Research Article

On Solutions of the Matrix Equation $A \circ_l X = B$ with respect to MM -2 Semitensor Product

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MM -2 semitensor product is a new and very useful mathematical tool, which breaks the limitation of traditional matrix multiplication on the dimension of matrices and has a wide application prospect. This article aims to investigate the solutions of the matrix equation $A \circ_l X = B$ with respect to MM -2 semitensor product. The case where the solutions of the equation are vectors is discussed first. Compatible conditions of matrices and the necessary and sufficient condition for the solvability is studied successively. Furthermore, concrete methods of solving the equation are provided. Then, the case where the solutions of the equation are matrices is studied in a similar way. Finally, several examples are given to illustrate the efficiency of the results.

1. Introduction

Matrix equations are a very important part of matrix theory [1], and they are often widely applied in many fields. For example, there have been a lot of researches about equations on economic theory [2], automation and information sciences [3–5], system and control theory [6–8], physics [9, 10], and computing sciences [11, 12]. For these fields, the numerical approximation solutions [13], least squares solutions [14], symmetric positive [15], and definite solutions [16] under various conditions can be obtained by direct or iterative methods. Furthermore, matrix equations are the basis of numerical calculation and are very good at dealing with arrays that are smaller than three dimensions.

But when dealing with high dimensional data arrangement, power system stability control algebraic, Boolean network, and other problems, the general matrix equation theory is hard to work. Cheng [17] proposed the semitensor product (STP) that successfully improved this shortcoming. As a convenient and powerful mathematical tool, it was quickly studied by scholars and applied to numerical mathematics, control system, Boolean

networks, and so on. In 2019, replacing I_k in the definition of STP by J_k , a new matrix product, called the second matrix-matrix semitensor product (MM -2 STP) of matrices is proposed by Cheng [18]. It provides a new way of solving problems in control systems and has a wide application prospect. For example, cross-dimensional systems are very important dimension-free systems in control theory [19]. There have been many mathematic models to describe a cross-dimensional system, such as electric power generators [20], spacecrafts [21], and biological systems [22], and switching is a classical method to settle dimension-varying system problems. But it has the disadvantage of neglecting the dynamics of the system in the dimension-varying process [23]. The MM -2 STP could supply a new way to establish unified form models for such switched systems so that we can discuss cross-dimensional systems better [24–27]. Based on this, the solution of the matrix equation $A \circ_l X = B$ with respect to MM -2 semitensor product is studied in this paper.

To achieve the goal, we will study the matrix equation $A \circ_l X = B$ with respect to MM -2 semitensor product by the following steps. At first, the definitions of the second semitensor product of matrices and some other related

conceptions are given briefly. After that, the solvability of the matrix equation $A \circ_l X = B$ with respect to MM -2 semitensor product will be discussed in matrix-vector and matrix cases, respectively. In both cases, we give the compatible conditions on matrices A and B first, and then we investigate the necessary and sufficient condition for the solvability. In addition, the concrete steps of solving the equation are clarified. Finally, we give some examples to verify the effectiveness of our results.

There are 5 sections contained in this article. Section 2 introduces some notations and definitions which will be used later. Section 3 explores the solvability of the matrix-vector equation $A \circ_l X = B$ with respect to MM -2 semitensor product. The compatible conditions of matrices are proposed and the necessary and sufficient condition for the solvability is established. Moreover, concrete solving methods are derived. Section 4 discusses the solvability of the matrix equation $A \circ_l X = B$ with respect to MM -2 semitensor product in the same way. Compatible conditions, solvability conditions, and concrete solving methods of the matrix equation have also been worked out. Section 5 gives some examples and Section 6 draws the conclusion.

2. Preliminaries

In this article, $\mathbb{C}^n$ denotes the vector space of complex n -tuples and $\mathcal{M}_{m \times n}$ denotes the vector space of $m \times n$ complex matrices. A^T stands for the transpose of a matrix A . $\{m, n\}$ and $\gcd\{m, n\}$ represent the least common multiple and the greatest common divisor of two positive integers m and n , respectively.

Definition 2.1. The Kronecker product of matrices A and B , denoted by $A \otimes B$, is defined as follows [28]:

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{21}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix} \in \mathcal{M}_{mp \times nq}. \quad (1)$$

Definition 2.2. The second left (right) MM -2 semitensor product of matrices A and B , denoted by $A \circ_l B$ ($A \circ_r B$), is defined as [18]

$$\begin{aligned} A \circ_l B &= (A \otimes J_{t/n})(B \otimes J_{t/p}) \in \mathcal{M}_{((mt)/n) \times ((qt)/p)}, \\ A \circ_r B &= (J_{t/n} \otimes A)(J_{t/p} \otimes B) \in \mathcal{M}_{((mt)/n) \times ((qt)/p)}, \end{aligned} \quad (2)$$

where $t = \{n, p\}$, $J_k = (1/k)1_{k \times k}$ is a $k \times k$ matrix with $(1/k)$ as its all entries.

Definition 2.3. The vectorization of $A \in \mathcal{M}_{m \times n}$, denoted by $V_c(A) \in \mathbb{C}^{mn}$, is defined as [28]

$$V_c(A) = (a_{11}, \dots, a_{1n}, a_{21}, \dots, a_{2n}, \dots, a_{m1}, \dots, a_{mn})^T. \quad (3)$$

3. Solution of $A \circ_l X = B$ with X Is a Vector

We now study the solvability of the matrix-vector equation under MM -2 semitensor product,

$$A \circ_l X = B, \quad (4)$$

where $A = [a_{ij}] \in \mathcal{M}_{m \times n}$, $B = [b_{ij}] \in \mathcal{M}_{h \times k}$, and $X \in \mathbb{C}^p$ is an unknown vector to be solved.

Initially, we consider the simple case $m = h$. Then, we will discuss the general case.

3.1. The Simple Case $m = h$. The solvability of the matrix-vector equation (4) under the condition that matrices $A \in \mathcal{M}_{m \times n}$, $B \in \mathcal{M}_{m \times k}$ is studied in this subsection.

At first, similar to the conclusion of Yao in [29], we have the following proposition.

Proposition 1. If matrix-vector equation (4) with $m = h$ has a solution, then (n/k) should be a positive integer, and $X \in \mathbb{C}^p$, $p = (n/k)$.

We call the conditions in Proposition 1 as compatible conditions for matrix-vector equation (4) with $m = h$. They are necessary conditions for matrix-vector equation (4). At this time, we say matrices A and B are compatible, and for facility, the matrices A and B are always assumed compatible in the remainder of this subsection.

