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*Retraction*

# Retracted: Ordered-Theoretic Fixed Point Results in Fuzzy *b*-Metric Spaces with an Application

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

## References

[1] K. Javed, F. Uddin, H. Aydi, A. Mukheimer, and M. Arshad, "Ordered-Theoretic Fixed Point Results in Fuzzy *b*-Metric Spaces with an Application," *Journal of Mathematics*, vol. 2021, Article ID 6663707, 7 pages, 2021.

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*Research Article*

# Ordered-Theoretic Fixed Point Results in Fuzzy *b*-Metric Spaces with an Application

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The aim of this manuscript is to initiate the study of the Banach contraction in $R$-fuzzy $b$-metric spaces and discuss some related fixed point results to ensure the existence and uniqueness of a fixed point. A nontrivial example is imparted to illustrate the feasibility of the proposed methods. Finally, to validate the superiority of the provided results, an application is presented to solve the first kind of a Fredholm-type integral equation.



## 1. Introduction and Preliminaries

Since the axiomatic interpretation of metric spaces and the inception of the Banach contraction principle, many authors have studied fixed point theory vividly. A number of results have been introduced, and metric fixed point has been generalized in different directions. In this connectedness, Bakhtin [1] and Czerwik [2] gave a generalization of a metric space and named it as a $b$-metric space. Zadeh [3] introduced the concept of fuzzy sets and generalized the concept of metric spaces and fuzzy sets and named them as fuzzy metric spaces, which became a point of interest for many authors [2, 4]. Nădăban [5] extended the concept of a fuzzy metric and introduced the notion of fuzzy $b$-metric spaces. For related works in this setting, refer to [6–9].

Recently, Baghani and Ramezani [10] tossed the concept of orthogonal sets and gave an extension of the Banach contraction principle. For more details, refer to [10–24].

In this article, we further aim to establish fixed point results in the setting of $R$-complete fuzzy $b$-metric spaces. We provide an example dealing with an $R$-fuzzy $b$-metric space, but it is not a fuzzy $b$-metric space. The presented results improve and generalize many results in the literature.

First, we recall some basic definitions and notions, which are essential for this work.

*Definition 1* (see [11]). A binary operation $*: [0,1]\times[0,1] \longrightarrow [0,1]$ is referred to as a continuous $t$-norm if the following assumptions hold:

(1) $e * f = f * e, \forall e, f \in [0,1]$
(2) $e * 1 = e, \forall e \in [0,1]$
(3) $(e * f) * s = e * (f * s), \forall e, f, s \in [0,1]$
(4) If $e \leq s$ and $f \leq u$, with $e, f, s, u \in [0,1]$, then $e * f \leq s * u$

Some fundamental examples of a $t$-norm are $e * f = e \cdot f, e * f = \min\{e, f\}$, and $e * f = \max\{e + f - 1, 0\}$.

*Definition 2* (see [12, 13]). A 3-tuple $(H, M, *)$ is said to be a fuzzy metric space if $H \neq M$ is an arbitrary set, $*$ is a continuous $t$-norm, and $M$ is a fuzzy set on $H \times H \times (0, \infty)$ meeting the following conditions for all $\sigma, H, z \in M, \tau, \mathbf{s} > 0$:

(B1) $M(\sigma, M, \tau) > 0$
(B2) $M(\sigma, M, \tau) = 1$ iff $\sigma = M$
(B3) $M(\sigma, M, \tau) = M(M, \sigma, \tau)$
(B4) $M(\sigma, z, \tau + \mathbf{s}) \geq M(\sigma, M, \tau) * M(M, z, \mathbf{s})$
(B5) $M(\sigma, M, M)$: $(0, \infty) \longrightarrow [0, 1]$ is continuous

*Example 1* (see [12]). Let $(H, d)$ be a metric space with a continuous $t$-norm $\mathsf{a} * M = \mathsf{a} \cdot M$, and let $M$ be a fuzzy set defined on $H \times H \times (0, \infty)$ by

$$M(\sigma, M, \tau) = \frac{\tau}{\tau + d(\sigma, M)}. \quad (1)$$

Then, $(H, M, *)$ is called a standard fuzzy metric space.

