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Research Article

An Inertial Method for Split Common Fixed Point Problems in Hilbert Spaces

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In this paper, we consider the split common fixed point problem in Hilbert spaces. By using the inertial technique, we propose a new algorithm for solving the problem. Under some mild conditions, we establish two weak convergence theorems of the proposed algorithm. Moreover, the stepsize in our algorithm is independent of the norm of the given linear mapping, which can further improve the performance of the algorithm.

1. Introduction

In recent years, there has been growing interest in the study of the split common fixed point problem because of its various applications in signal processing and image reconstruction [1–3]. More specifically, the problem consists in finding $\overline{x} \in H_1$ satisfying

$$\begin{aligned} &\overline{x} \in F(U), \\ &A\overline{x} \in F(T), \end{aligned} \tag{1}$$

where $F(U)$ and $F(T)$ stand for the fixed point sets of mappings $U\colon H_1 \longrightarrow H_1$ and $T\colon H_2 \longrightarrow H_2$, respectively, and $A\colon H_1 \longrightarrow H_2$ is a bounded linear mapping. Here, H_1 and H_2 are two Hilbert spaces. In particular, if we let the mappings in (1) be the projections, then it is reduced to the well-known split feasibility problem (SFP): find $\overline{x} \in H_1$ such that

$$\overline{x} \in C, A\overline{x} \in Q, \tag{2}$$

where $C \subseteq H_1$ and $Q \subseteq H_2$ are two nonempty closed convex subsets and $A\colon H_1 \longrightarrow H_2$ is a bounded linear mapping; see, e.g., [1, 4–7].

There are several algorithms for solving the split common fixed point problem. Among them, Censor and Segal [8] introduced an algorithm as

$$x^{k+1} = U\left(x^k - \tau A^* (I - T)Ax^k\right), \tag{3}$$

where I stands for the identity mapping, A^* is the adjoint mapping of A, and the stepsize τ is a constant in $(0, 2\|A\|^{-2})$. In particular, when $U = P_C$ and $T = P_Q$, then the above algorithm is reduced to the well-known CQ algorithm for solving the split feasibility problem [4]. Note that this choice of the stepsize requires the exact value or estimation of the norm $\|A\|$. To avoid the calculation of $\|A\|$, Cui and Wang [9] proposed a variable stepsize as

$$\tau_k = \frac{\left\|(I - T)Ax^k\right\|^2}{\left\|A^* (I - T)Ax^k\right\|^2}. \tag{4}$$

It is readily seen that the above choice of the stepsize does not need any prior knowledge of the linear operator. Recently, Wang [10] introduced a new method for solving (1) as

$$x^{k+1} = x^k - \tau_k\left[(I-U)x^k + A^*(I-T)Ax^k\right], \quad (5)$$

where the stepsize is set as

$$\tau_k = \frac{\left\|(I-U)x^k\right\|^2 + \left\|(I-T)Ax^k\right\|^2}{\left\|(I-U)x^k + A^*(I-T)Ax^k\right\|^2}. \quad (6)$$

Recently, the above algorithms were further extended to the general case; see, e.g., [2, 10–17].

The inertial method was first introduced in [18], and now, it has been successfully applied to solving various optimization problems arising from some applied sciences [19, 20]. In particular, this method was also applied for solving the split feasibility problem [21, 22]. By applying the inertial technique, Dang et al. [21] recently proposed the inertial relaxed CQ algorithm, which is defined as

$$\begin{cases} w^k = x^k + \theta_k(x^k - x^{k-1}), \\ x^{k+1} = P_C\left(w^k - \tau A^*(I-P_Q)Aw^k\right), \end{cases} \quad (7)$$

where $0 \le \theta_k < \theta < 1$ and $0 < \tau < (2/\|A\|^2)$. It is clear that the constant stepsize requires the estimation of the norm $\|A\|$. To avoid the estimation of the norm, Gibali et al. [23] modified the above stepsize as

