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Research Article

On Generalized Rational α–Geraghty Contraction Mappings in G–Metric Spaces

N. Priyobarta,[1] **Bulbul Khomdram**,[1] **Yumnam Rohen**,[1] **and Naeem Saleem**[2]

[1]*Department of Mathematics, National Institute of Technology Manipur, Langol, Imphal 795004, India*
[2]*Department of Mathematics, University of Management and Technology, Lahore, Pakistan*

Correspondence should be addressed to Naeem Saleem; naeem.saleem2@gmail.com

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In this paper, we discuss about various generalizations of α–admissible mappings. Furthermore, we extend the concept of α–admissible to generalize rational α–Geraghty contraction in G–metric space. With this new contraction mapping, we establish some fixed-point theorems in G–metric space. The obtained result is verified with an example.

1. Introduction and Preliminaries

Samet et al. [1] make a remarkable contribution by introducing α–admissible and $\alpha - \psi$–contractive mappings. They also showed that the most celebrated result, the Banach contraction principle and various other results, are consequences of their results. Geraghty [2] also made an improvement of the Banach contraction principle. Mustafa and Sims [3] introduced the concept of G–metric space and established the Banach contraction principle. In this paper, considering both the concepts of Samet et al. [1] and Geraghty [2], we introduce generalized rational α–Geraghty contraction in the framework of G–metric space and establish some theorems on fixed points.

Mustafa and Sims [3] give the following definition.

Definition 1 (see [3]). Let U be a nonempty set, and G: $U \times U \times U \longrightarrow \mathbb{R}^+$ satisfies the following:

(i) $G(\xi, \theta, \phi) = 0$ if and only if $\xi = \theta = \phi$.
(ii) $0 < G(\xi, \xi, \theta)$ for all $\xi, \theta \in U$ with $\xi \neq \theta$.
(iii) $G(\xi, \xi, \theta) \leq G(\xi, \theta, \phi)$ for all $\xi, \theta, \phi \in U$ with $\theta \neq \phi$.
(iv) $G(\xi, \theta, \phi) = G(\xi, \phi, \theta) = G(\theta, \phi, \xi) = \cdots$ (symmetry in all three variables).
(v) $G(\xi, \theta, \phi) \leq G(\xi, t, t) + G(t, \theta, \phi)$, for all $\xi, \theta, \phi, t \in U$.

Here, G is known as generalized metric or G–metric. The pair (U, G) is known as G–metric space.

The following example shows the relation between metric space and G–metric space given by Mustafa and Sims [3].

Example 1 (see [3])

(i) Let (U, m) be an ordinary metric space, then
$$G(\xi, \theta, \phi) = \frac{1}{3}\{m(\xi, \theta) + m(\theta, \phi) + m(\xi, \phi)\}, \tag{1}$$
is a G–metric on U.
(ii) Let (U, m) be an ordinary metric space, then
$$G(\xi, \theta, \phi) = \max\{m(\xi, \theta), m(\theta, \phi), m(\xi, \phi)\}, \tag{2}$$
is a G–metric on U.
(iii) Let G be a G–metric on U, then
$$m(\xi, \theta) = G(\xi, \theta, \theta) + G(\xi, \xi, \theta), \tag{3}$$
is a metric on U.

Definition 2 (see [3]). Consider a G–metric space (U, G) and a sequence $\{\xi_n\}$ of points of U. $\{\xi_n\}$ is said to be

G–convergent to $\xi \in U$ provided $\lim_{n,m\longrightarrow+\infty} G(\xi_n, \xi_m, \xi) = 0$; that is, there exists $K \in \mathbb{N}$ satisfying $G(\xi_n, \xi_m, \xi) < \varepsilon$ and $m, n \geq K$ where $\varepsilon > 0$. Here, ξ is known as the limit of the sequence $\{\xi_n\}$ and is denoted as $\xi_n \longrightarrow \xi$ or $\lim_{n\longrightarrow+\infty} \xi_n = \xi$.

Proposition 1 (see [3]). *In a G–metric space (U, G), we have the following equivalent statements:*

(i) $\{\xi_n\}$ is G convergent to ξ.

(ii) $G(\xi_n, \xi_n, \xi) \longrightarrow 0$ when $n \longrightarrow +\infty$.

(iii) $G(\xi_n, \xi, \xi) \longrightarrow 0$ when $n \longrightarrow +\infty$.

(iv) $G(\xi_n, \xi_m, \xi) \longrightarrow 0$ when $n, m \longrightarrow +\infty$.

Definition 3 (see [3]). In a G–metric space (U, G), the sequence $\{\xi_n\}$ is said to be G–Cauchy if for any $\varepsilon > 0$, there exists $K \in \mathbb{N}$ satisfying $G(\xi_n, \xi_m, \xi_l) < \varepsilon$ and $n, m, l \geq K$; that is, $G(\xi_n, \xi_m, \xi_l) \longrightarrow 0$ when $n, m, l \longrightarrow +\infty$.

Proposition 2 (see [3]). *In a G–metric space (U, G), we have the following equivalent statements:*

(i) The sequence $\{\xi_n\}$ is G–Cauchy.

(ii) For any $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that $G(\xi_n, \xi_n, \xi_m) < \varepsilon$ for all $n, m \geq K$.

Definition 4 (see [3]). A G–metric space (U, G) is said to be G–complete if every G–Cauchy sequence is G–convergent in (U, G).

Lemma 1 (see [3]). *In a G–metric space (U, G), for $\xi, \theta, \phi, t \in U$, we have the following:*

(i) If $G(\xi, \theta, \phi) = 0$, then $\xi = \theta = \phi$.

(ii) $G(\xi, \theta, \phi) \leq G(\xi, \xi, \theta) + G(\xi, \xi, \phi)$.

(iii) $G(\xi, \theta, \theta) \leq 2G(\theta, \xi, \xi)$.

(iv) $G(\xi, \theta, \phi) \leq G(\xi, t, \phi) + G(t, \theta, \phi)$.

(v) $G(\xi, \theta, \phi) \leq (2/3)[G(\xi, \theta, t) + G(\xi, t, \phi) + G(t, \theta, \phi)]$.

(vi) $G(\xi, \theta, \phi) \leq G(\xi, t, t) + G(\theta, t, t) + G(\phi, t, t)$.

Definition 5 (see [3]). In a G–metric space (U, G), a mapping $R: U \longrightarrow U$ is known as G–continuous if $\{R(\xi_n)\}$ is G–convergent to $R(\xi)$, where $\{\xi_n\}$ is any G–convergent sequence converging to ξ.

Here firstly, we recall the definition of α-admissible mappings and its generalizations in metric space and G-metric space.

Definition 6 (see [1]). Let R be a self-mapping on a metric space (U, d), and let $\alpha: U \times U \longrightarrow [0, +\infty)$ be a function. R is said to be an α-admissible if $\xi, \theta \in U$, and $\alpha(\xi, \theta) \geq 1$ makes $\alpha(R\xi, R\theta) \geq 1$.

Example 2 (see [1]). Consider $U = [0, +\infty)$ and define $R: U \longrightarrow U$ and $\alpha: U \times U \longrightarrow [0, +\infty)$ by $R\xi = 5\xi$ for all $\xi, \theta \in U$ and

$$\alpha(\xi, \theta) = \begin{cases} e^{(\theta/\xi)}, & \text{if } \xi \geq \theta, \xi \neq 0, \\ 0, & \text{if } \xi < \theta. \end{cases} \tag{4}$$

Then, R is α-admissible.

Definition 7 (see [4]). Let $R, S: U \longrightarrow U$ and $\alpha: U \times U \longrightarrow [0, +\infty)$. It is said that the pair (R, S) is α-admissible if $\xi, \theta \in U$ such that $\alpha(\xi, \theta) \geq 1$, then we have $\alpha(R\xi, S\theta) \geq 1$ and $\alpha(S\xi, R\theta) \geq 1$.

Definition 8 (see [5]). Let $R: U \longrightarrow U$ and $\alpha: U \times U \longrightarrow (-\infty, +\infty)$. It is said that R is a triangular α-admissible mapping if the following holds:

(T1) $\alpha(\xi, \theta) \geq 1$ makes $\alpha(R\xi, R\theta) \geq 1, \xi, \theta \in U$.

