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Research Article

Extended Perturbed Mixed Variational-Like Inequalities for Fuzzy Mappings

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In this article, our aim is to consider a class of fuzzy mixed variational-like inequalities (FMVLIs) for fuzzy mapping known as extended perturbed fuzzy mixed variational-like inequalities (EPFMVLIs). As exceptional cases, some new and classically defined "FMVLIs" are also attained. We have also studied the auxiliary principle technique of auxiliary "EPFMVLI" for "EPFMVLI." By using this technique and some new analytic results, some existence results and efficient numerical techniques of "EPFMVLI" are established. Some advanced and innovative iterative algorithms are also obtained, and the convergence criterion of iterative sequences generated by algorithms is also proven. In the end, some new and previously known existence results and algorithms are also studied. Results secured in this paper can be regarded as purification and development of previously familiar results.

1. Introduction

The boundless research work of fuzzy set and systems [1] has been devoted in advancement of different fields. It contributes to a vast class knowledge and appears in pure mathematics and applied sciences as well as operation research, computer science, managements sciences, artificial intelligence, control engineering, and decision sciences [2]. As a part of these knowledge developments, Chang and Zhu [3] initiated to introduce a new type of variational inequality for fuzzy mapping, which is known as fuzzy variational inequality. In fuzzy optimization, Noor [4–6] studied the characterization of minimum of convex fuzzy mapping through fuzzy variational inequality and fuzzy mixed variational inequality and obtained some advanced and effective iterative algorithms. Moreover, they showed the parallel correlation by linking fuzzy variational inequalities and fuzzy Wiener–Hopf equations. Similarly, they established parallel correspondence between fuzzy variational inequalities and the resolvent equations for fuzzy mappings and encouraged some important and novel new iterative algorithms and discussed their convergence criteria. They also introduced fuzzy mixed variational inequalities, and by using the classical auxiliary principle technique, some new existence theorems and iterative algorithms for fuzzy mixed variational inequalities are attained. It is worthy to mention here that one of the most considered generalizations of convex fuzzy mappings is preinvex fuzzy mapping. The idea of fuzzy preinvex mapping on the invex set was introduced and studied by Noor [7]. Moreover, any local minimum of a preinvex fuzzy mapping is a global minimum on invex set, and necessary and sufficient condition for fuzzy mapping is to be preinvex if its epigraph is an invex set.

Furthermore, it has been verified that fuzzy optimality conditions of differentiable fuzzy preinvex mappings can be distinguished by variational-like inequalities. Motivated and inspired by the ongoing research work, many authors discussed fuzzy variational inequalities and its important generalizations and its applications in different fields [8, 9]. In the subsequent text, we will review the applications and generalizations of variational inequalities for fuzzy mappings. First, we give some literature survey of variational inequality and its generalizations. In early 1960s, the idea of variational inequality was initiated by Hartman and Stampacchia [10]. The useful generalizations of variational inequality are variational-like inequality (in short, VLI) and generalized mixed variational-like inequality (in short, GMVLI). It is well known that in classical variational inequality theory, the projection method fails to discuss the existence of solutions of variational-like inequalities. This familiarity has attracted many authors to propose the auxiliary principle technique to acknowledge existence results for different variational inequalities. Glowinski et al. [11] suggested the auxiliary principle technique. For further information about the auxiliary principle technique, we refer the readers to [12–17] and the references therein. Similarly, the projection type methods can be used to recommend iterative methods for "VLI" and "GMVLI" for fuzzy mappings. To overcome this challenging task and with the help of classical auxiliary principle techniques, Chang [8], Chang et al. [18], and Kumam and Petrot [19] studied the idea of "GMVLI" and complementarity problems for ordinary set-valued mapping and fuzzy mappings in different contexts with compact and noncompact values. For applications of "GMVLI," see [3–7, 9, 12–14, 16, 19–26] and the references therein.

In this article, we shall introduce and discuss a new type of "GMVLI" for fuzzy mappings, which is known as extended perturbed fuzzy mixed variational-like inequality (EPFMVLI) for fuzzy mapping because by using this technique, we can easily handle the functional which is the sum of differentiable preinvex fuzzy mapping and strongly preinvex fuzzy mapping and their special cases like the single-valued functional which is the sum of the differentiable preinvex functions and strongly preinvex functions. By using this technique and some new analytic results, some existence results and efficient numerical techniques of "PFMVLI" are established. As a result of this technique, some advanced and innovative iterative algorithms can be obtained. Moreover, the convergence criteria of iterative sequences generated by algorithms can also be proven. At the end, we shall discuss some particular cases of "PFMVLI" for ordinary set-valued mappings and fuzzy mappings (see [3–7, 9, 12–14, 16, 19–22] and the references therein).

2. Preliminaries

Let $\mathfrak{J}$ be a real Hilbert space and $\varnothing \neq \mathfrak{C} \subset \mathfrak{J}$ be a convex set. We denote the collection $CB(\mathfrak{J})$ of all nonempty bounded and closed subsets of $\mathfrak{J}$, and $\mathfrak{D}(.,.)$is the Hausdorff metric on $CB(\mathfrak{J})$ defined by

$$\mathfrak{D}(\mathscr{K}, A) = \max\left\{\sup_{u\in A} d(u,\mathscr{K}), \sup_{\vartheta\in\mathscr{K}} d(\vartheta, A)\right\}, \quad A,\mathscr{K} \in CB(\mathfrak{J}). \tag{1}$$

A mapping $\psi\colon \mathfrak{J} \longrightarrow [0,1]$ is called fuzzy set. If $\mathrm{a} \in (0,1]$, then the set $\psi_{\mathrm{a}} = \{u \in \mathfrak{J} | \psi(u) \geq \mathrm{a}\}$ is known as a-level set of ψ. If $\mathrm{a} = 0$, then $\mathrm{supp}(\psi) = \{u \in \mathfrak{J} | \psi(u) > 0\}$ is called support of ψ. By $[\psi]^0$, we define the closure of $\mathrm{supp}(\psi)$.

In what follows, $\mathscr{F}(\mathfrak{J}) = \{\mathfrak{A}\colon \mathfrak{J} \longrightarrow I = [0,1]\}$ denotes the family of all fuzzy sets on $\mathfrak{J}$. A mapping $\mathscr{P}$ from $\mathfrak{J}$ to $\mathscr{F}(\mathfrak{J})$ is called a fuzzy mapping. If $\mathscr{P}\colon \mathfrak{J} \longrightarrow \mathscr{F}(\mathfrak{J})$is a fuzzy mapping, then the set $\mathscr{P}(u)$ for $u \in \mathfrak{J}$ is a fuzzy set in $\mathscr{F}(\mathfrak{J})$ (in the sequel we denote $\mathscr{P}(u)$ by $\mathscr{P}_u$) and $\mathscr{P}_u(\vartheta)\vartheta \in \mathfrak{J}$ is the degree of membership of ϑin $\mathscr{P}_u$.

Definition 1 (see [18]). If for each $u \in \mathfrak{J}$, the function $\vartheta \longrightarrow \mathscr{P}_u\vartheta$ is upper semicontinuous, then fuzzy mapping $\mathscr{P}$ is called closed. If $\{\vartheta_r\} \subset \mathfrak{J}$ is a net satisfying $\vartheta_r \longrightarrow \vartheta_0 \in \mathfrak{J}$, then $\mathscr{P}_u$ has the following property:

$$\mathrm{limt\,sup}_r \mathscr{P}_u(\vartheta_u) \leq \mathscr{P}_u(\vartheta_0). \tag{2}$$

Definition 2 (see [18]). A closed fuzzy mapping $\mathscr{P}\colon \mathfrak{J} \longrightarrow \mathscr{F}(\mathfrak{J})$ is said to satisfy the condition $(\mathfrak{B})$, if there exists a function $\mathscr{P}\colon \mathfrak{J} \longrightarrow [0,1]$ such that for each $u \in \mathfrak{J}$, the set

$$[\mathscr{P}_u]_{r(u)} = \{\vartheta \in \mathfrak{J}\colon \mathscr{P}_u(\vartheta) \geq r(u)\} \neq \varnothing, \tag{3}$$

is a bounded subset of $\mathfrak{J}$.

Remark 1 (see [18]). From Definitions 1 and 2, it can be easily seen that $[\mathscr{P}_u]_{r(u)} \in \mathrm{CB}(\mathfrak{J})$. Indeed, let $\{\vartheta_r\}_{r\in\Gamma} \subset [\mathscr{P}_u]_{r(u)}$ be a net and $\vartheta_r \longrightarrow \vartheta_0 \in \mathfrak{J}$. Then, from (2) and (3), we have $\mathrm{limt\,sup}_r \mathscr{P}_u(\vartheta_r) \leq \mathscr{P}_u(\vartheta_0)$ and $\mathscr{P}_u(\vartheta_r) \geq r(u)$ for each $r \in \Gamma$. Since $\mathscr{P}$ is closed with condition $(\mathfrak{B})$, we have

$$\mathscr{P}_u(\vartheta_0) \geq \mathrm{limt\,sup}_r \mathscr{P}_u(\vartheta_r) \geq r(u). \tag{4}$$

This implies that $\vartheta_0 \in [\mathscr{P}_u]_{r(u)}$, and so $[\mathscr{P}_u]_{r(u)} \in \mathrm{CB}(\mathfrak{J})$.

Problem 1. Let $\mathfrak{J}$ be a real Hilbert space. Then, for given nonlinear mappings $\xi(.,.)\colon \mathfrak{J} \times \mathfrak{J} \longrightarrow \mathfrak{J}$, $\mathscr{M}(.,.)\colon \mathfrak{J} \times \mathfrak{J} \longrightarrow \mathfrak{J}$, we consider the problem of finding $u \in \mathfrak{J}$, $p, q \in \mathfrak{J}$ and constant $\omega > 0$, such that

$$\begin{aligned} &\langle \mathscr{M}(p,q), \xi(\vartheta,u)\rangle + \mathscr{J}(\vartheta,u) - \mathscr{J}(u,u) \\ &\quad + \omega\|\xi(\vartheta,u)\|^2 \geq 0, \quad \forall \vartheta \in \mathfrak{J}, \end{aligned} \tag{5}$$

where $\mathscr{P},\mathscr{V}\colon \mathfrak{C} \longrightarrow \mathscr{F}(\mathfrak{J})$are two closed fuzzy mappings satisfying condition $(\mathfrak{B})$ with function $r, s\colon \mathfrak{J} \longrightarrow [0,1]$, respectively, such that

$$\mathscr{P}_u(p) \geq r(u), \mathscr{V}_u(q) \geq s(u), \quad \text{i.e.} \in [\mathscr{P}_u]_{r(u)}, q \in [\mathscr{V}_u]_{s(u)}, \tag{6}$$

and $\mathscr{J}(.,.)$: $\mathfrak{J} \times \mathfrak{J} \longrightarrow R$is a nondifferentiable function fulfilling the following assumption.