$$\tau_k = \rho_k \frac{\left\|\left(I-P_Q\right)Aw^k\right\|^2}{\eta_k^2}, \quad \eta_k = \max\left(1, \left\|A^*\left(I-P_Q\right)Aw^k\right\|\right), \quad (8)$$

with $0 < \rho_k < 4$. It is shown that the inertial relaxed CQ algorithm converges weakly toward a solution of the SFP provided that $\sum_{k=1}^{\infty}\theta_k\|x^k - x^{k-1}\|^2 < \infty$. The main advantage of the inertial method is that it can indeed speed up the convergence of the original algorithm. It is thus natural to extend it to the split common fixed point problem. Recently, Cui et al. [24] proposed a modified algorithm of (3) as

$$\begin{cases} w^k = x^k + \theta_k(x^k - x^{k-1}), \\ x^{k+1} = U\left(w^k - \tau_k A^*(I-T)Aw^k\right), \end{cases} \quad (9)$$

where $0 \le \theta_k < \theta < 1$ and τ_k is defined as in (6). It was shown that algorithm (9) converges weakly to a solution of the problem provided that $\sum_{k=1}^{\infty}\theta_k\|x^k - x^{k-1}\|^2 < \infty$.

In this paper, we aim to continue the study of the split common fixed point problem in Hilbert spaces. Motivated by the inertial method, we propose a new algorithm for solving the split common fixed point problem that greatly improves the performance of the original algorithm. Moreover, the stepsize in our algorithm is independent of the norm $\|A\|$. Under some mild conditions, we establish two weak convergence theorems of the proposed algorithm.

2. Preliminary

In the following, we shall assume that problem (1) is consistent, that is, its solution set denoted by $\mathcal{S}$ is nonempty. The notation " $\longrightarrow$ " stands for strong convergence, "$\rightharpoonup$" weak convergence, and $\omega_w\{x_n\}$ the set of weak cluster points of a sequence $\{x_n\}$. Let C be a nonempty closed convex subset. For a mapping T defined on C, we let $F(T) = \{x \in C\colon Tx = x\}$ be its fixed point set and $T' = I - T$ be its complement.

Definition 1. A mapping $T\colon C \longrightarrow H$ is said to be nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C. \quad (10)$$

T is called quasi-nonexpansive if $F(T) \ne \varnothing$, and

$$\|Tx - y\| \le \|x - y\|, \quad \forall x \in C, y \in F(T). \quad (11)$$

Definition 2. Let $T\colon C \longrightarrow H$ be a mapping with $F(T) \ne \varnothing$. Then, T' is said to be demiclosed at 0 if, for any $\{x^k\}$ in C, there holds the following implication:

$$\left.\begin{matrix} x^k \rightharpoonup x \\ T'x^k \longrightarrow 0 \end{matrix}\right] \Rightarrow x \in F(T). \quad (12)$$

It is well known that if T is a nonexpansive mapping, then T' is demiclosed at 0; see [25].

Lemma 1 (see [25]). *If $T\colon C \longrightarrow H$ is quasi-nonexpansive, then*

$$2\langle x - z, T'x\rangle \ge \left\|T'x\right\|^2, \quad \forall z \in F(T), x \in C. \quad (13)$$

Lemma 2 (see [25]). *Assume that $\{x^k\}$ is a sequence in H such that*

(i) For each $z \in C$, the limit of $\{\|x^k - z\|\}$ exists

(ii) Any weak cluster point of $\{x^k\}$ belongs to C

Then, $\{x^k\}$ is weakly convergent to an element in C.

Lemma 3 (see [18]). *Let $\{\phi_k\}$ and $\{\delta_k\}$ be two nonnegative real sequences such that $\sum_{k=0}^{\infty}\delta_k < \infty$ and*

$$\phi_{k+1} - \phi_k \le \theta_k\left(\phi_k - \phi_{k-1}\right) + \delta_k, \quad (14)$$

where $0 \le \theta_k \le \theta < 1$. Then, the sequence $\{\phi_k\}$ is convergent.