(T2) $\alpha(\xi, \phi) \geq 1$, $\alpha(\phi, \theta) \geq 1$, makes $\alpha(\xi, \theta) \geq 1, \xi, \theta, \phi \in U$.

Definition 9 (see [4]). Let $R, S: U \longrightarrow U$ and $\alpha: U \times U \longrightarrow [0, +\infty)$. It is said that a pair (R, S) is a triangular α-admissible mapping if the following holds:

(T1) $\alpha(\xi, \theta) \geq 1$ makes $\alpha(R\xi, S\theta) \geq 1$ and $\alpha(S\xi, R\theta) \geq 1, \xi, \theta \in U$.

(T2) $\alpha(\xi, \phi) \geq 1$, $\alpha(\phi, \theta) \geq 1$, makes $\alpha(\xi, \theta) \geq 1, \xi, \theta, \phi \in U$.

Definition 10 (see [6]). Let R be a self-mapping on a metric space (U, m) and let $\alpha, \eta: U \times U \longrightarrow [0, +\infty)$ be two functions. It is said that R is α-admissible mapping with respect to η if $\xi, \theta \in U$, and $\alpha(\xi, \theta) \geq \eta(\xi, \theta)$ implies $\alpha(R\xi, R\theta) \geq \eta(R\xi, R\theta)$.

It can be noted that if $\eta(\xi, \theta) = 1$, then the above definition becomes Definition 6. If we take $\alpha(\xi, \theta) = 1$, then R is said to be an η-subadmissible mapping.

Lemma 2 (see [5]). *Let $R: U \longrightarrow U$ be a triangular α-admissible mapping. Let us take $\xi_0 \in U$ such that $\alpha(\xi_0, R\xi_0) \geq 1$. Form a sequence $\{\xi_n\}$ as $\xi_{n+1} = R\xi_n$. Then, $\alpha(\xi_n, \xi_m) \geq 1$, where $m, n \in N \cup \{0\}$, $n < m$.*

Lemma 3 (see [7]). *Let $R, S: U \longrightarrow U$ be triangular α-admissible mapping. Let us take $\xi_0 \in U$ such that $\alpha(\xi_0, R\xi_0) \geq 1$. Form sequences $\xi_{2i+1} = R\xi_{2i}$ and $\xi_{2i+2} = S\xi_{2i+1}$, where $i = 0, 1, 2, \ldots$. Then, $\alpha(\xi_n, \xi_m) \geq 1$, where $m, n \in N \cup \{0\}$, $n < m$.*

Alghamdi and Karapinar [8] generalized the concept of α-admissible mappings in the context of G-metric space and called it β-admissible. The definition of β-admissible given by Alghamdi and Karapinar is defined as follows.

Definition 11 (see [8]). Let $R: U \longrightarrow U$ and $\beta: U \times U \times U \longrightarrow [0, +\infty)$, then R is said to be β-admissible if for all $\xi, \theta, \phi \in U$ then

$$\beta(\xi,\theta,\phi) \geq 1 \text{ implies } \beta(R\xi,R\theta,R\phi) \geq 1. \tag{5}$$

Alghamdi and Karapinar [8] introduced $G-\beta-\psi$ contractive mappings of type-I and type-II. They also introduced $G-\beta-\psi$ contractive mappings of type-A. They also gave the relation between these different types of $G-\beta-\psi$ contractions and equivalent Banach contractions.

Alghamdi and Karapinar [9] further generalized the results of Alghamdi and Karapinar [8] by introducing generalized $G-\beta-\psi$ contractive mappings of type-I and type-II.

Kutbi et al. [10] defined rectangular $G-\alpha$–admissible mapping. They also defined weak $\alpha-\psi-\phi$ contractive mappings to establish some coincidence point theorems for coupled and tripled in G_b-metric space.

Definition 12 (see [10]). Let (U,G) be a G-metric space and let $R,S\colon U \longrightarrow U$ and $\alpha\colon U^3 \longrightarrow [0,+\infty)$. R is said to be a rectangular $G-\alpha$–admissible mapping with respect to S if the following holds:

(i) $\alpha(S\xi,S\theta,S\phi) \geq 1$ implies $\alpha(R\xi,R\theta,R\phi) \geq 1$, $\xi,\theta,\phi \in U$.

(ii) $\alpha(S\xi,S\theta,S\theta) \geq 1$ and $\alpha(S\theta,S\phi,S\phi) \geq 1$ imply $\alpha(S\xi,S\theta,S\phi) \geq 1$, $\xi,\theta,\phi \in U$.

Hussain et al. [11] generalized the concept of rectangular $G-\alpha$–admissible mappings used to obtain coupled and tripled fixed-point theorems.

Hussain et al. [12] established a generalized form of α–admissible mappings in order to prove coincidence points and common fixed points in the framework of G-metric spaces. Furthermore, several authors obtained different kinds of generalization of Banach contraction principle in different spaces (see for details [13–20]).

Definition 13 (see [12]). Let U be an arbitrary set, $\alpha\colon U\times U\times U \longrightarrow [0,+\infty)$, and $R\colon U \longrightarrow U$. The mapping R is called an α-dominating map on U if $\alpha(\xi,R\xi,R\xi) \geq 1$ or $\alpha(\xi,\xi,R\xi) \geq 1$ for each ξ in U.

Definition 14 (see [12]). In an arbitrary set U, let $R,S\colon U \longrightarrow U$ be given mappings and $\alpha\colon U\times U\times U \longrightarrow [0,+\infty)$ be a function. The pair (R,S) is said to be partially weakly $G-\alpha$–admissible if and only if $\alpha(R\xi,SR\xi,SR\xi) \geq 1$ for all $\xi \in U$.

Definition 15 (see [12]). In an arbitrary set U, let $R,S\colon U \longrightarrow U$ be given mappings and $\alpha\colon U\times U\times U \longrightarrow [0,+\infty)$ be a function. The pair (R,S) is said to be partially weakly $G-\alpha$–admissible with respect to T if and only if for all $\xi \in U$, $\alpha(R\xi,S\theta,S\theta) \geq 1$, where $\theta \in T^{-1}(R\xi)$.

In the above definition, if $R=S$, R is said to be partially weakly $G-\alpha$–admissible (or α–admissible of rank 3) with respect to T.

If $T=I_U$ (the identity mapping on U), then the above definition becomes the definition of partially weakly $G-\alpha$–admissible pair.

Ansari et al. [21] also studied α-admissible mappings in G-metric space by introducing G-η-subadmissible mapping and α-dominating map. They also introduced η-subdominating map, α-regular in the framework of G-metric space, and partially weakly G-α-admissible and partially weakly G-η-subadmissible mappings.

Definition 16 (see [21]). Let (U,G) be a G-metric space, and let R be a self-mapping on U and $\eta\colon U\times U\times U \longrightarrow [0,+\infty)$ be a function. R is said to be a $G-\eta$–subadmissible (or η-subadmissible of rank 3) mapping if

$$\xi,\theta,\phi \in U, \eta(\xi,\theta,\phi) \leq 1, \text{ implies } \eta(R\xi,R\theta,R\phi) \leq 1. \tag{6}$$

Definition 17 (see [21]). Let U be an arbitrary set, $\eta\colon U\times U\times U \longrightarrow [0,+\infty)$, and $R\colon U \longrightarrow U$. A mapping R is called an η-subdominating map on U if $\eta(\xi,R\xi,R\xi) \leq 1$ or $\alpha(\xi,\xi,R\xi) \leq 1$ for each ξ in U.

Definition 18 (see [21]). In a G-metric space (U,G), let $R,S\colon U \longrightarrow U$ be given mappings and $\eta\colon U\times U\times U \longrightarrow [0,+\infty)$ be a function. The pair (R,S) is said to be partially weakly $G-\eta$–subadmissible (or η–subadmissible of rank 3) if and only if $\eta(R\xi,SR\xi,SR\xi) \leq 1$ for all $\xi \in U$.

