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Research Article

Imploring GE-Filters of GE-Algebras

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Relations between a transitive GE-algebra, a belligerent GE-algebra, an antisymmetric GE-algebra, and a left exchangeable GE-algebra are displayed. A new substructure, so called imploring GE-filter, is introduced, and its properties are investigated. The relationship between a GE-filter, an imploring GE-filter, a belligerent GE-filter, and a prominent GE-filter are considered. Conditions for an imploring GE-filter to be a belligerent GE-filter are given, and the conditions necessary for a (belligerent) GE-filter to be an imploring GE-filter are found. Relations between a prominent GE-filter and an imploring GE-filter are discussed, and a condition for an imploring GE-filter to be a prominent GE-filter is provided. Examples to show that a belligerent GE-filter and a prominent GE-filer are independent concepts are given. The extension property of the imploring GE-filter is established.

1. Introduction

Hilbert algebras were introduced by Henkin and Skolem in the middle of the last century. Since then, several scholars have participated in the study of Hilbert algebras, for example, the reader can refer to [1–12]. It is well known that these algebras are the algebraic counterpart of the fragment of the intuitionistic propositional calculus. The study of generalization of one known algebraic structure is also an important research task. Bandaru et al. introduced GE-algebra as a generalization of Hilbert algebra and studied its properties (see [13]). As a follow-up study to [13], Bandaru et al. introduced two kinds of GE-filters called voluntary GE-filter and belligerent GE-filters in GE-algebras and investigated related properties (see [14, 15]). Rezaei et al. [16] introduced prominent GE-filters in GE-algebras and studied its properties. As a continuing follow-up to the papers [13–16], we introduce a new substructure called imploring GE-filter and study its properties. We first discuss relations between a transitive GE-algebra, a belligerent GE-algebra, an antisymmetric GE-algebra, and a left exchangeable GE-algebra. We establish the relationship between a GE-filter, an imploring GE-filter, a belligerent GE-filter, and a prominent GE-filter. We discuss conditions for an imploring GE-filter to be a belligerent GE-filter and look at the conditions necessary for a (belligerent) GE-filter to be an imploring GE-filter. We consider relations between a prominent GE-filter and an imploring GE-filter and provide a condition for an imploring GE-filter to be a prominent GE-filter. We give examples to show that a belligerent GE-filter and a prominent GE-filer are independent concepts. We finally establish the extension property of the imploring GE-filter.

2. Preliminaries

Definition 1 (see [13]). A GE-algebra is a nonempty set X with a constant 1 and a binary operation *, satisfying the following axioms:

(i) (GE1) $u * u = 1$,
(ii) (GE2) $1 * u = u$,
(iii) (GE3) $u * (v * w) = u * (v * (u * w))$,
for all $u, v, w \in X$.

In a GE-algebra X, a binary relation "$\leq$" is defined by

$$(\forall x, y \in X), \quad (x \leq y \Leftrightarrow x * y = 1). \tag{1}$$

Definition 2. (see [13–15]). A GE-algebra X is said to be

(i) Transitive if it satisfies

$$(\forall x, y, z \in X), \quad (x * y \leq (z * x) * (z * y)). \tag{2}$$

(ii) Commutative if it satisfies

$$(\forall x, y \in X), \quad ((x * y) * y = (y * x) * x). \tag{3}$$

(iii) Left exchangeable if it satisfies

$$(\forall x, y, z \in X), \quad (x * (y * z) = y * (x * z)). \tag{4}$$

(iv) Belligerent if it satisfies

$$(\forall x, y, z \in X), \quad (x * (y * z) = (x * y) * (x * z)). \tag{5}$$

(v) Antisymmetric if the binary relation "$\leq$" is antisymmetric.

