Research Article

Computing FGZ Index of Sum Graphs under Strong Product

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Received 2 January 2021; Revised 10 February 2021; Accepted 11 March 2021; Published 30 March 2021

Academic Editor: Ahmet Sinan Cevik

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Topological index (TI) is a function that assigns a numeric value to a (molecular) graph that predicts its various physical and structural properties. In this paper, we study the sum graphs (S-sum, R-sum, Q-sum and T-sum) using the subdivision related operations and strong product of graphs which create hexagonal chains isomorphic to many chemical compounds. Mainly, the exact values of first general Zagreb index (FGZI) for four sum graphs are obtained. At the end, FGZI of the two particular families of sum graphs are also computed as applications of the main results. Moreover, the dominating role of the FGZI among these sum graphs is also shown using the numerical values and their graphical presentations.

1. Introduction

Let $G = (V(G), E(G))$ be a (molecular) graph with $V(G)$ and $E(G)$ as sets of vertex and edge respectively. The degree of a vertex $v \in V(G)$ is the number of edges which are incident to v . A topological index (TI) is a function that assigns a numeric value to the under study (molecular) graph, see [1–3]. TIs are used to predict the various chemical and structural properties such as octane isomers including entropy, acentric factor, density, total surface area, molar volume, boiling point, capacity of heat at temperature and pressure, enthalpy of formation, connectivity of compounds and octanol water partition, see [4]. These are also used to study the quantitative structure property and activity relationships which are very important in the subject of cheminformatics, see [5].

Wiener see [6] defined a distance-based TI to compute the boiling point of paraffin. Gutman and Trinajstić see [7] investigated total π -electron energy of the molecule graphs using a degree-based TI called by first Zagreb index nowadays. Later on, various Zagreb indices (hyper, forgotten, multiplicative and augmented) are defined to predict the different physicochemical properties chirality, heterosystems, complexity, branching and ZE-isomerism of the (molecular) graphs see [8]. Li and Zheng defined the first general Zagreb index (FGZI) and studied its different properties see [9].

The operations of graphs (addition, deletion, complement, product, union and intersection) also play a very

important role for the construction of new graphs, see [10]. In particular, for a graph G Yan et al. [11] introduced the five new graphs $S(G)$, $R(G)$, $L(G)$, $Q(G)$ and $T(G)$ are defined with the help of the operations of subdivision S , semitotal point R , Line graph L , semitotal edge Q and total vertex and edge T respectively. Also, Wiener index of these $\phi(G)$ graphs is computed, where $\Phi \in \{S, R, L, Q, T\}$. Eliasi and Taeri [12] constructed the Φ -sum graphs $G_\Phi + H$ by the Cartesian product of the graphs $\Phi(G)$ and H , where $\Phi(G)$ is obtained after applying the Φ on G for $\Phi \in \{S, R, Q, T\}$. Moreover, they computed the Wiener indices of the Φ -sums graphs $G_S + H$, $G_R + H$, $G_Q + H$ and $G_T + H$. Deng et al. [13] computed the first and second Zagreb indices of four operations on graphs using the concept of Cartesian product. Akhter and Imran [14] computed the forgotten topological index of four operations on graphs under Cartesian product. Liu et al. [15] computed the FGZI of Φ -sum graphs under the operation of Cartesian product. Sarala et al. [16] computed the F -index of Φ -sum graphs under the operation of strong product. For further study, we refer to [17–29].

In this paper, we extend this study and compute FGZI of the Φ -sums graphs $(G_\Phi \boxtimes H)$ under the operation of strong product of $\Phi(G)$ and H in term of FGZI of its factor graphs G and H , where $\Phi \in \{S, R, Q, T\}$, G and H are any two connected graphs. The obtained results are general extension of the results of Deng et al. [13], Akhter and Imran [14], Liu et al. [15] and Sarala et al. [16] works. Forthcoming sections

are arranged as: Section 2 includes the basic formulae and results, Section 3 covers the main results and Section 4 includes the conclusion.

2. Preliminaries

An atom is presented by a vertex and bonding between atom is showed by edge in the molecular graphs. The first and second Zagreb indices $M_1(G)$ and $M_2(G)$ of G are defined as follows [5, 7, 22]:

$$\begin{aligned} M_1(G) &= \sum_{v \in V(G)} [d_G(v)]^2 = \sum_{uv \in E(G)} [d_G(u) + d_G(v)], \\ M_2(G) &= \sum_{uv \in E(G)} [d_G(u)d_G(v)]. \end{aligned} \quad (1)$$

For any real number α FGZ index and general Randic index are defined as

$$\begin{aligned} M^\alpha &= \sum_{v \in V(G)} d_G^\alpha(v) = \sum_{uv \in E(G)} [d_G^{\alpha-1}(u) + d_G^{\alpha-1}(v)], \\ R_\alpha(G) &= \sum_{uv \in E(G)} [d_G(u)d_G(v)]^\alpha. \end{aligned} \quad (2)$$

For more details of aforesaid TIs, see [30, 31]. The following graphs are defined with the help of the subdivision related operations (S, R, Q and T), see [16].

- (i) By inserting a new vertex in each $uv \in E(G)$, $S(G)$ is obtained.
- (ii) By joining the end (original) vertices of the edges being incidence on each new vertex in $S(G)$, $R(G)$ is obtained.
- (iii) By joining the new vertices which have common adjacent (original) vertices in $S(G)$, $Q(G)$ is obtained.
- (iv) By operating both $R(G)$ and $Q(G)$ on $S(G)$, $T(G)$ is obtained. For more explanation, see Figure 1.

In 1960 Sabidussi introduced the strong product for any two graphs $G \circ H$ has vertex set Cartesian product $V(G) \times V(H)$ such that (s_1, t_1) and (s_2, t_2) will be adjacent in $G \circ H$

iff: $[s_1 = s_2 \text{ and } t_1 \text{ is adjacent to } t_2]$ or $[t_1 = t_2 \text{ and } s_1 \text{ is adjacent to } s_2]$ or $[s_1 \text{ is adjacent to } s_2 \text{ and } t_1 \text{ is adjacent to } t_2]$. Now, the Φ -sum graph under the operation of strong product is defined in [16].

Let G and H be two graphs. For $\Phi \in \{S, R, Q, T\}$, $\Phi(G)$ is a graph constructed by the operation Φ on G with set of vertex $V(\Phi(G))$ and set of edge $E(\Phi(G))$. Then, Φ -sum graph $G_\Phi \boxtimes H$ under the strong product of graphs $\Phi(G)$ and H is a graph with vertex set

$$V(G_\Phi \boxtimes H) = V(\Phi(G)) \times V(H) = (V(G) \cup E(G)) \times V(H). \quad (3)$$

such that two vertex (s_1, t_1) and (s_2, t_2) of $V(G_\Phi \boxtimes H)$ are adjacent iff, either $[s_1 = s_2 \in V(G) \text{ and } t_1 t_2 \in E(H)]$ or $[t_1 = t_2 \in V(H) \text{ and } s_1 s_2 \in E(\Phi(G))]$ or $[s_1 s_2 \in E(\Phi(G)) \text{ and } t_1 t_2 \in E(H)]$.

It is observed that $G_\Phi \boxtimes H$ has $|V(H)|$ copies of $\Phi(G)$ which are labeled by the vertices of H . Also, $V(G)$ and $E(G)$ are shown as blue and red vertices in $G_\Phi \boxtimes H$ respectively, see Figure 2.