Definition 19 (see [21]). In a G-metric space (U,G), let $R,S\colon U \longrightarrow U$ be given mappings and $\eta\colon U\times U\times U \longrightarrow [0,+\infty)$ be a function. The pair (R,S) is said to be partially weakly $G-\eta$–subadmissible (or η–subadmissible of rank 3) with respect to T if and only if for all $\xi \in U$, $\alpha(R\xi,S\theta,S\theta) \geq 1$, where $\theta \in T^{-1}(R\xi)$.

Hussain et al. [22] defined $G-(\alpha,\psi)$–Meir–Keeler contractive mapping and used it in proving fixed-point theorems in the framework of G-metric spaces.

Definition 20 (see [22]). Let (U,G) be a G-metric space and $\psi \in \Psi$. Let $R\colon U \longrightarrow U$ be an α-admissible mapping satisfying the following: for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\varepsilon \leq \psi(G(\xi,\theta,\phi)) < \varepsilon + \delta$ implies $\alpha(\xi,\xi)\alpha(\theta,\theta)\alpha(\phi,\phi)$ $\psi(G(R\xi,R\theta,R\phi)) < \varepsilon$ for all $\xi,\theta,\phi \in U$. Then, R is known as a $G-(\alpha,\psi)$–Meir–Keeler contractive mapping.

In the above definition, Ψ is the collection of nondecreasing functions $\psi\colon [0,+\infty) \longrightarrow [0,+\infty)$ continuous in t such that $\psi(t)=0$ if and only if $t=0$ and $\psi(t+s) \leq \psi(t)+\psi(s)$.

The concept of α-admissible mappings is extended to S-metric space by Zhou et al. [23] and called it γ-admissible. They are defined as follows.

Definition 21 (see [23]). Let $R\colon U \longrightarrow U$ and $\gamma\colon U^3 \longrightarrow [0,+\infty)$, then R is said to be γ-admissible if for all $\xi,\theta,\phi \in U$:

$$\gamma(\xi,\theta,\phi) \geq 1 \text{ implies } \gamma(R\xi,R\theta,R\phi) \geq 1. \tag{7}$$

They also extended γ-admissibility for two mappings. Furthermore, they also introduced concepts of various contractive mappings viz. type A, type B, type C, type D, and type E.

Bulbul et al. [24] also derived the concept of generalized $S-\beta-\psi$ contractive-type mappings on the line of generalized G-β-γ contractive-type mappings. Nabil et al. [25] also defined the concept of α-admissible mappings in S_b-metric space.

From these, what we observe is that β-admissible was for the first time used by Samet et al. [1] to represent α-admissible while dealing with coupled fixed point-related problems. Phiangsungnoen et al. [26] also used the name β-admissible mapping in order to represent α-admissible for fuzzy mappings. On the contrary, β-admissible of Alghamdi and Karapinar [9] and γ-admissible of Zhou et al. [23] are all extended versions of α-admissible mappings in G-metric space and S-metric space, respectively. Thus, we can remark that α-admissible and its various forms can be extended to G-metric as well as S-metric spaces and further to G_b-metric and S_b-metric spaces. With this idea, we introduce various forms of α-admissible mappings in the context of G-metric space and present following definitions. For notation, we use α_G for α-admissible mappings in G-metric space.

Definition 22. Let $R: U \longrightarrow U$ and $\alpha_G: U\times U\times U \longrightarrow [0,+\infty)$, then R is said to be α_G-admissible if $\xi,\theta,\phi \in U$, $\alpha_G(\xi,\theta,\phi)\geq 1$ implies $\alpha_G(R\xi,R\theta,R\phi)\geq 1$.

Definition 23. Let $R,S: U\longrightarrow U$ and $\alpha_G: U\times U\times U\longrightarrow [0,+\infty)$. We say that the pair (R,S) is α_G−admissible if $\xi,\theta,\phi\in U$ such that $\alpha_G(\xi,\theta,\phi)\geq 1$, then we have $\alpha_G(R\xi,S\theta,S\phi)\geq 1$ and $\alpha_G(S\xi,R\theta,R\phi)\geq 1$.

Definition 24. Let $R: U\longrightarrow U$ and $\alpha_G: U\times U\times U\longrightarrow[0,+\infty)$. We say that R is triangular α_G−admissible mapping if the following holds:

(i) $\alpha_G(\xi,\theta,\phi)\geq 1$ implies $\alpha_G(R\xi,R\theta,R\phi)\geq 1, \xi,\theta,\phi\in U$.

(ii) $\alpha_G(\xi,t,t)\geq 1$ and $\alpha_G(t,\theta,\phi)\geq 1$ implies $\alpha_G(\xi,\theta,\phi)\geq 1, \xi,\theta,\phi,t\in U$.

Definition 25. Let $R: U\longrightarrow U$ and let $\alpha_G,\eta_G: U\times U\times U\longrightarrow[0,+\infty)$ be functions. We say that R is α_G−admissible mapping with respect to η_G if $\xi,\theta,\phi\in U$,

$$\alpha_G(\xi,\theta,\phi)\geq\eta_G(\xi,\theta,\phi)\text{ implies }\alpha_G(R\xi,R\theta,R\phi)\geq\eta_G(R\xi,R\theta,R\phi). \tag{8}$$

Note that if we take $\eta_G(\xi,\theta,\phi)=1$, then this definition becomes Definition 22. Also, if we take $\alpha_G(\xi,\theta,\phi)=1$, then it is said that R is an η_G−subadmissible mapping.

Definition 26. Let $R,S: U\longrightarrow U$ and $\alpha_G,\eta_G: U\times U\times U\longrightarrow[0,+\infty)$. We say that the pair (R,S) is α_G−admissible mapping with respect to η_G if $\xi,\theta,\phi\in U$ such that $\alpha_G(\xi,\theta,\phi)\geq\eta_G(\xi,\theta,\phi)$, then we have $\alpha_G(R\xi,S\theta,S\phi)\geq\eta_G(R\xi,S\theta,S\phi)$ and $\alpha_G(S\xi,R\theta,R\phi)\geq\eta_G(S\xi,R\theta,R\phi)$.

Lemma 4. *Let $R,S: U\longrightarrow U$ are triangular α_G-admissible mappings. Suppose that there exists $\xi_0\in U$ such that $\alpha_G(\xi_0,R\xi_0,R\xi_0)\geq 1$. Define sequences*

$$\begin{aligned}\xi_{2i+1}&=R\xi_{2i},\\ \xi_{2i+2}&=S\xi_{2i+1},\quad\text{where } i=0,1,2,\ldots.\end{aligned} \tag{9}$$

Then, we have $\alpha_G(\xi_n,\xi_m,\xi_m)\geq 1$, $m,n\in\mathbb{N}\cup\{0\}$, $n<m$.

2. Main Results

Let us take $\mathscr{G}$ as the collection of functions $g: [0,+\infty)\longrightarrow[0,1)$ such that $g(t_n)\longrightarrow 1$ gives $t_n\longrightarrow 0$, where $\{t_n\}$ is a bounded sequence of positive real numbers.

We start our results with the following definitions.

Definition 27. In a G−metric space (U,G), let $\alpha_G: U\times U\times U\longrightarrow[0,+\infty)$ be a function. We say that mappings $R,S: U\longrightarrow U$ is a pair of generalized rational α_G−Geraghty contraction mappings of type-I if for all $\xi,\theta,\phi\in U$ and $g\in\mathscr{G}$,

$$\alpha_G(\xi,\theta,\phi)G(R\xi,S\theta,S\phi)\leq g(\nabla_1(\xi,\theta,\phi))\nabla_1(\xi,\theta,\phi), \tag{10}$$

where

$$\begin{aligned}&\nabla_1(\xi,\theta,\phi)\\ &=\max\left\{G(\xi,\theta,\phi),G(R\xi,S\theta,S\phi),\frac{G(\xi,R\xi,R\xi)G(\theta,S\theta,S\theta)}{1+G(\xi,\theta,\phi)+G(R\xi,S\theta,S\phi)},\frac{G(\theta,S\theta,S\theta)G(\phi,S\phi,S\phi)}{1+G(\xi,\theta,\phi)+G(R\xi,S\theta,S\phi)},\frac{G(\phi,S\phi,S\phi)G(\xi,R\xi,R\xi)}{1+G(\xi,\theta,\phi)+G(R\xi,S\theta,S\phi)}\right\}.\end{aligned} \tag{11}$$