By Proposition 1, if matrix-vector equation (4) with $m = h$ has a solution, then $X \in \mathbb{C}^p$, $p = (n/k)$. Therefore, we should inspect whether there is a p -dimension vector that satisfies matrix-vector (4) with $m = h$. Let $X = [x_1 \ x_2 \ \cdots \ x_p]^T \in \mathbb{C}^p$. Then, according to the definition, we have

$$\begin{aligned}
A \circ_1 X &= \frac{1}{(t/n)} (A \otimes 1_{(t/n)}) \cdot \frac{1}{(t/p)} (X \otimes 1_{(t/p)}) \stackrel{(t=n)}{\underset{([p=n/k])}{=}} A \cdot \frac{1}{k} (X \otimes 1_k) = \frac{1}{k} A \cdot \begin{bmatrix} x_1 1_k \\ x_2 1_k \\ \vdots \\ x_p 1_k \end{bmatrix} \\
&= \frac{1}{k} \begin{bmatrix} a_{1,1} & \cdots & a_{1,k} & \vdots & a_{1,k+1} & \cdots & a_{1,2k} & \vdots & \cdots & \cdots & \cdots & \vdots & a_{1,(p-1)k+1} & \cdots & a_{s,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \cdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,k} & \vdots & a_{m,k+1} & \cdots & a_{m,2k} & \vdots & \cdots & \cdots & \cdots & \vdots & a_{m,(p-1)k+1} & \cdots & a_{m,n} \end{bmatrix} \cdot \begin{bmatrix} x_1 1_k \\ x_2 1_k \\ \vdots \\ x_p 1_k \end{bmatrix} \\
&= \frac{1}{k} \begin{bmatrix} \dot{A}_1 & \vdots & \dot{A}_2 & \vdots & \cdots & \vdots & \dot{A}_p \end{bmatrix} \cdot \begin{bmatrix} x_1 1_k \\ x_2 1_k \\ \vdots \\ x_p 1_k \end{bmatrix} \\
&= \frac{1}{k} (x_1 \dot{A}_1 \cdot 1_k + x_2 \dot{A}_2 \cdot 1_k + \cdots + x_p \dot{A}_p \cdot 1_k) = \frac{1}{k} (x_1 \ddot{A}_1 + x_2 \ddot{A}_2 + \cdots + x_p \ddot{A}_p) \\
&= B,
\end{aligned} \tag{5}$$

where $\dot{A}_1, \dot{A}_2, \dots, \dot{A}_p \in \mathcal{M}_{m \times k}$ are the p equal-size blocks of matrix A , and

$$\ddot{A}_i = \dot{A}_i 1_k, \quad i = 1, 2, \dots, p. \tag{6}$$

Let w_{ij} be the sum of all the elements in row i of $\dot{A}_j$, then

$$\ddot{A}_j = \begin{bmatrix} w_{1j} & w_{1j} & \cdots & w_{1j} \\ w_{2j} & w_{2j} & \cdots & w_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ w_{mj} & w_{mj} & \cdots & w_{mj} \end{bmatrix}, \quad i = 1, 2, \dots, m, \quad j = 1, 2, \dots, p. \tag{7}$$

On the one hand, the following theorem can be obtained.

Theorem 1. *Matrix-vector equation (4) with $m = h$ has a solution if and only if $\ddot{A}_1, \ddot{A}_2, \dots, \ddot{A}_p$ and B are linearly dependent in vector space $\mathcal{M}_{m \times k}$. Furthermore, when $\ddot{A}_1, \ddot{A}_2, \dots, \ddot{A}_p$ are linearly independent, the solution would be unique.*

Also, the following corollary can be obtained.

Corollary 1. *If matrix-vector equation (4) with $m = h$ has a solution, it must satisfy*

$$\text{rank} [\ddot{A}_1 \ddot{A}_2 \cdots \ddot{A}_p] = \text{rank} [\ddot{A}_1 \ddot{A}_2 \cdots \ddot{A}_p B]. \tag{8}$$

Through the solving process, we see that equation (5) with $m = h$ is equivalent to the equation as follows:

$$\begin{aligned}
&x_1 V_c(\ddot{A}_1) + x_2 V_c(\ddot{A}_2) + \cdots + x_p V_c(\ddot{A}_p) \\
&= [V_c(\ddot{A}_1) \quad V_c(\ddot{A}_2) \quad \cdots \quad V_c(\ddot{A}_p)] X = k V_c(B) \in \mathbb{C}^{mk}.
\end{aligned} \tag{9}$$

On the other hand, equation (5) can be rewritten as

$$\frac{1}{k} \left(x_1 \begin{bmatrix} w_{11} & w_{11} & \cdots & w_{11} \\ w_{21} & w_{21} & \cdots & w_{21} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m1} & w_{m1} & \cdots & w_{m1} \end{bmatrix} + x_2 \begin{bmatrix} w_{12} & w_{12} & \cdots & w_{12} \\ w_{22} & w_{22} & \cdots & w_{22} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m2} & w_{m2} & \cdots & w_{m2} \end{bmatrix} + \cdots + x_p \begin{bmatrix} w_{1p} & w_{1p} & \cdots & w_{1p} \\ w_{2p} & w_{2p} & \cdots & w_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ w_{mp} & w_{mp} & \cdots & w_{mp} \end{bmatrix} \right) = B. \tag{10}$$

Let matrix $B = (b_{ij})$; it is easy to know that matrix $B = (b_{ij})$ must satisfy $b_{ij} = b_{i1}$, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, k$. That is, B is a matrix in which the elements in the same row are equal.

Then, we can draw a necessary condition as follows.

Theorem 2. *If matrix-vector equation (4) with $m = h$ has a solution, then matrix B must have the following form:*

$$B = \begin{bmatrix} b_{11} & b_{11} & \cdots & b_{11} \\ b_{21} & b_{21} & \cdots & b_{21} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m1} & \cdots & b_{m1} \end{bmatrix} = [C_1(B) \ C_1(B) \ \cdots \ C_1(B)], \quad (11)$$

where $C_j(B)$ denotes the j -th column of B , $j = 1, 2, \dots, k$.

The matrix B in Theorem 2 is said to have the proper form.

Matrix-vector equation (4) with $m = h$ is equivalent to the equation as follows:

$$x_1 \begin{bmatrix} w_{11} \\ w_{21} \\ \vdots \\ w_{m1} \end{bmatrix} + x_2 \begin{bmatrix} w_{12} \\ w_{22} \\ \vdots \\ w_{m2} \end{bmatrix} + \cdots + x_p \begin{bmatrix} w_{1p} \\ w_{2p} \\ \vdots \\ w_{mp} \end{bmatrix} = k \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{h1} \end{bmatrix}. \quad (12)$$

Let

$$W = \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1p} \\ w_{21} & w_{22} & \cdots & w_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m1} & w_{m2} & \cdots & w_{mp} \end{bmatrix}, \quad (13)$$

then we can easily get an equivalent form of matrix-vector equation (4) with $m = h$.

Theorem 3. *The matrix-vector equation (4) with $m = h$ is equivalent to the matrix-vector equation with conventional matrix product as follows:*

$$\frac{1}{k}WX = C_1(B). \quad (14)$$

Also, we can get another corollary.

Corollary 2. *If matrix-vector equation (4) with $m = h$ has a solution, the rank condition should satisfy*

$$\text{rank } W = \text{rank}[WC_1(B)]. \quad (15)$$

The solvability condition in Theorem 1 is consistent with condition (15).