3. Main Results

Now, we compute the analytical closed expression for FGZ index of the $G_S \boxtimes H$, $G_R \boxtimes H$, $G_Q \boxtimes H$ and $G_T \boxtimes H$.

Theorem 1. Let G and H be two graphs. For $\alpha \in \mathbb{N}^+$ and $n = \alpha - 1$, FGZ index of S -sum graph is

$$\begin{aligned} M^\alpha(G_S \boxtimes H) &= \sum_{m=0}^n \sum_{k=0}^m \binom{n}{m} \binom{m}{k} [M_1^{n-m+k}(G) M_1^{m+1}(H) \\ &\quad + M_1^{\alpha-m+k}(G) M_1^m(H) + M_1^{\alpha-m+k}(G) M_1^{m+1}(H)] \\ &\quad + 2^{n+1} e_G \sum_{m=0}^n \binom{n}{m} M_1^m(H) [1 + M_1(H)]. \end{aligned} \quad (4)$$

Proof. By the definition for any vertex $(u, v) \in V(G_S \boxtimes H)$, the degree $d(u, v)$ of (u, v) is

$$\begin{aligned} d(u, v) &= \begin{cases} d(u) + d(v) + d(u)d(v) & \text{if } u \in V(G) \\ d(u) + d(u)d(v) & \text{if } u \in V(S(G)) - V(G) \end{cases} \\ d(u, v) &= \begin{cases} d(u) + d(v) + d(u)d(v) & \text{if } u \in V(G) \\ 2 + 2d(v) & \text{if } u \in V(S(G)) - V(G), \end{cases} \\ M^\alpha(G_S \boxtimes H) &= \sum_{(u,v) \in V(G_S \boxtimes H)} d_{G_S \boxtimes H}^\alpha(u, v) = \sum_{(s_1, t_1), (s_2, t_2) \in E(G_S \boxtimes H)} [d^n(s_1, t_1) + d^n(s_2, t_2)] \\ &= \sum_{s_1=s_2 \in V(G)} \sum_{(t_1, t_2) \in E(H)} [d^n(s_1, t_1) + d^n(s_2, t_2)] + \sum_{t_1=t_2 \in V(H)} \sum_{(s_1, s_2) \in E(S(G))} [d^n(s_1, t_1) + d^n(s_2, t_2)] \\ &\quad + \sum_{s_1, s_2 \in E(S(G))} \sum_{(t_1, t_2) \in E(H)} [d^n(s_1, t_1) + d^n(s_2, t_2)] = \sum_1 + \sum_2 + \sum_3. \end{aligned} \quad (5)$$

For each vertex $s_1 = s_2 \in V(G)$ and edge $(t_1, t_2) \in E(H)$, we have

FIGURE 1: (a) $G \cong P_4$, (b) $S(G)$, (c) $R(G)$, (d) $Q(G)$, and (e) $T(G)$.

FIGURE 2: For $G \cong P_3$ and $H \cong P_2$, we have (a) $P_{3_s} \boxtimes P_2$, (b) $P_{3_r} \boxtimes P_2$, (c) $P_{3_q} \boxtimes P_2$, and (d) $P_{3_t} \boxtimes P_2$.

$$\begin{aligned}
\sum_1 &= \sum_{s_1=s_2 \in V(G)} \sum_{(t_1, t_2) \in E(H)} [d^n(s_1, t_1) + d^n(s_2, t_2)] \\
&= \sum_{s \in V(G)} \sum_{(t_1, t_2) \in E(H)} [\{d(s) + d(t_1) + d(s)d(t_1)\}^n + \{d(s) + d(t_2) + d(s)d(t_2)\}^n] \\
&= \sum_{s \in V(G)} \sum_{(t_1, t_2) \in E(H)} \sum_{m=0}^n \sum_{k=0}^m \binom{n}{m} \binom{m}{k} [d^{n-m}(s) d^{m-k}(t_1) d^k(s) d^k(t_1) + d^{n-m}(s) d^{m-k}(t_2) d^k(s) d^k(t_2)] \\
&= \sum_{s \in V(G)} \sum_{(t_1, t_2) \in E(H)} \sum_{m=0}^n \sum_{k=0}^m \binom{n}{m} \binom{m}{k} [d^{n-m+k}(s) d^k(t_1) + d^{n-m+k}(s) d^m(t_2)] \\
&= \sum_{s \in V(G)} \sum_{(t_1, t_2) \in E(H)} \sum_{m=0}^n \sum_{k=0}^m \binom{n}{m} \binom{m}{k} d^{n-m+k}(s) [d^m(t_1) + d^m(t_2)], \\
\sum_1 &= \sum_{m=0}^n \sum_{k=0}^m \binom{n}{m} \binom{m}{k} M_1^{n-m+k}(G) M_1^{m+1}(H).
\end{aligned} \tag{6}$$

We know that $|E(S(G))| = 2|E(G)|$. Also, for every $t_1 = t_2 \in V(H)$ and $(s_1, s_2) \in E(S(G))$ with $s_1 \in V(G)$, $s_2 \in V(S(G)) - V(G)$ then,