Definition 28. In a G-metric space (U,G), let $\alpha_G: U\times U\times U\longrightarrow[0,+\infty)$ be a function. We say that the mappings $R,S: U\longrightarrow U$ are a pair of generalized rational α_G−Geraghty contraction mappings of type-II if for all $\xi,\theta\in U$ and $g\in\mathscr{G}$,

$$\alpha_G(\xi,\theta,\theta)G(R\xi,S\theta,S\theta)\leq g(\nabla_2(\xi,\theta,\theta))\nabla_2(\xi,\theta,\theta), \tag{12}$$

where

$$\nabla_2(\xi,\theta,\theta) = \max\left\{G(\xi,\theta,\theta), G(R\xi,S\theta,S\theta), \frac{G(\xi,R\xi,R\xi)G(\theta,S\theta,S\theta)}{1+G(\xi,\theta,\theta)+G(R\xi,S\theta,S\theta)}, \frac{G(\theta,S\theta,S\theta)G(\theta,S\theta,S\theta)}{1+G(\xi,\theta,\theta)+G(R\xi,S\theta,S\theta)}\right\}. \quad (13)$$

If $R = S$, then we have the following.

Definition 29. In a G–metric space (U, G), let $\alpha_G\colon U\times U\times U \longrightarrow [0,+\infty)$ be a function. We say that mapping $R\colon U \longrightarrow U$ is a generalized rational α_G–Geraghty contraction mappings of type-I if there exists $g \in \mathcal{G}$ such that for all $\xi,\theta,\phi \in U$,

$$\alpha_G(\xi,\theta,\phi)G(R\xi,R\theta,R\phi) \le g(\nabla_1(\xi,\theta,\phi))\nabla_1(\xi,\theta,\phi), \quad (14)$$

where

$$\begin{aligned}&\nabla_1(\xi,\theta,\phi)\\ &= \max\left\{G(\xi,\theta,\phi), G(R\xi,R\theta,R\phi), \frac{G(\xi,R\xi,R\xi)G(\theta,R\theta,R\theta)}{1+G(\xi,\theta,\phi)+G(R\xi,R\theta,R\phi)}, \frac{G(\theta,R\theta,R\theta)G(\phi,R\phi,R\phi)}{1+G(\xi,\theta,\phi)+G(R\xi,R\theta,R\phi)}, \frac{G(\phi,R\phi,R\phi)G(\xi,R\xi,R\xi)}{1+G(\xi,\theta,\phi)+G(R\xi,R\theta,R\phi)}\right\}.\end{aligned} \quad (15)$$

Definition 30. In a G-metric space (U, G), let $\alpha_G\colon U\times U\times U \longrightarrow [0,+\infty)$ be a function. We say that the mapping $R\colon U \longrightarrow U$ is a generalized rational α_G–Geraghty contraction mappings of type-II if there exists $g \in \mathcal{G}$ such that for all $\xi,\theta \in U$,

$$\alpha_G(\xi,\theta,\theta)G(R\xi,R\theta,R\theta) \le g(\nabla_2(\xi,\theta,\theta))\nabla_2(\xi,\theta,\theta), \quad (16)$$

where

$$\nabla_2(\xi,\theta,\theta) = \max\left\{G(\xi,\theta,\theta), G(R\xi,R\theta,R\theta), \frac{G(\xi,R\xi,R\xi)G(\theta,R\theta,R\theta)}{1+G(\xi,\theta,\theta)+G(R\xi,R\theta,R\theta)}, \frac{G(\theta,R\theta,R\theta)G(\theta,R\theta,R\theta)}{1+G(\xi,\theta,\theta)+G(R\xi,R\theta,R\theta)}\right\}. \quad (17)$$

Theorem 1. *In a complete G-metric space (U, G), let $\alpha_G\colon U\times U\times U \longrightarrow [0,+\infty)$ be a function. Let $R, S\colon U \longrightarrow U$ be two mappings satisfying the following:*

(i) R, S is pair of generalized rational α_G-Geraghty contraction mappings of type-I.

(ii) (R, S) is a pair of triangular α_G-admissible mappings.

(iii) There exists $\xi_0 \in U$ such that $\alpha_G(\xi_0, R\xi_0, R\xi_0) \ge 1$.

(iv) R and S are continuous.

Then, a common fixed point exists for the pair (R, S).

Proof. Let $\xi_1 \in U$ be such that $\xi_1 = R\xi_0$ and $\xi_2 = S\xi_1$. Inductively, we construct a sequence $\{\xi_n\}$ in U as follows:

$$\begin{aligned}\xi_{2i+1} &= R\xi_{2i},\\ \xi_{2i+2} &= S\xi_{2i+1},\end{aligned} \quad (18)$$

where $i = 0, 1, 2, 3, \ldots$.

By assumption $\alpha_G(\xi_0,\xi_1,\xi_1) \ge 1$ and the pair (R, S) is α_G-admissible, by Lemma 4, we have

$$\alpha_G(\xi_n,\xi_{n+1},\xi_{n+1}) \ge 1, \quad \text{for all } n \in \mathbb{N}\cup\{0\}. \quad (19)$$

Then,

$$\begin{aligned}G(\xi_{2i+1},\xi_{2i+2},\xi_{2i+2}) &= G(R\xi_{2i}, S\xi_{2i+1}, S\xi_{2i+1})\\ &\le \alpha_G(\xi_{2i},\xi_{2i+1},\xi_{2i+1})G(R\xi_{2i}, S\xi_{2i+1}, S\xi_{2i+1})\\ &\le g(\nabla_1(\xi_{2i},\xi_{2i+1},\xi_{2i+1}))\nabla_1(\xi_{2i},\xi_{2i+1},\xi_{2i+1}),\end{aligned} \quad (20)$$

for all $i \in \mathbb{N}\cup\{0\}$.

Now,

$$\begin{aligned}
&\nabla_1(\xi_{2i},\xi_{2i+1},\xi_{2i+1})\\
&=\max\Bigg\{G(\xi_{2i},\xi_{2i+1},\xi_{2i+1}),G(R\xi_{2i},S\xi_{2i+1},S\xi_{2i+1}),\frac{G(\xi_{2i},R\xi_{2i},R\xi_{2i})G(\xi_{2i+1},S\xi_{2i+1},S\xi_{2i+1})}{1+G(\xi_{2i},\xi_{2i+1},\xi_{2i+1})+G(R\xi_{2i},S\xi_{2i+1},S\xi_{2i+1})},\frac{G(\xi_{2i+1},S\xi_{2i+1},S\xi_{2i+1})G(\xi_{2i+1},S\xi_{2i+1},S\xi_{2i+1})}{1+G(\xi_{2i},\xi_{2i+1},\xi_{2i+1})+G(R\xi_{2i},S\xi_{2i+1},S\xi_{2i+1})},\\
&\quad\frac{G(\xi_{2i+1},S\xi_{2i+1},S\xi_{2i+1})G(\xi_{2i},R\xi_{2i},R\xi_{2i})}{1+G(\xi_{2i},\xi_{2i+1},\xi_{2i+1})+G(R\xi_{2i},S\xi_{2i+1},S\xi_{2i+1})}\Bigg\}\\
&=\max\Bigg\{G(\xi_{2i},\xi_{2i+1},\xi_{2i+1}),G(\xi_{2i+1},\xi_{2i+2},\xi_{2i+2}),\frac{G(\xi_{2i},\xi_{2i+1},\xi_{2i+1})G(\xi_{2i},\xi_{2i+1},\xi_{2i+1})}{1+G(\xi_{2i},\xi_{2i+1},\xi_{2i+1})+G(\xi_{2i+1},\xi_{2i+2},\xi_{2i+2})},\frac{G(\xi_{2i+1},\xi_{2i+2},\xi_{2i+2})G(\xi_{2i+1},\xi_{2i+2},\xi_{2i+2})}{1+G(\xi_{2i},\xi_{2i+1},\xi_{2i+1})+G(\xi_{2i+1},\xi_{2i+2},\xi_{2i+2})},\\
&\quad\frac{G(\xi_{2i+1},\xi_{2i+2},\xi_{2i+2})G(\xi_{2i},\xi_{2i+1},\xi_{2i+1})}{1+G(\xi_{2i},\xi_{2i+1},\xi_{2i+1})+G(\xi_{2i+1},\xi_{2i+2},\xi_{2i+2})}\Bigg\}\\
&=\max\{G(\xi_{2i},\xi_{2i+1},\xi_{2i+1}),G(\xi_{2i+1},\xi_{2i+2},\xi_{2i+2})\}.
\end{aligned}\tag{21}$$

