

## Research Article

# Ordering of Transformed Recorded Electroencephalography (EEG) Signals by a Novel Precede Operator

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Recorded electroencephalography (EEG) signals can be represented as square matrices, which have been extensively analyzed using mathematical methods to extract invaluable information concerning brain functions in terms of observed electrical potentials; such information is critical for diagnosing brain disorders. Several studies have revealed that certain such square matrices—in particular, those related to so-called “elementary EEG signals”—exhibit properties similar to those of prime numbers in which every square EEG matrix can be regarded as a composite of these signals. A new approach to ordering square matrices is pivotal to extending the idea of square matrices as composite numbers. In this paper, several ordering concepts are investigated and a new technique for ordering matrices is introduced. Finally, some properties of this matrix order are presented, and the potential applications of this technique to analyzing EEG signals are discussed.

## 1. Introduction and Motivation

Epilepsy is a common neurological disease that affects 1% of the world’s population [1]. People of all ages can be affected by this chronic brain disorder [2], which lowers sufferers’ quality of life with the possibility of a seizure occurring at any time. Early diagnosis can not only improve quality of life but also prevent patients from experiencing accidents. The diagnosis of epilepsy is often not straightforward, and misdiagnosis occasionally does occur [3]. A detailed and reliable eyewitness account of the event is the most crucial piece of information for indicative assessment, but this may not be accessible [4]. In most cases, electroencephalography (EEG) is an essential diagnostic test for assessing patients with possible epilepsy. Besides providing diagnostic support [5], EEG can also assist in classifying the underlying epileptic syndrome [6].

Mathematical analysis of EEG signals offers medical professionals vital information regarding brain activity during a seizure, thus increasing the understanding of complex brain function [7, 8]. In general, relevant information is extracted from EEG signals via two main methods:

linear and nonlinear. Linear analysis (e.g., Fourier and wavelet transforms) has been successfully executed and has produced several good results [9–15]. However, since linear methods disregard the underlying nonlinear EEG dynamics, the results can only provide a limited amount of information concerning the brain’s electrical activity. By contrast, it is commonly accepted that the brain is a chaotic dynamical system; therefore, EEG signals generated by the brain are considered to be chaotic in another sense [16–18]. Additional information can be extracted from EEG signals by progressively incorporating a nonlinear analysis that reveals features that cannot be measured via linear methods [19].

Crucial to diagnosing the disorder is to solve the neuromagnetic inverse problem to identify the location of epileptic foci [20]. Therefore, fuzzy topographic topological mapping (FTTM) was introduced in [21] to determine the epileptic foci. More recently, FTTM has been extensively utilized to study the features of the recorded EEG signals of seizure patients (see [22–27]). Most notably, Yun in [28] claimed that one of the components of FTTM—namely, the magnetic contour (MC)—obeys the associative law, which is also satisfied in turn by events in time [29]. The author

concluded by stating that the MC is a plane containing information. This prompted Binjadhnan in [25, 30] to perform Krohn–Rhodes decomposition on a set of square matrices of EEG signals,  $MC_n(\mathbb{R})$ . The scholar found a remarkable result, namely, the EEG signals taken during an epileptic seizure (henceforth called EEG-signal square matrices for the remainder of this paper) are not chaotic, but rather exhibit ordered patterns in the form of simple algebraic structures, as expressed by Theorem 1.

**Theorem 1** (see [30]). *Any invertible square matrix of EEG-signal readings during an epileptic seizure at time  $t$  can be written as a product of elementary EEG signals during an epileptic seizure in one and only one way.*

Theorem 1 states that the elementary EEG signals (i.e., unipotent and diagonal EEG signals) constitute the building blocks of all EEG signals. This theorem, to a certain extent, is similar to the fundamental theorem of arithmetic, which holds that prime numbers are the multiplicative building blocks of the integers [31]. Equally significant are the results that indicate that  $MC_n(\mathbb{R})$  has properties resembling those of prime numbers via the Jordan–Chevalley decomposition [32]. The well-ordering property of positive integers is vital in producing one of the most beautiful results in the study of prime numbers, namely, the infinitude of prime numbers. Therefore, a technique of ordering matrices is required to extend the work of viewing the elementary EEG signals as prime numbers. The analogy of elementary EEG signals as prime numbers is of vital importance since the pattern of EEG signals can be investigated in terms of the pattern of prime numbers. Hence, the goal of this paper is to introduce a technique of ordering transformed EEG signals (in terms of square matrices).

The remainder of this paper is organized as follows. In Section 2, a brief review of a few concepts and techniques for ordering matrices is presented, along with their viability for ordering transformed EEG signals. In Section 3, a new technique for ordering matrices, namely, the precede operator, is introduced, allowing several ordering properties to be obtained. Next, this binary relationship is shown to fulfill the partial-order properties; beyond that, it is shown to be totally ordered in Section 4. In Section 5, several results are obtained when the order is applied to symmetric matrices. Then, the implementation of the precede operator to the real data of EEG signals is presented in Section 6. The interpretation of the results and their connection with the prime numbers are discussed in Section 7. Finally, we bring the paper to a close with concluding remarks concerning the need for such a partial order. Throughout the following sections, every matrix is considered to be a square matrix unless otherwise stated.

## 2. Concepts for Ordering Matrices

Over the past few decades, mathematicians and applied scientists alike have taken a deep interest in the ordering of matrices. Several order relations for matrix algebra have been produced in connection to a series of applications relevant to

different branches of mathematics and its applications. These order relations include minus partial order [33], star partial order [34], sharp partial order [35], and matrix majorization [36]. Mitra et al. [37] wrote a comprehensive monograph in which they presented developments in the field of matrix ordering and shorted operators for finite matrices in a unified way, thus sparking research interest in this topic.

Matrix partial ordering has applications in many different areas; for instance, Liu [38] developed applications for comparing linear models. Moreover, in the field of statistics, Baksalary and Puntanen [39] presented the best linear unbiased estimator in a general Gauss-Markov model. At the same time, Dahl et al. [40] characterized a binary relation involving stochastic matrices (namely, matrix majorization), which is very useful for comparisons of statistical experiments. Additionally, in the field of finance, Fontanari et al. [41] proposed a technique called quantum majorization to compare and rank correlation matrices such that portfolio risk can be more significantly assessed.

The minus partial order is the fundamental matrix partial order, of which almost all subsequent partial orders (including the star and sharp orders) constitute extensions. Such extensions have been created through the addition of restrictions to the minus partial order. The minus partial order (which was originally called the plus order) was established by Hartwig in [33] and independently by Nambooripad in [42] to generalize conventional partial orders on semigroups. Antezana et al. [43] and Šemrl [44] extended this partial order such that it could be applied in an objective way to operators on infinite-dimensional spaces. Djikić et al. [45] documented a new representation of the minus order on the algebra of bounded linear operators on a Hilbert space. The natural partial order of Vagner on inverse semigroups and the star order of Drazin can be extended through minus order [37]. One key feature to note is that these partial orders are defined via the method of generalized inverses.

Another essential ordering concept for matrices is majorization, which has been applied across many fields including economics [46], statistics [47, 48], and, most recently, quantum mechanics [49]. This concept was first introduced in a classical book by Hardy et al. [50]. Later, Marshall et al. [36] extensively treated both the theory and application of majorization. Torgersen in [51, 52] studied the generalization of vector majorization and developed the theory of statistical-experiment comparison. This theory is intended to answer the question “What conditions must be fulfilled in order to say that one statistical experiment provides more information than another?” A simple experiment can be found in [53], in which the conventional notion introduced by the author is closely related to that of vector majorization. However, while these generalizations evolved from statistical studies, they are not regularly discussed in the linear-algebra literature. Dahl [54] introduced and studied the generalization of (vector) majorization as it applies in the notable case of matrices with  $m$  rows. The classical concept of majorization between vectors can be generalized via matrix majorization [55].