Lemma 4 (see [25]). *Let $s, t \in \mathbb{R}$ and $x, y \in H$. It then follows that*

$$\|tx + sy\|^2 = t(t+s)\|x\|^2 + s(t+s)\|y\|^2 - ts\|x - y\|^2. \quad (15)$$

3. The Proposed Algorithm

Algorithm 1. Let x^0, x^1 be arbitrary. Given x^k, x^{k-1}, choose $\theta_k \in [0,1]$, and set

$$w^k = x^k + \theta_k\left(x^k - x^{k-1}\right). \quad (16)$$

If $\|U'w^k + A^*T'Aw^k\| = 0$, then stop; otherwise, update the next iteration via

$$x^{k+1} = w^k - \tau_k\left[U' w^k + A^* T' A w^k\right], \tag{17}$$

where

$$\tau_k = \frac{\left\|U' w^k\right\|^2 + \left\|T' A w^k\right\|^2}{2\left\|U' w^k + A^* T' A w^k\right\|^2}. \tag{18}$$

Remark 1. In comparison, our stepsize (18) is independent of the norm $\|A\|$ so that the calculation or estimation of $\|A\|$ is avoided.

Remark 2. If $\|U' w^k + A^* T' A w^k\| = 0$ for some $k \in \mathbb{N}$, then w^k is a solution of the problem. To see this, let $z \in \mathcal{S}$. It then follows from Lemma 1 that $\|U' w^k\|^2 \le 2\langle U' w^k, w^k - z\rangle$, and

$$\left\|T' A w^k\right\|^2 \le 2\langle T' A w^k, A w^k - Az\rangle = \langle A^* T' A w^k, w^k - z\rangle. \tag{19}$$

Combining these inequalities yields

$$\begin{aligned} \left\|U' w^k\right\|^2 + \left\|T' A w^k\right\|^2 &\le 2\langle U' w^k + A^* T' A w^k, w^k - z\rangle \\ &\le 2\left\|U' w^k + A^* T' A w^k\right\|\left\|w^k - z\right\|. \end{aligned} \tag{20}$$

This yields $\|U' w^k\| = \|T' A w^k\| = 0$, which implies $w^k \in \mathcal{S}$.

If we let $\theta_k \equiv 0$ in (16), then we get a new algorithm for problem (1).

Algorithm 2. Let x^0 be arbitrary. Given x^k, if $\|U' x^k + A^* T' A x^k\| = 0$, then stop; otherwise, update the next iteration via

$$x^{k+1} = x^k - \tau_k\left[U' x^k + A^* T' A x^k\right], \tag{21}$$

where

$$\tau_k = \frac{\left\|U' x^k\right\|^2 + \left\|T' A x^k\right\|^2}{2\left\|U' x^k + A^* T' A x^k\right\|^2}. \tag{22}$$

4. Convergence Analysis

In this section, we shall establish the convergence of the proposed algorithm. By Remark 2, we may assume that Algorithm 1 generates an infinite iterative sequence. To proceed, we first prove the following lemma.

Lemma 5. *Let $\{x^k\}$ and $\{w^k\}$ be the sequences generated by Algorithm 1. Let $\delta_k = (1/(4(1 + \|A\|^2)))(\|U' w^k\|^2 + \|T' A w^k\|^2)$. Then, for any $z \in S$, it follows that*

$$\left\|x^{k+1} - z\right\|^2 \le \left\|w^k - z\right\|^2 - \delta_k. \tag{23}$$

Proof. Since U is quasi-nonexpansive, we have

$$\begin{aligned} \left\|x^{k+1} - z\right\|^2 &= \left\|w^k - \tau_k\left[U' w^k + A^* T' A w^k\right] - z\right\|^2 \\ &= \left\|w^k - z\right\|^2 + \tau_k^2\left\|U' w^k + A^* T' A w^k\right\|^2 \\ &\quad - 2\tau_k\langle U' w^k, w^k - z\rangle - 2\tau_k\langle T' A w^k, A w^k - Az\rangle \\ &\le \left\|w^k - z\right\|^2 + \tau_k^2\left\|U' w^k + A^* T' A w^k\right\|^2 \\ &\quad - \tau_k\left\|U' w^k\right\|^2 - \tau_k\left\|T' A w^k\right\|^2. \end{aligned} \tag{24}$$