At the same time, let $C_j(\ddot{A}_i)$ to be the j -th column of $\ddot{A}_i$, $i = 1, 2, \dots, p$, $j = 1, 2, \dots, k$, then (8) can be rewritten as

$$\begin{aligned} & \text{rank} [C_1(\ddot{A}_1) \ C_1(\ddot{A}_2) \ \cdots \ C_1(\ddot{A}_p)] \\ &= \text{rank} [C_1(\ddot{A}_1) \ C_1(\ddot{A}_2) \ \cdots \ C_1(\ddot{A}_p) \ C_1(B)], \end{aligned} \quad (16)$$

and (9) can be rewritten as

$$\begin{aligned} & x_1 C_1(\ddot{A}_1) + x_2 C_1(\ddot{A}_2) + \cdots + x_p C_1(\ddot{A}_p) \\ &= [C_1(\ddot{A}_1) \ C_1(\ddot{A}_2) \ \cdots \ C_1(\ddot{A}_p)]X = kC_1(B) \in \mathbb{C}^m. \end{aligned} \quad (17)$$

3.2. The General Case. The solvability of the matrix-vector equation (4) under the condition that matrices $A \in \mathcal{M}_{m \times n}$, $B \in \mathcal{M}_{h \times k}$ is studied in this subsection.

At first, similar to the conclusion of Yao in [29], we have the following proposition.

Proposition 2. *If matrix-vector equation (4) has a solution, then there will be the following: (i) (h/m) and (n/k) are positive integers; (ii) $\gcd\{k, (h/m)\} = 1$, and the solution $X \in \mathbb{C}^p$, $p = ((nh)/(mk))$.*

We call the conditions in Proposition 2 as compatible conditions for matrix-vector equation (4). They are necessary conditions. At this time, we say matrices A and B are compatible, and for facility, the matrices A and B are always assumed compatible in the remainder of this subsection.

Now, we explore the necessary condition for the matrix-vector equation first.

Theorem 4. *If matrix-vector equation (4) has a solution, then matrix $B = ((b_{ij}), i = 1, 2, \dots, m, j = 1, 2, \dots, k)$ must have the following form:*

$$B = \begin{bmatrix} \text{Row}_1(B) \\ \vdots \\ \text{Row}_{(h/m)}(B) \\ \cdots \\ \vdots \\ \cdots \\ \text{Row}_{h-(h/m)+1}(B) \\ \vdots \\ \text{Row}_h(B) \end{bmatrix} = \begin{bmatrix} \text{Block}_1(B) \\ \vdots \\ \text{Block}_m(B) \end{bmatrix}, \quad (18)$$

where

$$\text{Block}_s(B) = \begin{bmatrix} b'_s & b'_s & \cdots & \cdots & b'_s & b'_s \\ \vdots & \vdots & \cdots & \cdots & \vdots & \vdots \\ b'_s & b'_s & \cdots & \cdots & b'_s & b'_s \end{bmatrix}$$

$$= b'_s 1_{(h/m) \times k} \in \mathcal{M}_{(h/m) \times k}, \quad (19)$$

$$b'_s = b_{(s-1)(h/m)+i,j}, s = 1, \dots,$$

$$m, i = 1, \dots, \frac{h}{m}, j = 1, \dots, k.$$

That is

$$B = \begin{bmatrix} b'_1 \\ b'_2 \\ \vdots \\ b'_m \end{bmatrix} \otimes 1_{(h/m) \times k}. \quad (20)$$

Proof. According to Proposition 2, the solution X of matrix-vector equation (4) belongs to $\mathbb{C}^p$, as $p = ((nh)/(mk))$; let $k = l_1 \cdot (h/m) + l_2$, where l_1, l_2 are integers. Thus, for $s = 1, \dots, m$, we can get

$$\begin{aligned} \text{Row}_s(A) \circ_1 X &= \frac{m}{h} (\text{Row}_s(A) \otimes 1_{(h/m)}) \cdot \frac{1}{k} (X \otimes 1_k) \\ &= \frac{m}{hk} \begin{bmatrix} a_{s,1} 1_{(h/m)} & a_{s,2} 1_{(h/m)} & \cdots & a_{s,n} 1_{(h/m)} \end{bmatrix} \cdot \begin{bmatrix} x_1 1_k \\ x_2 1_k \\ \vdots \\ x_p 1_k \end{bmatrix} \\ &= \frac{m}{hk} \left(x_1 \begin{bmatrix} a_{s,1} 1_{(h/m)} & a_{s,2} 1_{(h/m)} & \cdots & a_{s,l_1} 1_{(h/m) \times l_2} \end{bmatrix} \cdot 1_k \right. \\ &\quad + x_2 \begin{bmatrix} a_{s,l_1+1} 1_{(h/m) \times (h/m) - l_2} & a_{s,l_1+2} 1_{(h/m)} & \cdots & a_{s,[(2k)/(h/m)]+1} 1_{(h/m) \times \text{mod}((2k)/(h/m))} \end{bmatrix} \cdot 1_k \\ &\quad + \cdots + x_{(h/m)} \begin{bmatrix} a_{s,k-l_1} 1_{(h/m) \times l_2} & \cdots & a_{s,k} 1_{(h/m)} \end{bmatrix} \cdot 1_k + x_{(h/m)+1} \begin{bmatrix} a_{s,k+1} 1_{(h/m)} & \cdots & a_{s,k+l_1} 1_{(h/m)} & a_{s,k+l_1+1} 1_{(h/m) \times l_2} \end{bmatrix} \cdot 1_k \\ &\quad \left. + \cdots + x_p \begin{bmatrix} a_{s,n-l_1} 1_{(h/m) \times l_2} & \cdots & a_{s,n} 1_{(h/m)} \end{bmatrix} \cdot 1_k \right) \\ &= \frac{m}{hk} \left(x_1 \begin{bmatrix} \frac{h}{m} a_{s,1} + \frac{h}{m} a_{s,2} + \cdots + \frac{h}{m} a_{s,l_1} + l_2 a_{s,l_1+1} & \cdots & \frac{h}{m} a_{s,1} + \frac{h}{m} a_{s,2} + \cdots + \frac{h}{m} a_{s,l_1} + l_2 a_{s,l_1+1} \\ \vdots & \ddots & \vdots \end{bmatrix} \right. \\ &\quad + x_2 \begin{bmatrix} \left(\frac{h}{m} - l_2 \right) a_{s,l_1+1} + \frac{h}{m} a_{s,l_1+2} + \cdots + \text{mod} \left(\frac{2k}{(h/m)} \right) a_{s,[(2k)/(h/m)]+1} & \cdots & \left(\frac{h}{m} - l_2 \right) a_{s,l_1+1} + \frac{h}{m} a_{s,l_1+2} + \cdots + \text{mod} \left(\frac{2k}{(h/m)} \right) a_{s,[(2k)/(h/m)]+1} \\ \vdots & \ddots & \vdots \end{bmatrix} \\ &\quad + \cdots \\ &\quad \left. + x_{(h/m)} \begin{bmatrix} l_2 a_{s,k-l_1} + \frac{h}{m} a_{s,k-l_1+1} + \cdots + \frac{h}{m} a_{s,k} & \cdots & l_2 a_{s,k-l_1} + \frac{h}{m} a_{s,k-l_1+1} + \cdots + \frac{h}{m} a_{s,k} \\ \vdots & \ddots & \vdots \end{bmatrix} \right. \\ &\quad \left. + x_{(h/m)+1} \begin{bmatrix} l_2 a_{s,k-l_1} + \frac{h}{m} a_{s,k-l_1+1} + \cdots + \frac{h}{m} a_{s,k} & \cdots & l_2 a_{s,k-l_1} + \frac{h}{m} a_{s,k-l_1+1} + \cdots + \frac{h}{m} a_{s,k} \end{bmatrix} \right) \end{aligned}$$