Proposition 1 (see [13]). Every GE-algebra X satisfies the following items:

$$(\forall u \in X), \quad (u * 1 = 1), \tag{6}$$

$$(\forall u, v \in X), \quad (u * (u * v) = u * v), \tag{7}$$

$$(\forall u, v \in X), \quad (u \leq v * u), \tag{8}$$

$$(\forall u, v, w \in X), \quad (u * (v * w) \leq v * (u * w)), \tag{9}$$

$$(\forall u \in X), \quad (1 \leq u \Rightarrow u = 1), \tag{10}$$

$$(\forall u, v \in X), \quad (u \leq (v * u) * u), \tag{11}$$

$$(\forall u, v \in X), \quad (u \leq (u * v) * v). \tag{12}$$

If X is transitive, then

$$(\forall u, v, w \in X), \quad (u \leq v \Rightarrow w * u \leq w * v,\ v * w \leq u * w), \tag{13}$$

$$(\forall u, v, w \in X), \quad (u * v \leq (v * w) * (u * w)). \tag{14}$$

Lemma 1 (see [13]). *In a GE-algebra X, the following facts are equivalent to each other:*

$$(\forall x, y, z \in X), \quad (x * y \leq (z * x) * (z * y)), \tag{15}$$

$$(\forall x, y, z \in X), \quad (x * y \leq (y * z) * (x * z)). \tag{16}$$

Definition 3 (see [13]). A subset F of a GE-algebra X is called a GE-filter of X if it satisfies

$$1 \in F, \tag{17}$$

$$(\forall x, y \in X), \quad (x * y \in F,\ x \in F \Rightarrow y \in F). \tag{18}$$

Lemma 2 (see [13]). *In a GE-algebra X, every GE-filter F of X satisfies*

$$(\forall x, y \in X), \quad (x \leq y,\ x \in F \Rightarrow y \in F). \tag{19}$$

Definition 4 (see [14]). A subset F of a GE-algebra X is called a belligerent GE-filter of X if it satisfies (17) and

$$(\forall x, y, z \in X), \quad (x * (y * z) \in F,\ x * y \in F \Rightarrow x * z \in F). \tag{20}$$

Lemma 3 (see [14]). *If a GE-filter F of a GE-algebra X satisfies*

$$(\forall x, y, z \in X), \quad (x * (y * z) \in F \Rightarrow (x * y) * (x * z) \in F), \tag{21}$$

then F is a belligerent GE-filter of X.

Definition 5 (see [16]). A subset F of a GE-algebra X is called a prominent GE-filter of X if it satisfies (17) and

$$(\forall x, y, z \in X), \quad (x * (y * z) \in F,\ x \in F \Rightarrow ((z * y) * y) * z \in F). \tag{22}$$

Lemma 4 (see [16]). *Let F be a GE-filter of a GE-algebra X. Then, F is a prominent GE-filter of X if and only if it satisfies*

$$(\forall x, y \in X), \quad (x * y \in F \Rightarrow ((y * x) * x) * y \in F). \tag{23}$$

3. Relationship between GE-Algebras in Different Forms

We discuss relations between a transitive GE-algebra, a belligerent GE-algebra, an antisymmetric GE-algebra, and a left exchangeable GE-algebra.

Every belligerent GE-algebra is a transitive GE-algebra, but the converse is not true (see [16]).

Theorem 1. *Every antisymmetric GE-algebra is a left exchangeable GE-algebra.*

Proof. It is straightforward by (9) and the antisymmetricity of X.

The converse of Theorem 1 may not be true as seen in the following example. □

Example 1. Let $X = \{1, a, b, c\}$ be a set with a binary operation "$*$" given in Table 1.

Then, X is a left exchangeable GE-algebra which is not antisymmetric, since $a * b = 1$ and $b * a = 1$, but $a \neq b$.

Table 1: Cayley table for the binary operation " $*$ ".

$*$	1	a	b	c
1	1	a	b	c
a	1	1	1	c
b	1	1	1	c
c	1	1	b	1

In the following example, we know that any transitive GE-algebra is neither a belligerent GE-algebra nor a left exchangeable GE-algebra.

Example 2. Let $X = \{1, a, b, c, d\}$ be a set with a binary operation $*$ given in Table 2.

Clearly X is a transitive GE-algebra. But it is not a left exchangeable GE-algebra since

$$a * (b * c) = a * d = c \neq d = b * c = b * (a * c). \quad (24)$$

Also, X is not a belligerent GE-algebra since

$$a * (b * c) = a * d = c \neq d = b * c = (a * b) * (a * c). \quad (25)$$

The following example shows that any antisymmetric GE-algebra may not be a belligerent GE-algebra.

Example 3. Let $X = \{1, a, b, c\}$ be a set with a binary operation $*$ given in Table 3.

Then, X is an antisymmetric GE-algebra which is not belligerent since

$$a * (b * c) = a * 1 = 1 \neq c = 1 * c = (a * b) * (a * c). \quad (26)$$

We look at the necessary conditions for a transitive GE-algebra or an antisymmetric GE-algebra to change to a belligerent GE-algebra.