*Definition 3* (see [6]). A 4-tuple $(H, M, *, u)$ is said to be a fuzzy $b$-metric space if $H \neq M$ is an arbitrary set, $*$ is a continuous $t$-norm, and $M$ is a fuzzy set on $H \times H \times (0, \infty)$ meeting the following conditions for all $\sigma, M, z \in H, \tau, \mathbf{s} > 0$ and for a given real number $u \geq 1$:

(B1) $M(\sigma, M, \tau) > 0$
(B2) $M(\sigma, M, \tau) = 1$ iff $\sigma = M$
(B3) $M(\sigma, M, \tau) = M(M, \sigma, \tau)$
(B4) $M(\sigma, z, \tau + \mathbf{s}) \geq M(\sigma, M, \tau/u) * M(M, z, \mathbf{s}/u)$
(B5) $M(\sigma, M, M)$: $(0, \infty) \longrightarrow [0, 1]$ is continuous

*Example 2* (see [7]). Let $M(\sigma, M, \tau) = e^{-|\sigma - M|^p/\tau}$, where $p > 1$ represents a real number. It is then simple to prove that $M$ is a fuzzy $b$-metric with $u = 2^{p-1}$. It should be noted that, for $p = 2$, $(H, M, *)$ is not a fuzzy metric space.

*Definition 4.* Assume $H \neq M$ and $R \in H \times H$ is a binary relation. Suppose there exists $\sigma_0 \in M$ such that $\sigma_0 R \sigma$ or $\sigma R \sigma_0$ for all $\sigma \in H$. Then, we say that $H$ is an $R$-set.

*Example 3*

(i) Let $H = [0, \infty)$ and define $\sigma R M$ if $\sigma M = \min\{\sigma, M\}$; then, by putting $\sigma_0 = 1$, $(H, R)$ is an $R$-set.
(ii) Suppose $M$ is a set of scalar matrices of order $2 \times 2$ with entries from natural numbers (i.e., $M = \begin{bmatrix} \mathsf{a} & 0 \\ 0 & \mathsf{a} \end{bmatrix}$, for all $\mathsf{a} \in N$). Define the relation $R$ by

$$ARB \text{ if } \quad \det(A) \leq \det(B). \quad (2)$$

Then, by taking $A = I$, $(M, R)$ is an $R$-set.

*Definition 5* (see [10]). Suppose that $(H, R)$ is an $R$-set. A sequence $\{\sigma_n\}$ for all $n \in \mathbb{N}$ is said to be an $R$-sequence if $(\forall n;\ \sigma_n R \sigma_{n+1})$ or $(\forall n;\ \sigma_{n+1} R \sigma_n)$.

*Definition 6* (see [14])

(a) A metric space $(H, d)$ is an $R$-metric space if $(H, R)$ is an $R$-set.
(b) A mapping $\mathsf{F}$: $H \longrightarrow H$ is $R$-continuous at $\sigma \in H$ if for each $R$-sequence $\{\sigma_n\}$ for all $n \in \mathbb{N}$ in $H$ if $\lim_{n \longrightarrow \infty} d(\sigma_n, \sigma) = 0$, then $\lim_{n \longrightarrow \infty} d(\mathsf{F}\sigma_n, \mathsf{F}\sigma) = 0$. Furthermore, $\mathsf{F}$ is $R$-continuous if $\mathsf{F}$ is $R$-continuous at each $\sigma \in H$.
(c) A mapping $\mathsf{F}$: $H \longrightarrow H$ is called $R$-preserving if $\sigma R \mathsf{F}$, then $\mathsf{F}\sigma R \mathsf{F} M$ for all $\sigma$, $M \in H$.
(d) An $R$-sequence $\{\sigma_n\}$ in $H$ is said to be an $R$-Cauchy sequence if for every $\varepsilon > 0$, there exists an integer $N$ such that $\mathrm{d}(\sigma_n, \sigma_m) < \varepsilon$ for all $n \geq \mathbb{N}$ and $m \geq \mathbb{N}$. It is clear that $\sigma_n R \sigma_m$ or $\sigma_m R \sigma_n$.
(e) $H$ is $R$-complete if every $R$-Cauchy sequence is convergent.

## 2. Main Results

We start this section with the introduction of $R$-fuzzy $b$-metric spaces.