$$\begin{aligned}
\sum_2 &= \sum_{t_1=t_2 \in V(H)} \sum_{(s_1, s_2) \in E(S(G))} [d^n(s_1, t_1) + d^n(s_2, t_2)] \\
&= \sum_{t \in V(H)} \sum_{(s_1, s_2) \in E(S(G))} [\{d(s_1) + d(t) + d(s_1)d(t)\}^n + \{d(s_2) + d(s_2)d(t)\}^n] \\
&= \sum_{t \in V(H)} \sum_{(s_1, s_2) \in E(S(G))} \left[\sum_{m=0}^n \sum_{k=0}^m \binom{n}{m} \binom{m}{k} \{d^{n-m}(s_1) d^{m-k}(t) d^k(s_1) d^k(t)\} + \sum_{m=0}^n \binom{n}{m} \{d^{n-m}(s_2) d^m(s_2) d^m(t)\} \right] \\
&= \sum_{t \in V(H)} \sum_{(s_1, s_2) \in E(S(G))} \sum_{m=0}^n \binom{n}{m} \left[\sum_{k=0}^m \binom{m}{k} d^{n-m+k}(s_1) d^m(t) + 2^{n-m} (2 d(t))^m \right] \\
&= \sum_{t \in V(H)} \sum_{(s_1, s_2) \in E(S(G))} \sum_{m=0}^n \binom{n}{m} \left[\sum_{k=0}^m \binom{m}{k} d^{n-m+k}(s_1) d^m(t) + 2^{n-m+m} d^m(t) \right] \\
&= \sum_{m=0}^n \sum_{k=0}^m \binom{n}{m} \binom{m}{k} \{M_1^{n-m+k+1}(G)\} \{M_1^m(H)\} + \sum_{m=0}^n \binom{n}{m} \{2^{n+1} e_G M_1^m(H)\} \\
&= \sum_{m=0}^n \sum_{k=0}^m \binom{n}{m} \binom{m}{k} M_1^{n-m+k}(G) M_1^m(H) + e_G \sum_{m=0}^n \binom{n}{m} 2^{n+1} M_1^m(H). \\
\sum_3 &= \sum_{s_1, s_2 \in E(S(G))} \sum_{(t_1, t_2) \in E(H)} [d^n(s_1, t_1) + d^n(s_2, t_2)] = \sum_{\substack{s_1 \in V(G), \\ s_2 \in V(S(G)) - V(G)}} \sum_{(t_1, t_2) \in E(H)} [d^n(s_1, t_1) + d^n(s_2, t_2)] \\
&= \sum_{\substack{s_1 \in V(G), \\ s_2 \in V(S(G)) - V(G)}} \sum_{(t_1, t_2) \in E(H)} [\{d(s_1) + d(t_1) + d(s_1)d(t_1)\}^n + \{d(s_2) + d(s_2)d(t_2)\}^n] \\
&= \sum_{\substack{s_1 \in V(G), \\ s_2 \in V(S(G)) - V(G)}} \sum_{(t_1, t_2) \in E(H)} \left[\sum_{m=0}^n \sum_{k=0}^m \binom{n}{m} \binom{m}{k} \{d^{n-m}(s_1) d^{m-k}(t_1) d^k(s_1) d^k(t_1)\} + \sum_{m=0}^n \binom{n}{m} \{d^{n-m}(s_2) d^m(s_2) d^m(t_2)\} \right] \\
&= \sum_{\substack{s_1 \in V(G), \\ s_2 \in V(S(G)) - V(G)}} \sum_{(t_1, t_2) \in E(H)} \sum_{m=0}^n \binom{n}{m} \left[\sum_{k=0}^m \binom{m}{k} d^{n-m+k}(s_1) d^{m-k}(t_1) + 2^{n-m} 2^m d^m(t_2) \right] \\
&= \sum_{m=0}^n \left(n \sum_{k=0}^m \binom{m}{k} \sum_{\substack{s_1 \in V(G), \\ s_2 \in V(S(G)) - V(G)}} d^{n-m+k}(s_1) \sum_{(t_1, t_2) \in E(H)} d^m(t_1) + 2^n (2e_G) \sum_{(t_1, t_2) \in E(H)} d^m(t_2) \right) \\
&= \sum_{m=0}^n \sum_{k=0}^m \binom{n}{m} \binom{m}{k} M_1^{n-m+k}(G) M_1^{m+1}(H) + e_G \sum_{m=0}^n \binom{n}{m} 2^{n+1} M_1^{m+1}(H).
\end{aligned} \tag{7}$$

Hence

$$M^\alpha(G_S \boxtimes H) = \sum_{m=0}^n \sum_{k=0}^m \binom{n}{m} \binom{m}{k} [M_1^{n-m+k}(G)M_1^{m+1}(H) + M_1^{\alpha-m+k}(G)M_1^m(H) + M_1^{\alpha-m+k}(G)M_1^{m+1}(H)] \\ + 2^{n+1}e_G \sum_{m=0}^n \binom{n}{m} M_1^m(H)[1 + M_1(H)]. \quad (8)$$

Theorem 2. Let G and H be two graphs. For $\alpha \in \mathbb{N}^+$ and $n = \alpha - 1$, FGZ index of R -sum graph is

$$M^\alpha(G_R \boxtimes H) = \sum_{m=0}^n \sum_{k=0}^m \binom{n}{m} \binom{m}{k} 2^{n-m+k} M_1^m(H) [M_1^{n-m+k}(G)M_1(H) + M_1^{\alpha-m+k}(G)(2 + M_1(H))] \\ + 2 \sum_{m=0}^n \binom{n}{m} 2^{n+1} e_G M_1^{m+1}(H). \quad (9)$$

Proof. By definition

$$M^\alpha(G_R \boxtimes H) = \sum_{(u,v) \in V(G_R \boxtimes H)} d^\alpha(u, v) = \sum_{(s_1, t_1)(s_2, t_2) \in E(G_R \boxtimes H)} [d^n(s_1, t_1) + d^n(s_2, t_2)] \\ = \sum_{s_1=s_2 \in V(G)} \sum_{(t_1, t_2) \in E(H)} [d^n(s_1, t_1) + d^n(s_2, t_2)] + \sum_{t_1=t_2 \in V(H)} \left(\sum_{(s_1, s_2) \in E(S(G))} [d^n(s_1, t_1) + d^n(s_2, t_2)] \right) \\ + \sum_{(s_1, s_2) \in E(S(G))} \sum_{(t_1, t_2) \in E(H)} [d^n(s_1, t_1) + d^n(s_2, t_2)] = \sum_1 + \sum_2 + \sum_3. \quad (10)$$

Consider,

$$\sum_1 = \sum_{s_1=s_2 \in V(G)} \sum_{(t_1, t_2) \in E(H)} [d^n(s_1, t_1) + d^n(s_2, t_2)] = \sum_{m=0}^n \sum_{k=0}^m \binom{n}{m} \binom{m}{k} 2^{n-m+k} M_1^{n-m+k}(G) M_1^{m+1}(H). \quad (11)$$

Since $d_{R(G)}(s) = 2d_G(s)$ For each vertex $t_1 = t_2 \in E(H)$ and edge $s_1 s_2 \in E(R(G))$ where $s_1, s_2 \in (V(G))$ then,

$$\begin{aligned}
\sum_2 &= \sum_{t_1=t_2 \in V(H)} \sum_{s_1 s_2 \in E(R(G))} [d^n(s_1, t_1) + d^n(s_2, t_2)] = \sum_{t_1=t_2 \in V(H)} \sum_{\substack{s_1 s_2 \in E(R(G)), \\ s_1, s_2 \in V(G)}} [d^n(s_1, t_1) + d^n(s_2, t_2)] \\
&\quad + \sum_{t_1=t_2 \in V(H)} \sum_{\substack{s_1 s_2 \in E(R(G)), \\ s_1 \in V(G), s_2 \in V(R(G)) - V(G)}} [d^n(s_1, t_1) + d^n(s_2, t_2)] = \sum_2^I + \sum_2^{II}. \\
\sum_2^I &= \sum_{t_1=t_2 \in V(H)} \sum_{s_1 s_2 \in E(R(G))} [d^n(s_1, t_1) + d^n(s_2, t_2)] \\
&= \sum_{t \in V(H)} \sum_{(s_1 s_2) \in E(R(G))} [\{d(s_1) + d(t) + d(s_1)d(t)\}^n + \{d(s_2) + d(t) + d(s_2)d(t)\}^n] \\
&= \sum_{t \in V(H)} \sum_{(s_1 s_2) \in E(R(G))} \sum_{m=0}^n \sum_{k=0}^m \binom{n}{m} \binom{m}{k} [d^{n-m}(s_1) d^{m-k}(t) (d(s_1)d(t))^k + d^{n-m}(s_2) d^{m-k}(t) (d(s_2)d(t))^k] \\
&= \sum_{t \in V(H)} \sum_{\substack{(s_1 s_2) \in E(G) s_1, \\ s_2 \in V(G)}} \sum_{m=0}^n \sum_{k=0}^m \binom{n}{m} \binom{m}{k} (2)^{n-m+k} [d^{n-m+k}(s_1) d^m(t) + d^{n-m+k}(s_2) d^m(t)] \\
&= \sum_{m=0}^n \sum_{k=0}^m \binom{n}{m} \binom{m}{k} 2^{n-m+k} M_1^{\alpha-m+k}(G) M_1^m(H).
\end{aligned} \tag{12}$$