Assumption 1

(a) For any $u, \vartheta, p \in \mathfrak{J}$, there exists constant $\partial > 0$, such that

(i) $|\mathscr{J}(u, \vartheta)| \leq \partial \|u\| \|\vartheta\|$

(ii) $|\mathscr{J}(u, \vartheta) - \mathscr{J}(u, p)| \leq \partial \|u\| \|u - p\|$

(b) $\mathscr{J}(u, \vartheta)$is linear in respect of u(i.e., for any $u, \vartheta \in \mathfrak{J} \mathscr{J}(-u, \vartheta) = -\mathscr{J}(u, \vartheta)$)

(c) $\mathscr{J}(u, \vartheta)$is convex in respect of ϑ

The inequality (5) is called "EPFMVLI." It is important to mention that the generalized fuzzy mixed variational-like inequality (in short, GFMVLI), classical mixed variational-like inequality (in short, MVLI), and associated fuzzy optimizations problems are particular cases of inequality (5). For applications, see [3, 8–10, 13] and the references therein.

Now, we study some certain cases of problem (5).

(1) If $\omega = 0$, then problem (5) is analogous to finding $u \in \mathfrak{J} p \in [\mathscr{P}_u]_{r(u)} q \in [\mathscr{V}_u]_{s(u)}$, such that

$$\langle \mathscr{M}(p, q), \xi(\vartheta, u) \rangle + \mathscr{J}(\vartheta, u) - \mathscr{J}(u, u) \geq 0, \quad \forall \vartheta \in \mathfrak{J}. \tag{7}$$

This type of problem is called "GFMVLI" and was studied and established by Chang et al. [18] and, under some added condition, improved by Kumam and Petrot [19].

(2) Let $\mathscr{P}, \mathscr{V}$: $\mathfrak{J} \longrightarrow \mathscr{F}(\mathfrak{J})$be an ordinary multi-valued mapping and $\mathscr{J}, \xi$ be the mapping in problem (5).

Now, we define a fuzzy mapping $\tilde{\mathscr{P}}(.), \tilde{\mathscr{V}}(.)$: $\mathfrak{J} \longrightarrow \mathscr{F}(\mathfrak{J})$as follows:

$$\begin{aligned} \tilde{\mathscr{P}}_u &= \mathscr{X}\mathscr{P}_u, \\ \tilde{\mathscr{V}}_u &= \mathscr{X}\mathscr{V}_u, \end{aligned} \tag{8}$$

where $\mathscr{X}\mathscr{P}_u \mathscr{X}\mathscr{V}_u$are two characteristic functions of $\mathscr{P}(u)$, $\mathscr{V}(u)$. From (8), it can straightforwardly be noticed that $\tilde{\mathscr{P}}$, $\tilde{\mathscr{V}}$ are two closed fuzzy mapping fulfilling condition ($\mathfrak{B}$) with constant function $r(u) = 1, s(u) = 1$, respectively, for all $u \in \mathfrak{J}$. Also,

$$\begin{aligned} \left[\tilde{\mathscr{P}}_u\right]_{r(u)} &= [\mathscr{X}\mathscr{P}_u]_1 = \{\vartheta \in \mathfrak{J}: \mathscr{P}_u(\vartheta) = 1\} = \mathscr{P}(u), \\ \left[\tilde{\mathscr{V}}_u\right]_{s(u)} &= [\mathscr{X}\mathscr{V}_u]_1 = \{\vartheta \in \mathfrak{J}: \mathscr{V}_u(\vartheta) = 1\} = \mathscr{V}(u). \end{aligned} \tag{9}$$

Then, problem (5) is parallel to detecting $u \in \mathfrak{J}$, such that

$$\begin{aligned} &\left(\tilde{\mathscr{P}}_u\right)(p) = 1, \left(\tilde{\mathscr{V}}_u\right)(q) = 1, \quad \text{i.e. } p \in \mathscr{P}(u), q \in \mathscr{V}(u), \\ &\langle \mathscr{M}(p, q), \xi(\vartheta, u) \rangle + \mathscr{J}(\vartheta, u) - \mathscr{J}(u, u) + \omega \|\xi(\vartheta, u)\|^2 \geq 0, \quad \forall \vartheta \in \mathfrak{J}. \end{aligned} \tag{10}$$

This kind of problem is called the set-valued "EPMVLI." This mixed variational-like inequality is also new one.

(3) If $\omega = 0$, then problem (10), is analogous to finding $u \in \mathfrak{J}$, $p \in \mathscr{P}(u)$, $q \in \mathscr{V}(u)$, such that

$$\langle \mathscr{M}(p, q), \xi(\vartheta, u) \rangle + \mathscr{J}(\vartheta, u) - \mathscr{J}(u, u) \geq 0, \quad \forall \vartheta \in \mathfrak{J}. \tag{11}$$

This type of problem is called "GMVLI" and was studied and established by Noor [18] and, under some added condition, improved by Zeng [17].

(4) If $\mathscr{M}(p, q) = \mathscr{M}(p)$and $\mathscr{P}$: $\mathfrak{J} \longrightarrow F(\mathfrak{J})$, then problem (5) is analogous to finding $u \in \mathfrak{J}$, $p \in [\mathscr{P}_u]_{r(u)}$, such that

$$\begin{aligned} &\langle \mathscr{M}(p), \xi(\vartheta, u) \rangle + \mathscr{J}(\vartheta) - \mathscr{J}(u) \\ &+ \omega \|\xi(\vartheta, u)\|^2 \geq 0, \quad \forall \vartheta \in \mathfrak{J}. \end{aligned} \tag{12}$$

Then, problem (12) is called "PFMVLI." This class of fuzzy variational-like inequality is also new one.

Note that problem (12) is a particular case of problem (5); therefore, we discuss some other special cases of problem (5) using problem (12).

(5) If $\mathscr{M} = I$(identity mapping), $p \in \mathfrak{J}$, and $\mathscr{P}$: $\mathfrak{J} \longrightarrow \mathfrak{J}$ is single valued, then problem (10) is parallel to finding $u \in \mathfrak{J}$such that

$$\begin{aligned} &\langle \mathscr{P}(u), \xi(\vartheta, u) \rangle + \mathscr{J}(\vartheta) - \mathscr{J}(u) \\ &+ \omega \|\xi(\vartheta, u)\|^2 \geq 0, \quad \forall \vartheta \in \mathfrak{J}. \end{aligned} \tag{13}$$

This is known as "PMVLI" and was studied by Noor et al. [12].

(6) If $\xi(\vartheta, u) = \vartheta - u$, then problem (12) is called strongly fuzzy mixed variational inequality and is parallel to finding $u \in \mathfrak{J}$, $p \in [\mathscr{P}_u]_{r(u)}$such that

$$\langle \mathcal{M}(p), \vartheta - u\rangle + \mathcal{J}(\vartheta) - \mathcal{J}(u) + \omega\|\vartheta - u\|^2 \geq 0, \quad \forall \vartheta \in \mathfrak{J}. \tag{14}$$

This class of "FMVI" is also new one. The classical FMVLI and associated fuzzy optimizations problems are special cases of inequality (15) [21].

(7) If $\omega = 0$, then problem (12) is parallel to finding $u \in \mathfrak{J}$, $p \in [\mathcal{P}_u]_{r(u)}$ such that

$$\langle \mathcal{M}(p), \xi(\vartheta, u)\rangle + \mathcal{J}(\vartheta) - \mathcal{J}(u) \geq 0, \quad \forall \vartheta \in \mathfrak{J}, \tag{15}$$

which is known as "FMVLI." The FMVLI is a generalised form of Chang et al. [18] and Kumam and Petrot [19]. It is also an ordinary set-valued mapping of Noor [22]. For the development of numerical methods and applications of (15), see [9, 15, 16, 27] and the references therein.

(8) For $\omega = 0$,problem (14) reduces to

$$u \in \mathfrak{J}, p \in [\mathcal{P}_u]_{r(u)}, \quad \langle \mathcal{M}(p), \vartheta - u\rangle + \mathcal{J}(\vartheta) - \mathcal{J}(u) \geq 0, \quad \forall \vartheta \in \mathfrak{J}, \tag{16}$$

which is known as "FMVI." Chang et al. [18] and Kumam and Petrot [19] studied it as a special case. In case of ordinary set-valued mapping as developed by Lions and Stampacchia [20], for applications, see [11, 15, 22, 27] and the references therein.

(9) When $\mathcal{M} = I$and $\mathcal{P}: \mathfrak{J} \longrightarrow \mathcal{F}(\mathfrak{J})$is a fuzzy mapping, then problem (16) is parallel to finding $u \in \mathfrak{J}$, $p \in [\mathcal{P}_u]_{r(u)}$.

$$\langle p, \vartheta - u\rangle + \mathcal{J}(\vartheta) - \mathcal{J}(u) \geq 0, \quad \forall \vartheta \in \mathfrak{J}. \tag{17}$$

This is known as "FMVI" (see [5]).

(10) If $\mathcal{J}(.)$ is an indicator mapping of a closed convex set $\mathfrak{C}$in $\mathfrak{J}$, that is,

$$I_{\mathcal{J}}(u) = \begin{cases} 0, & u \in \mathfrak{C}, \\ \infty, & \text{otherwise}, \end{cases} \tag{18}$$

then problem (15) is analogous to finding $p \in \mathfrak{J}$, $p \in [\mathcal{P}_u]_{r(u)}$such that

$$\langle \mathcal{M}(p), \xi(\vartheta, u)\rangle \geq 0, \quad \forall \vartheta \in \mathfrak{J}, \tag{19}$$

which is known as "FVLI."

(11) When $\mathcal{M} = I$, then problem (19) is analogous to finding $u \in \mathfrak{J}$, $p \in [\mathcal{P}_u]_{r(u)}$ such that

$$\langle p, \xi(\vartheta, u)\rangle \geq 0, \quad \forall \vartheta \in \mathfrak{J}, \tag{20}$$

which is also known as "FVLI." (see [7]). For the applications of "FVLI" and fuzzy optimization problem, see [9, 16, 22, 27] and the references therein.

(12) If $\xi(\vartheta, u) = \vartheta - u$, then problem (19) is analogous to finding $u \in \mathfrak{J}$, $p \in [\mathcal{P}_u]_{r(u)}$ such that

$$\langle \mathcal{M}(p), \vartheta - u\rangle \geq 0, \quad \forall \vartheta \in \mathfrak{J}. \tag{21}$$

This is known as "FVI."

(13) If $\mathcal{M} = I$, then problem (21) is analogous to finding $u \in \mathfrak{J}$, $p \in [\mathcal{P}_u]_{r(u)}$ such that

$$\langle p, \vartheta - u\rangle \geq 0, \quad \forall \vartheta \in \mathfrak{J}. \tag{22}$$

This is also known as "FVI" (see [3, 4]). The "VI" is mainly due to Stampacchia and Guido [28] and Lions and Stampacchia [20], as developed and studied by Noor [4]. For the applications of problem (22), see [9, 11, 20, 21] and the references therein.

From the above discussion, it can be easily seen that problems (7)–(22) are particular cases of "EPFMVLI" (5). In fact, "EPFMVLI" is more generalized and unifying one, which is main motivation of our work. For a proper and suitable choice of $\mathcal{P}$, ξ, and $\mathcal{J}$, we can choose a number of known and unknown "PFVLI" and complementary problems.