In view of (18), we have

$$\left\|x^{k+1} - z\right\|^2 \le \left\|w^k - z\right\|^2 - \frac{\left(\left\|U' w^k\right\|^2 + \left\|T' A w^k\right\|^2\right)^2}{4\left\|U' w^k + A^* T' A w^k\right\|^2}. \tag{25}$$

To finish the proof, it suffices to note that

$$\begin{aligned} &\frac{\left(\left\|U' w^k\right\|^2 + \left\|T' A w^k\right\|^2\right)^2}{\left\|U' w^k + A^* T' A w^k\right\|^2} \\ &\ge \frac{\left(\left\|U' w^k\right\|^2 + \left\|T' A w^k\right\|^2\right)^2}{\left(\left\|U' w^k\right\| + \|A\|\left\|T' A w^k\right\|\right)^2} \\ &\ge \frac{\left(\left\|U' w^k\right\|^2 + \left\|T' A w^k\right\|^2\right)^2}{\left(1 + \|A\|^2\right)\left(\left\|U' w^k\right\|^2 + \left\|T' A w^k\right\|^2\right)} \\ &= \frac{1}{1 + \|A\|^2}\left(\left\|U' w^k\right\|^2 + \left\|T' A w^k\right\|^2\right). \end{aligned} \tag{26}$$

This completes the proof. □

Theorem 1. *Assume that U is quasi-nonexpansive such that U' is demiclosed at 0, and T is quasi-nonexpansive such that T' is demiclosed at 0. If, for each $k \in \mathbb{N}$, $\theta_k \le \theta < 1$ such that*

(c1) $\sum_{k=1}^{\infty} \theta_k\|x^k - x^{k-1}\|^2 < \infty$,
then the sequence $\{x^k\}$ generated by Algorithm 1 converges weakly to an element in $\mathcal{S}$.

Proof. We first show that the sequence $\{\|x^k - z\|\}$ is convergent for any $z \in \mathcal{S}$. From Lemma 4, we deduce

$$\begin{aligned} \left\|w^k - z\right\|^2 &= \left\|(1 + \theta_k)\left(x^k - z\right) - \theta_k\left(x^{k-1} - z\right)\right\|^2 \\ &= (1 + \theta_k)\left\|x^k - z\right\|^2 - \theta_k\left\|x^{k-1} - z\right\|^2 \\ &\quad + \theta_k(1 + \theta_k)\left\|x^k - x^{k-1}\right\|^2. \end{aligned} \tag{27}$$

By Lemma 5, this yields

$$\begin{aligned} \left\|x^{k+1} - z\right\|^2 &\le (1 + \theta_k)\left\|x^k - z\right\|^2 - \theta_k\left\|x^{k-1} - z\right\|^2 \\ &\quad + 2\theta_k\left\|x^k - x^{k-1}\right\|^2 - \delta_k. \end{aligned} \tag{28}$$

Let $\phi_k := \|x^k - z\|^2$. Then, the above inequality can be rewritten as

$$\phi_{k+1} - \phi_k \le \theta_k(\phi_k - \phi_{k-1}) + 2\theta_k\left\|x^k - x^{k-1}\right\|^2 - \delta_k. \quad (29)$$

By condition (c1), we then apply Lemma 3 to deduce that $\{\phi_k\}$ is convergent, and so is the sequence $\{\|x^k - z\|\}$.