For each $t_1 = t_2 \in E(H)$ and $s_1 s_2 \in E(R(G))$ where $s_1 \in (V(G))$, $s_2 \in (V(R(G)) - V(G))$ then,

$$\begin{aligned}
\sum_2^{II} &= \sum_{t_1=t_2 \in V(H)} \sum_{(s_1 s_2) \in E(R(G))} [d^n(s_1, t_1) + d^n(s_2, t_2)] \\
&= \sum_{t \in E(H)} \sum_{(s_1 s_2) \in E(R(G))} [\{d(s_1) + d(t) + d(s_1)d(t)\}^n + \{d(s_2) + d(s_2)d(t)\}^n] \\
&= \sum_{t \in V(H)} \sum_{(s_1 s_2) \in E(R(G))} \sum_{m=0}^n \binom{n}{m} \left[\sum_{k=0}^m \binom{m}{k} \{d^{n-m}(s_1) d^{m-k}(t) (d(s_1)d(t))^k\} + \{d^{n-m}(s_2) (d(s_2)d(t))^m\} \right] \\
&= \sum_{t \in V(H)} \sum_{s_1 \in V(G)} \sum_{m=0}^n \binom{n}{m} \left[\sum_{k=0}^m \binom{m}{k} \{2^{n-m+k} d^{n-m+k}(s_1) d^m(t) + 2^{n-m} 2^m d^m(t)\} \right] \\
&= \sum_{t \in V(H)} \sum_{s_1 \in (V(G))} \sum_{m=0}^n \binom{n}{m} \left[\sum_{k=0}^m \binom{m}{k} \{2^{n-m+k} d^{n-m+k}(s_1) d^m(t)\} + \{2^n d^m(t)\} \right] \\
&= \sum_{m=0}^n \binom{n}{m} \left[\sum_{k=0}^m \binom{m}{k} 2^{n-m+k} \sum_{t \in V(H)} d^m(t) \sum_{s_1 \in (V(G))} d^{n-m+k}(s_1) + 2^n \sum_{t \in E(H)} d^m(t) (2e_G) \right] \\
&= \sum_{m=0}^n \sum_{k=0}^m \binom{n}{m} \binom{m}{k} 2^{n-m+k} M_1^m(H) M_1^{\alpha-m+k}(G) + 2^{n+1} e_G \sum_{m=0}^n \binom{n}{m} M_1^{m+1}(H).
\end{aligned} \tag{13}$$

For each $t_1 t_2 \in E(H)$ and $(s_1 s_2) \in E(R(G))$, where $s_1, s_2 \in V(G)$ then,

$$\begin{aligned}
\sum_3 &= \sum_{s_1 s_2 \in E(R(G))} \sum_{(t_1 t_2) \in E(H)} [d^n(s_1, t_1) + d^n(s_2, t_2)] \\
\sum_3 &= \sum_{\substack{s_1 s_2 \in E(R(G)) \\ s_1, s_2 \in V(G)}} \sum_{(t_1 t_2) \in E(H)} [d^n(s_1, t_1) + d^n(s_2, t_2)] + \sum_{\substack{s_1 s_2 \in E(R(G)), s_1 \in V(G) \\ s_2 \in V(R(G)) - V(G)}} \sum_{(t_1 t_2) \in E(H)} [d^n(s_1, t_1) + d^n(s_2, t_2)] \\
&= \sum_{(t_1 t_2) \in E(H)} \sum_{s_1, s_2 \in V(G)} \sum_{m=0}^n \sum_{k=0}^m \binom{n}{m} \binom{m}{k} [\{(2 d(s_1))^{n-m+k} d^m(t_1)\} + \{(2 d(s_2))^{n-m+k} d^m(t_2)\}] \\
&= \sum_{(t_1 t_2) \in E(H)} \sum_{s_1, s_2 \in V(G)} \sum_{m=0}^n \sum_{k=0}^m \binom{n}{m} \binom{m}{k} 2^{n-m+k} [d^{n-m+k}(s_1) d^m(t_1) + d^{n-m+k}(s_2) d^m(t_2)] = \sum_3' + \sum_3'', \\
\sum_3' &= \sum_{(t_1 t_2) \in E(H)} \sum_{(s_1 s_2) \in E(R(G))} [d^n(s_1, t_1) + d^n(s_2, t_2)] \\
&= \sum_{(t_1 t_2) \in E(H)} \sum_{(s_1 s_2) \in E(R(G))} [\{d(s_1) + d(t_1) + d(s_1)d(t_1)\}^n + \{d(s_2) + d(t_2) + d(s_2)d(t_2)\}^n] \\
&= \sum_{(t_1 t_2) \in E(H)} \sum_{(s_1 s_2) \in E(R(G))} \sum_{m=0}^n \sum_{k=0}^m \binom{n}{m} \binom{m}{k} [\{d^{n-m+k}(s_1) d^m(t_1)\} + \{d^{n-m+k}(s_2) d^m(t_2)\}] \\
&= \sum_{m=0}^n \sum_{k=0}^m \binom{n}{m} \binom{m}{k} 2^{n-m+k} M_1^{\alpha-m+k}(G) M_1^{m+1}(H).
\end{aligned} \tag{14}$$

For each $t_1 t_2 \in E(H)$ and $(s_1 s_2) \in E(R(G))$, where $s_1 \in V(G), s_2 \in V(R(G)) - V(G)$ then,

$$\begin{aligned}
\sum_3'' &= \sum_{t_1 t_2 \in E(H)} \sum_{(s_1 s_2) \in E(R(G))} [d^n(s_1, t_1) + d^n(s_2, t_2)] \\
&= \sum_{t_1 t_2 \in E(H)} \sum_{(s_1 s_2) \in E(R(G))} [\{d(s_1) + d(t_1) + d(s_1)d(t_1)\}^n + \{2 + 2 d(t_2)\}^n] \\
&= \sum_{t_1 t_2 \in E(H)} \sum_{(s_1 s_2) \in E(R(G))} \sum_{m=0}^n \binom{n}{m} \left[\sum_{k=0}^m \binom{m}{k} d^{n-m}(s_1) d^{m-k}(t_1) (d(s_1)d(t_1))^k + 2^{n-m} (2 d(t_2))^m \right] \\
&= \sum_{t_1 t_2 \in E(H)} \sum_{s_1, s_2 \in V(G)} \sum_{m=0}^n \binom{n}{m} \left[\sum_{k=0}^m \binom{m}{k} (2 d(s_1))^{n-m+k} d^m(t_1) + 2^{n-m} 2 d(t_2)^m \right] \\
&= \sum_{t_1 t_2 \in E(H)} \sum_{s_1, s_2 \in V(G)} \sum_{m=0}^n \binom{n}{m} \left[\sum_{k=0}^m \binom{m}{k} 2^{n-m+k} d^{n-m+k}(s_1) d^m(t_1) + 2^n d^m(t_2) \right] \\
&= \sum_{m=0}^n \binom{n}{m} \left[\sum_{k=0}^m \binom{m}{k} 2^{n-m+k} \sum_{s_1, s_2 \in V(G)} d_G^{n-m+k}(s_1) \sum_{t_1 t_2 \in E(H)} d^m(t_1) + 2^n \sum_{t_1 t_2 \in E(H)} d^m(t_2) (2e_G) \right] \\
&= \sum_{m=0}^n \binom{n}{m} \left[\sum_{k=0}^m \binom{m}{k} 2^{n-m+k} M_1^{\alpha-m+k}(G) M_1^{m+1}(H) + 2^{n+1} e_G M_1^{m+1}(H) \right] \\
\sum_3'' &= \sum_{m=0}^n \binom{n}{m} \left[\sum_{k=0}^m \binom{m}{k} 2^{n-m+k} M_1^{\alpha-m+k}(G) M_1^{m+1}(H) + 2^{n+1} e_G M_1^{m+1}(H) \right].
\end{aligned} \tag{15}$$

