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Research Article

A Reliable Treatment for Nonlinear Differential Equations

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In this paper, we use the concept of homotopy, Laplace transform, and He's polynomials, to propose the auxiliary Laplace homotopy parameter method (ALHPM). We construct a homotopy equation consisting on two auxiliary parameters for solving nonlinear differential equations, which switch nonlinear terms with He's polynomials. The existence of two auxiliary parameters in the homotopy equation allows us to guarantee the convergence of the obtained series. Compared with numerical techniques, the method solves nonlinear problems without any discretization and is capable to reduce computational work. We use the method for different types of singular Emden–Fowler equations. The solutions, constructed in the form of a convergent series, are in excellent agreement with the existing solutions.

1. Introduction

Initial value problems with singularity and of type Lane–Emden differential equation,

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + y^r = 0, \tag{1}$$

have been used to model a large category of phenomena in various science, such as mathematical physics and astrophysics. The first studies on these equations have been published by Lane in 1870 [1]. A further research was continued by Emden in [2]. Then, this equation was used in the modeling of some problems, such as the thermal behavior of spherical cloud of gas.

In astrophysics and in the study of a self-gravitating spherically symmetric polytropic fluid, this equation appears as its gravitational Poisson's equation. There are several phenomena, such as astrophysics, aerodynamics, stellar structure, chemistry, biochemistry, and many others which can be modeled by the Lane–Emden equation [3–5]. Fowler [6, 7] generalized the Lane–Emden equation to the Emden–Fowler equation:

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + af(x)g(y) = 0, \tag{2}$$

for some given functions $f(x)$ and $g(y)$. In this connection, we note that the following heat equation,

$$\frac{\partial^2 y}{\partial x^2} + \frac{r}{x}\frac{\partial y}{\partial x} + af(x,t)g(y) + h(x,t) = \frac{\partial y}{\partial t}, \tag{3}$$

where $0 < x \le L, 0 < t < T$, and $r > 0$, appears in the modeling of the diffusion of heat perpendicular to surfaces of parallel planes. For $r = 2$ and $h(x,t) = 0$ and for the steady state, equation (3) is the Emden–Fowler equation. Singularity behavior that occurs at $x = 0$ is a difficult element in the analysis of this type of equations, and due to this problem, common methods need to be reviewed. For example, Wazwaz [8] used the Adomian decomposition method to solve these equations; however, the appropriate choices of operator L were required to overcome the singularity behavior at the origin. In this paper and in our proposed method, taking the Laplace transform from both sides of the equation solves this difficulty. Indeed, in vast majority of cases in differential equations with variable coefficients, we cannot obtain an exact solution, so we must look for approximate solutions, such as asymptotic techniques [9, 10], analytical [11–14], and numerical methods [15–19]. Emden–Fowler-type equations have been solved using the Adomian polynomials [20] by Wazwaz in [8], using the homotopy perturbation method by Chowdhury and Hashim in [21], and using the variational iteration method by Wazwaz in [22]. In this paper, we use a new auxiliary homotopy parameter method using He's polynomials, called the auxiliary Laplace homotopy parameter method (ALHPM) for solving Emden–Fowler equations.

2. Main Results

In this section, we present the new auxiliary Laplace homotopy parameter method (ALHPM). For this purpose, let us consider the following nonlinear nonhomogeneous PDE:

$$\mathscr{D}u(x,t) + \mathscr{N}u(x,t) = h(x,t), \tag{4}$$

which subject to

$$u(x,0) = g_1(x), \quad u_t(x,0) = g_2(x), \tag{5}$$

where $\mathscr{D} = \partial^2/\partial t^2$, $\mathscr{N}$ is a general nonlinear term, which may include nonlinear differential operators, and $h(x,t)$ denotes the source term. Using the Laplace transform [23] and by its applying to the both sides of (4), we deduce

$$\mathrm{L}[\mathscr{D}u(x,t)] + \mathrm{L}\mathscr{N}[u(x,t)] = \mathrm{L}[h(x,t)]. \tag{6}$$

This quickly yields that

$$L[u(x,t)] - \frac{1}{s}g_1(x) - \frac{1}{s^2}g_2(x) + \frac{1}{s^2}L[\mathscr{N}u(x,t) - h(x,t)] = 0. \tag{7}$$