Some of the techniques of ordering matrices found in the literature, along with their real-world applications, advantages of the techniques, and their limitations (with respect to

the purpose of ordering transformed EEG signals), are summarized in Table 1.

The techniques of ordering matrices summarized in Table 1 have been deemed unfit to be used to extend the work of Binjadhnan and Ahmad [25, 30] and Fuad and Ahmad [32] since there are some conditions required to be fulfilled, and some are limited to the special matrices. This offers the possibility of introducing and investigating a new partial order of square matrices as discussed in Section 3.

### 3. Precede Operator

As mentioned in Section 2, the set of square matrices of EEG signals during a seizure,  $MC_n(\mathbb{R})$ , has properties similar to those of prime numbers. Therefore, EEG-signal square matrices can be assumed to be analogs of natural numbers. It can be said that one matrix is “greater” than another matrix, just as any natural number can be either greater than or less than another natural number since  $\mathbb{R}$  is a complete ordered field and  $\mathbb{N} \subset \mathbb{R}$ . With this in mind, the precede operator, denoted by  $\succ$ , is introduced as defined by Definition 1.

*Definition 1.* Let  $C$  and  $C'$  be  $n \times n$  matrices and  $C \neq C'$ . Matrix  $C$  is said to precede  $C'$ , written as  $C \succ C'$ , whenever the first  $c_{ij} > c'_{ij}$  exists for some  $i, j$ . The comparison must be made in the sequence of rows, i.e.,  $R_1, R_2, \dots, R_n$ , until the first  $c_{ij} > c'_{ij}$  is discovered and denoted as  $\succ(C, C') = c_{ij}$ . Otherwise, if  $c'_{ij} > c_{ij}$ , then  $C' \succ C$ . When  $C = C'$ , i.e., all the

corresponding entries for each matrix are the same, then  $\succ(C, C') = c_{11}$ .

In other words, let us consider  $C, C' \in M_n$  such that

$$C = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{2n} & \cdots & c_{nn} \end{pmatrix}, \tag{1}$$

$$C' = \begin{pmatrix} c'_{11} & c'_{12} & \cdots & c'_{1n} \\ c'_{21} & c'_{22} & \cdots & c'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c'_{n1} & c'_{2n} & \cdots & c'_{nn} \end{pmatrix}.$$

Define  $R_i: M_n \times M_n \longrightarrow \text{Row}_i(C) \times \text{Row}_i(C')$ , such that  $\text{Row}_i(C) = (c_{i1}, c_{i2}, \dots, c_{in})$  and  $\text{Row}_i(C') = (c'_{i1}, c'_{i2}, \dots, c'_{in})$  denote row  $i$  in  $C$  and  $C'$ , respectively.

Next, define  $\omega_{ij}: \text{Row}_i(C) \times \text{Row}_i(C') \longrightarrow M_n$  such that

$$\omega_{ij} = \begin{cases} c_{ij}, & \text{when } c_{ij} \geq c'_{ij}, \\ c'_{ij}, & \text{when } c'_{ij} > c_{ij}, \end{cases} \text{ for } i, j = 1, 2, 3, \dots, n. \tag{2}$$

Hence, matrix  $[\omega_{ij}] = \begin{pmatrix} \omega_{11} & \omega_{12} & \cdots & \omega_{1n} \\ \omega_{21} & \omega_{22} & \cdots & \omega_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{n1} & \omega_{n2} & \cdots & \omega_{nn} \end{pmatrix}.$

Now, define  $\omega'_{1j}: M_n \longrightarrow M_{1 \times n}$  such that

$$\begin{aligned} \omega'_{1j}([\omega_{ij}]) &= (\text{Row}_1([\omega_{ij}]) \text{ Row}_2([\omega_{ij}]) \text{ Row}_3([\omega_{ij}]) \cdots \text{Row}_n([\omega_{ij}])) \\ &= (\omega_{11} \ \omega_{12} \ \omega_{13} \ \cdots \ \omega_{1n} \ \omega_{21} \ \omega_{22} \ \omega_{23} \ \cdots \ \omega_{2n} \ \cdots \ \omega_{n1} \ \omega_{n2} \ \omega_{n3} \ \cdots \ \omega_{nn}) \\ &= (\omega'_{11} \ \omega'_{12} \ \omega'_{13} \ \cdots \ \omega'_{1n} \ \omega'_{1(n+1)} \ \omega'_{1(n+2)} \ \cdots \ \omega'_{1(2n)} \ \cdots \ \omega'_{1(n^2-n)} \ \cdots \ \omega'_{1(n^2-2)} \ \omega'_{1(n^2-1)} \ \omega'_{1(n^2)}). \end{aligned} \tag{3}$$

In other words, rearrange and reidentify the entries of  $[\omega_{ij}]$ .

Next, define  $\omega^*: M_{1 \times n} \longrightarrow \mathbb{R}$  such that

$$\omega^* = \begin{cases} \omega'_{1j}, & \text{when } c_{ij} \neq c'_{ij}, \\ \omega'_{1(j'+1)}, & \text{when } c_{ij} = c'_{ij}, \end{cases} \text{ for } i, j = 1, 2, 3, \dots, n, \ j' = 1, 2, 3, \dots, n^2. \tag{4}$$

TABLE 1: Summary of the techniques of ordering matrices available in the literature.

Ordering technique	Real-world applications	Advantages	Limitations
Multivariate majorization	(i) Measuring income inequalities and comparing the contents of experiments [36, 40] (ii) Comparing the information of classical or quantum physical states [56] (iii) Network flow theory [55] (iv) Measuring income inequalities [57–59] (v) Measuring experimental designs and survey sampling [60]	(i) More than one attribute of a system, such as income inequality, can be compared (ii) Comparison between matrices that have different dimensions	Requires the existence of a doubly stochastic matrix
Quantum majorization	(i) Comparing and ranking correlation matrices to assess portfolio risk in a unified framework [41] (ii) Comparing quantum processes in which a complete set of entropic conditions for state transformations in resources theories of asymmetry and quantum thermodynamics is derived [49]	(i) It is a generalization of matrix majorization (ii) The technique can be applied to all quantum states, whereas the previous results are limited to a restricted family of states (iii) Quantum majorization is preferred mainly for two reasons: (a) verification in the data can be easily done and (b) the axiomatic approach commonly used in financial and actuarial mathematics is satisfied	(i) Requires the existence of completely positive and trace-preserving (CPTP) maps (ii) Additional tools are required for the case of approximate transformations
Loewner's ordering	Multivariate analysis [61]  Matrix ordering of special $C$ -matrices for statistical analysis [62]	Generalization of univariate statistical analysis Facilitate the comparison of information matrices between corresponding block designs and dispersion of two multinomial distributions	Limited to symmetric matrices of the same order  Limited to the special case of a $C$ -matrix in experimental design theory
Partial order induced by affine-invariant geometry	Image processing [63–65]  Information geometry to perform statistical analysis [66]	(i) Fundamental concepts of mathematical morphology could be transferred to matrix-valued data (ii) The ordering technique can be applied to higher-dimensional tensor data (i) The ordering technique is critical to study the monotony of functions (ii) The ordering technique can be applied to study dynamical systems and convergence analysis of algorithms defined on matrices	Limited to the set of symmetric matrices  Limited to the set of positive definite matrices of dimension $n$ derived from the affine-invariant geometry
Sharp partial order	Autonomous linear systems [67, 68]	Enables a comparison between two autonomous systems, and extraction of much more information (i) The compartmental control system models' performance and efficiency, such as infectious disease evolution, are improved	Requires the existence of group inverses
Minus partial order	Compartmental control systems [69]	(ii) A reachable successor system can be obtained from a nonreachable one	Requires the existence of generalized inverses

Finally,  $\succ(C, C') = \omega^*$ .