Hence

$$M^\alpha(G_R \boxtimes H) = \sum_{m=0}^n \sum_{k=0}^m \binom{n}{m} \binom{m}{k} 2^{n-m+k} M_1^m(H) [M_1^{n-m+k}(G) M_1(H) + M_1^{\alpha-m+k}(G) (2 + M_1(H))] \\ + 2 \sum_{m=0}^n \binom{n}{m} 2^{n+1} e_G M_1^{m+1}(H). \quad (16)$$

□

Theorem 3. Let G and H be two graphs. For $\alpha \in \mathbb{N}^+$ and $n = \alpha - 1$, FGZ index of Q -sum graph is

$$M^\alpha(G_Q \boxtimes H) = \sum_{m=0}^n \sum_{k=0}^m \binom{n}{m} \binom{m}{k} M_1^{n+k-m}(G) M_1^m(H) [M_1(H) + M_1(G) + M_1(G) M_1(H)] \\ + \sum_{m=0}^n \sum_{i=0}^n \binom{n}{m} \binom{n}{i} \sum_{uv \in E(G)} M_1^m(H) [d_G^{n-i}(u) d_G^i(v)] [2 + 2M_1(H)] + \sum_{m=0}^n \binom{n}{m} \sum_{i=0}^n \binom{n}{i} \\ \cdot \sum_{\substack{uv, vw \\ \in E(G)}} M_1^m(H) [d_G^{n-i}(u) d_G^i(v) + d_G^{n-i}(v) d_G^i(w)] [1 + M_1(H)]. \quad (17)$$

Proof. By definition

$$M^\alpha(G_Q \boxtimes H) = \sum_{(u,v) \in V(G_Q \boxtimes H)} d^\alpha(u, v) = \sum_{(s_1, t_1) (s_2, t_2) \in E(G_Q \boxtimes H)} [d^n(s_1, t_1) + d^n(s_2, t_2)] \\ = \sum_{s_1=s_2 \in V(G)} \sum_{t_1 t_2 \in E(H)} [d^n(s_1, t_1) + d^n(s_2, t_2)] + \sum_{t_1 t_2 \in E(H)} \sum_{s_1 s_2 \in E(Q(G))} [d^n(s_1, t_1) + d^n(s_2, t_2)] \\ + \sum_{s_1 s_2 \in E(Q(G))} \sum_{(t_1 t_2) \in E(H)} [d^n(s_1, t_1) + d^n(s_2, t_2)] = \sum_1 + \sum_2 + \sum_3. \quad (18)$$

Consider,

$$\sum_1 = \sum_{s_1=s_2 \in V(G)} \sum_{(t_1 t_2) \in E(H)} [d^n(s_1, t_1) + d^n(s_2, t_2)] = \sum_{m=0}^n \sum_{k=0}^m \binom{n}{m} \binom{m}{k} M_1^{n-m+k}(G) M_1^{m+1}(H). \quad (19)$$

For each vertex $t_1 = t_2 \in V(H)$ and edge $s_1 s_2 \in E(Q(G))$ where $s_1 \in V(G)$, $s_2 \in V(Q(G)) - V(G)$. Note that $d_{Q(G)s_2}$

$= d_G(u) + d_G(v)$ for $s_2 \in [V(Q(G)) - V(G)]$, s_2 is the vertex inserted into the edge uv of G for all $u, v \in V(G)$, we have

$$\begin{aligned}
\sum_2 &= \sum_{t_1=t_2 \in V(H)} \sum_{s_1 s_2 \in E(Q(G))} [d^n(s_1, t_1) + d^n(s_2, t_2)] \\
\sum_2 &= \sum_{t_1=t_2 \in V(H)} \sum_{\substack{(s_1 s_2) \in E(Q(G)), \\ s_1 \in V(G) s_2 \in V(Q(G)) - V(G)}} [d^n(s_1, t_1) + d^n(s_2, t_2)] \\
&\quad + \sum_{t_1=t_2 \in V(H)} \sum_{\substack{s_1 s_2 \in E(Q(G)), \\ s_1, s_2 \in V(Q(G)) - V(G)}} [d^n(s_1, t_1) + d^n(s_2, t_2)] = \sum_{2'} + \sum_{2''}, \\
\sum_{2'} &= \sum_{t_1=t_2 \in V(H)} \sum_{(s_1 s_2) \in E(Q(G))} [d^n(s_1, t_1) + d^n(s_2, t_2)] \\
&= \sum_{t \in V(H)} \sum_{(s_1 s_2) \in E(Q(G))} [\{d(s_1) + d(t) + d(s_1)d(t)\}^n + \{d(s_2) + d(s_2)d(t)\}^n] \\
&= \sum_{t \in V(H)} \sum_{(s_1 s_2) \in E(Q(G))} \sum_{m=0}^n \binom{n}{m} \left[\sum_{k=0}^m \binom{m}{k} d^{n-m}(s_1) d^{m-k}(t) (d(s_1)d(t))^k + d^{n-m}(s_2) (d(s_2)d(t))^m \right] \\
&= \sum_{t \in V(H)} \sum_{(s_1 s_2) \in E(Q(G))} \sum_{m=0}^n \binom{n}{m} \left[\sum_{k=0}^m \binom{m}{k} d^{n-m}(s_1) d^{m-k}(t) (d(s_1)d(t))^k + d^n(s_2) d^m(t) \right],
\end{aligned} \tag{20}$$

consider

$$C_1 = \sum_{\substack{s_1 s_2 \in E(Q(G)), \\ s_1 \in V(G) s_2 \in V(Q(G)) - V(G)}} d^n(s_1). \tag{21}$$

In C_1 , $s_1 \in V(G)$ and $d^n(s_1)$ occurs $d(s_1)$ times. Thus

$$C_1 = \sum_{\substack{s_1 s_2 \in E(Q(G)), \\ s_1 \in V(G) s_2 \in V(Q(G)) - V(G)}} d^{n+1}(s_1). \tag{22}$$

Let

$$C_2 = \sum_{\substack{s_1 s_2 \in E(Q(G)), \\ s_1 \in V(G) s_2 \in V(Q(G)) - V(G)}} d^n(s_2). \tag{23}$$