Now, we use the homotopy concept to deduce the following homotopy equation of two parameters:

$$L[v(x,t) - u_0] + \frac{\hbar}{s^2}pL[\mathscr{N}v(x,t) - h(x,t)] = 0, \tag{8}$$

where $\hbar$ and p are nonzero auxiliary parameters, L is the Laplace transform, and u_0 is an initial guess of the solution. The result of applying Laplace inverse on both sides of (8) is as follows:

$$v(x,t) = I(x,t) - \frac{\hbar}{s^2}L^{-1}[L[p\mathscr{N}v(x,t) - ph(x,t)]] = 0, \tag{9}$$

where

$$I(x,t) = g_1(x) + tg_2(x). \tag{10}$$

Now, we use the basis of the homotopy perturbation method to have a series expansion for $v(x,t)$ as follows:

$$v(x,t) = \sum_{i=0}^{\infty} p^i v_i, \tag{11}$$

where p is an embedding parameter. To switch nonlinear operator N, we use He's polynomials [24] to obtain

$$\mathscr{N}(v(x,t)) = \Sigma_{n=0}^{\infty} p^n H_n, \tag{12}$$

where

$$H_n(v_0, v_1, \ldots, v_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty} p^i v_i\right)\right]_{p=o}. \tag{13}$$

Substituting (11) and (12) in (9) and comparing the coefficient of like powers of p, we obtain

$$\begin{aligned} v_0(x,t) &= I(x,t) = g_1(x) + tg_2(x), \\ v_1(x,t) &= -\hbar L^{-1}\left[\frac{1}{s^2}L[H_0 - f]\right], \\ v_{n+1} &= -\hbar L^{-1}\left[\frac{1}{s^2}L[H_n]\right], \quad n = 1, 2, \ldots \end{aligned} \tag{14}$$

3. Emden–Fowler Equations

Due to the singularity behavior at the origin, as well as other various linear and nonlinear singular IVPs, numerical solution of Emden–Fowler equations is a challenging issue. For numerical treatment, some authors have been forced to propose alternative approaches. These techniques often focus on the removal of the singularity of this equation. For example, by expanding the unknown function as different basis functions, the problem reduces to a set of algebraic equations to using operational matrices. However, it is clear that equivalent numerical techniques are more computationally expensive. So, often analytical methods have been considered. In this section, in order to show the efficiency of the auxiliary Laplace homotopy parameter method, presented in the previous section, here, we use this method to solve the initial value problems related to second order singular Emden–Fowler differential equations. The examples show that our method leads to the exact solution series of the problem, and in this case, it is possible to guess the closed form of the solution.

Example 3.1. Let us consider the following nonlinear singular Lane–Emden equation:

$$y'' + \frac{2}{x}y' + y^5 = 0, \tag{15}$$

with

$$y(0) = 1, \quad y'(0) = 0. \tag{16}$$

Using the Laplace transform and by applying to the both sides of (15), we deduce

$$\frac{d}{ds}Y(s)+\frac{1}{s^2}+\frac{1}{s^2}\frac{d}{ds}L(y^n)=0, \tag{17}$$

where $Y(s)=L[y(x)]$. As the ALHPM technique, using the concept of the homotopy perturbation method and He's polynomials for nonlinear term y^5 and constructing two parameters homotopy of equation (17), we get the following:

$$\frac{d}{ds}\left(\widehat{y}_0+\widehat{y}_1p+\widehat{y}_2p^2+\cdots\right)+\frac{1}{s^2}+\hbar\frac{1}{s^2}\frac{d}{ds}L\left(H_0p+H_1p^2+H_2p^3+\cdots\right)=0, \tag{18}$$

and the first few components of He's polynomials are

$$\begin{aligned}H_0&=y_0^5,\\ H_1&=5y_0^4y_1,\\ H_2&=5y_0^4y_2+10y_0^3y_1^2,\\ &\vdots\end{aligned} \tag{19}$$

Using (12) and by comparing the same powers of p, it is clear that the recursive relation is

$$\begin{aligned}&\frac{d}{ds}\left(\widehat{y}_0(s)\right)=-\frac{1}{s^2},\\ &\frac{d}{ds}\left(\widehat{y}_{n+1}(s)\right)=-\hbar\frac{1}{s^2}L[H_n],\quad n=0,1,\ldots\end{aligned} \tag{20}$$

Therefore, we have

$$\widehat{y}_0(s)=\frac{1}{s},\quad \widehat{y}_1(s)=-\hbar\frac{1}{3s^3},\quad \widehat{y}_2(s)=\hbar\frac{1}{s^5},\quad \widehat{y}_3(s)=-\hbar\frac{25}{3s^7}, \tag{21}$$

and so on. Consequently, using the Mathematica symbolic code, the series solution of (13) and (14) is given by

$$y(x)=1-\hbar\frac{x^2}{6}+\hbar\frac{x^4}{24}-\hbar\frac{5x^6}{432}+\cdots. \tag{22}$$

By putting $\hbar=1$, we get a series providing a closed form of the exact solution:

$$y(x)=\frac{1}{\sqrt{1+x^2/3}}. \tag{23}$$

Example 3.2. Now, in this example, we consider the following Emden–Fowler equation:

$$y''+\frac{2}{x}y'+ax^my^n=0, \tag{24}$$

with

$$y(0)=1\quad\text{and}\quad y'(0)=0, \tag{25}$$

which has been investigated in several studies from various points of view because of its interesting mathematical and physical properties.