$$\begin{aligned}
& + x_{(h/m)+1} \begin{bmatrix} \frac{h}{m}a_{s,k+1} + \dots + \frac{h}{m}a_{s,k+l_1} + l_2a_{s,k+l_1+1} & \dots & \frac{h}{m}a_{s,k+1} + \dots + \frac{h}{m}a_{s,k+l_1} + l_2a_{s,k+l_1+1} \\ \vdots & \ddots & \vdots \\ \frac{h}{m}a_{s,k+1} + \dots + \frac{h}{m}a_{s,k+l_1} + l_2a_{s,k+l_1+1} & \dots & \frac{h}{m}a_{s,k+1} + \dots + \frac{h}{m}a_{s,k+l_1} + l_2a_{s,k+l_1+1} \end{bmatrix} \\
& + \dots \\
& + x_p \begin{bmatrix} l_2a_{s,n-l_1} + \frac{h}{m}a_{s,n-l_1+1} + \dots + \frac{h}{m}a_{s,n} & \dots & l_2a_{s,n-l_1} + \frac{h}{m}a_{s,n-l_1+1} + \dots + \frac{h}{m}a_{s,n} \\ \vdots & \ddots & \vdots \\ l_2a_{s,n-l_1} + \frac{h}{m}a_{s,n-l_1+1} + \dots + \frac{h}{m}a_{s,n} & \dots & l_2a_{s,n-l_1} + \frac{h}{m}a_{s,n-l_1+1} + \dots + \frac{h}{m}a_{s,n} \end{bmatrix} \\
& = \frac{m}{hk} \left[x_1 \left(\frac{h}{m}a_{s,1} + \frac{h}{m}a_{s,2} + \dots + \frac{h}{m}a_{s,l_1} + l_2a_{s,l_1+1} \right) \cdot 1_k \right. \\
& \quad + x_2 \left[\left(\frac{h}{m} - l_2 \right) a_{s,l_1+1} + \frac{h}{m}a_{s,l_1+2} + \dots + \text{mod} \left(\frac{2k}{(h/m)} \right) a_{s,[(2k)/(h/m)]+1} \right] \cdot 1_k \\
& \quad + \dots + x_{(h/m)} \left(l_2a_{s,k-l_1} + \frac{h}{m}a_{s,k-l_1+1} + \dots + \frac{h}{m}a_{s,k} \right) \cdot 1_k \\
& \quad + x_{(h/m)+1} \left(\frac{h}{m}a_{s,k+1} + \dots + \frac{h}{m}a_{s,k+l_1} + l_2a_{s,k+l_1+1} \right) \cdot 1_k \\
& \quad + \dots + x_p \left(l_2a_{s,n-l_1} + \frac{h}{m}a_{s,n-l_1+1} + \dots + \frac{h}{m}a_{s,n} \right) \cdot 1_k \Big] \\
& = \frac{m}{hk} \left[\left(\frac{h}{m}a_{s,1} + \frac{h}{m}a_{s,2} + \dots + \frac{h}{m}a_{s,l_1} + l_2a_{s,l_1+1} \right) x_1 \right. \\
& \quad + \left[\left(\frac{h}{m} - l_2 \right) a_{s,l_1+1} + \frac{h}{m}a_{s,l_1+2} + \dots + \text{mod} \left(\frac{2k}{(h/m)} \right) a_{s,[(2k)/(h/m)]+1} \right] x_2 \\
& \quad + \dots \\
& \quad + \left(l_2a_{s,k-l_1} + \frac{h}{m}a_{s,k-l_1+1} + \dots + \frac{h}{m}a_{s,k} \right) x_{(h/m)} \\
& \quad + \left(\frac{h}{m}a_{s,k+1} + \dots + \frac{h}{m}a_{s,k+l_1} + l_2a_{s,k+l_1+1} \right) x_{(h/m)+1} \\
& \quad + \dots + \left(l_2a_{s,n-l_1} + \frac{h}{m}a_{s,n-l_1+1} + \dots + \frac{h}{m}a_{s,n} \right) x_p \Big] \cdot 1_k \\
& = \text{Block}_s(B).
\end{aligned} \tag{21}$$

Thus, $\text{Block}_s(B)$ are matrices having the form as follows:

$$\text{Block}_s(B) = \begin{bmatrix} b'_s & b'_s & \dots & \dots & b'_s & b'_s \\ \vdots & \vdots & \dots & \dots & \vdots & \vdots \\ b'_s & b'_s & \dots & \dots & b'_s & b'_s \end{bmatrix}. \tag{22}$$

The proof is completed. $\square$

The matrix B in Theorem 4 is said to have the proper form.

Then, by the proof of Theorem 1, we can draw the following theorem.

Theorem 5. Solving matrix-vector equation (4) is equivalent to solving the following equation system:

$$\begin{aligned}
& \frac{m}{hk} \left[\left(\frac{h}{m} a_{s,1} + \frac{h}{m} a_{s,2} + \cdots + \frac{h}{m} a_{s,l_1} + l_2 a_{s,l_1+1} \right) x_1 \right. \\
& + \left[\left(\frac{h}{m} - l_2 \right) a_{s,l_1+1} + \frac{h}{m} a_{s,l_1+2} \right. \\
& + \cdots + \left. \left. \text{mod} \left(\frac{2k}{(h/m)} \right) a_{s,[(2k)/(h/m)]+1} \right] x_2 + \cdots \right. \\
& + \left(l_2 a_{s,k-l_1} + \frac{h}{m} a_{s,k-l_1+1} + \cdots + \frac{h}{m} a_{s,k} \right) x_{(h/m)} \\
& + \left(\frac{h}{m} a_{s,k+1} + \cdots + \frac{h}{m} a_{s,k+l_1} + l_2 a_{s,k+l_1+1} \right) x_{(h/m)+1} \\
& + \cdots + \left(l_2 a_{s,n-l_1} + \frac{h}{m} a_{s,n-l_1+1} + \cdots + \frac{h}{m} a_{s,n} \right) x_p \left. \right] \\
& = b'_s, \quad s = 1, \dots, m.
\end{aligned} \tag{23}$$

4. Solution of $A \circ_l X = B$ with X Is a Matrix

We now study the solvability of the matrix equation under MM -2 semitensor product,

$$A \circ_l X = B, \tag{24}$$

where $A = [a_{ij}] \in \mathcal{M}_{m \times n}$, $B = [b_{ij}] \in \mathcal{M}_{h \times k}$, and $X \in \mathcal{M}_{p \times q}$ is an unknown matrix to be solved.

4.1. The Simple Case $m = h$. The solvability of the matrix equation under the condition that matrices $A \in \mathcal{M}_{m \times n}$, $B \in \mathcal{M}_{m \times k}$ is studied in this subsection.

At first, similar to the conclusion of Yao in [29], we have the following proposition.

Proposition 3. *If matrix equation (24) with $m = h$ has a solution, then $X \in \mathcal{M}_{p \times q}$, where $p = (n/\alpha)$, $q = (k/\alpha)$, α is a common divisor of n and k .*

We call the conditions in Proposition 3 as compatible conditions for matrix-vector equation (24) with $m = h$. They are necessary conditions for matrix-vector equation (24). At this time, we say matrices A and B are compatible, and for

facility, the matrices A and B are always assumed compatible in the remainder of this subsection.