In C_1 , $s_2 = uv \in E(G)$ and $d^n(s_2)$ occurs two times. Therefore

$$\begin{aligned}
C_2 &= 2 \sum_{\substack{s_2=uv \in \\ V(Q(G)) - V(G)}} [d(u) + d(v)]^n = 2 \sum_{uv \in E(G)} \sum_{i=0}^n \binom{n}{i} d_G^{n-i}(u) d_G^i(v), \\
\sum_{2'} &= \sum_{t \in V(H)} \sum_{(s_1 s_2) \in E(Q(G))} \sum_{m=0}^n \binom{n}{m} \left[\sum_{k=0}^m \binom{m}{k} d^{n-m+k}(s_1) d^m(t) + \sum_{t \in V(H)} d^m(t) \sum_{(s_1 s_2) \in E(Q(G))} d^n(s_2) \right] \\
&= \sum_{m=0}^n \sum_{k=0}^m \binom{n}{m} \binom{m}{k} M_1^{\alpha-m+k}(G) M_1^m(H) + 2 \sum_{m=0}^n \binom{n}{m} M_1^m(H) \sum_{uv \in G} \sum_{i=0}^n \binom{n}{i} d^{n-i}(u) d^i(v).
\end{aligned} \tag{24}$$

Now $\forall t_1 = t_2 \in V(H)$ and edge $s_1 s_2 \in E(Q(G))$ where $s_1, s_2 \in V(Q(G)) - V(G)$, we have

$$\begin{aligned}
\sum_2^n &= \sum_{t_1=t_2 \in V(H)} \sum_{s_1, s_2 \in E(Q(G))} [d^n(s_1, t_1) + d^n(s_2, t_2)] \\
&= \sum_{t \in V(H)} \sum_{s_1, s_2 \in E(Q(G))} [\{d(s_1) + d(s_1)d(t)\}^n + \{d(s_2) + d(s_2)d(t)\}^n] \\
&= \sum_{t \in V(H)} \sum_{s_1, s_2 \in E(Q(G))} \left[\left\{ \sum_{m=0}^n \binom{n}{m} d^{n-m}(s_1) d^m(s_1) d^m(t) \right\} + \left\{ \sum_{m=0}^n \binom{n}{m} d^{n-m}(s_2) d^m(s_2) d^m(t) \right\} \right] \\
&= \sum_{t \in V(H)} \sum_{s_1, s_2 \in E(Q(G))} \left[\left\{ \sum_{m=0}^n \binom{n}{m} d^n(s_1) d^m(t) \right\} + \left\{ \sum_{m=0}^n \binom{n}{m} d^n(s_2) d^m(t) \right\} \right] \\
&= \sum_{t \in V(H)} \sum_{s_1, s_2 \in E(Q(G))} \sum_{m=0}^n \binom{n}{m} d^m(t) [d^n(s_1) + d^n(s_2)] \\
&= \sum_{t \in V(H)} \sum_{uv, vw \in E(G)} \sum_{m=0}^n \binom{n}{m} d^m(t) [(d(u) + d(v))^n + (d(v) + d(w))^n] \tag{25} \\
&= \sum_{t \in V(H)} \sum_{uv, vw \in E(G)} \sum_{m=0}^n \binom{n}{m} d^m(t) \left[\sum_{i=0}^n \binom{n}{i} (d^{n-i}(u) d^i(v)) + \sum_{i=0}^n \binom{n}{i} d^{n-i}(v) d^i(w) \right] \\
&= \sum_{m=0}^n \binom{n}{m} \sum_{t \in V(H)} d^m(t) \sum_{uv, vw \in E(G)} \sum_{i=0}^n \binom{n}{i} [d^{n-i}(u) d^i(v) + d^{n-i}(v) d^i(w)] \\
&= \sum_{m=0}^n \binom{n}{m} \sum_{t \in V(H)} d^m(t) \sum_{uv, vw \in E(G)} \sum_{i=0}^n \binom{n}{i} [d^{n-i}(u) d^i(v) + d^{n-i}(v) d^i(w)] \\
&= \sum_{m=0}^n \binom{n}{m} \sum_{t \in V(H)} d^m(t) \sum_{uv, vw \in E(G)} \sum_{i=0}^n \binom{n}{i} [d^{n-i}(u) d^i(v) + d^{n-i}(v) d^i(w)] \\
&= \sum_{m=0}^n \binom{n}{m} M_1^m(H) \sum_{uv, vw \in E(G)} \sum_{i=0}^n \binom{n}{i} [d^{n-i}(u) d^i(v) + d^{n-i}(v) d^i(w)].
\end{aligned}$$

For each $t_1 t_2 \in E(H)$ and $(s_1 s_2) \in E(Q(G))$, where $s_1, s_2 \in V(G)$, we have

$$\begin{aligned}
\sum_3 &= \sum_{s_1 s_2 \in E(Q(G))} \sum_{(t_1 t_2) \in E(H)} [d^n(s_1, t_1) + d^n(s_2, t_2)] \\
\sum_3 &= \sum_{\substack{s_1 s_2 \in E(Q(G)) \\ s_1 \in V(G), s_2 \in V(Q(G)) - V(G)}} \sum_{(t_1 t_2) \in E(H)} [d^n(s_1, t_1) + d^n(s_2, t_2)] \\
&\quad + \sum_{\substack{s_1 s_2 \in E(Q(G)) \\ s_1, s_2 \in V(Q(G)) - V(G)}} \sum_{(t_1 t_2) \in E(H)} [d^n(s_1, t_1) + d^n(s_2, t_2)] = \sum_3' + \sum_3'' \\
\sum_3' &= \sum_{t_1 t_2 \in E(H)} \sum_{(s_1 s_2) \in E(Q(G))} [d^n(s_1, t_1) + d^n(s_2, t_2)] \\
&= \sum_{t_1 t_2 \in E(H)} \sum_{(s_1 s_2) \in E(Q(G))} [\{d(s_1) + d(t_1) + d(s_1)d(t_1)\}^n + \{d(s_2) + d(s_2)d(t_2)\}^n] \\
&= \sum_{t_1 t_2 \in E(H)} \sum_{(s_1 s_2) \in E(Q(G))} \sum_{m=0}^n \binom{n}{m} \left[\sum_{k=0}^m \binom{m}{k} d^{n-m}(s_1) d^{m-k}(t_1) (d(s_1)d(t_1))^k + d^{n-m}(s_2) (d(s_2)d(t_2))^m \right] \\
&= \sum_{t_1 t_2 \in E(H)} \sum_{(s_1 s_2) \in E(Q(G))} \sum_{m=0}^n \binom{n}{m} \left[\sum_{k=0}^m \binom{m}{k} d^{n-m+k}(s_1) d^m(t_1) + d^n(s_2) d^m(t_2) \right] \\
&= \sum_{m=0}^n \binom{n}{m} \left[\sum_{k=0}^m \binom{m}{k} M_1^{\alpha-m+k}(G) M_1^{m+1}(H) + \sum_{t_1 t_2 \in E(H)} d^m(t_2) \sum_{(s_1 s_2) \in E(Q(G))} d^n(s_2) \right] \\
&= \sum_{m=0}^n \binom{n}{m} \left[\sum_{k=0}^m \binom{m}{k} M_1^{\alpha-m+k}(G) M_1^{m+1}(H) + 2 \sum_{t_1 t_2 \in E(H)} d^m(t_2) \sum_{u, v \in V(G)} (d(u) + d(v))^n \right] \\
&= \sum_{m=0}^n \binom{n}{m} \left[\sum_{k=0}^m \binom{m}{k} M_1^{\alpha-m+k}(G) M_1^{m+1}(H) + 2 M_1^{m+1}(H) \sum_{u, v \in V(G)} (d(u) + d(v))^n \right] \\
&= \sum_{m=0}^n \binom{n}{m} \left[\sum_{k=0}^m \binom{m}{k} M_1^{\alpha-m+k}(G) M_1^{m+1}(H) + 2 M_1^{m+1}(H) \sum_{u, v \in V(G)} \sum_{i=0}^n d^{n-i}(u) d^i(v) \right] \\
&= \sum_{m=0}^n \binom{n}{m} \left[\sum_{k=0}^m \binom{m}{k} M_1^{\alpha-m+k}(G) M_1^{m+1}(H) + 2 M_1^{m+1}(H) \sum_{i=0}^n \sum_{u, v \in V(G)} d^{n-i}(u) d^i(v) \right].
\end{aligned} \tag{26}$$