Using the Laplace transform and by applying to the both sides of (24), we deduce

$$\frac{d}{ds}Y(s)+\frac{1}{s^2}-(-1)^{m+1}\frac{a}{s^2}\frac{d^{m+1}}{ds^{m+1}}L(y^n)=0, \tag{26}$$

where $Y(s)=L[y(x)]$. By constructing two parameters' homotopy of equation (26) and applying the aforesaid method, we get the following:

$$\frac{d}{ds}\left(\widehat{y}_0+\widetilde{y}_1p+\widehat{y}_2p^2+\cdots\right)+\frac{1}{s^2}-(-1)^{m+1}a\hbar\frac{1}{s^2}\frac{d^{m+1}}{ds^{m+1}}L\left(H_0p+H_1p^2+H_2p^3+\cdots\right)=0, \tag{27}$$

where H_n are He's polynomials. The first few components of He's polynomials are as follows:

$$\begin{aligned}H_0&=y_0^n,\\ H_1&=ny_0^{n-1}y_1,\\ H_2&=ny_0^{n-1}y_2+\frac{n(n-1)}{2!}y_0^{n-2}y_1^2.\\ &\vdots\end{aligned} \tag{28}$$

Comparing the similar powers of p, we deduce

$$\begin{aligned}&\widehat{y}_0(0)=\frac{1}{s},\quad \widehat{y}_1=-\hbar\frac{(m+1)!}{(m+3)}\frac{1}{s^{m+3}},\\ &\widehat{y}_2=-\hbar\frac{na^2(2m+3)!}{(m+2)(m+3)(2m+5)}\frac{1}{s^{2m+5}},\ldots,\end{aligned} \tag{29}$$

so that, using Laplace inverse, we obtain

$$y(x)=1-\hbar\frac{a}{(m+3)(m+2)}x^{m+2}+\hbar\frac{a^2n}{2(2m+5)(m+3)(m+2)^2}y^{2m+4}+\cdots. \tag{30}$$

Substituting $\hbar=1$ in equation (30), one has

$$y(x)=1-\frac{a}{(m+3)(m+2)}x^{m+2}+\frac{a^2n}{2(2m+5)(m+3)(m+2)^2}x^{2m+4}+\cdots, \tag{31}$$

which is the same as that obtained in [8]. We can easily show that if $m\notin\{-3,-2,-5/2,-7/37/3,-9/4,\cdots\}$, one can obtain the exact solution of (24) from (31). As an example, substituting $m=0$ and $n=0$ into (31), we obtain

$$y(x)=1-\frac{a}{3!}x^2, \tag{32}$$

which is the exact solution of the following equation:

Table 1: Comparison of the numerical solutions and absolute errors obtained by our ALHPM method, ALHPMP, and collocation scheme for Example 3.3.

x	ALHPM	Collocation scheme [25]	ALHPMP [11]	err1	err2	err3
0.1	−0.0016658423	−0.0016658367	−0.0016658417	7.77600, −12	3.00000, −10	4.78000, −14
0.2	−0.0066533652	−0.0066533643	−0.0066533621	3.52000, −12	0.00000, 00	2.42000, −14
0.3	−0.0149328847	−0.0149328883	−0.0149328822	2.07700, −11	2.00000, −10	1.07000, −13
0.4	−0.0264554801	−0.0264554779	−0.0264554792	1.90900, −10	3.70000, −9	3.92300, −13
0.5	−0.0411539505	−0.0411539500	−0.0411539533	3.17060, −9	3.24000, −8	7.70100, −11
0.6	−0.0589440799	−0.0589440832	−0.0589440787	2.42100, −9	1.98900, −7	2.12100, −10
0.7	−0.0797260101	−0.0797260049	−0.0797260088	5.78890, −8	9.17600, −7	4.12100, −9
0.8	−0.1033860573	−0.1033860422	−0.1033860677	6.61120, −8	3.43540, −6	4.41000, −7
0.9	−0.1297985455	−0.1297985388	−0.1297985675	5.67120, −7	1.09758, −5	6.71000, −7
1	−0.1588278601	−0.1588278334	−0.1588278677	6.77000, −7	3.09241, −5	9.89090, −7