Lemma 5. *Every transitive GE-algebra X satisfies*

$$(\forall x, y, z \in X)(x * (y * z) \leq (x * y) * (x * z)). \quad (27)$$

Proof. For every $x, y, z \in X$, we have

$$\begin{aligned} 1 &= (x * (y * z)) * (y * (x * z)) \\ &\leq (x * (y * z)) * ((x * y) * (x * (x * z))) \\ &= (x * (y * z)) * ((x * y) * (x * z)). \end{aligned} \quad (28)$$

By (2), (7), (9), and (13), it follows from (10) that $(x * (y * z)) * ((x * y) * (x * z)) = 1$, i.e., $x * (y * z) \leq (x * y) * (x * z)$. □

Corollary 1. *Every belligerent GE-algebra X satisfies (27).*

Theorem 2. *Every antisymmetric and transitive GE-algebra is a belligerent GE-algebra.*

Proof. Let X be an antisymmetric and transitive GE-algebra. Then, X is left exchangeable GE-algebra by Theorem 1 and $x * (y * z) \leq (x * y) * (x * z)$ for all $x, y, z \in X$ by Lemma 5. Using (GE2), (4), (8), and Lemma 1, we have

Table 2: Cayley table for the binary operation " $*$ ".

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	c
b	1	a	1	d	d
c	1	a	1	1	1
d	1	a	1	1	1

Table 3: Cayley table for the binary operation " $*$ ".

$*$	1	a	b	c
1	1	a	b	c
a	1	1	1	c
b	1	a	1	1
c	1	a	b	1

$$\begin{aligned} 1 &= ((x * y) * (x * z)) * (y * (x * y)) * (y * (x * z))) \\ &= ((x * y) * (x * z)) * (1 * (y * (x * z))) \\ &= ((x * y) * (x * z)) * (y * (x * z)) \\ &= ((x * y) * (x * z)) * (x * (y * z)), \end{aligned} \quad (29)$$

that is, $(x * y) * (x * z) \leq x * (y * z)$. Since X is antisymmetric, it follows that $x * (y * z) = (x * y) * (x * z)$ for all $x, y, z \in X$. Therefore, X is a belligerent GE-algebra. □

4. Imploring GE-Filters

Definition 6. A subset F of a GE-algebra X is called an imploring GE-filter of X if it satisfies (17) and

$$(\forall x, y, z \in X)(x * ((y * z) * y) \in F, \quad x \in F \Rightarrow y \in F). \quad (30)$$

Example 4. Let $X = \{1, a, b, c, d, e\}$ be a set with a binary operation $*$ given in Table 4.

Then, X is a GE-algebra which is not transitive, not antisymmetric, not left exchangeable, not commutative, and not belligerent. It is easy to check that $F\colon = \{1, a, b, e\}$ is an imploring GE-filter of X.

Given a subset F of a GE-algebra X, consider the following assertion:

$$(\forall x, y \in X), \quad ((x * y) * y \in F \Rightarrow (y * x) * x \in F). \quad (31)$$

In general, any GE-filter F of a GE-algebra X does not satisfy condition (31) as seen in the following example.

Example 5. In Example 4, the set $F\colon = \{1, e\}$ is a GE-filter of X which does not satisfy condition (31) since $(b * a) * a = a * a = 1 \in F$, but $(a * b) * b = 1 * b = b \notin F$.

The following example shows that any imploring GE-filter of a GE-algebra X may not satisfy condition (31).

Example 6. Let $X = \{1, a, b, c, d, e, f\}$ be a set with a binary operation $*$ given in Table 5.

Table 4: Cayley table for the binary operation “ $*$ ”.

$*$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	1	c	c	e
b	1	a	1	d	d	e
c	1	1	b	1	1	e
d	1	1	1	1	1	e
e	1	a	b	c	d	1

Table 5: Cayley table for the binary operation “ $*$ ”.

$*$	1	a	b	c	d	e	f
1	1	a	b	c	d	e	f
a	1	1	1	d	d	f	f
b	1	1	1	e	e	e	e
c	1	1	b	1	1	1	1
d	1	a	1	a	1	1	1
e	1	a	b	a	1	1	1
f	1	a	1	a	1	1	1

Then, X is a GE-algebra which is not transitive, not left exchangeable, and not belligerent. It is easy to verify that $F:=\{1\}$ is an imploring GE-filter of X. But it does not satisfy (31) since $(b*d)*d = e*d = 1 \in F$ and $(d*b)*b = 1*b = b \notin F$.