Next, we will use mathematical terminologies S-mixed monotone and L-continuous for strongly mixed monotone and Lipschitz continuous, respectively.

Definition 3 (see [18]). If fuzzy mappings $\mathcal{P}, \mathcal{V}: \mathfrak{J} \longrightarrow \mathcal{F}(\mathfrak{J})$ are closed and fulfil the condition $(\mathfrak{B})$ with functions $r, s: \mathfrak{J} \longrightarrow [0,1]$, then nonlinear mapping $\mathcal{M}(.,.): \mathfrak{J} \times \mathfrak{J} \longrightarrow \mathfrak{J}$ is said to be

(i) γ-L-continuous in respect of first argument if for any $u_1, u_2 \in \mathfrak{J}$and $p_1 \in [\mathcal{P}_{u_1}]_{r(u_1)}$, $u_2 \in [\mathcal{P}_{u_2}]_{r(u_2)}$, there exists a constant $\gamma > 0$ such that

$$\|(\mathcal{M}(p_1,.) - \mathcal{M}(p_2,.))\| \leq \gamma\|p_1 - p_2\|, \quad \text{for all } p_1 \in [\mathcal{P}_{u_1}]_{r(u_1)}, \ p_2 \in [\mathcal{P}_{u_2}]_{r(u_2)}. \tag{23}$$

(ii) β-strongly mixed monotone in respect of $\mathcal{P}$ and $\mathcal{V}$ if there exists a constant $\beta > 0$ such that

$$\langle \mathcal{M}(p_1, q_1) - \mathcal{M}(p_2, q_2), u_1 - u_2\rangle \geq \beta\|u_1 - u_2\|^2, \tag{24}$$

for all $p_1 \in [\mathscr{P}_{u_1}]_{r(u_1)}$, $p_2 \in [\mathscr{P}_{u_2}]_{r(u_2)}$, $q_1 \in [\mathscr{V}_{u_1}]_s(u_1)$, $q_2 \in [\mathscr{V}_{u_2}]_{s(u_2)}$.

(iii) α-L-continuous in respect of second argument if for any $\vartheta_1, \vartheta_2 \in \mathfrak{J}$ and $q_1 \in [\mathscr{V}_{\vartheta_1}]_{s(\vartheta_1)}$, $q_2 \in [\mathscr{V}_{\vartheta_2}]_{s(\vartheta_2)}$, there exists a constant $\alpha > 0$, such that

$$\|(\mathscr{M}(.,q_1) - \mathscr{M}(.,q_2))\| \leq \alpha\|q_1 - q_2\|. \quad (25)$$

Definition 4 (see [18]). Let $\mathscr{P}, \mathscr{V}: \mathfrak{J} \longrightarrow \mathscr{F}(\mathfrak{J})$ be two fuzzy mappings. Then,

(iv) $\mathscr{P}$ is said to be $\mathscr{P}$-L-continuous if for any $u, \vartheta \in \mathfrak{J}$, there exists a function $r: \mathfrak{C} \longrightarrow [0,1]$and constant $\lambda > 0$, such that

$$\mathfrak{D}\left([\mathscr{P}_u]_{r(u)}, [\mathscr{P}_\vartheta]_{r(\vartheta)}\right) \leq \lambda\|\vartheta - u\|. \quad (26)$$

(v) $\mathscr{V}$ is said to be $\mathscr{V}$-L-continuous: if there exists a function $s: \mathfrak{C} \longrightarrow [0,1]$and constant $\xi > 0$, such that

$$\mathfrak{D}\left([\mathscr{V}_u]_{s(u)}, [\mathscr{V}_\vartheta]_{s(\vartheta)}\right) \leq \xi\|\vartheta - u\|, \quad (27)$$

where $\mathfrak{D}(.,.)$ is the Hausdorff metric on $\mathscr{F}(\mathfrak{J})$.

In particular, from (i) and (ii), we have

$$\begin{aligned}\beta\|u_1 - u_2\|^2 &\leq \langle \mathscr{M}(p_1,q_1) - \mathscr{M}(p_2,q_2), u_1 - u_2\rangle \leq \|\mathscr{M}(p_1,q_1) - \mathscr{M}(p_2,q_2)\|\|u_1 - u_2\| \\ &\leq \gamma\|p_1 - p_2\|\|u_1 - u_2\| \\ &\leq \gamma\mathfrak{D}\left([\mathscr{P}_{u_1}]_{r(u_1)}, [\mathscr{P}_{u_2}]_{r(u_2)}\right)\|u_1 - u_2\|.\end{aligned} \quad (28)$$

From (iv), we have

$$\prec \gamma\lambda\|u_1 - u_2\|^2, \quad (29)$$

which implies that $\beta \prec \gamma\lambda$. Similarly, from (ii), (iii), and (v), we can observe that $\beta \prec \alpha\xi$.

Definition 5 (see [12]). The bifunction $\xi(.,.): \mathfrak{J} \times \mathfrak{J} \longrightarrow \mathfrak{J}$ is said to be

(vi) S-monotone: if there exists constant $\mu > 0$, such that

$$\langle \xi(\vartheta,u), \vartheta - u\rangle \geq \mu\|\vartheta - u\|^2, \quad \text{for all } u, \vartheta \in \mathfrak{J}. \quad (30)$$

(vii) L-continuous: if there exists constant $\delta > 0$, such that

$$\|\xi(\vartheta,u)\| \prec \delta\|\vartheta - u\|, \quad \text{for all } u, \vartheta \in \mathfrak{J}. \quad (31)$$

From (vi) and (vii), we can observe that $\mu \prec \delta$.

For any $\mathscr{K} \subseteq \mathfrak{J}$, we denote the conv($\mathscr{K}$), the convex hull of $\mathscr{K}$. A set-valued mapping $T: \mathfrak{J} \longrightarrow 2^{\mathfrak{J}}$is called a KKM mapping if, for every finite subset $\{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \ldots, \vartheta_n\}$of $\mathfrak{J}$,

$$\operatorname{conv}\{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4 \ldots, \vartheta_n\} \subseteq \bigcup_{i=1}^{n} T(\vartheta_i). \quad (32)$$

Lemma 1 (see [29]). *Let $\mathscr{K}$ be arbitrary nonempty in a topological vector space G and let $T: \mathscr{K} \longrightarrow 2^{\mathfrak{J}}$ be a KKM mapping. If $T(\vartheta)$ is closed for all $\vartheta \in \mathscr{K}$ and is compact for at least one $\vartheta \in \mathscr{K}$, then*

$$\bigcap_{\vartheta \in \mathscr{K}} T(\vartheta) \neq \varnothing. \quad (33)$$

Theorem 1 (see [8]). *Let G be a locally convex Hausdorff topologically vector space and $g: G \longrightarrow R \cup \{+\infty\}$ be a properly convex functional (i.e., a functional $g(.)$ is called proper, if $g(u) > -\infty$ for all $u \in G$ and $\mathscr{J}(u) \equiv +\infty$). Then, g is lower semicontinuous on G if and only if, g is weakly lower semicontinuous on G.*

Assumption 2. Let $\mathscr{M}: \mathfrak{J} \times \mathfrak{J} \longrightarrow \mathfrak{J}$ and $\xi(.,.): \mathfrak{J} \times \mathfrak{J} \longrightarrow \mathfrak{J}$ be two mappings satisfying the following condition:

(a) $\xi(\vartheta,u) + \xi(u,\vartheta) = 0$ (and so $\xi(u,u) = 0$, for all $u \in \mathfrak{J}$), for all $u, \vartheta \in \mathfrak{J}$.

(b) For any given $u \in \mathfrak{J}$, the mapping $\vartheta \longmapsto \langle \mathscr{M}(p,q), \xi(\vartheta,u)\rangle$is concave, where $p \in [\mathscr{P}_u]_{r(u)}, q \in [\mathscr{V}_u]_{s(u)}$.

(c) For any given $u \in \mathfrak{J}$, the mapping $\vartheta \mapsto \langle \mathscr{M}(p,q), \xi(\vartheta,u)\rangle$is lower semicontinuous, where $p \in [\mathscr{P}_u]_{r(u)}, q \in [\mathscr{V}_u]_{s(u)}$.

$$u_n \longrightarrow u, p_n \longrightarrow p, q_n \longrightarrow q \text{ imply } \langle \mathscr{M}(p,q), \xi(\vartheta,u)\rangle \leq \liminf_{n\longrightarrow\infty} \langle \mathscr{M}(p_n,q_n), \xi(\vartheta,u_n)\rangle. \quad (34)$$

Lemma 2 (see [30]). *Let (X, d) be a complete metric space and $\mathscr{K}_1, \mathscr{K}_2 \in CB(X)$ and $r \geq 1$ be any real number. Then, for every $k_1 \in \mathscr{K}_1$, there exist $k_2 \in \mathscr{K}_2$ such that $d(k_1,k_2) \leq r\mathfrak{D}(\mathscr{K}_1, \mathscr{K}_2)$.*

In the next sections, we will use the above preliminaries.

3. Auxiliary Principle and Algorithm

Now, we review the Auxiliary problem associated with "EPFMVLI" (5) and prove an existence theorem for the auxiliary problem. Furthermore, based on this existence result, we suggest an iterative algorithm for the "PFMVLI" (5).

Auxiliary problem: for a given $u \in \mathfrak{C}, p \in [\mathscr{P}_u]_{r(u)}, q \in [\mathscr{V}_u]_{s(u)}$ satisfying problem (5), we consider the problem of finding $u \in \mathfrak{J}$, such that

$$\langle z, \vartheta - z\rangle \geq \langle u, \vartheta - z\rangle - \rho\langle \mathscr{M}(p,q), \xi(\vartheta,z)\rangle + \rho\mathscr{J}(u,z) - \rho\mathscr{J}(u,\vartheta) - \omega\rho\|\xi(\vartheta,z)\|^2, \tag{35}$$

for $\forall\vartheta \in \mathfrak{C}$, where $\rho > 0$ is a constant.

The inequality (35) is also called auxiliary "EPFMVLI."

Theorem 2. *Let $\mathfrak{J}$ be real Hilbert space and $\mathfrak{C} \neq \emptyset$ be a closed bounded subset of $\mathfrak{J}$. Let $\mathscr{P}, \mathscr{V}: \mathfrak{J} \times \mathfrak{J} \longrightarrow \mathscr{F}(\mathfrak{J})$ be two closed continuous fuzzy mappings satisfying condition ($\mathfrak{B}$) with function $r, s: \mathfrak{J} \longrightarrow [0,1]$. If Assumptions 1 and 2 are satisfied and bifunction $\xi(.,.): \mathfrak{C} \times \mathfrak{C} \longrightarrow \mathfrak{J}$ is L-continuous with constant $\delta > 0$, then auxiliary problem (35) has a unique solution.*

Proof. Let $u \in \mathfrak{C}$, $p \in [\mathscr{P}_u]_{r(u)}, q \in [\mathscr{V}_u]_{s(u)}$; we consider the mapping $T: \mathfrak{J} \longrightarrow 2^{\mathfrak{J}}$ defined by

$$T(\vartheta) = \{w \in \mathfrak{C}: \langle w - u, \vartheta - w\rangle \geq -\rho\langle \mathscr{M}(p,q), \xi(\vartheta,w)\rangle + \rho\mathscr{J}(u,w) - \rho\mathscr{J}(u,\vartheta) - \omega\rho\|\xi(\vartheta,w)\|^2\}, \tag{36}$$

for all $\vartheta \in \mathfrak{J}$.