If $\max\{G(\xi_{2i},\xi_{2i+1},\xi_{2i+1}),G(\xi_{2i+1},\xi_{2i+2},\xi_{2i+2})\}=G(\xi_{2i+1},\xi_{2i+2},\xi_{2i+2})$, then

$$\begin{aligned}G(\xi_{2i+1},\xi_{2i+2},\xi_{2i+2})&\leq g(G(\xi_{2i+1},\xi_{2i+2},\xi_{2i+2}))G(\xi_{2i+1},\xi_{2i+2},\xi_{2i+2})\\&<G(\xi_{2i+1},\xi_{2i+2},\xi_{2i+2}),\end{aligned}\tag{22}$$

which is a contradiction. Hence,

$$G(\xi_{2i+1},\xi_{2i+2},\xi_{2i+2})<G(\xi_{2i},\xi_{2i+1},\xi_{2i+1}).\tag{23}$$

This implies that

$$G(\xi_{n+1},\xi_{n+2},\xi_{n+2})<G(\xi_{n},\xi_{n+1},\xi_{n+1}),\tag{24}$$

for all $n\in\mathbb{N}\cup\{0\}$.

So the sequence $\{G(\xi_n,\xi_{n+1},\xi_{n+1})\}$ is nonnegative and nonincreasing. Now we prove that $G(\xi_n,\xi_{n+1},\xi_{n+1})\longrightarrow 0$. It is clear that $\{G(\xi_n,\xi_{n+1},\xi_{n+1})\}$ is a decreasing sequence. So, for some $r>0$, we have $\lim_{n\longrightarrow+\infty}G(\xi_n,\xi_{n+1},\xi_{n+1})=r$.

From (23),

$$\frac{G(\xi_{n+1},\xi_{n+2},\xi_{n+2})}{G(\xi_{n},\xi_{n+1},\xi_{n+1})}\leq g(G(\xi_n,\xi_{n+1},\xi_{n+1}))\leq 1.\tag{25}$$

Now, by taking limit as $n\longrightarrow+\infty$, we have

$$1\leq\lim_{n\longrightarrow+\infty}g(G(\xi_n,\xi_{n+1},\xi_{n+1}))<1,\tag{26}$$

that is,

$$\lim_{n\longrightarrow+\infty}g(G(\xi_n,\xi_{n+1},\xi_{n+1}))=1.\tag{27}$$

By the property of g, we have

$$\lim_{n\longrightarrow+\infty}G(\xi_n,\xi_{n+1},\xi_{n+1})=0.\tag{28}$$

We have to show that $\{\xi_n\}$ is a Cauchy sequence. If possible, let $\{\xi_n\}$ is not a Cauchy sequence. Then, there exist $\varepsilon>0$ and sequences $\{\xi_{m_k}\}$ and $\{\xi_{n_k}\}$ such that, for all positive integers k, we get $m_k>n_k>k$:

$$\begin{aligned}&G(\xi_{n_k},\xi_{m_k},\xi_{m_k})\geq\varepsilon,\\&G(\xi_{n_k},\xi_{m_k-1},\xi_{m_k-1})<\varepsilon.\end{aligned}\tag{29}$$

Therefore,

$$\begin{aligned}\varepsilon&\leq G(\xi_{n_k},\xi_{m_k},\xi_{m_k})\\&\leq G(\xi_{n_k},\xi_{m_k-1},\xi_{m_k-1})+G(\xi_{m_k-1},\xi_{m_k},\xi_{m_k})\\&<\varepsilon+G(\xi_{m_k-1},\xi_{m_k},\xi_{m_k}).\end{aligned}\tag{30}$$

Taking $k\longrightarrow+\infty$,

$$\varepsilon\leq\lim_{k\longrightarrow+\infty}\sup G(\xi_{n_k},\xi_{m_k},\xi_{m_k})<\varepsilon.\tag{31}$$

Therefore,

$$\lim_{k\longrightarrow+\infty}G(\xi_{n_k},\xi_{m_k},\xi_{m_k})=\varepsilon.\tag{32}$$

Also, from the triangular inequality, we have

$$\begin{aligned}G(\xi_{n_k},\xi_{m_k},\xi_{m_k})&\leq G(\xi_{n_k},\xi_{n_k+1},\xi_{n_k+1})+G(\xi_{n_k+1},\xi_{m_k},\xi_{m_k}),\\G(\xi_{n_k+1},\xi_{m_k},\xi_{m_k})&\leq G(\xi_{n_k+1},\xi_{n_k},\xi_{n_k})+G(\xi_{n_k},\xi_{m_k},\xi_{m_k})\\&\leq G(\xi_{n_k+1},\xi_{n_k+1},\xi_{n_k})+G(\xi_{n_k+1},\xi_{n_k+1},\xi_{n_k})\\&\quad+G(\xi_{n_k},\xi_{m_k},\xi_{m_k}).\end{aligned}\tag{33}$$

Taking upper limit as $k\longrightarrow+\infty$ above, we obtain

$$\begin{aligned}&\varepsilon\leq\lim_{k\longrightarrow+\infty}\sup G(\xi_{n_k},\xi_{m_k},\xi_{m_k})\leq\lim_{k\longrightarrow+\infty}\sup G(\xi_{n_k+1},\xi_{m_k},\xi_{m_k}),\\&\lim_{k\longrightarrow+\infty}\sup G(\xi_{n_k+1},\xi_{m_k},\xi_{m_k})\leq\lim_{k\longrightarrow+\infty}G(\xi_{n_k},\xi_{m_k},\xi_{m_k})\leq\varepsilon.\end{aligned}\tag{34}$$

Thus,

$$\begin{aligned}&\lim_{k\longrightarrow+\infty}\sup G(\xi_{n_k+1},\xi_{m_k},\xi_{m_k})=\varepsilon,\\&\lim_{k\longrightarrow+\infty}\sup G(\xi_{n_k},\xi_{m_k+1},\xi_{m_k+1})=\varepsilon.\end{aligned}\tag{35}$$

By triangle inequality, we have

$$G\left(\xi_{n_k+1},\xi_{m_k},\xi_{m_k}\right)\leq G\left(\xi_{n_k+1},\xi_{m_k+1},\xi_{m_k+1}\right)+G\left(\xi_{m_k+1},\xi_{m_k},\xi_{m_k}\right). \tag{36}$$

Taking limit as $k\longrightarrow+\infty$, we have

$$\varepsilon\leq\lim_{k\longrightarrow+\infty}\sup G\left(\xi_{n_k+1},\xi_{m_k+1},\xi_{m_k+1}\right). \tag{37}$$

Following the above process, we have

$$\lim_{k\longrightarrow+\infty}\sup G\left(\xi_{n_k+1},\xi_{m_k+1},\xi_{m_k+1}\right)\leq\varepsilon. \tag{38}$$

Combining, we have

$$\lim_{k\longrightarrow+\infty}G\left(\xi_{n_k+1},\xi_{m_k+1},\xi_{m_k+1}\right)=\varepsilon. \tag{39}$$

By Lemma 4, $\alpha_G(\xi_{n_k},\xi_{m_k},\xi_{m_k})\geq 1$, we have

$$\begin{aligned}G\left(\xi_{n_k+1},\xi_{m_k+1},\xi_{m_k+1}\right)&=G\left(R\xi_{n_k},S\xi_{m_k},S\xi_{m_k}\right)\\&\leq\alpha_G\left(\xi_{n_k},\xi_{m_k},\xi_{m_k}\right)G\left(R\xi_{n_k},S\xi_{m_k},S\xi_{m_k}\right)\\&\leq g\left(\nabla_1\left(\xi_{n_k},\xi_{m_k},\xi_{m_k}\right)\right)\nabla_1\left(\xi_{n_k},\xi_{m_k},\xi_{m_k}\right).\end{aligned} \tag{40}$$