Remark 1. Let $\alpha_i, i = 1, \dots, d$ to be all the common divisor of n and k . According to Proposition 3, matrix equation (24) with $m = h$ may have solution of size $p_i \times q_i$, where $p_i = (n/\alpha_i)$, $q_i = (k/\alpha_i)$, $i = 1, \dots, d$. At this time, these sizes are called admissible sizes, and we can see some relations between solutions of different admissible sizes.

- (1) Let $p_1 \times q_1, p_2 \times q_2$ be two admissible sizes and $1 < (p_2/p_1) = (q_2/q_1) \in \mathbb{Z}$, for the two equations, as follows:

$$A \circ_1 X = B, \quad X \in \mathcal{M}_{p_1 \times q_1}, \tag{25}$$

$$A \circ_1 X = B, \quad X \in \mathcal{M}_{p_2 \times q_2}. \tag{26}$$

If matrix equation (25) has a solution X , then $(p_1/p_2)(X \otimes 1_{(p_2/p_1)})$ is a solution of matrix equation (26); reversely, if matrix equation (26) has unique solution, the solution of equation (25), if exists, would be unique.

- (2) Let $\bar{\alpha} = \gcd\{n, k\}$, $\bar{p} = (n/\bar{\alpha})$, and $\bar{q} = (k/\bar{\alpha})$. If matrix equation (24) with $m = h$ has a minimum size $\bar{p} \times \bar{q}$ solution, then it has all admissible size solutions.
- (3) If $\bar{\alpha} = 1$, matrix equation (24) with $m = h$ will have only one admissible size $n \times k$, and at this time, it is a conventional one.
- (4) If $\text{rank}(A) = n$, every admissible size solution, if exists, would be unique.

Next, we will study the solvability of matrix equation (24) with $m = h$. According to Remark 1, the minimum size $\bar{p} \times \bar{q}$ solutions should be considered first, then the solutions for other admissible sizes can be derived in the same way.

Let $X = (x_{ij}) = [X_1 \ X_2 \ \dots \ X_{\bar{q}}]$, $i = 1, \dots, \bar{p}$, $j = 1, \dots, \bar{q}$. By the definition, matrix equation (24) with $m = h$ can be rewritten as

$$A \circ_1 X = A \circ_1 [X_1 \ X_2 \ \dots \ X_{\bar{q}}] = [\dot{B}_1 \ \dot{B}_2 \ \dots \ \dot{B}_{\bar{q}}], \tag{27}$$

where $\dot{B}_1 \ \dot{B}_2 \ \dots \ \dot{B}_{\bar{q}}$ are $\bar{q}$ equal-size blocks of matrix B , $\dot{B}_i \in \mathcal{M}_{m \times \bar{\alpha}}, i = 1, \dots, \bar{q}$. Thus, matrix equation (24) with $m = h$ is equivalent to the following matrix-vector equations under MM -2 semitensor product:

$$A^\circ_1 X_i = \dot{B}_i, \quad X_i \in \mathbb{C}^{\bar{p}}, i = 1, \dots, \bar{q}. \quad (28) \quad \text{As}$$

$$\begin{aligned} A^\circ_1 X_i &= A \cdot \frac{1}{(t/\bar{p})} (X_i \otimes 1_{(t/\bar{p})}) \stackrel{(t=n)}{=} \stackrel{(\bar{p}=n/\bar{\alpha})}{=} A \cdot \frac{1}{\bar{\alpha}} (X_i \otimes 1_{\bar{\alpha}}) = \frac{1}{\bar{\alpha}} A \cdot \begin{bmatrix} x_{1i} 1_{\bar{\alpha}} & x_{2i} 1_{\bar{\alpha}} & \cdots & x_{\bar{p}i} 1_{\bar{\alpha}} \end{bmatrix} \\ &= \frac{1}{\bar{\alpha}} \begin{bmatrix} a_{1,1} & \cdots & a_{1,\bar{\alpha}} & \vdots & a_{1,\bar{\alpha}+1} & \cdots & a_{1,2\bar{\alpha}} & \vdots & \cdots & \cdots & \cdots & \vdots & a_{1,(\bar{p}-1)\bar{\alpha}+1} & \cdots & a_{s,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \cdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,\bar{\alpha}} & \vdots & a_{m,\bar{\alpha}+1} & \cdots & a_{m,2\bar{\alpha}} & \vdots & \cdots & \cdots & \cdots & \vdots & a_{m,(\bar{p}-1)\bar{\alpha}+1} & \cdots & a_{m,n} \end{bmatrix} \cdot \begin{bmatrix} x_{1i} 1_{\bar{\alpha}} \\ x_{2i} 1_{\bar{\alpha}} \\ \vdots \\ x_{\bar{p}i} 1_{\bar{\alpha}} \end{bmatrix} \\ &= \frac{1}{\bar{\alpha}} \begin{bmatrix} \dot{A}_1 & \vdots & \dot{A}_2 & \vdots & \cdots & \vdots & \dot{A}_{\bar{p}} \end{bmatrix} \cdot \begin{bmatrix} x_{1i} 1_{\bar{\alpha}} \\ x_{2i} 1_{\bar{\alpha}} \\ \vdots \\ x_{\bar{p}i} 1_{\bar{\alpha}} \end{bmatrix} \\ &= \frac{1}{\bar{\alpha}} (x_{1i} \dot{A}_1 \cdot 1_{\bar{\alpha}} + x_{2i} \dot{A}_2 \cdot 1_{\bar{\alpha}} + \cdots + x_{\bar{p}i} \dot{A}_{\bar{p}} \cdot 1_{\bar{\alpha}}) (x_{1i} \ddot{A}_1 + x_{2i} \ddot{A}_2 + \cdots + x_{\bar{p}i} \ddot{A}_{\bar{p}}) \\ &= \dot{B}_i \in \mathcal{M}_{m \times \bar{\alpha}}. \end{aligned} \quad (29)$$

where $\dot{A}_1, \dot{A}_2, \dots, \dot{A}_{\bar{p}}$ are the p equal-size blocks of matrix A , belonging to $\mathcal{M}_{m \times \bar{\alpha}}$, and

$$\ddot{A}_s = \dot{A}_i 1_{\bar{\alpha}}, \quad s = 1, 2, \dots, \bar{p}. \quad (30)$$

Let w_{rs} be the sum of all the elements in row r of $\dot{A}_s$, then

$$\ddot{A}_s = \begin{bmatrix} w_{1s} & w_{1s} & \cdots & w_{1s} \\ w_{2s} & w_{2s} & \cdots & w_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ w_{ms} & w_{ms} & \cdots & w_{ms} \end{bmatrix}, \quad (31)$$

$$r = 1, 2, \dots, m, s = 1, 2, \dots, \bar{p}.$$

Thus, on the one hand, we have the following necessary and sufficient condition.

Theorem 6. For matrix equation (24) with $m = h$, it has a solution belonging to $\mathcal{M}_{\bar{p} \times \bar{q}}$ if and only if $\ddot{A}_1, \ddot{A}_2, \dots, \ddot{A}_{\bar{p}}$ and $\dot{B}_i, i = 1, 2, \dots, \bar{q}$ are linearly dependent in vector space $\mathcal{M}_{m \times \bar{\alpha}}$. Furthermore, when $\dot{A}_1, \dot{A}_2, \dots, \dot{A}_{\bar{p}}$ are linearly independent, the solution would be unique.