For each $t_1 t_2 \in E(H)$ and $(s_1 s_2) \in E(Q(G))$, where $s_1 \in V(G)$, $s_2 \in V(Q(G)) - V(G)$, we have

$$\begin{aligned}
\sum_3'' &= \sum_{t_1 t_2 \in E(H)} \sum_{(s_1 s_2) \in E(Q(G))} [d^n(s_1, t_1) + d^n(s_2, t_2)] \\
&= \sum_{t_1 t_2 \in E(H)} \sum_{(s_1 s_2) \in E(Q(G))} [\{d(s_1) + d(s_1)d(t_1)\}^n + \{d(s_2) + d(s_2)d(t_2)\}^n] \\
&= \sum_{t_1 t_2 \in E(H)} \sum_{(s_1 s_2) \in E(Q(G))} \left[\sum_{m=0}^n \binom{n}{m} \{d^{n-m}(s_1) (d(s_1)d(t_1))^m\} + \sum_{m=0}^n \binom{n}{m} \{d^{n-m}(s_2) (d(s_2)d(t_2))^m\} \right] \\
&= \sum_{m=0}^n \binom{n}{m} M_1^{m+1}(H) \left[\sum_{u, v \in V(G)} (d(u) + d(v))^n + \sum_{v, w \in V(G)} (d(v) + d(w))^n \right] \\
&= \sum_{m=0}^n \binom{n}{m} M_1^{m+1}(H) \sum_{i=0}^n \binom{n}{i} \left[\sum_{u, v \in V(G)} d^{n-i}(u) d^i(v) + \sum_{v, w \in V(G)} d^{n-i}(v) + d^i(w) \right].
\end{aligned} \tag{27}$$

Hence

$$\begin{aligned}
 M^\alpha(G_Q \boxtimes H) &= \sum_{m=0}^n \sum_{k=0}^m \binom{n}{m} \binom{m}{k} M_1^{n+k-m}(G) M_1^m(H) [M_1(H) + M_1(G) + M_1(G) M_1(H)] \\
 &+ \sum_{m=0}^n \sum_{i=0}^n \binom{n}{m} \binom{n}{i} \sum_{uv \in E(G)} M_1^m(H) [d_G^{n-i}(u) d_G^i(v) 2 + 2M_1(H)] + \sum_{m=0}^n \binom{n}{m} \sum_{i=0}^n \binom{n}{i} \\
 &\cdot \sum_{\substack{uv, vw \\ \in E(G)}} M_1^m(H) [d_G^{n-i}(u) d_G^i(v) + d_G^{n-i}(v) d_G^i(w)] [1 + M_1(H)].
 \end{aligned} \tag{28}$$

□

Theorem 4. Let G and H be two graphs. For $\alpha \in \mathbb{N}^+$ and $n = \alpha - 1$, FGZ index of T -sum graph is

$$\begin{aligned}
 M^\alpha(G_T \boxtimes H) &= \sum_{m=0}^n \sum_{k=0}^m \binom{n}{m} \binom{m}{k} 2^{n-m+k} M_1^{m+1}(H) M_1^{k-m}(G) [M_1^n(G) + 3M_1^\alpha(G)] \\
 &+ \sum_{m=0}^n \binom{n}{m} \sum_{uv \in V(G)} \sum_{i=0}^n \binom{n}{i} [d_G^{n-i}(u) d_G^i(v)] [3M_1^m(H) + 3M_1^{m+1}(H)] \\
 &+ \sum_{m=0}^n \binom{n}{m} \sum_{uv \in V(G)} \sum_{i=0}^n \binom{n}{i} [d_G^{n-i}(v) d_G^i(w)] [2M_1^{m+1}(H) + M_1^m(H)].
 \end{aligned} \tag{29}$$

Theorem 5. Let G and H be two graphs. Then, the FGZI of Φ -sum graphs where $\alpha \in \mathfrak{R}$, we have

$$\begin{aligned}
 (i) \quad M^\alpha(G_S \boxtimes H) &= \sum_{m=0}^\infty \sum_{k=0}^\infty \binom{n}{m} \binom{m}{k} [M_1^{n-m+k}(G) M_1^{m+1}(H) \\
 &+ M_1^{\alpha-m+k}(G) M_1^m(H) + M_1^{\alpha-m+k}(G) M_1^{m+1}(H)] \\
 &+ 2^{n+1} e_G \sum_{m=0}^\infty \binom{n}{m} M_1^m(H) [1 + M_1(H)], \\
 (ii) \quad M^\alpha(G_R \boxtimes H) &= \sum_{m=0}^\infty \sum_{k=0}^\infty \binom{n}{m} \binom{m}{k} 2^{n-m+k} M_1^m(H) [M_1^{n-m+k} \\
 &(G) M_1(H) + M_1^{\alpha-m+k}(G) (2 + M_1(H))] \\
 &+ 2 \sum_{m=0}^\infty \binom{n}{m} 2^{n+1} e_G M_1^{m+1}(H),
 \end{aligned}$$

$$\begin{aligned}
 (iii) \quad M^\alpha(G_Q \boxtimes H) &= \sum_{m=0}^\infty \sum_{k=0}^\infty \binom{n}{m} \binom{m}{k} M_1^{n+k-m}(G) M_1^m(H) \\
 &[M_1(H) + M_1(G) + M_1(G) M_1(H)] + \sum_{m=0}^\infty \sum_{i=0}^\infty \binom{n}{m} \binom{n}{i} [2 + \\
 &2M_1(H)] \sum_{uv \in E(G)} M_1^m(H) [d_G^{n-i}(u) d_G^i(v)] + \sum_{m=0}^\infty \\
 &\binom{n}{m} \sum_{i=0}^\infty \binom{n}{i} \sum_{uv, vw \in E(G)} M_1^m(H) [d_G^{n-i}(u) d_G^i(v) + \\
 &d_G^{n-i}(v) d_G^i(w)] [1 + M_1(H)], \\
 (iv) \quad M^\alpha(G_T \boxtimes H) &= \sum_{m=0}^\infty \sum_{k=0}^\infty \binom{n}{m} \binom{m}{k} 2^{n-m+k} M_1^{m+1}(H) \\
 &M_1^{k-m}(G) [M_1^n(G) + 3M_1^\alpha(G)] + \sum_{m=0}^\infty \binom{n}{m} \sum_{uv \in V(G)} \sum_{i=0}^\infty
 \end{aligned}$$

TABLE 1: Table for Example 1.