*Definition 7.* Let $H \neq M$ and $R$ be a reflexive binary relation on $H$. Let $*$ be a continuous $t$-norm and $H$ be a fuzzy set on $H \times H \times (0, \infty)$. Suppose that, for all $\tau, \mathbf{s} > 0$ and for all $\sigma, M, z \in H$, with either $(\sigma R z$ or $z R \sigma)$, either $(\sigma R M$ or $M R \sigma)$, and either $(M R z$ or $z R M)$, the following conditions hold:

(1) $M(\sigma, M, \tau) > 0$
(2) $M(\sigma, M, \tau) = 1$ if and only if $\sigma = M$
(3) $M(\sigma, M, \tau) = M(M, \sigma, \tau)$
(4) $M(\sigma, z, \tau + \mathbf{s}) \geq M(\sigma, M, \tau/u) * M(M, z, \mathbf{s}/u)$, where $u \geq 1$
(5) $M(\sigma, M, M)$: $(0, \infty) \longrightarrow [0, 1]$ is continuous

Then, $(H, M, *, u, R)$ is called an $R$-fuzzy $b$-metric space with the coefficient $u \geq 1$.

*Remark 1.* In the above definition, the set $H$ is endowed with a reflexive binary relation $R$, and $M$ is a fuzzy set on $H \times H \times (0, \infty)$ satisfying (1)–(5) for those comparable elements with respect to the reflexive binary relation $R$. An $R$-fuzzy $b$-metric may not be a fuzzy $b$-metric.

The following simplest example shows that the $R$-fuzzy $b$-metric with $u = 4$ does not need to be a fuzzy $b$-metric with $u = 4$.

*Example 4.* Let $H = [-1, 1]$ and $M(\sigma, M, \tau) = e^{-(\sigma - M)^3/\tau}$. Define a binary relation such that $\sigma R M$ iff $|\sigma| \geq |M|$. It is clear that $M(\sigma, M, \tau)$ is an $R$-fuzzy $b$-metric on $H$ with $u = 4$.

Note that for $\sigma = 0.1$, $M = 0.5$, and $z = 0.8$, the following condition does not hold:

$$M(\sigma, z, \tau + \mathbf{s}) \geq M\left(\sigma, M, \frac{\tau}{u}\right) * M\left(M, z, \frac{\mathbf{s}}{u}\right). \quad (3)$$

So, $M(\sigma, M, \tau)$ is not a fuzzy $b$-metric.

*Definition 8.* Let $(H, M, *, u, R)$ represent an $R$-fuzzy $b$-metric space.

(a) A sequence $\{\sigma_n\}$ for all $n \in \mathbb{N}$ is said to be an $R$-sequence if $(\forall n;\ \sigma_n R \sigma_{n+1})$ or $(\forall n;\ \sigma_{n+1} R \sigma_n)$.

(b) A Cauchy sequence $\{\sigma_n\}$ is said to be an $R$-Cauchy sequence if $(\forall n;\ \sigma_n R \sigma_{n+1})$ or $(\forall n;\ \sigma_{n+1} R \sigma_n)$.

(c) A mapping $\digamma\colon H \longrightarrow H$ is $R$-continuous at $\sigma \in M$ if for each $R$-sequence $\{\sigma_n\}$ for all $n \in \mathbb{N}$ in $M$ with $\lim_{n\longrightarrow\infty} M(\sigma_n, \sigma, \tau) = 1$ for all $\tau > 0$, then $\lim_{n\longrightarrow\infty} M(\digamma\sigma_n, \digamma\sigma, \tau) = 1$ for all $\tau > 0$. Furthermore, $\digamma$ is $R$-continuous if $\digamma$ is $R$-continuous at each $\sigma \in H$.

(d) A mapping $\digamma\colon H \longrightarrow H$ is called $R$-preserving if $\sigma R M$, then $\digamma\sigma R \digamma M$ for all $\sigma, M \in H$.

(e) If each $R$-Cauchy sequence is convergent, then $M$ is $R$-complete.

Motivated by the work of Baghani and Ramezani [10] and Hezarjaribi et al. [14], we introduce the concept of Banach contraction principle in the setting of $R$-fuzzy $b$-metric spaces.