From (36), it can be easily seen that for each $\vartheta \in \mathfrak{C}$, $T(\vartheta) \neq \emptyset$, since $\vartheta \in T(\vartheta)$.To obtain the solution, firstly we show that $T: \mathfrak{J} \longrightarrow 2^{\mathfrak{J}}$is a KKM mapping. Suppose the contrary, that is, $T: \mathfrak{J} \longrightarrow 2^{\mathfrak{J}}$is not KKM mapping, then there exists a finite subset $\{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \ldots, \vartheta_n\}$of $\mathfrak{C}$ and constant $\tau_i \geq 0$, $i = 1,2,3,4,\ldots,k$ with $\sum_{i=1}^{k} \tau_i = 1$such that

$$w_* = \sum_{i=1}^{k} \tau_i\vartheta_i \notin \bigcup_{i=1}^{k} T(\vartheta_i), \quad \text{for all } 1,2,3,4,\ldots,k. \tag{37}$$

Then, we have

$$\langle w_* - u, \vartheta_i - w_*\rangle + \rho\langle \mathscr{M}(p,q), \xi(\vartheta_i, w_*)\rangle + \rho\mathscr{J}(u,\vartheta_i) - \rho\mathscr{J}(u,w_*) + \omega\rho\left\|\xi(\vartheta_i,w_*)\right\|^2 < 0. \tag{38}$$

From Assumptions 1 and 2, the above inequality yields

$$\begin{aligned} 0 &> \sum_{i=1}^{k} \tau_i\langle w_* - u, \vartheta_i - w_*\rangle + \rho\sum_{i=1}^{k} \tau_i\langle \mathscr{M}(p,q), \xi(\vartheta_i,w_*)\rangle + \rho\sum_{i=1}^{k} \tau_i\mathscr{J}(u,\vartheta_i) \\ &\quad - \rho\mathscr{J}(w_*) + \omega\rho\sum_{i=1}^{k} \tau_i\left\|\xi(\vartheta_i,w_*)^2\right\| \\ &\geq \langle w_* - u, w_* - w_*\rangle + \rho\langle \mathscr{M}(p,q), \xi(w_*,w_*)\rangle - \rho\mathscr{J}(u,w_*) + \rho\mathscr{J}(u,w_*) \\ &\quad + \omega\rho\left\|\xi(w_*,w_*)\right\|^2 = 0. \end{aligned} \tag{39}$$

This is a contradiction. Hence, $T: \mathfrak{J} \longrightarrow 2^{\mathfrak{J}}$ is a KKM mapping.

Since $\overline{T(\vartheta)}^p$, the weak closure of $T(\vartheta)$ is weakly closed subset of a bounded set $\mathfrak{C}\subseteq\mathfrak{J}$, so it is weakly compact. Hence, by Lemma 1,

$$\bigcap_{\vartheta\in\mathfrak{C}} \overline{T(\vartheta)^p} \neq \emptyset. \tag{40}$$

Let

$$z \in \bigcap_{\vartheta\in\mathfrak{C}} \overline{T(\vartheta)^p}. \tag{41}$$

Then, there exists a sequence $z_n$in $T(\vartheta)$(fix ϑ), such that $z_n \longrightarrow z$. Then,

$$\langle z_n - u, \vartheta - z_n\rangle + \rho\langle \mathscr{M}(p,q), \xi(\vartheta,z_n)\rangle + \rho\mathscr{J}(u,\vartheta) - \rho\mathscr{J}(u,z_n) + \omega\rho\left\|\xi(\vartheta,z_n)\right\|^2 \geq 0. \tag{42}$$

Now, by using the property of inner product,

$$\begin{aligned} \langle z_n - u, \vartheta - z_n\rangle &= \langle z_n - u, \vartheta\rangle + \langle u, z_n\rangle - \langle z_n, z_n\rangle \\ &= \langle z_n - u, \vartheta\rangle + \langle u, z_n\rangle - \left\|z_n\right\|^2. \end{aligned} \tag{43}$$

Since $\|.\|$ is weakly lower semicontinuous, we have

$$\begin{aligned}\limsup_{n\longrightarrow\infty}\langle z_n-u,\vartheta-z_n\rangle &= \limsup_{n\longrightarrow\infty}\left\{\langle z_n-u,\vartheta\rangle+\langle u,z_n\rangle-\|z_n\|^2\right\}\\ &= \limsup_{n\longrightarrow\infty}\langle z_n-u,\vartheta\rangle+\limsup_{n\longrightarrow\infty}\langle u,z_n\rangle-\liminf_{n\longrightarrow\infty}\|z_n\|^2\\ &\leq\langle z-u,\vartheta-z\rangle.\end{aligned}\tag{44}$$

Again, by Assumptions 1 and 2, we get

$$\begin{aligned}0&\leq\limsup_{n\longrightarrow\infty}\left\{\langle z_n-u,\vartheta-z_n\rangle+\rho\langle\mathscr{M}(p,q),\xi(\vartheta,z_n)\rangle+\rho\mathscr{J}(u,\vartheta)-\rho\mathscr{J}(u,z_n)+\omega\rho\|\xi(\vartheta,z_n)\|^2\right\}\\ &=\limsup_{n\longrightarrow\infty}\left\{\langle z_n-u,\vartheta-z_n\rangle+\rho\limsup_{n\longrightarrow\infty}\langle\mathscr{M}(p,q),\xi(\vartheta,z_n)\rangle+\rho\limsup_{n\longrightarrow\infty}\mathscr{J}(u,\vartheta)-\rho\liminf_{n\longrightarrow\infty}\mathscr{J}(u,z_n)+\omega\rho\limsup_{n\longrightarrow\infty}\|\xi(\vartheta,z_n)\|^2\right\}\\ &\leq\langle z-u,\ \vartheta-z\rangle+\rho\langle\mathscr{M}(p,q),\xi(\vartheta,z)\rangle+\rho\mathscr{J}(u,\vartheta)+\rho\mathscr{J}(u,z)+\omega\rho\|\xi(\vartheta,z)\|^2.\end{aligned}\tag{45}$$

This implies that

$$\begin{aligned}\langle z,\vartheta-z\rangle\geq{}&\langle u,\vartheta-z\rangle-\rho\langle\mathscr{M}(p,q),\ \xi(\vartheta,z)\rangle\\ &+\rho\mathscr{J}(\vartheta)-\rho\mathscr{J}(z)-\omega\rho\|\xi(\vartheta,z)\|^2,\end{aligned}\tag{46}$$

for all $\vartheta\in\mathfrak{J}$.

Hence, z is solution of auxiliary problem (35).

Uniqueness: Let $z_1\in\mathfrak{C}$ be also a solution of Auxiliary problem. Then, we have

$$\begin{aligned}\langle z_1,\vartheta-z_1\rangle\geq{}&\langle u,\vartheta-z_1\rangle-\rho\langle\mathscr{M}(p,q),\xi(\vartheta,z_1)\rangle\\ &-\rho\mathscr{J}(u,\vartheta)+\rho\mathscr{J}(u,z_1)-\rho\omega\|\xi(\vartheta,z_1)\|^2,\end{aligned}\tag{47}$$

for all $\vartheta\in\mathfrak{J}$.

Replacing ϑ by $z_1$in (46) and ϑ by $z$in (47), we have

$$\begin{aligned}\langle z,z_1-z\rangle\geq{}&\langle u,z_1-z\rangle-\rho\langle\mathscr{M}(p,q),\xi(z_1,z)\rangle\\ &+\rho\mathscr{J}(u,z_1)-\rho\mathscr{J}(u,z)-\omega\rho\|\xi(z_1,z)\|^2,\end{aligned}\tag{48}$$

$$\begin{aligned}\langle z_1,z-z_1\rangle\geq{}&\langle u,z-z_1\rangle-\rho\langle\mathscr{M}(p,q),\xi(z,z_1)\rangle-\rho\mathscr{J}(u,z)\\ &+\rho\mathscr{J}(u,z_1)-\rho\omega\|\xi(z,z_1)\|^2.\end{aligned}\tag{49}$$

Using Assumption 2 ($\xi(u_1,u_2)=-\xi(u_2,u_1)$) and then adding (48) and (49), we have

$$\langle z_1-z,z-z_1\rangle\geq 0,\tag{50}$$

which implies that $z_1=z$ is the uniqueness of the solution of auxiliary problem (35).

This completes the proof of Theorem 2. □

Algorithm 1. At $n=0$, start with initial value $u_0\in\mathfrak{J}$, $p_0\in[\mathscr{P}_{u_0}]_{r(u_0)}, q_0\in[\mathscr{V}_{u_0}]_{s(u_0)}$; from Theorem 2, auxiliary problem (35) has a unique solution $u_1\in\mathfrak{J}$, such that

$$\begin{aligned}\langle u_1,\ \vartheta-u_1\rangle\geq{}&\langle u_0,\vartheta-u_1\rangle-\rho\langle\mathscr{M}(p_0,q_0),\xi(\vartheta,u_1)\rangle\\ &-\rho\mathscr{J}(u_0,\vartheta)+\rho\mathscr{J}(u_0,u_1)\\ &-\rho\omega\|\xi(\vartheta,u_1)\|^2,\quad\forall\vartheta\in\mathfrak{C}.\end{aligned}\tag{51}$$

Since $p_0\in[\mathscr{P}_{u_0}]_{r(u_0)}, q_0\in[\mathscr{V}_{u_0}]_{s(u_0)}$, then by Nadler's Lemma 2, there exist $p_1\in[\mathscr{P}_{u_1}]_{r(u_1)}$, $q_1\in[\mathscr{V}_{u_1}]_{s(u_1)}$such that

$$\begin{aligned}\|p_0-p_1\|&\leq(1+1)\mathfrak{D}\left([\mathscr{P}_{u_0}]_{r(u_0)},[\mathscr{P}_{u_1}]_{r(u_1)}\right),\\ \|q_0-q_1\|&\leq(1+1)\mathfrak{D}\left([\mathscr{V}_{u_0}]_{s(u_0)},[\mathscr{V}_{u_1}]_{s(u_1)}\right).\end{aligned}\tag{52}$$