Definition 1 is introduced as a map between a matrix and a real number in  $\mathbb{R}$ , since  $\mathbb{R}$  is a complete ordered set.

*Example 1.* Consider two matrices  $A$  and  $B$  such that

$$A = \begin{pmatrix} 0.8147 & 0.0975 & 0.1576 & 0.1419 & 0.6557 \\ 0.9058 & 0.2785 & 0.9706 & 0.4218 & 0.0357 \\ 0.1270 & 0.5469 & 0.09572 & 0.9157 & 0.8491 \\ 0.9134 & 0.9575 & 0.4854 & 0.7922 & 0.9340 \\ 0.6324 & 0.9649 & 0.8003 & 0.9595 & 0.6787 \end{pmatrix}, \quad (5)$$

$$B = \begin{pmatrix} 0.8147 & 0.0975 & 0.1576 & 0.1419 & 0.6557 \\ 0.9058 & 0.0318 & 0.6948 & 0.3816 & 0.4456 \\ 0.3922 & 0.2769 & 0.3171 & 0.7655 & 0.6463 \\ 0.6555 & 0.0462 & 0.9502 & 0.7952 & 0.7094 \\ 0.1712 & 0.0971 & 0.0344 & 0.1869 & 0.7547 \end{pmatrix}.$$

As can be clearly seen, the first  $a_{ij} > b_{ij}$  is found. In this case,  $a_{22} > b_{22}$  or  $0.2785 > 0.0318$ . Then,  $A > B$ .

**Theorem 2.** *The mapping  $\succ: M_n \times M_n \rightarrow \mathbb{R}$  is well-defined.*

*Proof.* Let  $(A, B) = (C, D)$ , such that  $(A, B), (C, D) \in M_n \times M_n$ . Therefore,  $A = C$  and  $B = D$  imply that

$$\begin{aligned} \succ(A, B) &= \succ(C, B) \text{ (since } A = C) \\ &= \succ(C, D) \text{ (since } B = D). \end{aligned} \quad (6)$$

Hence, the mapping of  $\succ: M_n \times M_n \rightarrow \mathbb{R}$  is well-defined.

The execution of this definition can be summarized by Algorithm 1.

In short,  $\succ: M_n \times M_n \rightarrow \mathbb{R}$ , where  $\succ$  is a composition of mappings such that

$$\succ = \omega^* \odot \omega'_{1j} \odot \omega_{ij} \odot R_i. \quad (7)$$

The composition of mappings  $\succ$  is best illustrated by Figure 1.  $\square$

#### 4. Ordered Matrices

In this section, several results are obtained to show that any square matrices together with the precede operator are ordered matrices.

**Lemma 1.** *If  $A > B$ , then  $B \not> A$ .*

*Proof.* Let  $A > B$ . Then, there exists a first  $a_{ij}$  and a first  $b_{ij}$  for some  $i, j \in \mathbb{N}$  such that  $a_{ij} \geq b_{ij}$ . Suppose that  $B \not> A$  is false; then,  $B > A$  is true. Therefore, there exists a first  $b_{ij}$  and a first  $a_{ij}$  for some  $i, j \in \mathbb{N}$  such that  $b_{ij} \geq a_{ij}$ . This is impossible, since  $A \neq B$  according to Definition 1, and it also contradicts with the assumption that says there exists a first  $a_{ij}$  and a first  $b_{ij}$  for some  $i, j \in \mathbb{N}$  such that  $a_{ij} \geq b_{ij}$ , as noted earlier.  $\square$

**Lemma 2.** *If  $A > B$  and  $B > A$ , then  $\succ(A, B) = a_{11}$ .*

*Proof.* Suppose  $A > B$ ; then, there exists a first  $a_{ij}$  and a first  $b_{ij}$  for some  $i, j \in \mathbb{N}$ , such that  $a_{ij} \geq b_{ij}$ . Similarly, if  $B > A$ ,

then there exists a first  $b_{ij}$  and a first  $a_{ij}$  for some  $i, j \in \mathbb{N}$ , such that  $b_{ij} \geq a_{ij}$ . This is only possible if  $a_{ij} = b_{ij}, \forall i, j = 1, 2, \dots, n$ . In other words,  $\succ(A, B) = a_{11}$ , by Definition 1.  $\square$

**Lemma 3.** *If  $A > B$  and  $B > C$ , then  $A > C$ .*

*Proof.* Let  $A > B$ . Then, there exists a first  $a_{ij}$  and a first  $b_{ij}$  for some  $i, j \in \mathbb{N}$  such that  $a_{ij} \geq b_{ij}$ . If  $B > C$ , then there exists a first  $b'_{ij}$  and a first  $c'_{ij}$  for some  $i, j \in \mathbb{N}$  such that  $b'_{ij} \geq c'_{ij}$ . Suppose that  $A > C$  is false; therefore,  $C > A$ . In other words, there exists a first  $c''_{ij}$  and a first  $a''_{ij}$  for some  $i, j \in \mathbb{N}$ , such that  $c''_{ij} \geq a''_{ij}$ . There are three cases to consider:

- (i)  $c''_{ij} > c'_{ij}$ : if  $c''_{ij} > c'_{ij}$ , then  $b'_{ij} \geq c''_{ij} > c'_{ij}$  is a contradiction, since  $c'_{ij}$  is no longer the first to be found, such that  $b'_{ij} \geq c'_{ij}$  and  $B \neq C$ , according to Definition 1 ( $\rightarrow \leftarrow$ ).
- (ii)  $c'_{ij} > c''_{ij}$ : if  $c'_{ij} > c''_{ij}$ , then  $c''_{ij} > c'_{ij} \geq a''_{ij}$  is a contradiction, since  $c''_{ij}$  is no longer the first  $c''_{ij}$  to be found, such that  $c''_{ij} \geq a''_{ij}$  but  $c'_{ij} (\rightarrow \leftarrow)$ .
- (iii)  $c'_{ij} = c''_{ij}$ : if  $c'_{ij} = c''_{ij}$ , then  $b'_{ij} \geq c'_{ij} = c''_{ij}$ , since  $B > C$ , and  $c''_{ij} \geq a''_{ij}$ , since  $C > A$ . Therefore,  $b'_{ij} \geq c''_{ij} \geq a''_{ij}$ , which immediately implies  $b'_{ij} \geq a''_{ij}$ . In other words,  $B > A$ , which is a contradiction ( $\rightarrow \leftarrow$ ).

All three cases lead to contradictions; thus, if  $A > B$  and  $B > C$ , then  $A > C$ .