*Definition 9.* Let $(H, M, *, u, R)$ be an $R$-fuzzy $b$-metric space. A map $\digamma\colon H \longrightarrow H$ is an $R$-contraction if there exists $q \in (0, 1)$ such that, for every $\tau > 0$ and $\sigma, M \in H$ with $\sigma R M$, we have

$$M(\digamma\sigma, \digamma M, q\tau) \geq M(\sigma, M, \tau). \tag{4}$$

**Theorem 1.** *Assume that $(H, M, *, u, R)$ is an $R$-complete fuzzy $b$-metric space. Let $\digamma\colon H \longrightarrow H$ be an $R$-continuous, $R$-contraction, and $R$-preserving mapping. Thus, $\digamma$ has a unique fixed point $\sigma_* \in H$. Furthermore,*

$$\lim_{n\longrightarrow\infty} M(\digamma^n\sigma, \sigma_*, \tau) = 1, \text{ for all } \sigma \in H \text{ and } \tau > 0. \tag{5}$$

*Proof.* Since $(H, M, *, u, R)$ is an $R$-complete fuzzy $b$-metric space, there exists $\sigma_0 \in H$ such that

$$\sigma_0 R M, \text{ for all } M \in H. \tag{6}$$

This yields that $\sigma_0 R \digamma\sigma_0$. Assume that

$$\sigma_1 = \digamma\sigma_0, \sigma_2 = \digamma^2\sigma_0 = \digamma\sigma_1, \ldots, \sigma_n = \digamma^n\sigma_0 = \digamma\sigma_{n-1}, \text{ for all } n \in N. \tag{7}$$

Since $\digamma$ is $R$-preserving, $\{\sigma_n\}$ is an $R$-sequence and $\digamma$ is an $R$-contraction. Thus,

$$M(\sigma_{n+1}, \sigma_n, q\tau) = M(\digamma\sigma_n, \digamma\sigma_{n-1}, q\tau) \geq M(\sigma_n, \sigma_{n-1}, \tau), \tag{8}$$

for all $n \in N$ and $\tau > 0$. Therefore, by applying the above expression, we deduce

$$\begin{aligned} M(\sigma_{n+1}, \sigma_n, \tau) &\geq M(\sigma_{n+1}, \sigma_n, q\tau) = M(\digamma\sigma_n, \digamma\sigma_{n-1}, q\tau) \geq M(\sigma_n, \sigma_{n-1}, \tau) \\ &= M(\digamma\sigma_{n-1}, \digamma\sigma_{n-2}, \tau) \geq M\left(\sigma_{n-1}, \sigma_{n-2}, \frac{\tau}{q}\right) \geq \ldots \geq M\left(\sigma_1, \sigma_0, \frac{\tau}{q^n}\right), \end{aligned} \tag{9}$$

for all $n \in N$ and $\tau > 0$. Thus, from (9) and (B4), we have

$$\begin{aligned} M\left(\sigma_n, \sigma_{n+p}, \tau\right) &\geq M\left(\sigma_n, \sigma_{n+1}, \frac{\tau}{u}\right) * M\left(\sigma_{n+1}, \sigma_{n+p}, \frac{\tau}{u}\right) \\ &\geq M\left(\sigma_n, \sigma_{n+1}, \frac{\tau}{u}\right) * M\left(\sigma_{n+1}, \sigma_{n+2}, \frac{\tau}{u^2}\right) * M\left(\sigma_{n+2}, \sigma_{n+3}, \frac{\tau}{u^3}\right) * \ldots * M\left(\sigma_{n+p-1}, \sigma_{n+p}, \frac{\tau}{u^{n+p}}\right) \\ &\geq M\left(\sigma_1, \sigma_0, \frac{\tau}{uq^n}\right) * M\left(\sigma_1, \sigma_0, \frac{\tau}{u^2q^n}\right) * \ldots * M\left(\sigma_1, \sigma_0, \frac{\tau}{u^{n+p}q^n}\right). \end{aligned} \tag{10}$$

Here, $u$ is an arbitrary positive integer. We know that $\lim_{n\longrightarrow\infty} M(\sigma, M, \tau) = 1$ for all $\sigma, M \in H$ and $\tau > 0$. From (10), we get

$$\lim_{n\longrightarrow\infty} M\left(\sigma_n, \sigma_{n+p}, \tau\right) \geq 1 * 1 * \ldots * 1 = 1. \tag{11}$$