For $n=1$, $u_1\in\mathfrak{J}$, $p_1\in[\mathscr{P}_{u_1}]_{r(u_1)}, q_1\in[\mathscr{V}_{u_1}]_{s(u_1)}$, again from Theorem 2, auxiliary problem (35) has a unique solution $u_2\in\mathfrak{J}$, such that

$$\begin{aligned}\langle u_2,\vartheta-u_2\rangle\geq{}&\langle u_1,\vartheta-u_2\rangle-\rho\langle\mathscr{M}(p_1,q_1),\ \xi(\vartheta,u_2)\rangle\\ &-\rho\mathscr{J}(u_1,\vartheta)+\rho\mathscr{J}(u_1,u_2)\\ &-\rho\omega\|\xi(\vartheta,u_2)\|^2,\quad\forall\vartheta\in\mathfrak{C}.\end{aligned}\tag{53}$$

Since $p_1\in[\mathscr{P}_{u_1}]_{r(u_1)}$, $q_1\in[\mathscr{V}_{u_1}]_{s(u_1)}$, then by Nadler's Lemma 2, there exist $p_2\in[\mathscr{P}_{u_2}]_{r(u_2)}$, $q_2\in[\mathscr{V}_{u_2}]_{s(u_2)}$ such that

$$\begin{aligned}\|p_1-p_2\|&\leq\left(1+\frac{1}{2}\right)\mathfrak{D}\left([\mathscr{P}_{u_1}]_{r(u_1)},[\mathscr{P}_{u_2}]_{r(u_2)}\right),\\ \|q_1-q_2\|&\leq\left(1+\frac{1}{2}\right)\mathfrak{D}\left([\mathscr{V}_{u_1}]_{s(u_1)},[\mathscr{V}_{u_2}]_{s(u_2)}\right),\\ &\ \ .\\ &\ \ .\end{aligned}\tag{54}$$

At step n, we can obtain sequences $u_n\in\mathfrak{J}, p_n\in[\mathscr{P}_{u_n}]_{r(u_n)}, q_n\in[\mathscr{V}_{u_n}]_{s(u_n)}\in CB(\mathfrak{J})$, such that

$$\begin{aligned}&\text{(i) } \|p_n - p_{n+1}\| \le \left(1 + \frac{1}{1+n}\right)\mathfrak{D}\left(\left[\mathscr{P}_{u_n}\right]_{r(u_n)}, \left[\mathscr{P}_{u_{n+1}}\right]_{r(u_{n+1})}\right),\\ &\text{(ii) } \|q_n - q_{n+1}\| \le \left(1 + \frac{1}{1+n}\right)\mathfrak{D}\left(\left[\mathscr{V}_{u_n}\right]_{s(u_n)}, \left[\mathscr{V}_{u_{n+1}}\right]_{s(u_{n+1})}\right),\\ &\text{(iii) } \langle u_{n+1}, \vartheta - u_{n+1}\rangle \ge \langle u_n, \vartheta - u_{n+1}\rangle - \rho\langle \mathscr{M}(p_n, q_n), \xi(\vartheta, u_{n+1})\rangle - \rho\mathscr{J}(u_n, \vartheta) + \rho\mathscr{J}(u_n, u_{n+1})\\ &\quad - \rho\omega\|\xi(\vartheta, u_{n+1})\|^2, \quad \forall \vartheta \in \mathfrak{C}, \forall n \ge 0.\end{aligned} \tag{55}$$

4. Existence and Convergence Analysis

Theorem 3. *Let statement of Theorem 2 hold and let $\mathscr{P}, \mathscr{V}$ be $\mathscr{P}$-L-continuous and $\mathscr{V}$-L-continuous with constants $\lambda > 0$ and $\xi > 0$, respectively. Let nonlinear continuous mapping $\mathscr{M}$: $\mathfrak{J} \times \mathfrak{J} \longrightarrow \mathfrak{J}$ be β-S-mixed monotone with constant $\beta > 0$ and γ-L-continuous in respect of first argument and α-L-continuous in respect of second argument with constants $\gamma > 0$ and $\alpha > 0$, respectively. If $\mathscr{J}(.)$ is nondifferentiable and the bifunction $\xi(.,.)$: $\mathfrak{J} \times \mathfrak{J} \longrightarrow \mathfrak{J}$ is S-monotone with constant $\mu > 0$, respectively, then for constant $\rho > 0$,*

$$0 < \rho < 2\frac{\left(\beta - \left(\eta + 2\omega\delta^2\right)\right)}{\gamma^2\lambda^2 - \left(\eta + 2\omega\delta^2\right)^2}, \quad \rho < \frac{1}{\left(\eta + 2\omega\delta^2\right)}, \ \beta > \eta + 2\omega\delta^2, \tag{56}$$

where

$$\eta = \gamma\lambda\sqrt{1 - 2\mu + \delta^2} + \alpha\xi\delta + \partial. \tag{57}$$

Then, there exist $\in\in\mathfrak{C}$, $p \in [\mathscr{P}_u]_{r(u)}$, $q \in [\mathscr{V}_u]_{s(u)}$ satisfying "EPFMVLI" (5), and the sequences $\{u_n\}$, $\{p_n\}$, and $\{q_n\}$generated by (55) converge strongly to u, p, and q, respectively.

Proof. From Algorithm 1 and auxiliary problem (35), for any $\vartheta \in \mathfrak{J}$, we have

$$\begin{aligned}\langle u_n, \vartheta - u_n\rangle \ge{}& \langle u_{n-1}, \vartheta - u_n\rangle - \rho\langle \mathscr{M}(p_{n-1}, q_{n-1}), \xi(\vartheta, u_n)\rangle - \rho\mathscr{J}(u_{n-1}, \vartheta)\\ &+ \rho\mathscr{J}(u_{n-1}, u_n) - \rho\omega\|\xi(\vartheta, u_n)\|^2,\end{aligned} \tag{58}$$

$$\begin{aligned}\langle u_{n+1}, \vartheta - u_{n+1}\rangle \ge{}& \langle u_n, \vartheta - u_{n+1}\rangle - \rho\langle \mathscr{M}(p_n, q_n), \xi(\vartheta, u_{n+1})\rangle - \rho\mathscr{J}(u_n, \vartheta)\\ &+ \rho\mathscr{J}(u_n, u_{n+1}) - \rho\omega\|\xi(\vartheta, u_{n+1})\|^2.\end{aligned} \tag{59}$$

By taking $\vartheta = u_{n+1}$in (58) and $\vartheta = u_n$in (59), we get

$$\begin{aligned}\langle u_n, u_{n+1} - u_n\rangle \ge{}& \langle u_{n-1}, u_{n+1} - u_n\rangle - \rho\langle \mathscr{M}(p_{n-1}, q_{n-1}), \xi(u_{n+1}, u_n)\rangle - \rho\mathscr{J}(u_{n-1}, u_{n+1})\\ &+ \rho\mathscr{J}(u_{n-1}, u_n) - \rho\omega\|\xi(u_{n+1}, u_n)\|^2,\end{aligned} \tag{60}$$

$$\begin{aligned}\langle u_{n+1}, u_n - u_{n+1}\rangle \ge{}& \langle u_n, u_n - u_{n+1}\rangle - \rho\langle \mathscr{M}(p_n, q_n), \xi(u_n, u_{n+1})\rangle - \rho\mathscr{J}(u_n, u_n)\\ &+ \rho\mathscr{J}(u_n, u_{n+1}) - \rho\omega\|\xi(u_n, u_{n+1})\|^2.\end{aligned} \tag{61}$$

Adding (60) and (61) and from Assumption 2 $(\xi(\vartheta, u) + \xi(u, \vartheta) = 0)$, we have

$$\begin{aligned}\langle u_n - u_{n+1}, u_n - u_{n+1}\rangle &\leq \langle u_{n-1} - u_n, u_n - u_{n+1}\rangle - \rho\langle \mathscr{M}(p_n, q_n) - \mathscr{M}(p_{n-1}, q_{n-1}), \xi(u_n, u_{n+1})\rangle \\ &\quad + \rho\mathscr{J}(u_n - u_{n-1}, u_n) - \rho\mathscr{J}(u_{n-1} - u_n, u_{n+1}) + 2\rho\omega\|\xi(u_n, u_{n+1})\|^2 \\ &\leq \langle u_{n-1} - u_n, u_n - u_{n+1}\rangle + \rho\langle \mathscr{M}(p_n, q_n) - \mathscr{M}(p_{n-1}, q_{n-1}), \xi(u_n, u_{n+1})\rangle \\ &\quad + \rho\mathscr{J}(u_n - u_{n-1}, u_n - u_{n+1}) + 2\rho\omega\|\xi(u_n, u_{n+1})\|^2,\end{aligned} \tag{62}$$

and so

$$\begin{aligned}\langle u_n - u_{n+1}, u_n - u_{n+1}\rangle &\leq \langle u_{n-1} - u_n - \rho\{\mathscr{M}(p_{n-1}, q_{n-1}) - \mathscr{M}(p_n, q_{n-1})\}, u_n - u_{n+1}\rangle \\ &\quad + \rho\langle \mathscr{M}(p_{n-1}, q_{n-1}) - \mathscr{M}(p_n, q_{n-1}), u_n - u_{n+1} - \xi(u_n, u_{n+1})\rangle \\ &\quad + \rho\mathscr{M}(p_n, q_n) - \mathscr{M}(p_n, q_{n-1}), \xi(u_n, u_{n+1}) \\ &\quad + \rho\mathscr{J}(u_n - u_{n-1}, u_n - u_{n+1}) + 2\rho\omega\|\xi(u_n, u_{n+1})\|^2.\end{aligned} \tag{63}$$

It follows that

$$\begin{aligned}\|u_n - u_{n+1}\|^2 &\leq \|u_{n-1} - u_n - \rho\{\mathscr{M}(p_{n-1}, q_{n-1}) - \mathscr{M}(p_n, q_{n-1})\}\|\|u_n - u_{n+1}\| \\ &\quad + \rho\|\mathscr{M}(p_{n-1}, q_{n-1}) - \mathscr{M}(p_n)\|\|u_n - u_{n+1} - \xi(u_n, u_{n+1})\| \\ &\quad \cdot \rho\|\mathscr{M}(p_n, q_n) - \mathscr{M}(p_n, q_{n-1})\|\|\xi(u_n, u_{n+1})\| + \rho\,\partial\|u_{n-1} - u_n\|\|u_n - u_{n+1}\| \\ &\quad + 2\omega\rho\|\xi(u_n, u_{n+1})\|^2.\end{aligned} \tag{64}$$