Simultaneously, we can get a necessary but not sufficient condition.

Corollary 3. If matrix-vector equation (24) with $m = h$ has a solution belonging to $\mathcal{M}_{\bar{p} \times \bar{q}}$, it must satisfy

$$\text{rank}[\ddot{A}_1 \ddot{A}_2 \cdot \ddot{A}_{\bar{p}}] = \text{rank}[\ddot{A}_1 \ddot{A}_2 \cdot \ddot{A}_{\bar{p}} B]. \quad (32)$$

On the other hand, equation (29) can be rewritten as

$$\frac{1}{\bar{\alpha}} \left(x_{1i} \begin{bmatrix} w_{11} & w_{11} & \cdots & w_{11} \\ w_{21} & w_{21} & \cdots & w_{21} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m1} & w_{m1} & \cdots & w_{m1} \end{bmatrix} + x_{2i} \begin{bmatrix} w_{12} & w_{12} & \cdots & w_{12} \\ w_{22} & w_{22} & \cdots & w_{22} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m2} & w_{m2} & \cdots & w_{m2} \end{bmatrix} + \cdots + x_{\bar{p}i} \begin{bmatrix} w_{1\bar{p}} & w_{1\bar{p}} & \cdots & w_{1\bar{p}} \\ w_{2\bar{p}} & w_{2\bar{p}} & \cdots & w_{2\bar{p}} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m\bar{p}} & w_{m\bar{p}} & \cdots & w_{m\bar{p}} \end{bmatrix} \right) = \dot{B}_i, \quad (33)$$

$i = 1, 2, \dots, \bar{q}$.

By equation (33), it is easy to know that $\dot{B}_i$ are matrices in which the elements in the same row are equal. Let $C_j(\dot{B}_i) \in \mathbb{C}^m$ denote the j -th column of $\dot{B}_i$, $j = 1, 2, \dots, \bar{\alpha}$. That is,

$$\dot{B}_i = [C_1(\dot{B}_i) \ C_1(\dot{B}_i) \ \dots \ C_1(\dot{B}_i)]. \quad (34)$$

Let

$$C_1(\dot{B}_i) = \begin{bmatrix} b_1^{(i)} \\ b_2^{(i)} \\ \vdots \\ b_m^{(i)} \end{bmatrix}, \quad (35)$$

and matrix $B = (b_{u,v})$, $u = 1, 2, \dots, m$, $v = 1, 2, \dots, k$. By comparison, we can get $b_t^{(i)} = b_{t,(i-1)\bar{\alpha}+v} + v$, $i = 1, \dots, \bar{q}$, $t = 1, \dots, m$, $v = 1, \dots, \bar{\alpha}$, and therefore, we have

$$\dot{B}_i = [C_1(\dot{B}_i) \ \dots \ C_1(\dot{B}_i)] = \begin{bmatrix} b_1^{(i)} & \dots & b_1^{(i)} \\ b_2^{(i)} & \dots & b_2^{(i)} \\ \vdots & \ddots & \vdots \\ b_m^{(i)} & \dots & b_m^{(i)} \end{bmatrix} = \begin{bmatrix} b_{1,(i-1)\bar{\alpha}+1} & \dots & b_{1,(i-1)\bar{\alpha}+1} \\ b_{2,(i-1)\bar{\alpha}+1} & \dots & b_{2,(i-1)\bar{\alpha}+1} \\ \vdots & \ddots & \vdots \\ b_{m,(i-1)\bar{\alpha}+1} & \dots & b_{m,(i-1)\bar{\alpha}+1} \end{bmatrix}. \quad (36)$$

Then, we can draw a necessary condition as follows.

$\bar{q}$ blocks, and furthermore, matrix B must have the following form:

Theorem 7. If matrix-vector equation (24) with $m = h$ has a solution belonging to $\mathcal{M}_{p \times \bar{q}}$, then matrix B can be divided into

$$\begin{aligned} B &= [\text{Block}_1(B) \ \dots \ \text{Block}_i(B) \ \dots \ \text{Block}_{\bar{q}}(B)] \\ &= [\dot{B}_1 \ \dots \ \dot{B}_i \ \dots \ \dot{B}_{\bar{q}}] \\ &= [C_1(\dot{B}_1) \ \dots \ C_1(\dot{B}_1) \ : \ \dots \ \dots \ : \ C_1(\dot{B}_i) \ \dots \ C_1(\dot{B}_i) \ : \ \dots \ \dots \ : \ C_1(\dot{B}_{\bar{q}}) \ \dots \ C_1(\dot{B}_{\bar{q}})] \\ &= \begin{bmatrix} b_1^{(1)} & \dots & b_1^{(1)} & : & \dots & \dots & : & b_1^{(i)} & \dots & b_1^{(i)} & : & \dots & \dots & : & b_1^{(\bar{q})} & \dots & b_1^{(\bar{q})} \\ \vdots & \ddots & \vdots & : & \dots & \dots & : & \vdots & \ddots & \vdots & : & \dots & \dots & : & \vdots & \ddots & \vdots \\ b_m^{(1)} & \dots & b_m^{(1)} & : & \dots & \dots & : & b_m^{(i)} & \dots & b_m^{(i)} & : & \dots & \dots & : & b_m^{(\bar{q})} & \dots & b_m^{(\bar{q})} \end{bmatrix} \\ &= \begin{bmatrix} b_{1,1} & \dots & b_{1,1} & : & \dots & \dots & : & b_{1,(i-1)\bar{\alpha}+1} & \dots & b_{1,(i-1)\bar{\alpha}+1} & : & \dots & \dots & : & b_{1,(\bar{q}-1)\bar{\alpha}+1} & \dots & b_{1,(\bar{q}-1)\bar{\alpha}+1} \\ \vdots & \ddots & \vdots & : & \dots & \dots & : & \vdots & \ddots & \vdots & : & \dots & \dots & : & \vdots & \ddots & \vdots \\ b_{m,1} & \dots & b_{m,1} & : & \dots & \dots & : & b_{m,(i-1)\bar{\alpha}+1} & \dots & b_{m,(i-1)\bar{\alpha}+1} & : & \dots & \dots & : & b_{m,(\bar{q}-1)\bar{\alpha}+1} & \dots & b_{m,(\bar{q}-1)\bar{\alpha}+1} \end{bmatrix}. \end{aligned} \quad (37)$$

The matrix B in Theorem 7 is said to have the proper form. and vector equation (24) with $m = h$ is equivalent to the equation as follows:

$$x_{1i} \begin{bmatrix} w_{11} \\ w_{21} \\ \vdots \\ w_{m1} \end{bmatrix} + x_{2i} \begin{bmatrix} w_{12} \\ w_{22} \\ \vdots \\ w_{m2} \end{bmatrix} + \dots + x_{\bar{p}i} \begin{bmatrix} w_{1\bar{p}} \\ w_{2\bar{p}} \\ \vdots \\ w_{m\bar{p}} \end{bmatrix} = \bar{\alpha} \begin{bmatrix} b_{1,(i-1)\bar{\alpha}+1} \\ b_{2,(i-1)\bar{\alpha}+1} \\ \vdots \\ b_{m,(i-1)\bar{\alpha}+1} \end{bmatrix}, \quad i = 1, \dots, \bar{q}. \quad (38)$$

Let

$$W = \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1\bar{p}} \\ w_{21} & w_{22} & \cdots & w_{2\bar{p}} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m1} & w_{m2} & \cdots & w_{m\bar{p}} \end{bmatrix}, \quad (39)$$

then we can easily get an equivalent form of matrix-vector equation (24) with $m = h$.