Proposition 2. *Every imploring GE-filter F of a transitive GE-algebra X satisfies condition (31).*

Proof. Let X be a transitive GE-algebra. Suppose F is an imploring GE-filter of X. Then, F is a GE-filter of X. Let $x, y \in F$ be such that $(x*y)*y \in F$. Combining (11) and (13) induces

$$((y*x)*x)*y \leq x*y. \tag{32}$$

It follows from (9), (13), and (16) that

$$\begin{aligned}(x*y)*y &\leq (y*x)*((x*y)*x) \leq (x*y)*((y*x)*x) \\ &\leq (((y*x)*x)*y)*((y*x)*x).\end{aligned} \tag{33}$$

Hence, $1*((((y*x)*x)*y)*((y*x)*x)) = (((y*x)*x)*y)*((y*x)*x) \in F$ and so $(y*x)*x \in F$. □

Corollary 2. *Every imploring GE-filter F of a belligerent GE-algebra X satisfies condition (31).*

We establish the relationship between a GE-filter, an imploring GE-filter, a belligerent GE-filter, and a prominent GE-filter.

Theorem 3. *Every imploring GE-filter is a GE-filter.*

Proof. Let F be an imploring GE-filter of a GE-algebra X. Let $x, y \in X$ be such that $x*y \in F$ and $x \in F$. If we put $z = y$ in (30) and use (GE1) and (GE2), then $x*((y*y)*y) = x*y \in F$ and so $y \in F$ by (30). Therefore, F is a GE-filter of X.

In the following example, we know that the converse of Theorem 3 is not true. □

Example 7. If we consider the GE-algebra in Example 4, then the set $F:=\{1, e\}$ is a GE-filter of X. But it is not an imploring GE-filter of X since $e \in F$ and $e*((b*a)*b) = e*(a*b) = e*1 = 1 \in F$, but $b \notin F$.

We list the conditions necessary for the converse of Theorem 3 to be established.

Theorem 4. *Let F be a GE-filter of a GE-algebra X. Then, F is an imploring GE-filter of X if and only if the following implication is valid:*

$$(\forall y, z \in X), \quad ((y*z)*y \in F \Rightarrow y \in F). \tag{34}$$

Proof. Assume that F is an imploring GE-filter of X. Then, (34) is induced by taking $x = 1$ in (30).

Conversely, let F be a GE-filter of X which satisfies (34). Let $x, y, z \in X$ be such that $x*((y*z)*y) \in F$ and $x \in F$. Then, $(y*z)*y \in F$ by (18). It follows from (34) that $y \in F$. Therefore, F is an imploring GE-filter of X.

The following example shows that a belligerent GE-filter and an imploring GE-filter are independent concepts in a GE-algebra, i.e., any belligerent GE-filter is not an imploring GE-filter, and any imploring GE-filter is not a belligerent GE-filter. □

Example 8.

(i) In Example 6, the imploring GE-filter $F:=\{1\}$ is not a belligerent GE-filter of X since $c*(d*b) = c*1 = 1 \in F$ and $c*d = 1 \in F$, but $c*b = b \notin F$.

(ii) Let $X = \{1, a, b, c, d, e, f, g\}$ be a set with a binary operation $*$ given in Table 6.

Then, X is a GE-algebra which is not transitive, not left exchangeable, and not belligerent. The set $F:=\{1\}$ is a belligerent GE-filter of X. But, it is not an imploring GE-filter of X since $1 \in F$ and $1*((b*a)*b) = 1*(a*b) = 1*1 = 1 \in F$, but $b \notin F$.

We discuss conditions for an imploring GE-filter to be a belligerent GE-filter.

Note that in GE-algebras, every belligerent GE-filter is a GE-filter, but not converse (see [14]).

Lemma 6. *In a transitive GE-algebra, every GE-filter is a belligerent GE-filter.*

Proof. Let F be a GE-filter of a transitive GE-algebra X. Let $x, y, z \in X$ be such that $x*(y*z) \in F$ and $x*y \in F$. Using Lemma 2 and Lemma 5 induces $(x*y)*(x*z) \in F$ which implies from (18) that $x*z \in F$. Therefore, F is a belligerent GE-filter of X. □

Table 6: Cayley table for the binary operation “ $*$ ”.