We next show that each weak cluster point of $\{x^k\}$ belongs to $\mathscr{S}$. Since $\{\phi_k\}$ is convergent, this implies that $\phi_k - \phi_{k+1}$ converges to 0 as $n \longrightarrow \infty$. It then follows from (29) that

$$\begin{aligned}\delta_k &\le (\phi_k - \phi_{k+1}) + \theta_k(\phi_k - \phi_{k-1}) + 2\theta_k\left\|x^k - x^{k-1}\right\|^2 \\ &\le |\phi_k - \phi_{k+1}| + |\phi_k - \phi_{k-1}| + 2\theta_k\left\|x^k - x^{k-1}\right\|^2.\end{aligned} \quad (30)$$

Note that $\lim_k \theta_k\|x^k - x^{k-1}\|^2 = 0$ by condition (c1). By passing to the limit in the above inequality, we have δ_k converging to 0 so that

$$\lim_{k\longrightarrow\infty}\left\|U'w^k\right\| = \lim_{k\longrightarrow\infty}\left\|T'Aw^k\right\| = 0. \quad (31)$$

Moreover, it is clear that $\{x^k\}$ is bounded; thus, the set $\omega_w(x_n)$ is nonempty. Now, take any $x \in \omega_w(x^k)$, and take a subsequence $\{x^{k_l}\}$ such that it weakly converges to x. On the contrary, we deduce from (c1) that

$$\left\|w^k - x^k\right\|^2 = \theta_k^2\left\|x^k - x^{k-1}\right\| \le \theta_k\left\|x^k - x^{k-1}\right\| \longrightarrow 0 \quad (32)$$

so that $\{w^{k_l}\}$ also weakly converges to x and $\{Aw^{k_l}\}$ weakly converges to Ax. Since U' and T' are both demiclosed at 0, this together with (31) indicates $x \in F(U)$ and $Ax \in F(T)$; that is, x is an element in $\mathscr{S}$.

Finally, by Lemma 2, the sequence $\{x^k\}$ converges weakly to a solution of problem (1). ☐

Remark 3. We now construct a sequence satisfying condition (c1). For each $k \in \mathbb{N}$, let

$$\theta_k = \begin{cases} \min\left(0.5, \dfrac{1}{(k+1)^2\left\|x^k - x^{k-1}\right\|^2}\right), & x^k \ne x^{k-1}, \\ 0.5, & x^k = x^{k-1}. \end{cases} \quad (33)$$

We next study the convergence of Algorithm 1 under another condition. To proceed, we need the following lemma.

Lemma 6. *Let $\{x^k\}$ and $\{w^k\}$ be the sequences generated by Algorithm 1. For any $z \in \mathscr{S}$, let $\phi_k = \|x^k - z\|^2 - \theta_k \|x^{k-1} - z\|^2 + (\theta_k/2)(3 + \theta_k)\|x^k - x^{k-1}\|^2$. If $\{\theta_k\}$ is nondecreasing, then*

$$2(\phi_k - \phi_{k+1}) \ge \left(1 - 4\theta_{k+1} - \theta_{k+1}^2\right)\left\|x^k - x^{k+1}\right\|^2 + \delta_k, \quad (34)$$

where δ_k is defined as in Lemma 5.

Proof. In view of (17) and (18), we get

$$\left\|x^{k+1} - w^k\right\|^2 = \frac{\left(\left\|U'w^k\right\|^2 + \left\|T'Aw^k\right\|^2\right)^2}{4\left\|U'w^k + A^*T'Aw^k\right\|^2}. \quad (35)$$

It then follows from inequality (25) that

$$\left\|x^{k+1} - z\right\|^2 \le \left\|w^k - z\right\|^2 - \frac{1}{2}\left\|x^{k+1} - w^k\right\|^2 - \frac{1}{2}\delta_k. \quad (36)$$

Moreover, it follows from (27) that

$$\begin{aligned}\left\|x^{k+1} - z\right\|^2 &\le (1 + \theta_k)\left\|x^k - z\right\|^2 - \theta_k\left\|x^{k-1} - z\right\|^2 \\ &\quad + 2\theta_k\left\|x^k - x^{k-1}\right\|^2 - \frac{1}{2}\left\|x^{k+1} - w^k\right\|^2 - \frac{1}{2}\delta_k.\end{aligned} \quad (37)$$