Theorem 8. *The matrix-vector equation (24) with $m = h$ is equivalent to the matrix-vector equation with conventional matrix product as follows:*

$$\frac{1}{\alpha}WX = [C_1(\dot{B}_1) \ C_1(\dot{B}_2) \ \cdots \ \cdots \ C_1(\dot{B}_{\bar{q}})]. \quad (40)$$

$$\begin{aligned} & \text{rank}[C_1(\ddot{A}_1) \ C_1(\ddot{A}_2) \ \cdots \ C_1(\ddot{A}_{\bar{p}})] \\ &= \text{rank}[C_1(\ddot{A}_1) \ C_1(\ddot{A}_2) \ \cdots \ C_1(\ddot{A}_{\bar{p}}) \ C_1(\dot{B}_1) \ C_1(\dot{B}_2) \ \cdots \ C_1(\dot{B}_{\bar{q}})]. \end{aligned} \quad (42)$$

4.2. The General Case. The solvability of the matrix-vector equation (24) under the condition that matrices $A \in \mathcal{M}_{m \times n}$, $B \in \mathcal{M}_{h \times k}$ is studied in this subsection.

At first, similar to the conclusion of Yao et al. in [29], we have the following proposition.

Proposition 4. *If matrix equation (24) has a solution, then there will be the following: (i) (h/m) is a positive integer; (ii) the solution $X \in \mathcal{M}_{p \times q}$, $p = ((nh)/(\alpha m))$, $q = (k/\alpha)$, where α is a common divisor of n and k , which satisfies $\gcd\{\alpha, (h/m)\} = 1$.*

We call the conditions in Proposition 4 as compatible conditions for matrix equation (24). They are necessary conditions. At this time, we say matrices A and B are compatible, and for facility, the matrices A and B are always assumed compatible in the remainder of this subsection.

Remark 2

- (1) The condition $m|h$ in Proposition 4 is just a necessary condition.
- (2) The sizes in Proposition 4 are called admissible sizes. When $\gcd\{n, k\} | (h/m)$, it only has one admissible size. At this time, we see that $\alpha = 1$, $p = ((nh)/m)$, $q = k$, and matrix equation (14) is just a conventional matrix equation $(m/h)(A \otimes 1_{(h/m)})X = B$.
- (3) Supposing that $p_1 \times q_1, p_2 \times q_2$ are two admissible sizes and $1 < (p_2/p_1) = (q_2/q_1) \in \mathbb{Z}$, for the two equations, as follows:

Also, we can get another corollary.

Corollary 4. *If matrix-vector equation (24) with $m = h$ has a solution, the rank condition should satisfy*

$$\text{rank } W = \text{rank}[WC_1(\dot{B}_1) \ C_1(\dot{B}_2) \ \cdots \ \cdots \ C_1(\dot{B}_{\bar{q}})]. \quad (41)$$

The solvability condition in Theorem 7 is consistent with condition (41).

At the same time, let $C_r(\ddot{A}_s)$ to be the r -th column of $\ddot{A}_s$, $s = 1, 2, \dots, \bar{p}$, $r = 1, 2, \dots, \bar{\alpha}$, then equations (32) and (40) can be rewritten as

$$A^{\circ_1}X = B, \quad X \in \mathcal{M}_{p_1 \times q_1}, \quad (43)$$

$$A^{\circ_1}X = B, \quad X \in \mathcal{M}_{p_2 \times q_2}. \quad (44)$$

- (4) If matrix equation (43) has a solution X , then $(p_1/p_2)(X \otimes 1_{(p_2/p_1)})$ is a solution of matrix equation (44); reversely, if equation (44) has unique solution, the solution of equation (43), if exists, would be unique.
- (5) Denote $\bar{p} = ((\gcd\{n, k\})/(\gcd\{n, k, (h/m)\}))$, $\bar{p} = ((nh)/(m\bar{\beta}))$, $\bar{q} = (k/\bar{\beta})$. If matrix equation (24) has a minimum size $\bar{p} \times \bar{q}$ solution, then it has all admissible size solutions.
- (6) If $\text{rank}(A) = n$, every admissible size solution, if exists, would be unique.

Similarly, in this case, we can also get a necessary condition for the solvability of matrix equation (24).

Theorem 9. *If matrix-vector equation (24) has a solution belonging to $\mathcal{M}_{p \times q}$, then matrix $B = ((b_{u,v}), u = 1, 2, \dots, m, v = 1, 2, \dots, k)$ can be divided into $m \times q$ blocks, and furthermore, matrix B must have the following form:*

$$B = \begin{bmatrix} \text{Block}_{11}(B) & \cdots & \text{Block}_{1q}(B) \\ \vdots & \ddots & \vdots \\ \text{Block}_{m1}(B) & \cdots & \text{Block}_{mq}(B) \end{bmatrix}, \quad (45)$$

where $\text{Block}_{ij}(B)$ have the form as follows:

$$\begin{aligned}
\text{Block}_{ij}(B) &= b'_{ij} 1_{(h/m) \times (k/q)} \\
&= b_{(i-1)(h/m)+u, (j-1)(k/q)+v} \in \mathcal{M}_{(h/m) \times (k/q)}, \\
i &= 1, 2, \dots, m, j = 1, 2, \dots, \\
q, u &= 1, 2, \dots, \frac{h}{m}, v = 1, 2, \dots, \frac{k}{q}.
\end{aligned} \tag{46}$$

The matrix B in Theorem 9 is said to have the proper form.

Now, we give the following algorithm for matrix equation (24).

Step 1. To see whether matrix equation (24) suits the compatible conditions or not, that is, $m|h$ and B , in the proper form,

$$B = \begin{bmatrix} \text{Block}_{11}(B) & \cdots & \text{Block}_{1q}(B) \\ \vdots & \ddots & \vdots \\ \text{Block}_{m1}(B) & \cdots & \text{Block}_{mq}(B) \end{bmatrix}, \tag{47}$$

where $\text{Block}_{ij}(B)$ have the form as $\text{Block}_{ij}(B) = b'_{ij} 1_{(h/m) \times (k/q)} \in \mathcal{M}_{(h/m) \times (k/q)}, i=1,2,\dots,m, j=1,2,\dots,q$.

Step 2. Figure out all the admissible sizes $p \times q$ meet the conditions in Proposition 4.

Step 3. For each size $p \times q$, we can solve q matrix-vector equations under MM -2 semitensor product to get the solutions of this matrix equation.

5. Examples

In this section, we give two cases numerical examples.