Finally, we conclude that

$$\frac{G\left(\xi_{n_k+1},\xi_{m_k+1},\xi_{m_k+1}\right)}{\nabla_1\left(\xi_{n_k},\xi_{m_k},\xi_{m_k}\right)}\leq g\left(\nabla_1\left(\xi_{n_k},\xi_{m_k},\xi_{m_k}\right)\right). \tag{41}$$

Applying $k\longrightarrow+\infty$, we obtain

$$\lim_{k\longrightarrow+\infty}g\left(G\left(\xi_{n_k},\xi_{m_k},\xi_{m_k}\right)\right)=1. \tag{42}$$

So,

$$\lim_{k\longrightarrow+\infty}G\left(\xi_{n_k},\xi_{m_k},\xi_{m_k}\right)=0<\varepsilon, \tag{43}$$

a contradiction. Thus, $\{\xi_n\}$ is a Cauchy sequence. By completeness of U, there exists $a\in U$ such that $\xi_n\longrightarrow a$ implies that $\xi_{2i+1}\longrightarrow a$ and $\xi_{2i+2}\longrightarrow a$. As R and S are continuous, we get $S\xi_{2i+1}\longrightarrow Sa$ and $R\xi_{2i+2}\longrightarrow Ra$. Thus, $a=Sa$. Similarly, $a=Ra$, and we have $Ra=Sa=a$. Then, (R,S) have common fixed point.

In the next theorem, we dropped the continuity condition. □

Theorem 2. *In a complete G-metric space (U,G), let $\alpha_G\colon U\times U\times U\longrightarrow\mathbb{R}$ be a function. Let $R,S\colon U\longrightarrow U$ be two mappings satisfying the following:*

(i) (R,S) is a pair of generalized rational α_G-Geraghty contraction mappings of type-I.

(ii) (R,S) is a pair of triangular α_G-admissible mappings.

(iii) There exists $\xi_0\in U$ such that $\alpha_G(\xi_0,R\xi_0,R\xi_0)\geq 1$.

(iv) If $\{\xi_n\}$ is a sequence in U such that $\alpha_G(\xi_n,\xi_{n+1},\xi_{n+1})\geq 1$ for all $n\in\mathbb{N}\cup\{0\}$ and $\xi_n\longrightarrow a\in U$ as $n\longrightarrow+\infty$, then a subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ exists satisfying $\alpha_G(\xi_{n_k},a,a)\geq 1$ for all k.

Then, (R,S) have a common fixed point.

Proof. It follows the similar lines of Theorem 1. Define a sequence $\xi_{2i+1}=R\xi_{2i}$ and $\xi_{2i+2}=S\xi_{2i+1}$, where $i=0,1,2,\ldots$ converges to $a\in U$. By the hypothesis of (iv), a subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ exists satisfying $\alpha_G(\xi_{2n_k},a,a)\geq 1$ for all k. Now, we have

$$\begin{aligned}G\left(\xi_{2n_k+1},Sa,Sa\right)&=G\left(R\xi_{2n_k},Sa,Sa\right)\\&\leq\alpha_G\left(\xi_{2n_k},a,a\right)G\left(R\xi_{2n_k},Sa,Sa\right)\\&\leq g\left(\nabla_1\left(\xi_{2n_k},a,a\right)\right)\nabla_1\left(\xi_{2n_k},a,a\right),\end{aligned} \tag{44}$$

so that

$$G\left(\xi_{2n_k+1},Sa,Sa\right)\leq g\left(\nabla_1\left(\xi_{2n_k},a,a\right)\right)\nabla_1\left(\xi_{2n_k},a,a\right). \tag{45}$$

On the contrary, we obtain

$$\begin{aligned}&\nabla_1\left(\xi_{2n_k},a,a\right)\\&=\max\left\{G\left(\xi_{2n_k},a,a\right),G\left(R\xi_{2n_k},Sa,Sa\right),\frac{G\left(\xi_{2n_k},R\xi_{2n_k},R\xi_{2n_k}\right)G(a,Sa,Sa)}{1+G\left(\xi_{2n_k},a,a\right)+G\left(R\xi_{2n_k},Sa,Sa\right)},\frac{G(a,Sa,Sa)G(a,Sa,Sa)}{1+G\left(\xi_{2n_k},a,a\right)+G\left(R\xi_{2n_k},Sa,Sa\right)},\right.\\&\left.\frac{G(a,Sa,Sa)G\left(\xi_{2n_k},R\xi_{2n_k},R\xi_{2n_k}\right)}{1+G\left(\xi_{2n_k},a,a\right)+G\left(R\xi_{2n_k},Sa,Sa\right)}\right\}\\&=\max\left\{G\left(\xi_{2n_k},a,a\right),G\left(\xi_{2n_k+1},Sa,Sa\right),\frac{G\left(\xi_{2n_k},\xi_{2n_k+1},\xi_{2n_k+1}\right)G(a,Sa,Sa)}{1+G\left(\xi_{2n_k},a,a\right)+G\left(\xi_{2n_k+1},Sa,Sa\right)},\frac{G(a,Sa,Sa)G(a,Sa,Sa)}{1+G\left(\xi_{2n_k},a,a\right)+G\left(\xi_{2n_k+1},Sa,Sa\right)},\right.\\&\left.\frac{G(a,Sa,Sa)G\left(\xi_{2n_k},\xi_{2n_k+1},\xi_{2n_k+1}\right)}{1+G\left(\xi_{2n_k},a,a\right)+G\left(\xi_{2n_k+1},Sa,Sa\right)}\right\}.\end{aligned} \tag{46}$$

Letting $k \longrightarrow +\infty$, then we have

$$\lim_{k\longrightarrow+\infty} \nabla_1\left(\xi_{2n_k}, a, a\right) = G(a, Sa, Sa). \tag{47}$$

Suppose that $G(a, Sa, Sa) > 0$. From (47), for a large k, we have $\nabla_1(\xi_{2n_k}, a, a) > 0$, which implies that

$$g\left(\nabla_1\left(\xi_{2n_k}, a, a\right)\right) < 1. \tag{48}$$

Then, we have

$$G\left(\xi_{2n_k+1}, Sa, Sa\right) < \nabla_1\left(\xi_{2n_k}, a, a\right). \tag{49}$$

Letting $k \longrightarrow +\infty$ in (49), we claim that

$$G(a, Sa, Sa) < G(a, Sa, Sa), \tag{50}$$

which is a contradiction. Thus, we find that $G(a, Sa, Sa) = 0$ implies $a = Sa$.

Also $a = Ra$ showing that a in U is a common fixed point of R and S. □

3. Consequences

If

$$\nabla_1(\xi,\theta,\phi) = \max\left\{G(\xi,\theta,\phi), G(R\xi,S\theta,S\phi), \frac{G(\xi,R\xi,R\xi)G(\theta,S\theta,S\theta)}{1+G(\xi,\theta,\phi)+G(R\xi,S\theta,S\phi)}, \frac{G(\theta,S\theta,S\theta)G(\phi,S\phi,S\phi)}{1+G(\xi,\theta,\phi)+G(R\xi,S\theta,S\phi)}, \frac{G(\phi,S\phi,S\phi)G(\xi,R\xi,R\xi)}{1+G(\xi,\theta,\phi)+G(R\xi,S\theta,S\phi)}\right\}, \tag{51}$$

and $R = S$ in Theorems 1 and 2, we have the following corollaries.

Corollary 1. *In a complete G-metric space (U, G), let R be α_G-admissible mapping satisfying the following:*

(i) R is generalized rational α_G-Geraghty contraction mappings of type-I.

(ii) R is triangular α_G-admissible.

(iii) There exists $\xi_0 \in U$ such that $\alpha_G(\xi_0, R\xi_0, R\xi_0) \geq 1$.

(iv) R is continuous.

Then, R has a fixed point $a \in U$, and R is a Picard operator; that is, $\{R^n\xi_0\}$ converges to a.