Consequently, the binary relation is a partial order.  $\square$

**Theorem 3.** *The set  $(M_n \times M_n, \succ)$  is a partially ordered set.*

*Proof.* The set  $(M_n \times M_n, \succ)$  is a partially ordered set since

- (i) It is reflexive, since  $C > C \uparrow = \succ(C, C \uparrow) = c_{11}$  by Definition 1
- (ii) By Lemma 2,  $\succ$  is antisymmetric
- (iii) By Lemma 3,  $\succ$  is transitive

Of equal importance,  $\succ$  is totally ordered as well.  $\square$

**Theorem 4.** *The set  $(M_n \times M_n, \succ)$  is totally ordered.*

*Proof.* According to Theorem 3, the set  $(M_n \times M_n, \succ)$  is partially ordered. Next,  $A = B$  or  $A \neq B$ . Consider the case where  $A \neq B$ , and a contrapositive for Lemma 2 is applied. In other words,  $(A = B) \rightarrow (A > B \text{ and } B > A, )$ ; this means that  $A \neq B \rightarrow (A > B) \text{ or } (B > A)$ ;  $A \neq B \rightarrow (B > A) \text{ or } (A > B)$ . In summary,  $A = B$  or  $A > B$  or  $B > A$ . Hence,  $(M_n \times M_n, \succ)$  is totally ordered.  $\square$

#### 5. Precede Ordering for Symmetric Matrices

When precede operator is applied to special matrices (namely, symmetric matrices), several properties are obtained.

**Proposition 1.** *If  $A$  and  $B$  are symmetric matrices and  $A > B$ , then  $A^T > B^T$  (i.e., matrix transposition preserves  $\succ$  for symmetric matrices).*

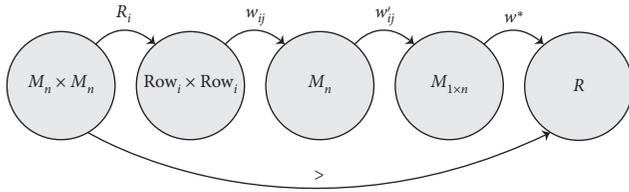
Step 1: given matrices  $C, C' \in M_n$  and  $C \neq C'$ , determine row  $i$  for  $C$  and  $C'$ , i.e.,  $R_i(C) = (c_{i1}, c_{i2}, \dots, c_{in})$  and  $R_i(C') = (c'_{i1}, c'_{i2}, \dots, c'_{in})$ .

Step 2: determine matrix  $[\omega_{ij}]$  whereby  $\omega_{ij}: R_i(C) \times R_i(C') \rightarrow M_n$ , such that  $\omega_{ij} = \begin{cases} c_{ij}, & \text{when } c_{ij} > c'_{ij} \\ c'_{ij}, & \text{when } c'_{ij} > c_{ij} \end{cases}$  for  $i, j = 1, 2, \dots, n$ .

Step 3: determine matrix  $[\omega'_{ij}]$  whereby  $\omega'_{ij}: M_n \rightarrow M_{1 \times n}$ , such that  $\omega'_{ij}([\omega_{ij}]) = (\text{Row}_1([\omega_{ij}]) \text{ Row}_2([\omega_{ij}]) \text{ Row}_3([\omega_{ij}]) \dots \text{Row}_n([\omega_{ij}])) = (\omega_{11} \ \omega_{12} \ \omega_{13} \ \dots \ \omega_{1n} \ \omega_{21} \ \omega_{22} \ \omega_{23} \ \dots \ \omega_{2n} \ \dots \ \omega_{n1} \ \omega_{n2} \ \omega_{n3} \ \dots \ \omega_{nm})$   
 $= (\omega'_{11} \ \omega'_{12} \ \omega'_{13} \ \dots \ \omega'_{1n} \ \omega'_{1(n+1)} \ \omega'_{1(n+2)} \ \dots \ \omega'_{1(2n)} \ \dots \ \omega'_{1(n^2-n)} \ \dots \ \omega'_{1(n^2-2)} \ \omega'_{1(n^2-1)} \ \omega'_{1(n^2)})$ .

Step 4: determine  $\omega^*: M_{1 \times n} \rightarrow \mathbb{R}$  such that  $\omega^* = \begin{cases} \omega'_{ij}, & \text{when } c_{ij} \neq c'_{ij} \\ \omega'_{1(j'+1)}, & \text{when } c_{ij} = c'_{ij} \end{cases}$  for  $i, j = 1, 2, \dots, n$  and  $j' = 1, 2, \dots, n^2$ .

ALGORITHM 1: Precede operator.

FIGURE 1: Composition of mapping  $\succ$ .

*Proof.* Suppose that  $A$  and  $B$  are symmetric matrices and  $A > B$ . It immediately follows that  $A^T > B^T$ , since  $A^T = A$  and  $B^T = B$ .  $\square$

**Proposition 2.** If  $A$  and  $B$  are symmetric matrices and  $A > B$ , then  $-B > -A$ .

*Proof.* Suppose that  $A > B$ ; then, there exists a first  $a_{ij}$  and a first  $b_{ij}$  for some  $i, j \in \mathbb{N}$ . Nevertheless,  $-b_{ij} \geq -a_{ij}$ , since  $a_{ij}, b_{ij} \in \mathbb{R}$  for some  $i, j \in \mathbb{N}$ , and  $a_{ij}$  and  $b_{ij}$  are the first terms of  $-A$  and  $-B$ , respectively, that exhibit such conditions. Consequently,  $-B > -A$ .  $\square$

**Proposition 3.** If  $A > B$ , then  $kA > kB$  for  $k \in \mathbb{R}^+$  (i.e., matrix-positive-scalar multiplication preserves  $\succ$ ).

*Proof.* Suppose that  $A > B$ ; then, there exists a first  $a_{ij}$  and a first  $b_{ij}$  for some  $i, j \in \mathbb{N}$  such that  $a_{ij} \geq b_{ij}$ . Then,  $ka_{ij} \geq kb_{ij}$ , since  $k \in \mathbb{R}^+$ . Hence,  $kA > kB$ .  $\square$

**Proposition 4.** If  $A$  and  $B$  are skew-symmetric matrices and  $A > B$ , then  $B^T > A^T$ .

*Proof.* Suppose that  $A$  and  $B$  are skew-symmetric matrices; therefore,  $A^T = -A$  and  $B^T = -B$ . Nevertheless, if  $A > B$ , then  $-B > -A$  by Proposition 2. Consequently,  $B^T > A^T$ , since  $A$  and  $B$  are skew-symmetric matrices.  $\square$

**Theorem 5.** Suppose that  $A, B, C$ , and  $D$  are positive symmetric matrices, such that  $A > B$  and  $C > D$ ; then,  $A + C > B + D$ .

*Proof.* Suppose that  $A = [a_{ij}]$ ,  $B = [b_{ij}]$ ,  $C = [c_{ij}]$ , and  $D = [d_{ij}]$  are positive symmetric matrices.  $A > B$  implies that there exists a first  $a_{ij}^*$  and a first  $b_{ij}^*$  for some  $i', j' \in \mathbb{N}$ , such that  $a_{ij}^* \geq b_{ij}^*$ . Similarly,  $C > D$  implies that there exists a first  $c_{ij}^*$

and a first  $d_{ij}^*$  for some  $i'', j'' \in \mathbb{N}$ , such that  $c_{ij}^*$  and a first  $d_{ij}^*$  for some  $i'', j'' \in \mathbb{N}$ , such that  $c_{ij}^* \geq d_{ij}^*$ . Now,  $a_{i'j'}^* + c_{i''j''}^* \geq b_{i'j'}^* + d_{i''j''}^*$ , since  $a_{i'j'}^*, c_{i''j''}^*, b_{i'j'}^*, d_{i''j''}^* \in \mathbb{R}$  and  $\mathbb{R}$  are the fields. In short, the first  $[a_{i'j'}^* + c_{i''j''}^*] \geq$  the first  $[b_{i'j'}^* + d_{i''j''}^*]$  for  $i'', j'', i', j' \in \mathbb{N}$ . This implies that  $A + C > B + D$ .