On the contrary, we have

$$\begin{aligned}\left\|w^k - x^{k+1}\right\|^2 &= \left\|x^k - x^{k+1} + \theta_k\left(x^k - x^{k-1}\right)\right\|^2 \\ &= \left\|x^k - x^{k+1}\right\|^2 + \theta_k^2\left\|x^k - x^{k-1}\right\|^2 \\ &\quad + 2\theta_k\langle x^k - x^{k+1}, x^k - x^{k-1}\rangle \\ &\ge \left\|x^k - x^{k+1}\right\|^2 + \theta_k^2\left\|x^k - x^{k-1}\right\|^2 \\ &\quad - 2\theta_k\left\|x^k - x^{k+1}\right\|\left\|x^k - x^{k-1}\right\| \\ &\ge \left\|x^k - x^{k+1}\right\|^2 + \theta_k^2\left\|x^k - x^{k-1}\right\|^2 \\ &\quad - \theta_k\left(\left\|x^k - x^{k+1}\right\|^2 + \left\|x^k - x^{k-1}\right\|^2\right) \\ &= (1 - \theta_k)\left\|x^k - x^{k+1}\right\|^2 - \theta_k(1 - \theta_k)\left\|x^k - x^{k-1}\right\|^2.\end{aligned} \quad (38)$$

Substituting this into (21), we have

$$\begin{aligned}\left\|x^{k+1} - z\right\|^2 &\le (1 + \theta_k)\left\|x^k - z\right\|^2 - \theta_k\left\|x^{k-1} - z\right\|^2 \\ &\quad + \theta_k(1 + \theta_k)\left\|x^k - x^{k-1}\right\|^2 \\ &\quad - \frac{1}{2}(1 - \theta_k)\left\|x^k - x^{k+1}\right\|^2 \\ &\quad + \frac{\theta_k}{2}(1 - \theta_k)\left\|x^k - x^{k-1}\right\|^2 - \frac{1}{2}\delta_k.\end{aligned} \quad (39)$$

Since $\{\theta_k\}$ is nondecreasing, this implies

$$\begin{aligned}\left\|x^{k+1}-z\right\|^{2} \leq&\ \left(1+\theta_{k}\right)\left\|x^{k}-z\right\|^{2} \\ &-\theta_{k}\left\|x^{k-1}-z\right\|^{2}+\frac{\theta_{k}}{2}\left(3+\theta_{k}\right)\left\|x^{k}-x^{k-1}\right\|^{2} \\ &-\frac{1}{2}\left(1-\theta_{k}\right)\left\|x^{k}-x^{k+1}\right\|^{2}-\frac{1}{2}\delta_{k} \\ \leq&\ \left(1+\theta_{k+1}\right)\left\|x^{k}-z\right\|^{2}-\theta_{k}\left\|x^{k-1}-z\right\|^{2} \\ &+\frac{\theta_{k}}{2}\left(3+\theta_{k}\right)\left\|x^{k}-x^{k-1}\right\|^{2} \\ &-\frac{1}{2}\left(1-\theta_{k+1}\right)\left\|x^{k}-x^{k+1}\right\|^{2}-\frac{1}{2}\delta_{k}.\end{aligned} \tag{40}$$

From the definition of ϕ_k, we get the desired inequality. □

Theorem 2. *Assume that U is quasi-nonexpansive such that U' is demiclosed at 0, and T is quasi-nonexpansive such that T' is demiclosed at 0. If*

(c2) $\{\theta_k\}$ is nondecreasing and converges to $\theta \in [0, \sqrt{5}-2)$,

then the sequence $\{x^k\}$ generated by Algorithm 1 converges weakly to an element in $\mathcal{S}$.

Proof. We first show that $\{\|x^k - z\|\}$ is convergent for each $z \in \mathcal{S}$. It then follows from Lemma 6 and the range of θ_k that

$$2\left(\phi_{k}-\phi_{k+1}\right) \geq\left(1-4\theta-\theta^{2}\right)\left\|x^{k}-x^{k+1}\right\|^{2}+\delta_{k} \geq 0 \tag{41}$$

so that $\{\phi_k\}$ is nonincreasing. From the definition of ϕ_k, we get

$$\left\|x^{k}-z\right\|^{2} \leq \theta_{k}\left\|x^{k-1}-z\right\|^{2}+\phi_{k} \leq \theta\left\|x^{k-1}-z\right\|^{2}+\phi_{1}. \tag{42}$$