Corollary 2. *In a complete G-metric space (U, G), let R be α_G-admissible mapping satisfying the following:*

(i) R is a generalized rational α_G-Geraghty contraction mappings of type-I.

(ii) R is triangular α_G-admissible.

(iii) There exists $\xi_0 \in U$ such that $\alpha_G(\xi_0, R\xi_0, R\xi_0) \geq 1$.

(iv) If $\{\xi_n\}$ is a sequence in U such that $\alpha(\xi_n, \xi_{n+1}, \xi_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\xi_n \longrightarrow a \in U$ as $n \longrightarrow +\infty$, then a subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ exists satisfying $\alpha(\xi_{n_k}, a, a) \geq 1$ for all k.

Then, R has a fixed point $a \in U$ and R is a Picard operator; that is, $\{R^n\xi_0\}$ converges to a.

If

$$\nabla_1(\xi,\theta,\phi) = \max\{G(\xi,\theta,\phi), G(\xi,R\xi,R\xi), G(\theta,S\theta,S\theta), G(\phi,S\phi,S\phi)\} \tag{52}$$

in Theorems 1 and 2, we can have another result.

Let (U, G) be a G-metric space and let $\alpha_G, \eta_G\colon U \times U \times U \longrightarrow [0, +\infty)$ be functions. Mappings $R, S\colon U \longrightarrow U$ are called a pair of generalized rational α_G-Geraghty contraction-type mappings with respect to η_G if there exists $g \in \mathcal{G}$ such that for all $\xi, \theta, \phi \in U$:

$$\alpha_G(\xi,\theta,\phi) \geq \eta_G(\xi,\theta,\phi) \Longrightarrow G(R\xi,S\theta,S\phi) \leq g(\nabla_1(\xi,\theta,\phi))\nabla_1(\xi,\theta,\phi), \tag{53}$$

where

$$\nabla_1(\xi,\theta,\phi) = \max\left\{G(\xi,\theta,\phi), G(R\xi,S\theta,S\phi), \frac{G(\xi,R\xi,R\xi)G(\theta,S\theta,S\theta)}{1+G(\xi,\theta,\phi)+G(R\xi,S\theta,S\phi)}, \frac{G(\theta,S\theta,S\theta)G(\phi,S\phi,S\phi)}{1+G(\xi,\theta,\phi)+G(R\xi,S\theta,S\phi)}, \frac{G(\phi,S\phi,S\phi)G(\xi,R\xi,R\xi)}{1+G(\xi,\theta,\phi)+G(R\xi,S\theta,S\phi)}\right\}. \tag{54}$$

Theorem 3. *In a complete G-metric space (U, G), let R be α_G-admissible mapping with respect to η_G satisfying the following:*

(i) (R, S) is a pair of a generalized rational α_G-Geraghty contraction type mapping.

(ii) (R, S) is a pair of triangular α_G-admissible mappings.

(iii) There exists $\xi_0 \in U$ such that $\alpha_G(\xi_0, R\xi_0, R\xi_0) \geq \eta_G(\xi_0, R\xi_0, R\xi_0)$.

(iv) R and S are continuous.

Then, (R, S) have common fixed point.

Proof. Let $\xi_1 \in U$ be such that $\xi_1 = R\xi_0$ and $\xi_2 = S\xi_1$. Inductively, we form a sequence $\{\xi_n\}$ in U as follows:

$$\begin{aligned}\xi_{2i+1} &= R\xi_{2i},\\ \xi_{2i+2} &= S\xi_{2i+1},\end{aligned} \tag{55}$$

where $i = 0, 1, 2, 3, \ldots$.

By assumption $\alpha_G(\xi_0, \xi_1, \xi_1) \geq \eta_G(\xi_0, \xi_1, \xi_1)$ and the pair (R, S) is α_G-admissible with respect to η_G, we have $\alpha_G(R\xi_0, S\xi_1, S\xi_1) \geq \eta_G(R\xi_0, S\xi_1, S\xi_1)$ from which we deduce that $\alpha_G(\xi_1, \xi_2, \xi_2) \geq \eta_G(\xi_1, \xi_2, \xi_2)$ which also implies that $\alpha_G(S\xi_1, R\xi_2, R\xi_2) \geq \eta_G(S\xi_1, R\xi_2, R\xi_2)$. Continuing in this way, we obtain $\alpha_G(\xi_n, \xi_{n+1}, \xi_{n+1}) \geq \eta_G(\xi_n, \xi_{n+1}, \xi_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$:

$$\begin{aligned}G(\xi_{2i+1}, \xi_{2i+2}, \xi_{2i+2}) &= G(R\xi_{2i}, S\xi_{2i+1}, S\xi_{2i+1})\\ &\leq g(\nabla_1(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}))\nabla_1(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}).\end{aligned} \tag{56}$$

Therefore,

$$G(\xi_{2i+1}, \xi_{2i+2}, \xi_{2i+2}) \leq \alpha_G(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}) G(R\xi_{2i}, S\xi_{2i+1}, S\xi_{2i+1}), \tag{57}$$

for all $i \in \mathbb{N} \cup \{0\}$.

Now,

$$\begin{aligned}\nabla_1(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}) &= \max\Bigg\{G(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}), G(R\xi_{2i}, S\xi_{2i+1}, S\xi_{2i+1}), \frac{G(\xi_{2i}, R\xi_{2i}, R\xi_{2i})G(\xi_{2i+1}, S\xi_{2i+1}, S\xi_{2i+1})}{1 + G(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}) + G(R\xi_{2i}, S\xi_{2i+1}, S\xi_{2i+1})},\\ &\quad \frac{G(\xi_{2i+1}, S\xi_{2i+1}, S\xi_{2i+1})G(\xi_{2i+1}, S\xi_{2i+1}, S\xi_{2i+1})}{1 + G(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}) + G(R\xi_{2i}, S\xi_{2i+1}, S\xi_{2i+1})}, \frac{G(\xi_{2i+1}, S\xi_{2i+1}, S\xi_{2i+1})G(\xi_{2i}, R\xi_{2i}, R\xi_{2i})}{1 + G(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}) + G(R\xi_{2i}, S\xi_{2i+1}, S\xi_{2i+1})}\Bigg\}\\ &= \max\Bigg\{G(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}), G(\xi_{2i+1}, \xi_{2i+2}, \xi_{2i+2}), \frac{G(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1})G(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1})}{1 + G(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}) + G(\xi_{2i+1}, \xi_{2i+2}, \xi_{2i+2})},\\ &\quad \frac{G(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1})G(\xi_{2i+1}, \xi_{2i+2}, \xi_{2i+2})}{1 + G(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}) + G(\xi_{2i+1}, \xi_{2i+2}, \xi_{2i+2})}, \frac{G(\xi_{2i+1}, \xi_{2i+2}, \xi_{2i+2})G(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1})}{1 + G(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}) + G(\xi_{2i+1}, \xi_{2i+2}, \xi_{2i+2})}\Bigg\}\\ &= \max\{G(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}), G(\xi_{2i+1}, \xi_{2i+2}, \xi_{2i+2})\}.\end{aligned} \tag{58}$$

From the definition of g, the case $\nabla_1(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}) = G(\xi_{2i+1}, \xi_{2i+2}, \xi_{2i+2})$ is impossible.

$$\begin{aligned}G(\xi_{2i+1}, \xi_{2i+2}, \xi_{2i+2}) &\leq g(\nabla_1(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}))\nabla_1(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1})\\ &\leq g(G(\xi_{2i+1}, \xi_{2i+2}, \xi_{2i+2}))G(\xi_{2i+1}, \xi_{2i+2}, \xi_{2i+2})\\ &< G(\xi_{2i+1}, \xi_{2i+2}, \xi_{2i+2}),\end{aligned} \tag{59}$$

which is a contradiction. Otherwise, in other case,

$$\begin{aligned}G(\xi_{2i+1}, \xi_{2i+2}, \xi_{2i+2}) &\leq g(\nabla_1(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}))\nabla_1(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1})\\ &\leq g(G(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}))G(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1})\\ &< G(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}).\end{aligned} \tag{60}$$

This implies that

$$G(\xi_{n+1}, \xi_{n+2}, \xi_{n+2}) < G(\xi_n, \xi_{n+1}, \xi_{n+1}), \tag{61}$$

for all $n \in \mathbb{N} \cup \{0\}$.