Theorem 5 is best illustrated by Example 2.  $\square$

*Example 2.* Consider the positive symmetric matrices  $A, B, C$ , and  $D$  such that

$$\begin{aligned} A &= \begin{pmatrix} 56 & 12 & 32 \\ 12 & 24 & 16 \\ 32 & 16 & 23 \end{pmatrix}, \\ B &= \begin{pmatrix} 15 & 11 & 21 \\ 11 & 3 & 7 \\ 21 & 7 & 43 \end{pmatrix}, \\ C &= \begin{pmatrix} 4 & 17 & 10 \\ 17 & 2 & 1 \\ 10 & 1 & 19 \end{pmatrix}, \\ D &= \begin{pmatrix} 1 & 16 & 3 \\ 16 & 1 & 1 \\ 3 & 1 & 9 \end{pmatrix}. \end{aligned} \tag{8}$$

By Definition 1,  $A > B$  and  $C > D$  is obtained. Next,

$$\begin{aligned} A + C &= \begin{pmatrix} 60 & 29 & 42 \\ 29 & 26 & 17 \\ 42 & 17 & 42 \end{pmatrix}, \\ B + D &= \begin{pmatrix} 16 & 27 & 24 \\ 27 & 4 & 8 \\ 24 & 8 & 52 \end{pmatrix}. \end{aligned} \tag{9}$$

Similarly, by Definition 1,  $A + C > B + D$  is obtained.

**Corollary 1.** Suppose that  $A$  and  $B$  are positive symmetric matrices. If  $A > B$ , then  $A + A^T > B + B^T$ .

*Proof.* Suppose that  $A$  and  $B$  are positive symmetric matrices and that  $A > B$ . Then,  $A^T > B^T$ , according to Proposition 1. By replacing  $C = A^T$  and  $D = B^T$  in Theorem 5, we obtain  $A + A^T > B + B^T$ .  $\square$

**Corollary 2.** Suppose that  $A$  and  $B$  are positive symmetric matrices such that  $A \succ B$ ; then,  $A - B \succ B - A$ .

*Proof.* Suppose that  $A$  and  $B$  are positive symmetric matrices and  $A \succ B$ . Then,  $-B \succ -A$ , by Proposition 2. By replacing  $C = -B$  and  $D = -A$  in Theorem 5, we obtain  $A - B \succ B - A$ .  $\square$

*Example 3.* Suppose that  $A$  and  $B$  are two symmetric matrices, such that

$$\begin{aligned} A &= \begin{pmatrix} 6 & 3 & 17 \\ 3 & 11 & 3 \\ 17 & 3 & 10 \end{pmatrix}, \\ B &= \begin{pmatrix} 1 & 10 & 13 \\ 10 & 5 & 1 \\ 13 & 1 & 17 \end{pmatrix}. \end{aligned} \tag{10}$$

Then, by Definition 1,  $A \succ B$ . Next,

$$\begin{aligned} A - B &= \begin{pmatrix} 4 & -7 & 4 \\ -7 & 6 & 2 \\ 4 & 2 & -7 \end{pmatrix}, \\ B - A &= \begin{pmatrix} -4 & 7 & -4 \\ 7 & -6 & -2 \\ -4 & -2 & 7 \end{pmatrix}, \end{aligned} \tag{11}$$

which, by Definition 1, implies that  $A - B \succ B - A$ .

**Theorem 6.**  $C \succ -C$  for every positive symmetric matrix  $C$  (i.e., a positive symmetric matrix precedes its negation).

*Proof.* Suppose that  $A$  and  $B$  are positive symmetric matrices and that  $A \succ B$ . Notice that,  $A + A^T$  is a positive symmetric matrix since

$$\begin{aligned} (A + A^T)^T &= A^T + (A^T)^T \\ &= A^T + A \\ &= A + A^T \\ &= (A + A^T). \end{aligned} \tag{12}$$

Similarly,  $B + B^T$  is also a positive symmetric matrix. Now,  $A + A^T \succ B + B^T$ , according to Corollary 1, and  $-(B + B^T) \succ -(A + A^T)$ , according to Proposition 2. Consequently,  $(A + A^T) + (-(B + B^T)) \succ (B + B^T) - (-(A + A^T))$ , by Theorem 5. Equivalently,

$$\begin{aligned} (A + A^T) - (B + B^T) &\succ (B + B^T) - (A + A^T) \\ \Rightarrow (A + A^T) - (B + B^T) &\succ -[(A + A^T) - (B + B^T)] - (*). \end{aligned} \tag{13}$$

Notice that if  $F$  and  $G$  are positive symmetric matrices, then  $F - G$  is a symmetric matrix, since  $(F - G)^T = (F + (-G))^T = (F^T + (-1)G^T) = (F - G)$ . Next, let  $F = (A + A^T)$  and  $G = (B + B^T)$ ; therefore,  $F - G = (A +$

$A^T) - (B + B^T)$  is symmetric. Replace  $C = F - G$  in (\*); then,  $C \succ -C$  is obtained as required.  $\square$

*Example 4.* Let  $C$  be a  $3 \times 3$  symmetric matrix such that

$$\begin{aligned} C &= \begin{pmatrix} 7 & 3 & 12 \\ 3 & 1 & 3 \\ 12 & 3 & 21 \end{pmatrix}, \\ -C &= \begin{pmatrix} -7 & -3 & -12 \\ -3 & -1 & -3 \\ -12 & -3 & -21 \end{pmatrix}. \end{aligned} \tag{14}$$

Then, by Definition 1, we obtain  $C \succ -C$ .

**Theorem 7.** Suppose that  $A, B$ , and  $C$  are positive symmetric matrices. If  $C \succ A + B$  and  $C \succ A$ , then  $C \succ (1/2)B$ .

*Proof.* Suppose that  $A, B$ , and  $C$  are positive symmetric matrices. Assume that  $C \succ A + B$  and  $C \succ A$ . According to Theorem 6,  $A \succ -A$ . Since  $C \succ A$  and  $A \succ -A$  imply that  $C \succ -A$  by Lemma 3, we consequently obtain  $C \succ A + B$  and  $C \succ -A \Rightarrow C + C \succ (A + B) + (-A)$  by Theorem 3. In other words,  $2C \succ B$ . Therefore, by Proposition 3,  $(1/2)(2C) \succ (1/2)B$  which implies that  $C \succ (1/2)B$ .  $\square$

*Example 5.* Suppose that  $A, B$ , and  $C$  are symmetric matrices such that

$$\begin{aligned} A &= \begin{pmatrix} 4 & 12 & 3 \\ 12 & 10 & 6 \\ 3 & 6 & 8 \end{pmatrix}, \\ B &= \begin{pmatrix} 3 & 10 & 4 \\ 10 & 6 & 8 \\ 4 & 8 & 2 \end{pmatrix}, \\ C &= \begin{pmatrix} 13 & 22 & 31 \\ 22 & 12 & 21 \\ 31 & 21 & 14 \end{pmatrix}. \end{aligned} \tag{15}$$

Clearly, by Definition 1,  $C \succ A$ . Next,

$$\begin{aligned} A + B &= \begin{pmatrix} 7 & 22 & 7 \\ 22 & 16 & 14 \\ 7 & 14 & 10 \end{pmatrix} \Rightarrow C \succ A + B, \\ \frac{1}{2}B &= \begin{pmatrix} \frac{3}{2} & 5 & 2 \\ 5 & 3 & 4 \\ 2 & 4 & 1 \end{pmatrix} \Rightarrow C \succ \frac{1}{2}B. \end{aligned} \tag{16}$$

## 6. Implementation

As an example of the implementation of the precede operator on real data, two readings of EEG signals' square matrices are presented. The EEG signals of epileptic seizure patients could be recorded and composed into a set of square matrices (see Figure 2). Firstly, the EEG data is gathered from the hospital by the EEG technologists and the three-dimensional data are transformed into two-dimensional data. The transformation of the EEG data into a lower-dimensional MC is executed via a novel technique called flattening the EEG, where the information can be preserved and conveniently analyzed [70].