$*$	1	a	b	c	d	e	f	g
1	1	a	b	c	d	e	f	g
a	1	1	1	c	e	e	1	1
b	1	a	1	d	d	d	f	g
c	1	f	b	1	1	1	f	1
d	1	f	b	1	1	1	f	1
e	1	f	b	1	1	1	f	1
f	1	g	b	c	c	c	1	g
g	1	f	b	c	c	c	f	1

Corollary 3. *In a belligerent GE-algebra, every GE-filter is a belligerent GE-filter.*

The following corollaries are direct results of Theorem 3, Lemma 6, and Corollary 3.

Corollary 4. *In a transitive GE-algebra, every imploring GE-filter is a belligerent GE-filter.*

Corollary 5. *In a belligerent GE-algebra, every imploring GE-filter is a belligerent GE-filter.*

The converse of Corollary 4 may not be true as seen in the following example.

Example 9. Consider a transitive GE-algebra $(X, *, 1)$ where $X = \{1, a, b, c, d, e\}$ and the binary operation $*$ is given by Table 7.

Then, $F{:}\ = \{1, e\}$ is a belligerent GE-filter of X but not an imploring GE-filter of X since $e \in F$ and $e * ((a * c) * a) = e * (c * a) = e * 1 = 1 \in F$, but $a \notin F$.

The following example shows that the converse of Corollary 5 may not be true.

Example 10. Let $X = \{1, a, b, c, d, e\}$ be a set with a binary operation $*$ given in Table 8.

Then, X is a belligerent GE-algebra and $F{:}\ = \{1\}$ is a belligerent GE-filter of X. But F is not an imploring GE-filter of X since $1 \in F$ and $1 * ((a * b) * a) = 1 * (d * a) = 1 * 1 = 1$, but $a \notin F$.

We now consider relations between a prominent GE-filter and an imploring GE-filter.

Theorem 5. *In a GE-algebra, every prominent GE-filter is an imploring GE-filter.*

Proof. Suppose F is a prominent GE-filter of X. Then, F is a GE-filter of X and $1 \in F$. Assume that $z * ((x * y) * x) \in F$ and $z \in F$. Then, $(x * y) * x \in F$, which implies from (GE1), (GE2), (7), and Lemma 4 that

$$x = 1 * x = ((x * y) * (x * y)) * x = (((x * (x * y)) * (x * y)) * x \in F. \tag{35}$$

Therefore, F is an imploring GE-filter of X. □

Corollary 6. *In a transitive GE-algebra, every prominent GE-filter is a belligerent GE-filter.*

Table 7: Cayley table for the binary operation “ $*$ ”.

$*$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	b	c	c	e
b	1	a	1	d	d	e
c	1	1	1	1	1	e
d	1	1	1	1	1	e
e	1	a	b	c	d	1

Table 8: Cayley table for the binary operation “ $*$ ”.

$*$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	d	c	d	c
b	1	1	1	e	1	e
c	1	1	d	1	d	1
d	1	1	1	c	1	c
e	1	1	b	1	b	1

The following example shows that any imploring GE-filter is not a prominent GE-filter in general.

Example 11. Consider a GE-algebra $(X, *, 1)$ where $X = \{1, a, b, c, d, e\}$ and the binary operation $*$ is given by Table 9.

Then, $F{:}\ = \{1\}$ is an imploring GE-filter of X but not a prominent GE-filter of X since $a * b = 1 \in F$, but $((b * a) * a) * b = (c * a) * b = 1 * b = b \notin F$.

We provide a condition for an imploring GE-filter to be a prominent GE-filter.