Following the similar lines of the Theorem 1, we can prove that R and S have a common fixed point. □

Theorem 4. *In a complete G-metric space (U, G), let (R, S) be a pair of α_G-admissible mappings with respect to η_G satisfying the following:*

(i) The pair (R, S) is a generalized rational α_G-Geraghty contraction type mappings.

(ii) The pair (R, S) is triangular α_G-admissible.

(iii) There exists $\xi_0 \in U$ such that $\alpha_G(\xi_0, R\xi_0, R\xi_0) \geq \eta_G(\xi_0, R\xi_0, R\xi_0)$.

(iv) If $\{\xi_n\}$ is a sequence in U such that $\alpha_G(\xi_n, \xi_{n+1}, \xi_{n+1}) \geq \eta_G(\xi_n, \xi_{n+1}, \xi_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$ and $\xi_n \longrightarrow a \in U$ as $n \longrightarrow +\infty$, then a subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ exists satisfying $\alpha_G(\xi_{n_k}, a, a) \geq \eta_G(\xi_{n_k}, a, a)$ for all k.

Then, R and S have common fixed point.

Proof. It follows the similar line of Theorem 2.

If

$$\nabla_1(\xi,\theta,\phi) = \max\left\{G(\xi,\theta,\phi), G(R\xi,R\theta,R\phi), \frac{G(\xi,R\xi,R\xi)G(\theta,R\theta,R\theta)}{1+G(\xi,\theta,\phi)+G(R\xi,R\theta,R\phi)}, \frac{G(\theta,R\theta,R\theta)G(\phi,R\phi,R\phi)}{1+G(\xi,\theta,\phi)+G(R\xi,R\theta,R\phi)}, \frac{G(\phi,R\phi,R\phi)G(\xi,R\xi,R\xi)}{1+G(\xi,\theta,\phi)+G(R\xi,R\theta,R\phi)}\right\}, \tag{62}$$

and $R = S$ in Theorems 3 and 4, we get the following corollaries. □

Corollary 3. *In a complete G-metric space (U,G), let R be α_G-admissible mappings with respect to η_G satisfying the following:*

(i) R is a generalized rational α_G-Geraghty contraction type mapping.
(ii) R is triangular α_G-admissible.
(iii) There exists $\xi_0 \in U$ such that $\alpha_G(\xi_0, R\xi_0, R\xi_0) \geq \eta_G(\xi_0, R\xi_0, R\xi_0)$.
(iv) R is continuous.

Then, R has a fixed point $a \in U$ and R be a Picard operator; that is, $\{R^n\xi_0\}$ converges to a.

Corollary 4. *In a complete G-metric space (U,G), let R be α_G-admissible mapping with respect to η_G satisfying the following:*

(i) R is a generalized rational α_G-Geraghty contraction type mapping.
(ii) R is triangular α_G-admissible.
(iii) There exists $\xi_0 \in U$ such that $\alpha_G(\xi_0, R\xi_0, R\xi_0) \geq \eta_G(\xi_0, R\xi_0, R\xi_0)$.
(iv) There exists $\xi_0 \in U$ such that $\alpha_G(\xi_n, R\xi_{n+1}, R\xi_{n+1}) \geq \eta_G(\xi_n, R\xi_{n+1}, R\xi_{n+1})$ for all $n \in \mathbb{N}\cup\{0\}$ and $\xi_n \longrightarrow a \in U$ as $n \longrightarrow +\infty$, then a subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ exists satisfying $\alpha_G(\xi_{n_k}, a, a) \geq \eta_G(\xi_{n_k}, a, a)$ for all k.

Then, R has a fixed point $a \in U$, and R is a Picard operator; that is, $\{R^n\xi_0\}$ converges to a.

Example 3. Let $U = \{1,2,3\}$ with G-metric, then $G(1,3,3) = G(3,1,1) = (5/7)$, $G(1,1,1) = G(2,2,2) = G(3,3,3) = 0$, $G(1,2,2) = G(2,1,1) = 1$, $G(2,3,3) = G(3,2,2) = (4/7)$, and

$$\alpha_G(\xi,\theta,\phi) = \begin{cases} 1, & \text{if } \xi,\theta,\phi \in U, \\ 0, & \text{otherwise.} \end{cases} \tag{63}$$

Define the mappings $R, S\colon U \longrightarrow U$ as follows $R\xi = 1$ for each $\xi \in U$, $S(1) = S(3) = 1$, $S(2) = 3$, and $g\colon [0,+\infty) \longrightarrow [0,1)$, then

$$\alpha_G(\xi,\theta,\phi)G(S\xi,S\theta,S\phi) \leq g(\nabla_1(\xi,\theta,\phi))\nabla_1(\xi,\theta,\phi). \tag{64}$$

Let $\xi = 2$, $\theta = 3$, $\phi = 3$, then condition (i) is not satisfied by the mapping S as $G(S2,S3,S3) = G(3,1,1) = (5/7)$, where

$$\begin{aligned}
\nabla_1(\xi,\theta,\phi) &= \max\left\{G(2,3,3), G(S2,S3,S3), \frac{G(2,S2,S2)G(3,S3,S3)}{1+G(2,3,3)+G(S2,S3,S3)}, \frac{G(3,S3,S3)G(3,S3,S3)}{1+G(2,3,3)+G(S2,S3,S3)}\right\} \\
&= \max\left\{G(2,3,3), G(3,1,1), \frac{G(2,3,3)G(3,1,1)}{1+G(2,3,3)+G(3,1,1)}, \frac{G(3,1,1)G(3,1,1)}{1+G(2,3,3)+G(3,1,1)}\right\} \\
&= \max\left\{\frac{4}{7}, \frac{5}{7}, \frac{5}{28}, \frac{25}{112}\right\} \\
&= \frac{5}{7}.
\end{aligned} \tag{65}$$

Thus, $\alpha_G(\xi,\theta,\phi)G(S\xi,S\theta,S\phi) \leq g(\nabla_1(\xi,\theta,\phi))\nabla_1(\xi,\theta,\phi)$ is not true.

We prove that Theorem 1 can be applied to R and S. Let $\xi,\theta,\phi \in U$; clearly, (R,S) is α_G-admissible such that

$\alpha_G(\xi,\theta,\phi)\geq 1$. Let $\xi,\theta,\phi\in U$ so that $R\xi,S\theta,S\phi\in U$ and $\alpha_G(R\xi,S\theta,S\phi)=1$. Hence, (R,S) is α_G-admissible. We know that condition (i) of Theorem 1 is satisfied.

If $\xi,\theta,\phi\in U$, then $\alpha_G(\xi,\theta,\phi)=1$, and we have

$$\alpha_G(\xi,\theta,\phi)G(R\xi,S\theta,S\phi)\leq g(\nabla_1(\xi,\theta,\phi))\nabla_1(\xi,\theta,\phi), \quad (66)$$

where

$$\begin{aligned}\nabla_1(\xi,\theta,\phi) &= \max\left\{G(2,3,3),G(R2,S3,S3),\frac{G(2,R2,R2)G(3,S3,S3)}{1+G(2,3,3)+G(R2,S3,S3)},\frac{G(3,S3,S3)G(3,S3,S3)}{1+G(2,3,3)+G(R2,S3,S3)}\right\}\\ &= \max\left\{G(2,3,3),G(1,1,1),\frac{G(2,1,1)G(3,1,1)}{1+G(2,3,3)+G(1,1,1)},\frac{G(3,1,1)G(3,1,1)}{1+G(2,3,3)+G(1,1,1)}\right\}\\ &= \max\left\{\frac{4}{7},0,\frac{5}{11},\frac{25}{77}\right\}\\ &= \frac{4}{7},\end{aligned} \quad (67)$$

and $G(R2,S3,S3)=G(1,1,1)=0$.

$$\alpha_G(\xi,\theta,\phi)G(R\xi,S\theta,S\phi)\leq g(\nabla_1(\xi,\theta,\phi))\nabla_1(\xi,\theta,\phi). \quad (68)$$

Hence, the conditions of Theorem 1 are satisfied. So, R and S have a common fixed point.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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