Example 1. Considering the matrix-vector equation $A \circ_1 X = B$, where A and B are as follows (for convenience, we set $A \in \mathcal{M}_{m \times n}, B \in \mathcal{M}_{h \times k}$, and $X \in \mathbb{C}^p$):

$$\begin{aligned}
(1) \quad A &= \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \\
B &= \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}.
\end{aligned} \tag{48}$$

Noting that $m = h$, and $(n/k) = (4/3)$, thus the given matrices are not compatible, and by Proposition 1, the equation has no solution.

$$\begin{aligned}
(2) \quad A &= \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \\
B &= \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}.
\end{aligned} \tag{49}$$

Noting that $m = h$, although $(n/k) = 2$, but B does not have the proper form, so by Theorem 2, the equation has no solution.

$$\begin{aligned}
(3) \quad A &= \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 2 \end{bmatrix}, \\
B &= \begin{bmatrix} 3 & 3 \\ 5 & 5 \end{bmatrix}.
\end{aligned} \tag{50}$$

Noting that $m = h$, $(n/k) = 2$, and B has the proper form, so by Proposition 1, the equation may have a solution $X \in \mathbb{C}^2$. Let

$$\begin{aligned}
\dot{A}_1 &= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \\
\ddot{A}_2 &= \begin{bmatrix} 0 & 2 \\ 2 & 2 \end{bmatrix}.
\end{aligned} \tag{51}$$

We have

$$\begin{aligned}
A_1 &= \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}, \\
A_2 &= \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix}, \\
W &= \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix},
\end{aligned} \tag{52}$$

taking $X = [x_1 \ x_2]^T$.

(i) Method 1: by definition, we have

$$A \circ_1 X = \frac{1}{2} A (X \otimes 1_{2 \times 2}) = B. \tag{53}$$

Solving equation

$$\frac{1}{2} \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 & x_1 \\ x_1 & x_1 \\ x_2 & x_2 \\ x_2 & x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 & x_1 \\ x_1 & x_1 \\ x_2 & x_2 \\ x_2 & x_2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 5 & 5 \end{bmatrix}, \tag{54}$$

we can get the solution $X = [1 \ 2]^T$, and by Theorem 1, the solution is unique.

(ii) Method 2: by Theorem 3, we have

$$A \circ_1 X = B \iff \frac{1}{k} W X = C_1(B). \tag{55}$$

Solving equation

$$\frac{1}{2} \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \tag{56}$$

we can get the solution $X = [12]^T$, and by Theorem 1, the solution is unique.

(4)

$$\begin{aligned} A &= \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \\ B &= \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}. \end{aligned} \quad (57)$$

Noting that $m = h$, the given matrices are compatible and B has the proper form; therefore, by Proposition 1, we know the equation may have a solution $X \in \mathbb{C}^2$. But through calculating, we can see $\text{rank } W \neq \text{rank}[WC_1(B)]$, so this equation has no solution according to Corollary 2.

(5)

$$\begin{aligned} A &= \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}, \\ B &= \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 1 & 3 \\ 1 & 1 & 0 \end{bmatrix}. \end{aligned} \quad (58)$$

Noting that $m \neq h$, $(4/2)$, $(6/3)$ are positive integers, but B does not have the proper form. Hence, by Theorem 4, the equation has no solution.

(6)

$$\begin{aligned} A &= \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 \end{bmatrix}, \\ B &= \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}. \end{aligned} \quad (59)$$

Noting that $m \neq h$, because the given matrices are compatible, B has the proper form, so the equation may have a solution $X \in \mathbb{C}^4$ according to Proposition 2. Let $X = [x_1 \ x_2 \ x_3 \ x_4]^T$. Comparing with $k = l_1 \cdot (h/m) + l_2$, we see $l_1 = 1, l_2 = 1$, then according to Theorem 5, we just need to consider the following equation system:

$$\begin{cases} 2x_1 + 2x_3 = 12, \\ x_1 + x_2 + 4x_4 = 18. \end{cases} \quad (60)$$

Solving it, we get a solution $X = [4 \ 2 \ 2 \ 3]^T$, and $\{X + k_1[-1 \ 1 \ 1 \ 0]^T + k_2[0 \ -4 \ 0 \ 1]^T, k_1, k_2 \in \mathbb{C}\}$ are all the solutions.

Example 2. Considering the matrix equation $A \circ_1 X = B$, where A and B are as follows (for convenience, we set $A \in \mathcal{M}_{m \times n}, B \in \mathcal{M}_{h \times k}$, and $X \in \mathcal{M}_{p \times q}$):

(1)

$$\begin{aligned} A &= \begin{bmatrix} 2 & 0 \\ 0 & 3 \\ 2 & 1 \end{bmatrix}, \\ B &= \begin{bmatrix} 1 & 0 & 2 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 3 \end{bmatrix}. \end{aligned} \quad (61)$$

Noting that $m = h$, as B does not have the proper form, hence the equation has no solution according to Theorem 7.

(2)

$$\begin{aligned} A &= \begin{bmatrix} 1 & 0 & 0 & 2 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}, \\ B &= \begin{bmatrix} 1 & 1 & 3 & 3 \\ 3 & 3 & 2 & 2 \end{bmatrix}. \end{aligned} \quad (62)$$

Noting that $m = h$, and B has the proper form, hence the equation may have solutions and the admissible sizes are $3 \times 2, 6 \times 4$ according to Proposition 3. It is easy to verify that

$$X_1 = \begin{bmatrix} 4 & 2 \\ 2 & 4 \\ 6 & 4 \end{bmatrix} \quad (63)$$

is a solution. Thus, by Remark 1,

$$X_2 = \frac{1}{2}(X_1 \otimes I_2) = \begin{bmatrix} 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 3 & 3 & 2 & 2 \\ 3 & 3 & 2 & 2 \end{bmatrix} \quad (64)$$

is also a solution.

(3)

$$\begin{aligned} A &= \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \\ B &= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 3 \\ 1 & 1 & 3 & 3 \\ 1 & 1 & 3 & 3 \\ 1 & 1 & 3 & 3 \end{bmatrix}. \end{aligned} \quad (65)$$

Noting that $m \neq h$, the given matrices are compatible, and B has the proper form, so the equation may have solutions and the admissible sizes are $3 \times 2, 6 \times 4$ according to Proposition 4. It is easy to verify that

$$X_1 = \begin{bmatrix} 3 & 0 \\ 0 & 6 \\ -3 & 0 \end{bmatrix}, \quad (66)$$

is a solution. Thus, by Remark 2,

$$X_2 = \frac{1}{2} (X_1 \otimes I_2) = \begin{bmatrix} \frac{3}{2} & \frac{3}{2} & 0 & 0 \\ \frac{3}{2} & \frac{3}{2} & 0 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 3 & 3 \\ -\frac{3}{2} & -\frac{3}{2} & 0 & 0 \\ -\frac{3}{2} & -\frac{3}{2} & 0 & 0 \end{bmatrix}, \quad (67)$$

is also a solution.

6. Conclusion

In this article, we studied the solutions of the matrix equation $A \circ_1 X = B$ with respect to the MM -2 semitensor product. We discussed it in two ways: the solutions are matrices and the solutions are vectors. In each case, we first investigated the necessary conditions for the equation to have a solution. Then, we transformed the equation into the equivalent form of ordinary matrix multiplication according to the definition to study the solvability. Further, we obtained the necessary and sufficient conditions for the equation to have a solution and the specific steps to solve the equation. At last, we presented several examples to illustrate the efficiency of our results.

We expect the results obtained in this article to be useful. We are sure that they will have broad application prospects in control systems, engineering, computational mathematics, computer science, information science, etc.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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