4}\varphi\left(\dfrac{\varepsilon^2}{2}\right)\right), \\ \mathscr{D}^\alpha \psi(\varepsilon) = \dfrac{1}{15}\varepsilon^3[\varepsilon - \psi(\varepsilon)] - \dfrac{1}{20}\varepsilon^4\varphi^2(\varepsilon) + \dfrac{1}{15}\varepsilon^4\tan\left(\dfrac{\pi}{4}\psi\left(\dfrac{\varepsilon^2}{2}\right)\right), \\ \varepsilon^{1-\alpha}\varphi(\varepsilon)|_{\varepsilon=0} = \varepsilon^{1-\alpha}\displaystyle\int_{1/6}^{2/3}\left(\frac{1}{3} + s^2\right)\varphi(s)\mathrm{d}s, \\ \varepsilon^{1-\alpha}\psi(\varepsilon)|_{\varepsilon=0} = \varepsilon^{1-\alpha}\displaystyle\int_{1/6}^{2/3}\left(\frac{1}{3} + s^2\right)\psi(s)\mathrm{d}s, \end{cases} \tag{33}$$

where $\varepsilon \in \mathscr{J}, \alpha = 1/2$, and $\mathscr{D}^\alpha$ is the standard Riemann–Liouville fractional derivative.

Clearly,

$$\begin{cases} \mathscr{F}(\varepsilon, \varphi(\varepsilon), \psi(\varepsilon), \varphi(\theta(\varepsilon))) = \dfrac{1}{15}\varepsilon^3[\varepsilon - \varphi(\varepsilon)] - \dfrac{1}{20}\varepsilon^4\psi^2(\varepsilon) + \dfrac{1}{15}\varepsilon^4\tan\left(\dfrac{\pi}{4}\varphi(\theta(\varepsilon))\right), \\ \mathscr{G}(\varepsilon, \psi(\varepsilon), \varphi(\varepsilon), \psi(\theta(\varepsilon))) = \dfrac{1}{15}\varepsilon^3[\varepsilon - \psi(\varepsilon)] - \dfrac{1}{20}\varepsilon^4\varphi^2(\varepsilon) + \dfrac{1}{15}\varepsilon^4\tan\left(\dfrac{\pi}{4}\psi(\theta(\varepsilon))\right), \end{cases} \tag{34}$$

where $\theta(\varepsilon) = \varepsilon^2/2$.

Taking $\varphi_0(\varepsilon) = 0, \psi_0(\varepsilon) = 1$, it is easy to show that condition (H_1') of Theorem 2 holds.

On the contrary, we have

$$\begin{cases} \mathscr{F}(\varepsilon, \varphi(\varepsilon), \psi(\varepsilon), \varphi(\theta(\varepsilon))) - \mathscr{F}(\varepsilon, \mu(\varepsilon), \nu(\varepsilon), \mu(\theta(\varepsilon))) \geq -\dfrac{2}{15}\varepsilon^3(\varphi - \mu)(\varepsilon), \\ \mathscr{G}(\varepsilon, \varphi(\varepsilon), \psi(\varepsilon), \varphi(\theta(\varepsilon))) - \mathscr{G}(\varepsilon, \mu(\varepsilon), \nu(\varepsilon), \mu(\theta(\varepsilon))) \geq -\dfrac{2}{15}\varepsilon^3(\varphi - \mu)(\varepsilon), \\ \mathscr{G}(\varepsilon, \varphi(\varepsilon), \psi(\varepsilon), \varphi(\theta(\varepsilon))) - \mathscr{F}(\varepsilon, \mu(\varepsilon), \nu(\varepsilon), \mu(\theta(\varepsilon))) \geq -\dfrac{2}{15}\varepsilon^3(\varphi - \mu)(\varepsilon), \end{cases} \tag{35}$$

where $\varphi_0(\varepsilon) \leq \mu \leq \varphi \leq \psi_0(\varepsilon), \varphi_0(\varepsilon) \leq \psi \leq \nu \leq \psi_0(\varepsilon)$.

We see $I(\varepsilon) = 2/15\varepsilon^3, J(\varepsilon) = 0$, and $\sup_{\varepsilon\in\mathscr{J}}(I + J)(\varepsilon)\mathrm{L}^\alpha\Gamma(1 - \alpha) = 2\sqrt{\pi}/15 < 1$. So, (H_2) holds.

Thus, Theorem 2 is applied to the system (33), and we have the conclusion of Theorem 2.

5. Conclusion

In this paper, by employing the method of upper and lower solutions combined with the monotone iterative technique, we studied a class of nonlinear fractional differential system involving nonlocal strip and coupled integral boundary conditions. Precisely, we considered the following nonlinear Riemann–Liouville fractional differential system:

$$\begin{cases} \mathscr{D}^\alpha\varphi(\varepsilon) = \mathscr{F}(\varepsilon, \varphi(\varepsilon), \psi(\varepsilon), \varphi(\theta(\varepsilon))), \\ \mathscr{D}^\alpha\psi(\varepsilon) = \mathscr{G}(\varepsilon, \psi(\varepsilon), \varphi(\varepsilon), \psi(\theta(\varepsilon))), \end{cases} \tag{36}$$

with two types of integral boundary conditions:

(i) Nonlocal coupled integral boundary conditions of the form:

$$\begin{cases} \varepsilon^{1-\alpha}\varphi(\varepsilon)|_{\varepsilon=0} = \varepsilon^{1-\alpha}\displaystyle\int_0^\tau W(s)\psi(s)\mathrm{d}s + x_0, \\ \varepsilon^{1-\alpha}\psi(\varepsilon)|_{\varepsilon=0} = \varepsilon^{1-\alpha}\displaystyle\int_0^\tau W(s)\varphi(s)\mathrm{d}s + y_0. \end{cases} \tag{37}$$

(ii) Nonlocal strip condition of the form:

$$\begin{cases} \varepsilon^{1-\alpha}\varphi(\varepsilon)|_{\varepsilon=0} = \varepsilon^{1-\alpha}\displaystyle\int_\nu^\tau V(s)\varphi(s)\mathrm{d}s + x_0, \\ \varepsilon^{1-\alpha}\psi(\varepsilon)|_{\varepsilon=0} = \varepsilon^{1-\alpha}\displaystyle\int_\nu^\tau V(s)\psi(s)\mathrm{d}s + y_0. \end{cases} \tag{38}$$

We investigated the existence of extremal system of solutions for the above nonlinear fractional differential system involving nonlocal strip and coupled integral boundary conditions. A new comparison result for fractional differential system was also established, which played an important role in the proof of our main results. It is a contribution to the field of fractional differential system. As an extension of our conclusion, we present an open question, namely, how to develop the existence of extremal system of solutions for the above nonlinear fractional differential system with impulsive effect by the method of upper and lower solutions combined with the monotone iterative technique. The biggest difficulty for this is to perfectly establish new comparison result for fractional differential system with impulsive effect.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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