Theorem 6. *In a transitive GE-algebra, every imploring GE-filter is a prominent GE-filter.*

Proof. Let F be an imploring GE-filter of a transitive GE-algebra X. Then, F is a GE-filter of X. Let $x, y \in X$ be such that $y * x \in F$. Since $x \leq ((x * y) * y) * x$ by (8), we have $(((x * y) * y) * x) * y \leq x * y$ by (13). If we put $u{:}\ = ((x * y) * y) * x$, then $x \leq u$ by (8). It follows from (15), (9), and (13) that

$$\begin{aligned} y * x &\leq ((x * y) * y) * ((x * y) * x) \\ &\leq (x * y) * (((x * y) * y) * x) \\ &= (x * y) * u \leq (u * y) * u. \end{aligned} \tag{36}$$

Using Lemma 2, we get $(u * y) * u \in F$ and so $((x * y) * y) * x = u \in F$ by Theorem 4. Therefore, F is a prominent GE-filter of X by Lemma 4. □

Corollary 7. *In a belligerent GE-algebra, every imploring GE-filter is a prominent GE-filter.*

We reveal that a belligerent GE-filter and a prominent GE-filer are independent concepts in next examples. In other words, any belligerent GE-filter is not a prominent GE-filer and any prominent GE-filer is not a belligerent GE-filter.

Table 9: Cayley table for the binary operation "$*$".

$*$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	1	1	d	d
b	1	c	1	c	e	e
c	1	1	b	1	d	d
d	1	c	b	c	1	1
e	1	a	b	a	1	1

Example 12.

(i) Consider a GE-algebra $(X, *, 1)$ where $X = \{1, a, b, c, d, e\}$ and the binary operation $*$ is given by Table 10.

Then, $F\colon = \{1, b\}$ is a belligerent GE-filter of X. But, it is not a prominent GE-filter of X since $c * a = 1 \in F$, but $((a * c) * c) * a = (c * c) * a = 1 * a = a \notin F$.

(ii) Let $X = \{1, a, b, c, d, e, f\}$ be a set with the binary operation $*$ given by Table 11.

Then, X is a GE-algebra which is not transitive, not left exchangeable, and not belligerent. The set $F\colon = \{1\}$ is a prominent GE-filter of X. But it is not a belligerent GE-filter of X since $c * (a * b) = c * 1 = 1 \in F$ and $c * a = 1 \in F$, but $c * b = b \notin F$.

The following example shows that in a transitive GE-algebra, any GE-filter may not be an imploring GE-filter.

Example 13. Let $X = \{1, a, b, c, d, e, f, g\}$ and the binary operation $*$ is given by Table 12.

Then, X is a transitive GE-algebra. Clearly, $F = \{1, b\}$ is a GE-filter of X. But, F is not an imploring GE-filter of X since $b \in F$ and $b * ((a * c) * a) = b * (c * a) = b * 1 = 1 \in F$, but $a \notin F$.

As conditions are strengthened, a GE-filter can be an imploring GE-filter in a transitive GE-algebra.

Theorem 7. *Let F be a GE-filter of a transitive GE-algebra X, satisfying (31). Then, F is an imploring GE-filter of X.*

Proof. Suppose F is a GE-filter of a transitive GE-algebra X, satisfying (31). Let $x, y \in X$ be such that $(x * y) * x \in F$. Since $x \leq (x * y) * y$ by (12), it follows from (13) that $(x * y) * x \leq (x * y) * ((x * y) * y)$. Hence, $(x * y) * y = (x * y) * ((x * y) * y) \in F$ by (7) and Lemma 2, and so $(y * x)x \in F$ by (31). Since $y \leq x * y$ by (8), we get $(x * y) * x \leq y * x$ by (13) which implies from Lemma 2 that $y * x \in F$. Hence, $x \in F$, and therefore, F is an imploring GE-filter of X by Theorem 4. □

Corollary 8. *Let F be a GE-filter of a belligerent GE-algebra X, satisfying (31). Then, F is an imploring GE-filter of X.*

The following example shows that in a transitive GE-algebra, a belligerent GE-filter is neither an imploring GE-filter nor a prominent GE-filter, and a GE-filter is not a prominent GE-filter.

Table 10: Cayley table for the binary operation "$*$".

$*$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	b	c	c	e
b	1	a	1	d	d	e
c	1	1	b	1	1	e
d	1	1	1	1	1	e
e	1	a	b	c	d	1

Table 11: Cayley table for the binary operation "$*$".

$*$	1	a	b	c	d	e	f
1	1	a	b	c	d	e	f
a	1	1	1	d	d	f	f
b	1	1	1	e	e	e	e
c	1	1	b	1	1	1	1
d	1	a	b	a	1	1	1
e	1	a	b	a	1	1	1
f	1	a	b	a	1	1	1

Table 12: Cayley table for the binary operation "$*$".