By induction, we have

$$\left\|x^{k}-z\right\|^{2} \leq\left\|x^{0}-z\right\|^{2}+\frac{\phi_{1}}{1-\theta}. \tag{43}$$

Thus, $\{x^k\}$ is bounded. Moreover, from the definition of ϕ_k,

$$\phi_{k+1} \geq-\theta_{k+1}\left\|x^{k}-z\right\|^{2} \geq-\left\|x^{k}-z\right\|^{2} \geq-\left\|x^{0}-z\right\|^{2}-\frac{\phi_{1}}{1-\theta}, \tag{44}$$

which implies that $\{\phi_k\}$ is bounded from below, and thus, it is convergent. Passing to the limit in (41) yields

$$\lim_{k\longrightarrow\infty} \delta_{k}=\lim_{k\longrightarrow\infty}\left\|x^{k+1}-x^{k}\right\|=0. \tag{45}$$

On the contrary,

$$\begin{aligned}&\theta_{k}\left|\left\|x^{k-1}-z\right\|^{2}-\left\|x^{k}-z\right\|^{2}\right| \\ &=\theta_{k}\left|\left\|x^{k-1}-z\right\|-\left\|x^{k}-z\right\|\right|\left(\left\|x^{k-1}-z\right\|+\left\|x^{k}-z\right\|\right) \\ &\leq\left\|x^{k-1}-x^{k}\right\|\left(\left\|x^{k-1}-z\right\|+\left\|x^{k}-z\right\|\right) \longrightarrow 0,\end{aligned} \tag{46}$$

from which it follows that

$$\lim_{k\longrightarrow\infty}\left\|x^{k}-z\right\|^{2}=\frac{1}{1-\theta} \lim_{k\longrightarrow\infty} \phi_{k}. \tag{47}$$

Here, we used the fact (by the definition of ϕ_k) that

$$\left\|x^{k}-z\right\|^{2}=\frac{1}{1-\theta_{k}}\left(\phi_{k}+\theta_{k}\left(\left\|x^{k-1}-z\right\|^{2}-\left\|x^{k}-z\right\|^{2}\right)-\frac{\theta_{k}\left(3+\theta_{k}\right)}{2}\left\|x^{k}-x^{k-1}\right\|^{2}\right). \tag{48}$$

Thus, $\{\|x^k - z\|\}$ is convergent.

We next show that the sequence $\{x^k\}$ converges weakly to a solution of problem (1). By Lemma 2, it suffices to show that each weak cluster point of $\{x^k\}$ belongs to $\mathcal{S}$. Moreover, it is clear that $\{x^k\}$ is bounded; thus, the set $\omega_w(x_n)$ is nonempty. Now, take any $x \in \omega_w(x^k)$. On the contrary, we deduce from (16) and (45) that

$$\left\|w^{k}-x^{k}\right\|=\theta_{k}\left\|x^{k}-x^{k-1}\right\| \leq\left\|x^{k}-x^{k-1}\right\| \longrightarrow 0. \tag{49}$$

In a similar way, we deduce that $x \in F(U)$ and $Ax \in F(T)$; that is, x is an element in $\mathcal{S}$. Hence, the proof is complete.

If we let $\theta_k \equiv 0$, then it satisfies (c1) and (c2). As a result, we get the following conclusion. □

Corollary 1. *Assume that U is quasi-nonexpansive such that U' is demiclosed at 0, and T is quasi-nonexpansive such that T' is demiclosed at 0. Then, the sequence $\{x^k\}$ generated by Algorithm 2 converges weakly to an element in $\mathcal{S}$.*

5. Concluding Remarks

The main contribution of this paper is to propose a new algorithm for solving the split common fixed point problem in Hilbert spaces. There are two advantages of the proposed algorithm. Compared with the original algorithm for solving the problem, our proposed algorithm is faster in convergence rate. Furthermore, the stepsize in the proposed algorithm is independent of the norm of the given linear mapping, which can further improve its performance.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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