$[m, n, \alpha]$	$M^\alpha(P_{nS} \boxtimes P_m)$	$M^\alpha(P_{nR} \boxtimes P_m)$	$M^\alpha(P_{nQ} \boxtimes P_m)$	$M^\alpha(P_{nT} \boxtimes P_m)$
(3, 3, 3)	1808	6414	3442	8048
(4, 4, 4)	37260	208104	102800	258412
(5, 5, 5)	434888	5505322	1845182	6915616
(6, 6, 6)	5.42×10^6	127857836	28374140	150810676
(7, 7, 7)	6.25×10^7	2713045998	406381162	3056832424
(8, 8, 8)	6.84×10^7	5.39×10^{10}	560449256	5.88×10^{10}
(9, 9, 9)	7.11×10^{10}	1.02×10^{12}	7.56×10^{10}	1.088×10^{12}
(10, 10, 10)	7.34×10^{10}	1.85×10^{13}	1.01×10^{12}	1.94×10^{13}

FIGURE 3: Graphical representations for Table 1.

TABLE 2: Table for Example 2.

$[m, n, \alpha]$	$M^\alpha(C_{nS} \boxtimes C_m)$	$M^\alpha(C_{nR} \boxtimes C_m)$	$M^\alpha(C_{nQ} \boxtimes C_m)$	$M^\alpha(C_{nT} \boxtimes C_m)$
(3, 3, 3)	6552	26640	17568	37656
(4, 4, 3)	11648	47360	31232	66944
(5, 5, 3)	18200	74000	48800	104600

FIGURE 4: Graphical representations for Table 2.

$$\binom{n}{i} [d_G^{n-i}(u) d_G^i(v)] [3M_1^m(H) + 3M_1^{m+1}(H)] \\ + \sum_{m=0}^{\infty} \binom{n}{m} \sum_{vw \in V(G)} \sum_{i=0}^{\infty} \binom{n}{i} [d_G^{n-i}(v) d_G^i(w)] [2M_1^{m+1}(H) + M_1^m(H)].$$

4. Discussion and Conclusion

We conclude our work with applications of (Theorem 1–Theorem 4) in the following examples.

Example 1. For two graphs P_n and P_m of order $n \geq 3$ and $m \geq 2$, we have

$$\begin{aligned} M^\alpha(P_{nS} \boxtimes P_m) &= \sum_{m=0}^n \sum_{k=0}^m C_m^n C_k^m [2^{\alpha+k+2} (n-2)(m-2) + 5(2)^{\alpha-m+k} (n-2) + 5(2)^{m+1} (m-2) + 12] \\ &\quad + 2^{n+1} (n-1) \sum_{m=0}^n C_m^n [2^m (2m^2 - 5m + 2) + 4m - 2], \\ M^\alpha(P_{nR} \boxtimes P_m) &= \sum_{m=0}^n C_m^n \left[\sum_{k=0}^m C_k^m 2^{n-m+k} \{ (2^{m+1} (m+2) + 2) (2^{\alpha-m+k} (n-3) + 8) \} + 2^{n+3} (n-1) \{ 2^m (m-1) + 1 \} \right], \\ M^\alpha(P_{nQ} \boxtimes P_m) &= \sum_{m=0}^n \sum_{k=0}^m C_m^n C_k^m [2^{\alpha+k+2} (n-2)(m-2) + 5(2)^{\alpha-m+k} (n-2) + 5(2)(m-2) + 12] \\ &\quad + \sum_{m=0}^n \sum_{i=0}^n C_m^n C_i^n [2^m (m-2) + 2] [2^i + 2^n (n-2) + 2^{n-i}] [4(m-1) + 2^i + 2^n (n-2) + 2^{n-i} + 2m - 1], \\ M^\alpha(P_{nT} \boxtimes P_m) &= \sum_{m=0}^n \sum_{k=0}^m C_m^n C_k^m 2^{n-m+k} [2^{m+1} (m-2) + 2] [2^{k-m} (n-2) + 2] [2^n + 3(2)^\alpha] + \sum_{m=0}^n \sum_{i=0}^n C_m^n C_i^n \\ &\quad \left[\{ 9(2^m) (m-2) + 12 \} \{ 2^i + 2^n (n-2) + 2^{n-i} \} + \{ 2^m (5m - 10) + 6 \} \{ 2^i + 2^n (n-2) + 2^{n-i} \}^2 \right]. \end{aligned} \quad (30)$$

Example 2. For two graphs C_n and C_m of order $n, m \geq 3$, we have

$$\begin{aligned} M^\alpha(C_{nS} \boxtimes C_m) &= \sum_{m=0}^n \sum_{k=0}^m C_m^n C_k^m [2^{\alpha+k+2} (nm)] + \sum_{m=0}^n C_m^n [2^{n+m+1} m(n-1) (1 + 2^m m)], \\ M^\alpha(C_{nR} \boxtimes C_m) &= \sum_{m=0}^n \sum_{k=0}^m C_m^n C_k^m [2^{2n+2k-m+1} (9mn)] + \sum_{m=0}^n C_m^n [2^{n+m+3} m(n-1)], \\ M^\alpha(C_{nQ} \boxtimes C_m) &= \sum_{m=0}^n \sum_{k=0}^m C_m^n C_k^m [2^{\alpha+k+2} (nm)] + \sum_{m=0}^n \sum_{i=0}^n C_m^n C_i^n [2^{m+n+2} nm (1 + 2m)], \\ M^\alpha(C_{nT} \boxtimes C_m) &= \sum_{m=0}^n \sum_{k=0}^m C_m^n C_k^m [2^{2(n+k+1)-m} (nm)] + \sum_{m=0}^n \sum_{i=0}^n C_m^n C_i^n [2^{n+m} (14nm)]. \end{aligned} \quad (31)$$

- (i) In this paper, we computed the exact values of FGZ index of the four Φ -sum graphs ($G_S \boxtimes H$, $G_R \boxtimes H$, $G_Q \boxtimes H$ and $G_T \boxtimes H$) in terms of their factor graphs, where these graphs are obtained under the operation of the strong product.
- (ii) It is the extension of Deng et al. [13], Akhter and Imran [14], Liu et al. [15] and Sarala et al. [16] works.
- (iii) The FGZ index of the Φ -sum graphs obtained from the paths and cycles are also computed as applications of the obtained results.
- (iv) Table 1 and Figure 3 present that FGZ index of $P_{nT} \boxtimes P_m$ is dominant having ordering $M^\alpha(P_{nT} \boxtimes P_m) \geq M^\alpha(P_{nR} \boxtimes P_m) \geq M^\alpha(P_{nQ} \boxtimes P_m) \geq M^\alpha(P_{nS} \boxtimes P_m)$.
- (v) Table 2 and Figure 4 also presents that FGZ index of $C_{nT} \boxtimes C_m$ is dominant having ordering

$$M^\alpha(C_{nT} \boxtimes C_m) \geq M^\alpha(C_{nR} \boxtimes C_m) \geq M^\alpha(C_{nQ} \boxtimes C_m) \geq M^\alpha(C_{nS} \boxtimes C_m).$$

Data Availability

All data are included within this paper. However, the reader may contact the corresponding author for more details of the data.

Disclosure

The authors have no conflicts of interest.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The author is deeply thankful to the reviewers for their valuable suggestions to improve the quality of this manuscript. This work was supported by the Humanities and Social Science Project of Anhui Provincial Education Department. Research on social crisis management in Anhui cities under the background of big data (Subject No: SK2016A0233).

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