Then, $\{\sigma_n\}$ is an $R$-Cauchy sequence. The hypothesis of $R$-completeness of the fuzzy $b$-metric space $(H, M, *, u, R)$ ensures that there exists $\sigma_* \in H$ such that $M(\sigma_n, \sigma_*, \tau) \longrightarrow 1$ as $n \longrightarrow +\infty$ for all $\tau > 0$. Since $\digamma$ is an $R$-continuous mapping, one writes $M(\sigma_{n+1}, \digamma\sigma_*, \tau) = M(\digamma\sigma_n, \digamma\sigma_*, \tau) \longrightarrow 1$ as $n \longrightarrow +\infty$. Hence,

$$M(\sigma_*, \digamma\sigma_*, \tau) \geq M\left(\sigma_*, \sigma_{n+1}, \frac{\tau}{2u}\right) * M\left(\sigma_{n+1}, \digamma\sigma_*, \frac{\tau}{2u}\right). \tag{12}$$

As $n \longrightarrow +\infty$, we get $F(\sigma_*, F\sigma_*, \tau) = 1 * 1 = 1$; hence, $F\sigma_* = \sigma_*$.

To show the uniqueness of the fixed point for the mapping $F$, assume that $\sigma_*$ and $M_*$ are two fixed points of $F$ such that $\sigma_* \neq M_*$. We have

$$\sigma_0 R \sigma_* \text{ and } \sigma_0 R M_*. \tag{13}$$

Since $M$ is $R$-preserving, we can write

$$F^n \sigma_0 R F^n \sigma_* \text{ and } F^n \sigma_0 R F^n F v_*, \tag{14}$$

for all $n \in N$. Using (4), we have

$$\begin{aligned} M(F^n\sigma_0, F^n\sigma_*, \tau) &\geq M(F^n\sigma_0, F^n\sigma_*, q\tau) \geq M\left(\sigma_0, \sigma_*, \frac{\tau}{q^n}\right), \\ M(F^n\sigma_0, F^n v_*, \tau) &\geq M(F^n\sigma_0, F^n v_*, q\tau) \geq M\left(\sigma_0, v_*, \frac{\tau}{q^n}\right). \end{aligned} \tag{15}$$

Hence,

$$\begin{aligned} M(\sigma_*, v_*, \tau) = M(F^n\sigma_*, F^n v_*, \tau) &\geq M\left(F^n\sigma_0, F^n\sigma_*, \frac{\tau}{2u}\right) * M\left(F^n\sigma_0, F^n v_*, \frac{\tau}{2u}\right) \\ &\geq M\left(\sigma_0, \sigma_*, \frac{\tau}{2uq^n}\right) * M\left(\sigma_0, v_*, \frac{\tau}{2uq^n}\right) \longrightarrow 1 \text{ as } n \longrightarrow \infty. \end{aligned} \tag{16}$$

So, $\sigma_* = M_*$; hence, $\sigma_*$ is the unique fixed point.

**Corollary 1.** *Let $(H, M, *, u, R)$ be an $R$-complete fuzzy b-metric space. Let $H: H \longrightarrow H$ be an $R$-contraction and $R$-preserving. Also, if $\{\sigma_n\}$ is an $R$-sequence with $\sigma_n \longrightarrow \sigma \in F$, then $\sigma R \sigma_n$ for all $n \in \mathbb{N}$. Therefore, $M$ has a unique fixed point $\sigma_* \in H$. Furthermore, $\lim_{n\longrightarrow\infty} M(F^n\sigma, \sigma_{\∗}, \tau) = 1$, for all $\sigma \in H$ and $\tau > 0$.*

*Proof.* The proof of this result moves along the same lines as in Theorem 1, that is, $\{\sigma_n\}$ is an $R$-Cauchy sequence and converges to $\sigma_* \in H$. Hence, $\sigma_* R \sigma_n$ for all $n \in \mathbb{N}$. From (4), we have

$$\begin{aligned} M(F\sigma_*, \sigma_{n+1}, \tau) = M(F\sigma_*, F\sigma_n, \tau) &\geq M(F\sigma_*, F\sigma_n, \tau q) \\ &\geq M(\sigma_*, \sigma_n, \tau). \end{aligned} \tag{17}$$

Also,

$$\lim_{n\longrightarrow\infty} M(F\sigma_*, \sigma_{n+1}, \tau) = 1. \tag{18}$$

Hence,

$$M(\sigma_*, F\sigma_*, \tau) \geq M\left(\sigma_*, \sigma_{n+1}, \frac{\tau}{2u}\right) * M\left(\sigma_{n+1}, F\sigma_*, \frac{\tau}{2u}\right). \tag{19}$$

As $n \longrightarrow +\infty$, we get $M(\sigma_*, F\sigma_*, \tau) = 1 * 1 = 1$, and so, $F\sigma_* = \sigma_*$. The rest of the proof is the same as in Theorem 1.