By the β-S-mixed monotonicity, γ-L-continuity in respect of fist argument of $\mathscr{M}$, and $\mathscr{P}$-L-continuity of $\mathscr{P}$, we have

$$\begin{aligned}&\|u_{n-1} - u_n - \rho(\mathscr{M}(p_{n-1}, q_{n-1}) - \mathscr{M}(p_n, q_{n-1}))\|^2 \leq \|u_{n-1} - u_n\|^2 \\ &\quad - 2\rho\langle \mathscr{M}(p_{n-1}, q_{n-1}) - \mathscr{M}(p_n, q_{n-1}), u_{n-1} - u_n\rangle + \rho^2\|\mathscr{M}(p_{n-1}, q_{n-1}) - \mathscr{M}(p_n, q_{n-1})\|^2 \\ &\leq \|u_{n-1} - u_n\|^2 - 2\rho\beta\|u_{n-1} - u_n\|^2 + \rho^2\gamma^2\|p_{n-1} - p_n\|^2 \\ &\leq \|u_{n-1} - u_n\|^2 - 2\rho\beta\|u_{n-1} - u_n\|^2 \\ &\quad + \rho^2\gamma^2\left(\mathfrak{D}\left(\left[\mathscr{P}_{u_{n-1}}\right]_{r(u_n)}, \left[\mathscr{P}_{u_n}\right]_{r(u_n)}\right)\right)^2 \\ &\leq \left(1 - 2\rho\beta + \rho^2\gamma^2\lambda^2\left(1 + \frac{1}{n}\right)^2\right)\|u_{n-1} - u_n\|^2,\end{aligned} \tag{65}$$

$$\begin{aligned}&\|\mathscr{M}(p_{n-1}, q_{n-1}) - \mathscr{M}(p_n, q_{n-1})\| \leq \gamma\|p_{n-1} - p_n\| \\ &\leq \gamma\left(1 + \frac{1}{n}\right)\mathfrak{D}\left(\left[\mathscr{P}_{u_{n-1}}\right]_{r(u_{n-1})}, \left[\mathscr{P}_{u_n}\right]_{r(u_n)}\right) \leq \gamma\lambda\left(1 + \frac{1}{n}\right)\|u_{n-1} - u_n\|.\end{aligned} \tag{66}$$

Similarly, by the S-monotonicity and L-continuity of bifunction ξ, we have

$$\begin{aligned}\left\|u_n - u_{n+1} - \xi(u_n, u_{n+1})\right\|^2 &\le \left\|u_n - u_{n+1}\right\|^2 - 2\langle u_n - u_{n+1}, \xi(u_n, u_{n+1})\rangle + \left\|\xi(u_n, u_{n+1})\right\|^2 \\ &\le \left\|u_n - u_{n+1}\right\|^2 - 2\mu\left\|u_n - u_{n+1}\right\|^2 + \delta^2\left\|u_n - u_{n+1}\right\|^2 \\ &= \left(1 - 2\mu + \delta^2\right)\left\|u_n - u_{n+1}\right\|^2.\end{aligned} \tag{67}$$

By the α-L-continuity of $\mathscr{M}$ in respect of second argument and $\mathscr{V}$-L-continuity of $\mathscr{V}$, we get

$$\begin{aligned}&\left\|\mathscr{M}(p_n, q_n) - \mathscr{M}(p_n, q_{n-1})\right\| \le \alpha\left\|p_{n-1} - p_n\right\| \\ &\le \alpha\left(1 + \frac{1}{n}\right)\mathfrak{D}\left(\left[\mathscr{P}_{u_{n-1}}\right]_{r(u_{n-1})}, \left[\mathscr{P}_{u_n}\right]_{r(u_n)}\right) \le \alpha\xi\left(1 + \frac{1}{n}\right)\left\|u_{n-1} - u_n\right\|.\end{aligned} \tag{68}$$

Combining (64)–(68) and by the L-continuity of ξ, we have

$$\begin{aligned}&\left\|u_n - u_{n+1}\right\|^2 \le \\ &\left\{\sqrt{1 - 2\rho\beta + \rho^2(\gamma\lambda)^2\left(1 + \frac{1}{n}\right)^2} + \rho\gamma\lambda\left(1 + \frac{1}{n}\right)\sqrt{1 - 2\mu + \delta^2}\right\}\left\|u_{n-1} - u_n\right\|\left\|u_n - u_{n+1}\right\| \\ &\cdot \rho\alpha\xi\delta\left(1 + \frac{1}{n}\right)\left\|u_{n-1} - u_n\right\|\left\|u_n - u_{n+1}\right\| + \rho\,\partial\left\|u_{n-1} - u_n\right\|\left\|u_n - u_{n+1}\right\| \\ &\quad + 2\omega\rho\delta^2\left\|u_n - u_{n+1}\right\|^2,\end{aligned} \tag{69}$$

which implies that

$$\begin{aligned}&\left\|u_n - u_{n+1}\right\| \\ &\le \left\{\sqrt{1 - 2\rho\beta + \rho^2(\gamma\lambda)^2\left(1 + \frac{1}{n}\right)^2} + \rho\gamma\lambda\left(1 + \frac{1}{n}\right)\sqrt{1 - 2\mu + \delta^2} + \rho\alpha\xi\delta\left(1 + \frac{1}{n}\right) + \rho\,\partial\right\}\left\|u_{n-1} - u_n\right\| \\ &\quad + 2\omega\rho\delta^2\left\|u_n - u_{n+1}\right\|.\end{aligned} \tag{70}$$

It follows that

$$\begin{aligned}&\|u_n - u_{n+1}\|\\&\leq \frac{\sqrt{1-2\rho\beta+\rho^2(\gamma\lambda)^2(1+(1/n))^2}+\rho\gamma\lambda(1+(1/n))\sqrt{1-2\mu+\delta^2}+\rho\alpha\xi\delta(1+(1/n))+\rho\partial}{1-2\omega\rho\delta^2}\|u_{n-1}-u_n\|\\&=\varphi_n\|u_{n-1}-u_n\|,\end{aligned} \tag{71}$$

where

$$\begin{aligned}\varphi_n &= \frac{\sqrt{1-2\rho\beta+\rho^2(\gamma\lambda)^2(1+(1/n))^2}+\rho\gamma\lambda(1+(1/n))\sqrt{1-2\mu+\delta^2}+\rho\alpha\xi\delta(1+(1/n))+\rho\partial}{1-2\omega\rho\delta^2},\\ \varphi &= \frac{\sqrt{1-2\rho\beta+\rho^2(\gamma\lambda)^2}+\rho\gamma\lambda\sqrt{1-2\mu+\delta^2}+\rho\alpha\xi\delta+\rho\partial}{1-2\omega\rho\delta^2}.\end{aligned} \tag{72}$$

Clearly,

$$\varphi_n \longrightarrow \varphi = \frac{k(\rho)+\rho\eta}{1-2\omega\rho\delta^2}, \tag{73}$$

where

$$\begin{aligned}k(\rho) &= \sqrt{1-2\rho\beta+\rho^2(\gamma\lambda)^2},\\ \eta &= \gamma\lambda\sqrt{1-2\mu+\delta^2}+\rho\alpha\xi\delta+\rho\partial.\end{aligned} \tag{74}$$

From (56), it follows that $\varphi < 1$. Hence, it follows from (71) and (74) that $\{u_n\}$is Cauchy sequence in $\mathfrak{C}$. Hence, it converges to some point. Since $\mathfrak{C}$is closed convex set in $\mathfrak{C}$, there exists $u$in $\mathfrak{C}$such that $u_n \longrightarrow u$, which satisfies "EPFMVLI" (5).

On the other hand, from Algorithm 1, we have

$$\begin{aligned}\|p_n - p_{n+1}\| &\leq \left(1+\frac{1}{1+n}\right)\mathfrak{D}\left(\left[\mathscr{P}_{u_n}\right]_{r(u_n)}, \left[\mathscr{P}_{u_{n+1}}\right]_{r(u_{n+1})}\right)\\ &\leq \left(1+\frac{1}{1+n}\right)\gamma\|u_n-u_{n+1}\|^2\\ \|q_{n-1}-q_n\| &\leq \left(1+\frac{1}{n}\right)\mathfrak{D}\left(\left[\mathscr{V}_{u_{n-1}}\right]_{s(u_{n-1})}, \left[\mathscr{V}_{u_n}\right]_{s(u_n)}\right)\\ &\leq \left(1+\frac{1}{n}\right)\xi\|u_{n-1}-u_n\|.\end{aligned} \tag{75}$$

This implies that $\{p_n\}$, $\{q_n\}$both are Cauchy sequences in $\mathfrak{F}$, since $\{u_n\}$ is convergence sequence. Then, we consider that $p_n \longrightarrow p$ and $q_n \longrightarrow q$ when $n \longrightarrow \infty$. Since $p_n \in [\mathscr{P}_{u_n}]_{r(u_n)}$and $q_n \in [\mathscr{V}_{q_n}]_{s(u_n)}$, we have

$$\begin{aligned}d\left(p, \left[\mathscr{P}_{u_n}\right]_{r(u_n)}\right) &\leq \|p-p_n\| + d\left(p_n, \left[\mathscr{P}_{u_n}\right]_{r(u_n)}\right) + \mathfrak{D}\left(\left[\mathscr{P}_{u_n}\right]_{r(u_n)}, \left[\mathscr{P}_u\right]_{r(u)}\right)\\ &\leq \|p-p_n\| + 0 + \gamma\|u_n-u\|^2\\ d\left(q, \left[\mathscr{V}_{u_n}\right]_{s(u_n)}\right) &\leq \|q-q_n\| + d\left(q_n, \left[\mathscr{V}_{u_n}\right]_{s(u_n)}\right) + \mathfrak{D}\left(\left[\mathscr{V}_{u_n}\right]_{s(u_n)}, \left[\mathscr{V}_u\right]_{s(u)}\right)\\ &\leq \|q-q_n\| + 0 + \xi\|u_n-u\|^2.\end{aligned} \tag{76}$$

When $n \longrightarrow \infty$, we have

$$\begin{aligned}d\left(p, [\mathscr{P}_u]_{r(u)}\right) &\leq \|p-p_n\| + 0 + \gamma\|u_n-u\|^2 \longrightarrow 0,\\ d\left(q, [\mathscr{P}_u]_{r(u)}\right) &\leq \|p-p_n\| + 0 + \xi\|u_n-u\|^2 \longrightarrow 0.\end{aligned} \tag{77}$$

Hence, $p \in [\mathscr{P}_u]_{r(u)}$and $q \in [\mathscr{V}_u]_{s(u)}$.