The coordinate system of EEG signals, depicted in Figure 3(a), is defined as

$$C_{\text{EEG}} = \{(x, y, z, e_p) \mid x, y, z, e_p \in \mathbb{R} \text{ and } x^2 + y^2 + z^2 = r^2\}, \quad (17)$$

Moreover, a function  $S_t: C_{\text{EEG}} \rightarrow \text{MC}$  (see Figure 3(b)) is defined as

$$S_t((x, y, z, e_p)) = \left( \frac{rx + iry}{r+z}, e_p \right) = \left( \frac{rx}{r+z}, \frac{ry}{r+z} \right)_{e_p(x,y,z)}. \quad (18)$$

The mapping  $S_t$  is an injective mapping of a conformal structure since both  $C_{\text{EEG}}$  and MC were designed and proven as two manifolds [70]. Hence, the mapping  $S_t$  can keep up the data in a specific angle and orientation of the surface throughout the recorded EEG signals. The technique of flat EEG is executed on three groups of EEG signals recorded from three different epileptic patients [70]. The author digitized the EEG signals during epileptic seizures at 256 samples per second using the Nicolet One EEG software. Next, each APD at every second was stored in a file that contained the position of an electrode on a magnetic-contour (MC) plane. Subsequently, the stored data were used to compose a set of square matrices.

Differences in surface potential could be recorded using an array of electrodes appended to the scalp; the computed voltages between pairs of electrodes are then clarified, amplified, and recorded. The most widely used system of electrode placement is the International Ten-Twenty System; this is a recommended standard method for characterizing the locations of electrodes at particular time intervals along with the head for recording scalp EEG [71]. The Ten-Twenty system depends upon the connection between the position of an electrode and the underlying area of the cerebral cortex (the "Ten" and "Twenty" refer to 10% and 20% interelectrode distances, respectively) [72]. The electrode position of this system is shown systematically in Figure 4.

Figure 4(a) illustrates the case where almost all of the electrodes are positioned 40% or below from vertex  $C_z$ . On the contrary, Figure 4(b) shows the electrode position from

the top view of the head by modeling the head as a sphere. We assume that the hemisphere is 80% from the top of the head [70]. In other words, from the front to the back is from  $F_{pz}$  to  $O_z$  and from the left to the right is from  $T_3$  to  $T_4$ . In general, every APD at each second is stored in a file containing the position of electrodes on the MC plane, as tabulated in Table 2.

Then, the readings in Table 2 are rewritten in terms of a  $5 \times 5$  matrix, as shown below.

Let  $x_1 \leq x_2 \leq \dots \leq x_{21}$ ,  $i, j = \{1, 2, 3, 4, 5\}$  and a function  $\beta_{ij}$  be defined as

$$\beta_{ij} = \begin{cases} (x_{(i-1)5+j}, y_{(i-1)5+j}), & \text{for } (i-1)5 + j \leq 21, \\ 0, & \text{for } (i-1)5 + j > 21. \end{cases} \quad (19)$$

The mapping of  $\beta_{ij}$  can be rewritten as the following matrix:

$$\begin{pmatrix} \beta_{11} & \beta_{12} & \beta_{13} & \beta_{14} & \beta_{15} \\ \beta_{21} & \beta_{22} & \beta_{23} & \beta_{24} & \beta_{25} \\ \beta_{31} & \beta_{32} & \beta_{33} & \beta_{34} & \beta_{35} \\ \beta_{41} & \beta_{42} & \beta_{43} & \beta_{44} & \beta_{45} \\ \beta_{51} & \beta_{52} & \beta_{53} & \beta_{54} & \beta_{55} \end{pmatrix}. \quad (20)$$

Specifically,

$$\begin{pmatrix} (x_1, y_1) & (x_2, y_2) & (x_3, y_3) & (x_4, y_4) & (x_5, y_5) \\ (x_6, y_6) & (x_7, y_7) & (x_8, y_8) & (x_9, y_9) & (x_{10}, y_{10}) \\ (x_{11}, y_{11}) & (x_{12}, y_{12}) & (x_{13}, y_{13}) & (x_{14}, y_{14}) & (x_{15}, y_{15}) \\ (x_{16}, y_{16}) & (x_{17}, y_{17}) & (x_{18}, y_{18}) & (x_{19}, y_{19}) & (x_{20}, y_{20}) \\ (x_{21}, y_{21}) & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (21)$$

The corresponding square matrix is generated by substituting the analogous average potential difference of every element into the above matrix. In particular, every single second of the APD is stored in a square matrix that contains the positions of electrodes on the MC plane.

Therefore, the MC plane becomes a set of  $n \times n$  matrices (EEG signals), which is written as follows:

$$\text{MC}_n(\mathbb{R}) = \{[\beta_{ij}(z)_t]_{n \times n} \mid i, j \in \mathbb{Z}^+, \beta_{ij}(z)_t \in \mathbb{R}\}, \quad (22)$$

where  $\beta_{ij}(z)_t$  is a potential-difference reading for EEG signals from a particular  $ij$  sensor at time  $t$ . For instance, the recorded EEG signals data during seizures at times  $t = 2$  and  $t = 3$  are tabulated in Tables 3 and 4.

The data in Tables 3 and 4 are then reordered in ascending order of  $X$  values and tabulated in Tables 5 and 6, respectively, through the MATLAB programming developed by Binjadhnan [30]. Typically, the program exhibits twenty-one sensor readings and four readings of zero (i.e., no recorded data from the "ghost" sensors at the specified time  $t$ ); thus, a  $5 \times 5$  matrix is obtained from each program.

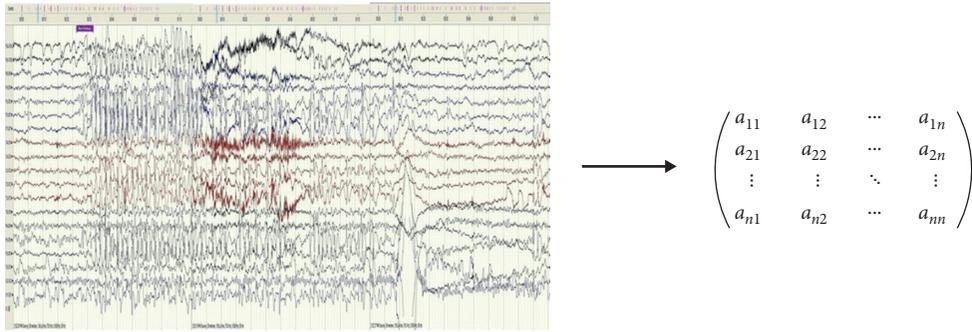


FIGURE 2: EEG signals during an epileptic seizure of Patient A at a time  $t$  is transformed into a square matrix [30].