**Corollary 2.** *Let $(F, M, *, u, R)$ be an $R$-complete fuzzy b-metric space and $F: H \longrightarrow H$ be an $R$-continuous and $R$-preserving mapping. Suppose that there exist $q \in (0, 1/2)$ and $\tau > 0$ such that*

$$M(F\sigma, Fv, q\tau) \geq M\left(F\sigma, \sigma, \frac{\tau}{2}\right) + M\left(Fv, v, \frac{\tau}{2}\right). \tag{20}$$

*Then, $M$ has a unique fixed point.*

**Corollary 3.** *Let $(H, M, *, u, R)$ be an $R$-complete fuzzy b-metric space and $F: H \longrightarrow H$ be an $R$-continuous and $R$-preserving mapping. Assume that there exist $q \in (0, 1/u)$ and $\tau > 0$ such that*

$$M(F\sigma, Fv, q\tau) \geq \min\{M(F\sigma, \sigma, \tau), M(Fv, v, \tau)\}. \tag{21}$$

*Then, $F$ has a unique fixed point.*

*Proof.* The proof is a part of the next corollary.

**Corollary 4.** *Let $(F, M, *, u, R)$ be an $R$-complete fuzzy b-metric space and $F: H \longrightarrow H$ be an $R$-continuous and $R$-preserving mapping. Assume that there exist $q \in (0, 1/u)$ and $\tau > 0$ such that*

$$M(\digamma\sigma, \digamma v, q\tau) \geq \min\{M(\digamma\sigma, \sigma, \tau), M(\digamma v, v, \tau), M(\sigma, v, \tau)\}. \tag{22}$$

*Then,* $\digamma$ *has a unique fixed point.*

*Proof.* This corollary is a generalization of Theorem 2.5 in [8]. It is easy to prove this result by the help of Theorem 1 of this article and Theorem 2.5 of [8].

*Example 5.* Let $H = [-1, 1]$. The relation on $H$ is defined as $\sigma R M M|\sigma| \geq |M|$. Define the $R$-fuzzy $b$-metric given as in Example 4:

$$M(\sigma, M, \tau) = \begin{cases} e^{-(\sigma-M)^3/\tau}, & \text{if } \tau > 0, \\ 0, & \text{if } \tau \leq 0, \end{cases} \tag{23}$$

with the $t$-norm $\alpha * M = \alpha M M$. Let $\{\sigma_n\}$ be an $R$-sequence in $H$ such that $\sigma_n = 1$. Hence, $\{\sigma_n\}$ converges to 1. Therefore, $(H, M, *, u, R)$ is an $R$-complete fuzzy $b$-metric space with $u = 4$.

Define $\digamma: H \longrightarrow H$ by

$$\digamma(\sigma) = \begin{cases} \dfrac{\sigma}{4} & \text{if } \sigma \in [0, 1], \\ 0 & \text{if } \sigma \in [-1, 0). \end{cases} \tag{24}$$

Note the following:

(1) If $\sigma \in [0, 1]$ and $v \in [0, 1]$, then $\digamma(\sigma) = \sigma/4$ and $\digamma(v) = v/4$

(2) If $\sigma \in [0, 1]$ and $v \in [-1, 0)$, then $\digamma(\sigma) = \sigma/4$ and $\digamma(v) = 0$

(3) If $\sigma \in [-1, 0)$ and $\sigma \in [-1, 0)$, then $\digamma(\sigma) = 0$ and $\digamma(v) = 0$

In all cases, we have $|\digamma(\sigma)| \geq |\digamma(M)|$. Thus, $\digamma$ is an $R$-preserving map.

Let $\{\sigma_n\}$ be an arbitrary $R$-sequence in $H$ so that $\{\sigma_n\}$ converges to $\sigma \in H$. Now,

$$\lim_{n\longrightarrow\infty} M(\sigma_n, \sigma, \tau) = \lim_{n\longrightarrow\infty} e^{-(\sigma_n-\sigma)^3/\tau}. \tag{25}$$

As $\{\sigma_n\}$ converges to $\sigma \in H$, we have $e^{-(0)^3/\tau} = e^0 = 1$.

Now, we need to show that $\lim_{n\longrightarrow\infty} M(\digamma\sigma_n, \digamma\sigma, q\tau) = 1$. For this purpose, there are some cases.