$$y'' + \frac{2}{x}y' + a = 0. \tag{33}$$

Similarly, for $m = 0$ and $n = 1$, we obtain a series as a closed form of

$$y(x) = \frac{\sin(\sqrt{a}\,x)}{\sqrt{a}x}, \tag{34}$$

which is the exact solution of the equation:

$$y'' + \frac{2}{x}y' + ay = 0. \tag{35}$$

More choices of m and n can be easily checked. However, for the values of m and n which the obtained series (31) is not defined, the solution of (21) must be investigated separately. If $m = -1$ and $n = 1$, equation (24) is the well-known Euler equation and the solution can be obtained easily. For $m \in \{-3, -5/2, -7/3, -9/4, \cdots\}$, $n = 1$, $x > 0$ (and generally $m = -2\eta + 1/\eta$), the exact solution of (21) can be represented by the Bessel functions of the first and the second kinds.

Example 3.3. Now, let us suppose $f(x) = x^m$ and $g(y) = e^y$, so we consider

$$y'' + \frac{2}{x}y' + ax^m e^y = 0, \tag{36}$$

with

$$y(0) = y'(0) = 0. \tag{37}$$

Similar to the previous examples, using the Laplace transform and by its applying to the both sides of (36), we deduce

$$\frac{d}{ds}Y(s) - (-1)^{m+1}\hbar a\frac{1}{s^2}\frac{d^{m+1}}{ds^{m+1}}L(e^y) = 0. \tag{38}$$

In a similar way as above, one writes

$$\frac{d}{ds}\left(\hat{y}_0 + \hat{y}_1 P + \hat{y}_2 p^2 + \cdots\right) - (-1)^{m+1}\frac{a}{s^2}\frac{d^{m+1}}{ds^{m+1}}L\left(H_0 p + H_1 p^2 + H_2 p^3 + \cdots\right) = 0. \tag{39}$$

By equating the powers of p, one can obtain

$$\hat{y}_0(0) = 0, \quad \hat{y}_1 = -\hbar\frac{a(m+1)!}{(m+3)}\frac{1}{s^{m+3}}, \quad \hat{y}_2 = -\hbar\frac{a^2(2m+4)!}{2(2m+5)(m+3)(m+2)^2}\frac{1}{s^{2m+5}}, \cdots. \tag{40}$$

Therefore, we have the following series solution:

$$y(x) = -\hbar\frac{a}{(m+3)(m+2)}x^{m+2} + \hbar\frac{a^2}{2(2m+5)(m+3)(m+2)^2}x^{2m+4} - \hbar\frac{a^3(3m+8)}{6(3m+7)(2m+5)(m+3)^2(m+2)^3}x^{3m+6}\cdots. \tag{41}$$

Putting $\hbar = 1$ in equation (41) recovers the solution obtained in [8]. Then,

$$\mathbf{y}(x) = -\frac{a}{(m+3)(m+2)}x^{m+2} + \frac{a^2}{2(2m+5)(m+3)(m+2)^2}x^{2m+4} - \frac{a^3(3m+8)}{6(3m+7)(2m+5)(m+3)^2(m+2)^3}x^{3m+6}\cdots. \tag{42}$$

In the direction of method efficiency, we consider the case $m = 0$ and $a = 1$. The numerical results are compared with the solutions obtained via collocation scheme in Table 1. On the contrary, it is possible to improve the results of the series solution by using the Pade' approximations. We have applied the Pade' [5, 5] approximation to the obtained series solution (ALHPMP [11]).

The accuracy is obtained when the computational cost of our method is much less than the comparative method.

4. Conclusion

The new auxiliary Laplace homotopy parameter method was presented. Using the concept of homotopy and Laplace transformation, a two-parameter depending homotopy equation was constructed. Nonlinear terms were dealt with by He's polynomials. Using of two auxiliary parameters in homotopy equation enables us to make the solutions more reliable. We used this method to solve the initial value problems related to second-order singular Emden–Fowler differential equations. In these equations, the singularity behavior at the origin prevents us from using the usual methods, and to overcome this problem, these methods must be changed. In our proposed method, this limitation is easily removed. Obtained solutions, constructed in the form of a convergent series, were in excellent agreement with the existing solutions.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare no conflicts of interest.

Authors' Contributions

All authors contributed equally in writing this article. All authors read and approved the final manuscript.

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