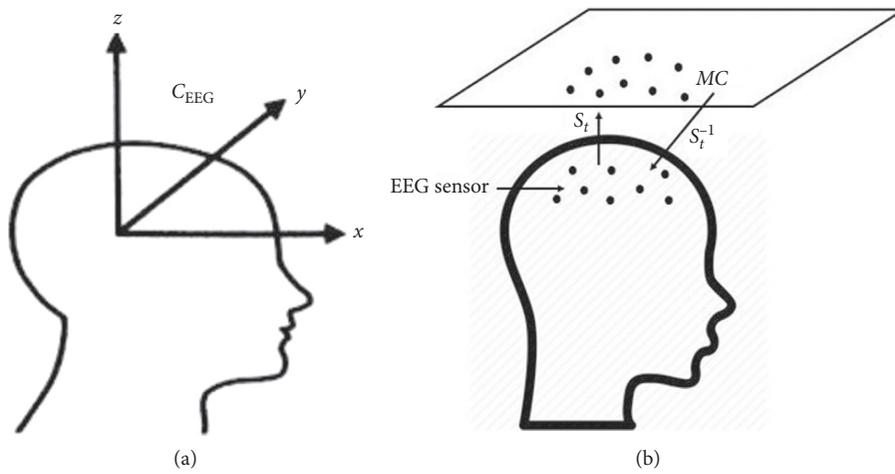


FIGURE 3: (a) EEG coordinate system and (b) EEG projection [70].

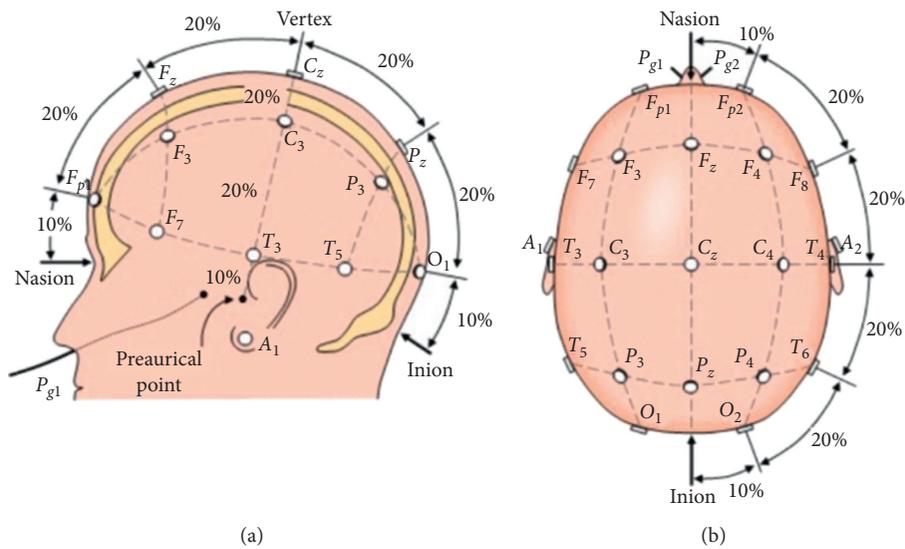


FIGURE 4: The international Ten-Twenty system seen from (a) left and (b) above the head [73].

TABLE 2: Average potential difference (APD) at the sensor on an MC [30].

Sensor	X	Y	APD
$F_{pz}$	$x_{21}$	$y_{21}$	$z_{21}$
$F_{p1}$	$x_{19}$	$y_{19}$	$z_{19}$
$F_{p2}$	$x_{20}$	$y_{20}$	$z_{20}$
$F_3$	$x_{15}$	$y_{15}$	$z_{15}$
$F_4$	$x_{16}$	$y_{16}$	$z_{16}$
$C_3$	$x_9$	$y_9$	$z_9$
$C_4$	$x_{10}$	$y_{10}$	$z_{10}$
$P_3$	$x_6$	$y_6$	$z_6$
$P_4$	$x_7$	$y_7$	$z_7$
$O_1$	$x_2$	$y_2$	$z_2$
$O_2$	$x_3$	$y_3$	$z_3$
$F_7$	$x_{17}$	$y_{17}$	$z_{17}$
$F_8$	$x_{18}$	$y_{18}$	$z_{18}$
$T_3$	$x_{11}$	$y_{11}$	$z_{11}$
$T_4$	$x_{12}$	$y_{12}$	$z_{12}$
$T_5$	$x_4$	$y_4$	$z_4$
$T_6$	$x_6$	$y_6$	$z_6$
$F_z$	$x_1$	$y_1$	$z_1$
$C_z$	$x_{13}$	$y_{13}$	$z_{13}$
$P_z$	$x_8$	$y_8$	$z_8$
$O_z$	$x_{14}$	$y_{14}$	$z_{14}$

TABLE 4: Average potential difference (APD) for patient A at the sensor on an MC at time  $t = 3$  [30].

Sensor	X	Y	APD
$F_{pz}$	7.68	0	0
$F_{p1}$	7.3041	3.3733	65.682265625
$F_{p2}$	7.3041	-2.3733	41.90457031
$F_3$	3.3691	3.3691	8.524765625
$F_4$	3.3691	-3.3691	0.0625
$C_3$	0	3.181	2.902265625
$C_4$	0	-3.1812	79.02796875
$P_3$	-3.3691	3.3691	110.2065234
$P_4$	-3.3691	-3.3691	63.74707031
$O_1$	-7.3041	2.3733	234.7169922
$O_2$	-7.3041	-2.3733	198.7374219
$F_7$	4.5142	6.2133	97.15632813
$F_8$	4.5142	-6.2133	76.86242188
$T_3$	0	7.68	138.7585156
$T_4$	0	-7.68	137.3880078
$T_5$	-4.5142	6.2133	209.6701172
$T_6$	-4.5142	-6.2133	230.6576563
$F_z$	3.1812	0	18.03042969
$C_z$	0	0	60.58832031
$P_z$	-3.1812	0	37.07589844
$O_z$	-7.68	0	0

TABLE 3: Average potential difference (APD) of patient A at the sensor on an MC at time  $t = 2$  [30].

Sensor	X	Y	APD
$F_{pz}$	7.68	0	0
$F_{p1}$	7.3041	3.3733	33.94984375
$F_{p2}$	7.3041	-2.3733	15.32792969
$F_3$	3.3691	3.3691	22.99746094
$F_4$	3.3691	-3.3691	0.707890625
$C_3$	0	3.181	27.5621875
$C_4$	0	-3.1812	23.01671875
$P_3$	-3.3691	3.3691	84.15105469
$P_4$	-3.3691	-3.3691	25.2309375
$O_1$	-7.3041	2.3733	112.3017578
$O_2$	-7.3041	-2.3733	70.05695313
$F_7$	4.5142	6.2133	31.42472656
$F_8$	4.5142	-6.2133	15.34089844
$T_3$	0	7.68	58.33558594
$T_4$	0	-7.68	53.67203125
$T_5$	-4.5142	6.2133	114.2680859
$T_6$	-4.5142	-6.2133	102.3164453
$F_z$	3.1812	0	10.21890625
$C_z$	0	0	12.73027344
$P_z$	-3.1812	0	3.456171875
$O_z$	-7.68	0	0

TABLE 5: Reordering the average potential difference (APD) at the sensor for a patient A at time  $t = 2$  [30].