$*$	1	a	b	c	d	e	f	g
1	1	a	b	c	d	e	f	g
a	1	1	1	c	e	e	1	1
b	1	a	1	d	d	d	g	g
c	1	1	1	1	1	1	1	1
d	1	1	1	1	1	1	1	1
e	1	1	1	1	1	1	1	1
f	1	a	1	e	e	e	1	1
g	1	a	b	d	d	d	b	1

Example 14. In Example 13, the set $F\colon = \{1, b\}$ is a belligerent GE-filter and so a GE-filter of X. But, it is not an imploring GE-filter of X since $b \in F$ and $b * ((a * c) * a) = b * (c * a) = b * 1 = 1 \in F$, but $a \notin F$. Also, since $c * a = 1 \in F$, but $((a * c) * c) * a = (c * c) * a = 1 * a = a \notin F$, F is not a prominent GE-filter of X. Since $c * a = 1 \in F$ and $((a * c) * c) * a = (c * c) * a = 1 * a = a \notin F$, we know that F is not a prominent GE-filter of X.

We finally establish the extension property of the imploring GE-filter.

Lemma 7. *In a transitive GE-algebra X, every GE-filter F satisfies*

$$(\forall x, y, z \in X), \quad (x * (y * z) \in F \Rightarrow (x * y) * (x * z) \in F). \tag{37}$$

Proof. Assume that $x * (y * z) \in F$ for all $x, y, z \in X$. Using (7), (9), (13), and (15), we have

$$\begin{aligned} x * (y * z) &\leq x * ((x * y) * (x * z)) \\ &\leq x * (x * ((x * y) * z)) \\ &= x * ((x * y) * z) \\ &\leq (x * y) * (x * z). \end{aligned} \tag{38}$$

It follows from Lemma 2 that $(x * y) * (x * z) \in F$. □

Theorem 8. *If F is an imploring GE-filter of a transitive GE-algebra X, then every GE-filter G containing F is an imploring GE-filter of X.*

Proof. Let F be an imploring GE-filter of a transitive GE-algebra X and let G be a GE-filter of X such that $F \subseteq G$. Assume that $v: = (x * y) * y \in G$ for all $x, y \in X$. Then, $v * ((x * y) * y) = 1 \in F$, which implies from Lemma 7 that $(v * (x * y)) * (v * y) \in F$. Since $x * (v * y) \leq v * (x * y)$ by (9), we get $(v * (x * y)) * (v * y) \leq (x * (v * y)) * (v * y)$ by (13). Hence, $(x * (v * y)) * (v * y) \in F$ by Lemma 2. It follows from Proposition 2 that $((v * y) * x) * x \in F \subseteq G$. Note that

$$\begin{aligned}(x * y) * y = v &\leq (v * y) * y \leq (y * x) * ((v * y) * x) \\ &\leq (((v * y) * x) * x) * ((y * x) * x).\end{aligned} \tag{39}$$

Hence, $(((v * y) * x) * x) * ((y * x) * x) \in G$, and so $(y * x) * x \in G$. Therefore, G is an imploring GE-filter of X by Theorem 7. □

Corollary 9. *If F is an imploring GE-filter of a belligerent GE-algebra X, then every GE-filter G containing F is an imploring GE-filter of X.*

5. Conclusions

As a generalization of Hilbert algebras, Bandaru et al. introduced the notion of GE-algebras. After that, Bandaru, Rezaei, Borumand Saeid, and Jun studied belligerent GE-filter, prominent GE-filter, and voluntary GE-filter as a study on the substructure of GE-algebras. As a follow-up to these studies, we studied a new substructure called imploring GE-filter. First of all, we discussed relations between a transitive GE-algebra, a belligerent GE-algebra, an antisymmetric GE-algebra, and a left exchangeable GE-algebra. We established the relationship between GE-filter, imploring GE-filter, belligerent GE-filter, and prominent GE-filter. We considered conditions for the imploring GE-filter to be the belligerent GE-filter and looked at the conditions necessary for the (belligerent) GE-filter to be the imploring GE-filter. We discussed relations between the prominent GE-filter and imploring GE-filter and gave a condition for the imploring GE-filter to be the prominent GE-filter. We cited examples to show that a belligerent GE-filter and a prominent GE-filer are independent concepts. We established the extended properties of the imploring GE-filter. The purpose of our research in the future is to continue to think about these things and define and study new substructures in this algebraic structure.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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