Finally, we show that

$$\langle \mathscr{M}(p), \xi(\vartheta, u)\rangle + \mathscr{J}(\vartheta) - \mathscr{J}(u) + \omega\|\xi(\vartheta, u)\|^2 \geq 0, \quad \forall \vartheta \in \mathfrak{F}. \tag{78}$$

We again study (55), as follows:

$$\langle u_{n+1}, \vartheta - u_{n+1}\rangle \geq \langle u_n, \vartheta - u_{n+1}\rangle - \rho\langle \mathcal{M}(p_n, q_n), \xi(\vartheta, u_{n+1})\rangle - \rho\mathcal{J}(u_n, \vartheta) + \rho\mathcal{J}(u_n, u_{n+1}) - \rho\omega\|\xi(\vartheta, u_{n+1})\|^2, \quad \forall\vartheta \in \mathfrak{J}, \quad \forall n \geq 0. \tag{79}$$

Now from Assumptions 1 and 2, we get

$$0 \leq \limsup_{n\longrightarrow\infty}\left[u_{n+1}, \vartheta - u_{n+1} - u_n, \vartheta - u_{n+1} + \rho\mathcal{M}(p_n, q_n), \xi(\vartheta, u_{n+1}) + \rho\mathcal{J}(u_n, \vartheta) - \rho\mathcal{J}(u_n, u_{n+1}) + \rho\omega\|\xi(\vartheta, u_{n+1})\|^2\right], \leq \rho[\mathcal{M}(p, q), \xi(\vartheta, u) + \mathcal{J}(u, \vartheta) - \mathcal{J}(u, u) + \omega\|\xi(\vartheta, u)\|^2. \tag{80}$$

This implies that

$$0 \leq [\langle \mathcal{M}(p, q)\rangle, \xi(\vartheta, u) + \mathcal{J}(u, \vartheta) - \mathcal{J}(u, u) + \omega\|\xi(\vartheta, u)\|^2. \tag{81}$$

This completes the proof. □

4.1. Special Cases

Theorem 4. *Let $\mathfrak{J}$ be real Hilbert space and $\mathfrak{C} \neq \emptyset$ be a closed bounded subset of $\mathfrak{J}$. Let $\mathcal{P}$: $\mathfrak{J} \longrightarrow \mathcal{F}(\mathfrak{J})$ be a closed continuous fuzzy mapping satisfying condition ($\mathfrak{B}$) with function α: $\mathfrak{J} \longrightarrow [0, 1]$ and $\mathcal{P}$-L-continuous with constant $\lambda > 0$. Let nonlinear continuous mapping $\mathcal{M}$: $\mathfrak{J} \longrightarrow \mathfrak{J}$ be β-S mixed monotone and γ-L-continuous with constants $\beta > 0$ and $\gamma > 0$, respectively. Let $\mathcal{J}$: $\mathfrak{C} \longrightarrow R$ be a nondifferentiable convex functional and ∂-L-continuous with constant ∂. Let the nonlinear mapping $\xi(.,.)$:J × J $\longrightarrow$ J is S-monotone and L-continuous with constants $\mu > 0$ and $\delta > 0$ respectively If Assumption 2 holds, then for constant $\rho > 0$,*

$$0 < \rho < \frac{2\left(\beta - \left(\eta + 2\omega\delta^2\right)\right)}{\gamma^2\lambda^2 - \left(\eta + 2\omega\delta^2\right)^2}, \quad \rho < \frac{1}{\left(\eta + 2\omega\delta^2\right)}, \beta > \eta + 2\omega\delta^2, \tag{82}$$

where

$$\eta = \gamma\lambda\sqrt{1 - 2\mu + \delta^2}. \tag{83}$$

Then, there exist unique $u \in \mathfrak{C}$, $u \in [\mathcal{P}_u]_{\alpha(u)}$satisfying the set-valued "PFMVLI" (12), and the sequences $\{u_n\}$and $\{p_n\}$ generated by (55) converge strongly to $u$and p, respectively.

Proof. Demonstration of Theorem 4 is similar to demonstration of Theorems 2 and 3. □

Algorithm 2. At $n = 0$, start with initial value $u_0 \in \mathfrak{J}$, $p_0 \in [\mathcal{P}_{u_0}]_{\alpha(u_0)}$; from Theorem 4, the auxiliary problem (35) has a unique solution $u_1 \in \mathfrak{J}$, such that

$$\langle u_1, \vartheta - u_1\rangle \geq \langle u_0, \vartheta - u_1\rangle - \rho\langle \mathcal{M}(p_0), \xi(\vartheta, u_1)\rangle - \rho\mathcal{J}(\vartheta) + \rho\mathcal{J}(u_1) - \rho\omega\|\xi(\vartheta, u_1)\|^2, \quad \forall\vartheta \in \mathfrak{J}. \tag{84}$$

Since $p_0 \in [\mathcal{P}_{u_0}]_{\alpha(u_0)}$, then by Nadler's Lemma 2, there exists $p_1 \in [\mathcal{P}_{u_1}]_{\alpha(u_1)}$, such that

$$\|p_0 - p_1\| \leq (1 + 1)\mathfrak{D}\left(\left[\mathcal{P}_{u_0}\right]_{\alpha(u_0)}, \left[\mathcal{P}_{u_1}\right]_{\alpha(u_1)}\right). \tag{85}$$

For $n = 1$, $u_1 \in \mathfrak{J}$, $p_1 \in [\mathcal{P}_{u_1}]_{\alpha(u_1)}$, again from Theorem 2, auxiliary problem (35) has a unique solution $u_2 \in \mathfrak{J}$, such that

$$\langle u_2, \vartheta - u_2\rangle \geq \langle u_1, \vartheta - u_2\rangle - \rho\langle \mathcal{M}(p_1), \xi(\vartheta, p_2)\rangle - \rho\mathcal{J}(\vartheta) + \rho\mathcal{J}(u_2) - \rho\omega\xi(\vartheta, u_2)^2, \quad \forall\vartheta \in \mathfrak{J}. \tag{86}$$

Since $p_1 \in [\mathcal{P}_{u_1}]_{\alpha(u_1)}$, then by Nadler's Lemma 2, there exists $p_2 \in [\mathcal{P}_{u_2}]_{\alpha(u_2)}$, such that

$$\|p_1 - p_2\| \leq \left(1 + \frac{1}{2}\right)\mathfrak{D}\left(\left[\mathcal{P}_{u_1}\right]_{\alpha(u_1)}, \left[\mathcal{P}_{u_2}\right]_{\alpha(u_2)}\right), \quad \vdots \tag{87}$$

At step n, we can obtain sequences $u_n \in \mathfrak{J}$,$p_n \in [\mathcal{P}_{u_n}]_{\alpha(u_n)} \in CB(\mathfrak{J})$, such that

$$\begin{aligned}&\text{(i) } \|p_n - p_{n+1}\| \le \left(1 + \frac{1}{1+n}\right)\mathfrak{D}\left(\left[\mathscr{P}_{u_n}\right]_{\alpha(u_n)}, \left[\mathscr{P}_{u_{n+1}}\right]_{\alpha(u_{n+1})}\right)\\ &\text{(ii) } \langle u_{n+1}, \vartheta - u_{n+1}\rangle \ge \langle u_n, \vartheta - u_{n+1}\rangle - \rho\langle \mathscr{M}(p_n), \xi(\vartheta, u_{n+1})\rangle - \rho\mathscr{J}(\vartheta) + \rho\mathscr{J}(u_{n+1})\\ &\quad - \rho\omega\|\xi(\vartheta, u_{n+1})\|^2, \quad \forall \vartheta \in \mathfrak{J}, \forall n \ge 0.\end{aligned} \tag{88}$$

Theorem 5. *Let the operator* $\mathscr{P}, \mathscr{V}: \mathfrak{J} \longrightarrow CB(\mathfrak{J})$ *be* $\mathscr{P}$*-L-continuous and* $\mathscr{V}$*-L-continuous with constants* $\lambda > 0$ *and* $\xi > 0$*, respectively, and nonlinear continuous mapping* $\mathscr{M}: \mathfrak{J} \longrightarrow \mathfrak{J}$ *be* β*-S-mixed monotone,* γ*-L-continuous in respect of first argument, and* α*-L-continuous in respect of second argument with constants* $\beta > 0$, $\gamma > 0$, *and* $\alpha > 0$, *respectively, and* $\mathscr{J}(.)$ *is nondifferentiable. Let bifunction* $\xi(.,.)$ *be S-monotone with constant* $\mu > 0$ *and L-continuous with constant* $\delta > 0$*, respectively. If Assumptions 1 and 2 hold, then for constant* $\rho > 0$,

$$0 < \rho < 2\frac{\left(\beta - \left(\eta + 2\omega\delta^2\right)\right)}{\gamma^2\lambda^2 - \left(\eta + 2\omega\delta^2\right)^2}, \quad \rho < \frac{1}{\left(\eta + 2\omega\delta^2\right)}, \beta > \eta + 2\omega\delta^2, \tag{89}$$

where

$$\eta = \gamma\lambda\sqrt{1 - 2\mu + \delta^2} + \alpha\xi\delta + \partial. \tag{90}$$

Then, there exist $u \in \mathfrak{C}$, $p \in \mathscr{P}(u), q \in \mathscr{V}(u)$satisfying the set-valued "EPMVLI" (10), and the sequences $\{u_n\}$, $\{p_n\}$, and $\{q_n\}$generated by (55) converge strongly to u, p, and q, respectively.

Proof. By using the set-valued mapping $F: \mathfrak{J} \longrightarrow CB(\mathfrak{J})$, we define the fuzzy mapping $\widetilde{\mathscr{P}}: \mathfrak{J} \longrightarrow \mathscr{F}(\mathfrak{J})$, as follows:

$$\begin{aligned}\widetilde{\mathscr{P}}_u &= \mathscr{X}\mathscr{P}(u),\\ \widetilde{\mathscr{V}}_u &= \mathscr{X}\mathscr{V}(u),\end{aligned} \tag{91}$$

where $\mathscr{X}\mathscr{P}(u)$and $\mathscr{X}\mathscr{V}(u)$are characteristic functions of the sets $\mathscr{P}(u)$. It is easy to see that $\widetilde{\mathscr{P}}, \widetilde{\mathscr{V}}$are closed fuzzy mappings satisfying condition ($\mathfrak{B}$) with constant functions $r(u) = 1, s(u) = 1$ for all $u \in \mathfrak{J}$. Also,

$$\begin{aligned}\left[\widetilde{\mathscr{P}}_u\right]_{r(u)} &= \left[\mathscr{X}\mathscr{P}_u\right]_1 = \{\vartheta \in \mathfrak{J}: \mathscr{P}_u(\vartheta) = 1\} = \mathscr{P}(u),\\ \left[\widetilde{\mathscr{V}}_u\right]_{s(u)} &= \left[\mathscr{X}\mathscr{V}_u\right]_1 = \{\vartheta \in \mathfrak{J}: \mathscr{V}_u(\vartheta) = 1\} = \mathscr{V}(u).\end{aligned} \tag{92}$$

(1) Then, problem (5) is analogous to finding $u \in \mathfrak{J}$, such that

$$\begin{aligned}&\left(\widetilde{\mathscr{P}}_u\right)(p) = 1, \quad \text{i.e. } p \in \mathscr{P}(u),\\ &\left(\widetilde{\mathscr{V}}_u\right)(q) = 1, \quad \text{i.e. } q \in \mathscr{V}(u),\\ &\langle \mathscr{M}(p,q), \xi(\vartheta, u)\rangle + \mathscr{J}(u, \vartheta) - \mathscr{J}(u,u) + \omega\|\xi(\vartheta, u)\|^2 \ge 0, \quad \forall \vartheta \in \mathfrak{J}.\end{aligned} \tag{93}$$

Hence, from Theorems 2 and 3, we can obtain the conclusion of Theorem 5 immediately. □

Algorithm 3. At $n = 0$, start with initial value $u_0 \in \mathfrak{J}$, $p_0 \in \mathscr{P}(u_0)$, $q_0 \in \mathscr{V}(u_0)$; from Theorem 5, auxiliary problem (35) has a unique solution $u_1 \in \mathfrak{J}$, such that

$$\begin{aligned}\langle u_1, \vartheta - u_1\rangle &\ge \langle u_0, \vartheta - u_1\rangle - \rho\langle \mathscr{M}(p_0, q_0), \xi(\vartheta, u_1)\rangle\\ &\quad - \rho\mathscr{J}(u_0, \vartheta) + \rho\mathscr{J}(u_0, u_1)\\ &\quad - \rho\omega\|\xi(\vartheta, u_1)\|^2, \quad \forall \vartheta \in \mathfrak{J}.\end{aligned} \tag{94}$$