Sensor	X	Y	APD
$F_{pz}$	-7.68	0	0
$F_{p1}$	-7.3041	2.3733	112.3017578
$F_{p2}$	-7.3041	-2.3733	70.05695313
$F_3$	-4.5142	6.2133	114.2680859
$F_4$	-4.5142	-6.2133	102.3164453
$C_3$	-3.3691	3.3691	84.15105469
$C_4$	-3.3691	-3.3691	25.2309375
$P_3$	-3.1812	0	3.456171875
$P_4$	0	3.1812	27.5621875
$O_1$	0	-3.1812	23.01671875
$O_2$	0	7.68	58.33558594
$F_7$	0	-7.68	53.67203125
$F_8$	0	0	12.73027344
$T_3$	3.1812	0	10.21890625
$T_4$	3.3691	3.3691	22.99746094
$T_5$	3.3691	-3.3691	0.707890625
$T_6$	4.5142	6.2133	31.42472656
$F_z$	4.5142	-6.2133	15.34089844
$C_z$	7.3041	2.3733	33.94984375
$P_z$	7.3041	-2.3733	15.32792969
$O_z$	7.68	0	0

Table 5 yields the matrix:

$$A(2) = \begin{pmatrix} 0 & 112.3018 & 70.05695 & 114.2681 & 102.3164 \\ 84.1511 & 25.2309 & 3.45617 & 27.5622 & 23.0167 \\ 58.3356 & 53.672 & 12.7303 & 10.2189 & 22.9975 \\ 0.707891 & 31.4247 & 15.3409 & 33.9498 & 15.3279 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{23}$$

Table 6 yields the matrix:

$$A(3) = \begin{pmatrix} 0 & 234.717 & 198.7374 & 209.6701 & 230.6577 \\ 110.2065 & 63.74707 & 37.0759 & 2.902266 & 79.02797 \\ 138.7585 & 137.388 & 60.58832 & 18.03043 & 8.524766 \\ 0.0625 & 97.1563 & 76.8624 & 65.6827 & 41.9046 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{24}$$

TABLE 6: Reordering the average potential difference (APD) at the sensor for a patient  $A$  at time  $t = 3$  [30].

Sensor	X	Y	APD
$F_{pz}$	-7.68	0	0
$F_{p1}$	-7.3041	2.3733	234.7169922
$F_{p2}$	-7.3041	-2.3733	198.7374219
$F_3$	-4.5142	6.2133	209.6701172
$F_4$	-4.5142	-6.2133	230.6576563
$C_3$	-3.3691	3.3691	110.2065234
$C_4$	-3.3691	-3.3691	63.74707031
$P_3$	-3.1812	0	37.07589844
$P_4$	0	3.1812	2.902265625
$O_1$	0	-3.1812	79.02796875
$O_2$	0	7.68	138.7585156
$F_7$	0	-7.68	137.3880078
$F_8$	0	0	60.58832031
$T_3$	3.1812	0	18.03042969
$T_4$	3.3691	3.3691	8.524765625
$T_5$	3.3691	-3.3691	0.0625
$T_6$	4.5142	6.2133	97.15632813
$F_z$	4.5142	-6.2133	76.86242188
$C_z$	7.3041	-2.3733	65.68265625
$P_z$	7.3041	2.3733	41.90457031
$O_z$	7.68	0	0

The two matrices,  $A(2)$  and  $A(3)$ , can be compared using the precede operator. Again,

$$\begin{aligned}
 A(2) &= \begin{pmatrix} 0 & 112.3018 & 70.05695 & 114.2681 & 102.3164 \\ 84.1511 & 25.2309 & 3.45617 & 27.5622 & 23.0167 \\ 58.3356 & 53.672 & 12.7303 & 10.2189 & 22.9975 \\ 0.707891 & 31.4247 & 15.3409 & 33.9498 & 15.3279 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
 A(3) &= \begin{pmatrix} 0 & 234.717 & 198.7374 & 209.6701 & 230.6577 \\ 110.2065 & 63.74707 & 37.0759 & 2.902266 & 79.02797 \\ 138.7585 & 137.388 & 60.58832 & 18.03043 & 8.524766 \\ 0.0625 & 97.1563 & 76.8624 & 65.6827 & 41.9046 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.
 \end{aligned}
 \tag{25}$$

Then, by Definition 1, the first  $a_{ij}$  that is greater than  $b_{ij}$  ( $a_{ij} > b_{ij}$ ) is  $a_{12} > b_{12}$  (i.e.,  $234.7169922 > 112.3018$ ); hence,  $A(3) > A(2)$ . Consequently, the precede operator can be used in the set of EEG signals' square matrices, allowing one matrix to be said to precede another. The set of EEG signals' square matrices, along with the precede operator, enables one to see the set of matrices as analogous to the set of real numbers, as related to each other with the "greater than,"  $>$  operator. This analogy is best demonstrated in Figure 5.

### 7. Discussion

The key advantage of using the precede operator over the techniques discussed in Section 2 is that it does not require to fulfill the necessary conditions (such as the existence of doubly stochastic matrix and generalized inverses), rather, the precede operator "inherits" the totally-ordered property of real numbers. Thus, a similar result provides a piece of evidence that the

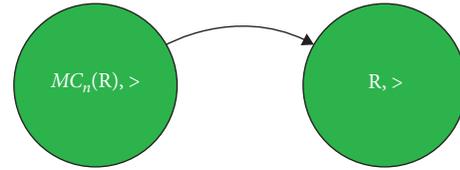


FIGURE 5: The analogy between the set of EEG-signal square matrices and the precede operator and the set of real numbers and the greater-than operator.

TABLE 7: The similarity between elementary EEG signals with the prime numbers.

	EEG signals	Positive integers
	Definition 2 (see [30]): if $A(t)$ and $B(t)$ are EEG signals, we say that $A(t)$ divides $B(t)$ , written as $A(t) B(t)$ if there exist EEG signals $M(t)$ such that $B(t) = A(t)M(t)$ , where $B(t)$ is an invertible EEG signal	Definition 3 (see [31]): if $a$ and $b$ are integers, we say that $a$ divides $b$ if there is an integer $c$ such that $b = ac$
Divisibility		
Unique factorization theorem	Theorem 1 [30]	Fundamental theorem of arithmetic [31]
Building blocks	Elementary EEG signals (1) Jordan–Chevalley decomposition of elementary EEG signals (see Theorem 10 and Theorem 11 in [74]) (2) Theorem 4 (totally ordered)	Prime numbers
Important properties		(1) Twin-prime conjecture [31] (2) Well-ordered

elementary EEG signals during seizure contain similar attributes to the distribution of prime numbers among positive integers. The similarity of elementary EEG signals with prime numbers is summarized in Table 7.

The resemblance of elementary EEG signals with the prime numbers in terms of its ordering properties corroborates the assertions made by Binjadhnan [30], Barja [27], and Ahmad Fuad and Ahmad [74] that the EEG signals during seizures contain a similar pattern to that of the prime number distribution among positive integers. The premise of viewing the EEG signals as prime numbers enables one to study the dynamics of EEG signals during seizures in terms of the pattern of prime numbers. More importantly, the deduced pattern of seizures is critical to devise a methodology that is capable of predicting a seizure, which in turn would significantly improve the patients' quality of life [75]. Conversely, it is instructive to explore the property of elementary EEG signals that could possibly exist in the distribution of prime numbers.

### 8. Conclusions

This study introduces a new technique for ordering square matrices, called precede operator, and this binary relation is

proven to exhibit partial ordering. In addition, several results obtained by using this new ordering technique are presented. Furthermore, the newly introduced matrix partial order is applied to the square matrices of EEG signals, giving opportunities to further develop EEG-signal square matrices' structure by studying those features that resemble some of the rich properties of prime numbers. In particular, it will be of interest to extend the totally ordered property of transformed EEG signals to the well-ordered as well. We are working on this problem and hoping to present our findings in a future paper. Furthermore, as this paper is limited to the precede operator implemented only on the transformed EEG signals, it would be intriguing to investigate the feasibility of the proposed technique to other structures, such as, among others, the measurement of income inequality, statistical experiment, and thermodynamics, as previously summarized in Table 1.

### Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

### Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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