(1) Take $\sigma_n, \sigma \in [-1, 0)$; then,

$$\lim_{n\longrightarrow\infty} M(\digamma\sigma_n, \digamma\sigma, q\tau) = \lim_{n\longrightarrow\infty} M(0, 0, q\tau) = \lim_{n\longrightarrow\infty} e^0 = 1. \tag{26}$$

(2) Take $\sigma_n, \sigma \in [0, 1]$; then,

$$\begin{aligned} \lim_{n\longrightarrow\infty} M(\digamma\sigma_n, \digamma\sigma, q\tau) &= \lim_{n\longrightarrow\infty} M\left(\frac{\sigma_n}{4}, \frac{\sigma}{4}, q\tau\right) \\ &= \lim_{n\longrightarrow\infty} e^{-(\sigma_n-\sigma)^3/64q\tau}. \end{aligned} \tag{27}$$

As $\{\sigma_n\}$ converges to $\sigma \in H$, we have $e^{-(0)^3/64\tau} = e^0 = 1$.

(3) Now, take $\sigma_n \in [0, 1]$ and $\sigma \in [-1, 0)$; then,

$$\begin{aligned} \lim_{n\longrightarrow\infty} M(\digamma\sigma_n, \digamma\sigma, q\tau) &= \lim_{n\longrightarrow\infty} M\left(\frac{\sigma_n}{4}, 0, q\tau\right) \\ &= \lim_{n\longrightarrow\infty} e^{-(\sigma_n)^3/64q\tau}. \end{aligned} \tag{28}$$

As $n \longrightarrow \infty$, we can easily see $\lim_{n\longrightarrow\infty} e^{-(\sigma_n)^3/64q\tau} = e^0 = 1$.

Hence, $\digamma$ is $R$-continuous.

For each $\sigma, M \in H$ with $\sigma R M$, we have the following.

Case (a) For $\sigma, H \in [0, 1]$, we have

$$\begin{aligned} M(\digamma\sigma, \digamma v, q\tau) &= M\left(\frac{\sigma}{4}, \frac{v}{4}, q\tau\right) = e^{-(\sigma-M)^3/64q\tau} \\ &\geq e^{-(\sigma-M)^3/\tau} = M(\sigma, v, \tau). \end{aligned} \tag{29}$$

Case (b) For $\sigma, v \in [-1, 0)$, we have

$$\begin{aligned} M(\digamma\sigma, \digamma v, q\tau) &= M(0, 0, q\tau) = e^0 \\ &\geq e^{-(\sigma-M)^3/\tau} = M(\sigma, v, \tau). \end{aligned} \tag{30}$$

Hence, $\digamma$ is an $R$-contraction. Hence, by Theorem 1, $\digamma$ has a unique fixed point.

## 3. An Application to an Integral Equation

Within this part, we apply Theorem 1.

Let $\hbar = C([\alpha, M], \mathbb{R})$ be the set of all continuous real-valued functions defined on $[\alpha, M]$.

Now, we consider the following Fredholm-type integral equation of first kind:

$$\sigma(l) = \int_{\mathsf{a}}^{M} F(l,\tau)\sigma(l)d\tau, \quad \text{for } l,\tau \in [\mathsf{a}, M], \tag{31}$$

where $F \in H$. Define $M$ as in Example 4, that is,

$$M(\sigma(l), v(l), \tau) = \sup_{l\in[\mathsf{a},M]}\left(e^{-(\sigma(l)-M(l))^3/\tau}\right), \quad \text{for all } \sigma, v \in \hbar \text{ and } \tau > 0. \tag{32}$$

Then, $(\hbar, M, *, u, R)$ is an $R$-complete fuzzy $b$-metric space.

**Theorem 2.** *Assume that* $(F(l,\tau)\sigma(l) - F(l,\tau)M(l)) \leq q(\sigma(l) - M(l))$ *for* $\sigma, M \in H$, $q \in (0,1)$, *and* $\forall l, \tau \in [\mathsf{a}, M]$. *Also, consider* $\int_{\mathsf{a}}^{M} d\tau = M - \mathsf{a} = 1$. *Let* $\mathsf{F}: H \longrightarrow H$ *be*

*(i) R-preserving*

*(ii) R-contraction*

*(iii) R-continuous*

*Then, the Fredholm-type integral equation of first kind in equation (31) has a unique solution.*