Since $p_0 \in \mathscr{P}(u_0)$, $q_0 \in \mathscr{V}(u_0)$, then by Nadler's Lemma 2, there exist $p_1 \in \mathscr{P}(u_1)$, $q_1 \in \mathscr{V}(u_1)$ such that

$$\begin{aligned}\|p_0 - p_1\| &\le (1+1)\mathfrak{D}(\mathscr{P}(u_0), \mathscr{P}(u_1)),\\ \|q_0 - q_1\| &\le (1+1)\mathfrak{D}(\mathscr{V}(u_0), \mathscr{V}(u_1)).\end{aligned} \tag{95}$$

For $n = 1$, $u_1 \in \mathfrak{J}, p_1 \in \mathscr{P}(u_1)$, $q_1 \in \mathscr{V}(u_1)$again from Theorem 5, the Auxiliary problem (35) has a unique solution $u_2 \in \mathfrak{J}$, such that

$$\begin{aligned}\langle u_2, \vartheta - u_2\rangle &\ge \langle u_1, \vartheta - u_2\rangle - \rho\langle \mathscr{M}(p_1, q_1), \xi(\vartheta, u_2)\rangle\\ &\quad - \rho\mathscr{J}(u_1, \vartheta) + \rho\mathscr{J}(u_1, u_2)\\ &\quad - \rho\omega\|\xi(\vartheta, u_2)\|^2, \quad \forall \vartheta \in \mathfrak{J}.\end{aligned} \tag{96}$$

Since $p_1 \in \mathscr{P}(u_1)$, $q_1 \in \mathscr{V}(u_1)$, then by Nadler's Lemma 2, there exist $p_2 \in \mathscr{P}(u_2)$, $q_2 \in \mathscr{V}(u_2)$ such that

$$\|p_1 - p_2\| \le \left(1 + \frac{1}{2}\right)\mathfrak{D}(\mathscr{P}(u_1), \mathscr{P}(u_2)),$$
$$\|q_1 - q_2\| \le \left(1 + \frac{1}{2}\right)\mathfrak{D}(\mathscr{V}(u_1), \mathscr{V}(u_2)), \tag{97}$$

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At step n, we can obtain sequences $u_n \in \mathfrak{J}, p_n \in \mathscr{P}(u_n), q_n \in \mathscr{V}(u_n) \in CB(\mathfrak{J})$, such that

$$\begin{aligned}
&\|p_n - p_{n+1}\| \le \left(1 + \frac{1}{1+n}\right)\mathfrak{D}(\mathscr{P}(u_n), \mathscr{P}(u_{n+1})),\\
&\|q_n - q_{n+1}\| \le \left(1 + \frac{1}{1+n}\right)\mathfrak{D}(\mathscr{V}(u_n), \mathscr{V}(u_{n+1})),\\
&\langle u_{n+1}, \vartheta - u_{n+1}\rangle \ge \langle u_n, \vartheta - u_{n+1}\rangle - \rho\langle \mathscr{M}(p_n, q_n), \xi(\vartheta, u_{n+1})\rangle - \rho\mathscr{J}(u_n, \vartheta) + \rho\mathscr{J}(u_n, u_{n+1})\\
&\quad - \rho\omega\|\xi(\vartheta, u_{n+1})\|^2, \quad \forall \vartheta \in \mathfrak{C}, \forall n \ge 0.
\end{aligned} \tag{98}$$

If $\omega = 0$, then Theorem 3 reduces to the following result (see [18]).

Corollary 1. *Let statement of Theorem 2 hold and let $\mathscr{P}, \mathscr{V}$ be $\mathscr{P}$-L-continuous and $\mathscr{V}$-L-continuous with constants $\lambda > 0$ and $\xi > 0$, respectively. Let nonlinear continuous mapping $\mathscr{M}: \mathfrak{J} \times \mathfrak{J} \longrightarrow \mathfrak{J}$ be β-S-mixed monotone with constant $\beta > 0$ and γ-L-continuous in respect of first argument and α-L-continuous in respect of second argument with constants $\gamma > 0$ and $\alpha > 0$, respectively. If $\mathscr{J}(.)$ is nondifferentiable and the bifunction $\xi(.,.)$: $\mathfrak{J} \times \mathfrak{J} \longrightarrow \mathfrak{J}$ is S-monotone with constant $\mu > 0$, respectively, then for constant $\rho > 0$,*

$$0 < \rho < 2\frac{\beta - \eta}{\gamma^2\lambda^2 - \eta^2}, \quad \rho < \frac{1}{\eta}, \beta > \eta, \tag{99}$$

where

$$\eta = \gamma\lambda\sqrt{1 - 2\mu + \delta^2} + \alpha\xi\delta + \partial. \tag{100}$$

Then, there exist $\in\in\mathfrak{C}$, $p \in [\mathscr{P}_u]_{r(u)}$, $q \in [\mathscr{V}_u]_{s(u)}$ satisfying "EPFMVLI" (5), and the sequences $\{u_n\}$, $\{p_n\}$, and $\{q_n\}$ generated by (55) converge strongly to u, p, and q, respectively.

If $\omega = 0$, then Theorem 5 reduces to the following result (see [13]).

Corollary 2. *Let the operator $\mathscr{P}, \mathscr{V}$: $\mathfrak{J} \longrightarrow CB(\mathfrak{J})$ be $\mathscr{P}$-L-continuous and $\mathscr{V}$-L-continuous with constants $\lambda > 0$ and $\xi > 0$, respectively, and nonlinear continuous mapping $\mathscr{M}: \mathfrak{J} \longrightarrow \mathfrak{J}$ be β-S-mixed monotone, γ-L-continuous in respect of first argument, and α-L-continuous in respect of second argument with constants $\beta > 0$, $\gamma > 0$, and $\alpha > 0$, respectively, and $\mathscr{J}(.)$ is nondifferentiable. Let bifunction $\xi(.,.)$ be S-monotone with constant $\mu > 0$ and L-continuous with constant $\delta > 0$, respectively. If Assumptions 1 and 2 hold, then for constant $\rho > 0$,*

$$0 < \rho < 2\frac{\beta - \eta}{\gamma^2\lambda^2 - \eta^2}, \quad \rho < \frac{1}{\eta}, \beta > \eta, \tag{101}$$

where

$$\eta = \gamma\lambda\sqrt{1 - 2\mu + \delta^2} + \alpha\xi\delta + \partial. \tag{102}$$

Then, there exist $u \in \mathfrak{C}$, $p \in \mathscr{P}(u), q \in \mathscr{V}(u)$ satisfying the set-valued "EPMVLI" (10), and the sequences $\{u_n\}$, $\{p_n\}$, and $\{q_n\}$ generated by (55) converge strongly to u, p, and q, respectively.

If $\mathscr{M} = I$(identity mapping), $p \in \mathfrak{J}$, and $\mathscr{P}$: $\mathfrak{J} \longrightarrow \mathfrak{J}$ is single valued, then Theorem 5 reduces to the following result (see [12]).

Corollary 3. *Let $\mathscr{P}$: $\mathfrak{J} \longrightarrow \mathfrak{J}$ be $\mathscr{P}$-S-monotone with constant $\beta > 0$ and $\mathscr{P}$-L-continuous with constant $\lambda > 0$, respectively. Let $\mathscr{J}$: $\mathfrak{J} \longrightarrow R$ be a nondifferentiable mapping and the bifunction $\xi(.,.)$ be S-monotone with constant $\mu > 0$ and L-continuous with constant $\delta > 0$, respectively. If $\xi(.,.)$ satisfies following condition:*

$$\xi(\vartheta, u) + \xi(u, \vartheta) = 0 \,(\text{and so}\, \xi(u, u) = 0, \text{ for all}\, u \in \mathfrak{J}), \quad \text{for all}\, u, \vartheta \in \mathfrak{J}, \tag{103}$$

then for constant $\rho > 0$,

$$0 < \rho < 2\frac{(\beta - (\eta + 2\omega\delta^2))}{(\gamma)^2 - (\eta + 2\omega\delta^2)^2}, \quad \rho < \frac{1}{(\eta + 2\omega\delta^2)}, \beta > \eta + 2\omega\delta^2, \tag{104}$$

where

$$\eta = \gamma\sqrt{1 - 2\mu + \delta^2}. \tag{105}$$

Then, there exists $u \in \mathfrak{C}$, satisfying "PMVLI" (13), and the sequence $\{u_n\}$ generated by (55) converges strongly to u.

If $\mathscr{M} = I$ and $\omega = 0$, $p \in \mathfrak{J}$, and $\mathscr{P}: \mathfrak{J} \longrightarrow \mathfrak{J}$ is single valued, then Theorem 5 reduces to the following result (see [22]).

Corollary 4. *Let $\mathscr{P}: \mathfrak{J} \longrightarrow \mathfrak{J}$ be $\mathscr{P}$-S-monotone with constant $\beta > 0$ and $\mathscr{P}$-L-continuous with constant $\lambda > 0$, respectively. Let $\mathscr{J}: \mathfrak{J} \longrightarrow R$ be a nondifferentiable mapping and the bifunction $\xi(.,.)$ be S-monotone with constant $\mu > 0$ and L-continuous with constant $\delta > 0$, respectively. If $\xi(.,.)$ satisfies following condition:*

$$\xi(\vartheta, u) + \xi(u, \vartheta) = 0 \,(\text{and so}\, \xi(u, u) = 0, \text{ for all}\, u \in \mathfrak{J}), \quad \text{for all}\, u, \vartheta \in \mathfrak{J}, \tag{106}$$

then for constant $\rho > 0$,

$$0 < \rho < 2\frac{(\beta - \eta)}{(\gamma)^2 - \left(\eta + 2\omega\delta^2\right)^2}, \quad \rho\eta < 1, \beta > \eta, \tag{107}$$

where

$$\eta = \gamma\sqrt{1 - 2\mu + \delta^2}. \tag{108}$$

Then, there exists $u \in \mathfrak{C}$, satisfying "MVLI" (15), and the sequence $\{u_n\}$generated by (55) converges strongly to u.

5. Conclusion

In this paper, we have proposed the idea of "EPFMVLI." As a particular case of "EPFMVLI," extended perturbed mixed variational-like inequalities are also introduced. With the help of the extended auxiliary principle technique and some new analytic techniques, some existence theorems of auxiliary "EPFMVLI" are studied for "EPFMVLI," and some iterative methods are obtained for the solution of "EPFMVLI." Then, we have obtained some known and new results. We would like to mention that many earlier defined familiar methods as well as decent, projection techniques and its mixed forms such as relaxation and Newton's methods can be obtained from auxiliary "SPFMVLI." There are much rooms for exploring of this concept, as the suitable choices of fuzzy mappings can be obtained from EPFMVLI (5), see [3–8, 12–14, 18–22].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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