*Proof.* Define $\mathsf{F}: \hbar \longrightarrow \hbar$ by

$$\mathsf{F}\sigma(l) = \int_{\mathsf{a}}^{M} F(l,\tau)\sigma(l)d\tau, \quad \text{for all } l, \tau \in [\mathsf{a}, M]. \tag{33}$$

(i) Define $R$ as $\sigma(l)\,R\,M(l)M|\sigma(l)| \geq |M(l)|$. We have $\sigma(l)\,R\,M(l)$, and we want to show that $\mathsf{F}\sigma(l)\,R\,\mathsf{F}M(l)$. We see that $\sigma(l)$ and $\mathsf{F}\sigma(l)$ belong to $H$ because $\mathsf{F}$ is a self-map. So, observe that if $\sigma(l)\,R\,M(l)$, then it must be $\mathsf{F}\sigma(l)\,R\,\mathsf{F}M(l)$. Hence, $\mathsf{F}$ is $R$-preserving.

(ii) Observe that the existence of a fixed point of $\mathsf{F}$ is equivalent to the existence of a solution of Fredholm-type integral equation (31). Now, for all $\sigma, M \in M$, we have

$$\begin{aligned}
M(M\sigma(l), \mathsf{F}M(l), q\tau) &= e^{-(\mathsf{F}\sigma(l)-\mathsf{F}M(l))^3/q\tau}\\
&= e^{-\left(\int_{\mathsf{a}}^{M} F(l,\tau)\sigma(l)\,d\tau\right) - \int_{\mathsf{a}}^{M} \mathrm{F}(l,\tau)\mathrm{M}(l)\mathrm{d}\tau^3/q\tau}\\
&= e^{-\int_{\mathsf{a}}^{M}(F(l,\tau)\sigma(l) - \mathrm{F}(l,\tau)\mathrm{M}(l))\mathrm{d}\tau/q\tau}\\
&\geq e^{-\int_{\mathsf{a}}^{M} q(\sigma(l)-\mathrm{M}(l))\mathrm{d}\tau/q\tau} \geq \sup_{l\in[\mathsf{a},M]}\left(e^{-q(\sigma(l)-M(l))^3\int_{\mathsf{a}}^{M} d\tau/q\tau}\right)\\
&= \sup_{l\in[\mathsf{a},M]}\left(e^{-(\sigma(l)-M(l))^3/\tau}\right)\\
&= M(\sigma(l), M(l), \tau).
\end{aligned} \tag{34}$$

Hence, $\mathsf{F}$ is an $R$-contraction.

(iii) Suppose $\{p_n\}$ is an $R$-sequence in $H$ such that $\{p_n\}$ converges to $p \in H$. Because $\mathsf{F}$ is $R$-preserving, $\{\mathsf{F}p_n\}$ is an $R$-sequence for each $n \in \mathbb{N}$. From (ii), we have

$$M(\mathsf{F}p_n(l), \mathsf{F}p(l), q\tau) \geq \mathsf{F}(p_n(l), p(l), \tau). \tag{35}$$

As $\lim_{n\longrightarrow\infty} M(p_n(l), p(l), \tau) = 1$, for all $\tau > 0$, it is clear that

$$\lim_{n\longrightarrow\infty} \mathsf{F}(Mp_n(l), \mathsf{F}p(l), q\tau) = 1. \tag{36}$$

Hence, $\mathsf{F}$ is $R$-continuous.

Now, assume that $\sigma$ and $M$ are two fixed points of $\mathscr{F}$; then, we have

$$M(\sigma(l), M(l), \tau) \geq M\left(\sigma(l), M(l), \frac{\tau}{q}\right). \tag{37}$$

Thus, for all $n \in \mathbb{N}$,

$$M(\sigma(l), M(l), \tau) \geq M\left(\sigma(l), M(l), \frac{\tau}{q^n}\right) = e^{-(\sigma(l) - M(l))^3/\tau/q^n}. \tag{38}$$

Taking the limit as $n \longrightarrow \infty$ and using the fact $\lim_{n\longrightarrow\infty} M(\sigma(l), M(l), \tau) = 1$, we get $\sigma(l) = M(l)$.

Consequently, all the conditions of Theorem 1 hold. The operator $\digamma$ therefore has a unique fixed point.

## 4. Conclusion

Herein, we introduced the notion of $R$-fuzzy $b$-metric spaces and we proved some related fixed point results. Moreover, we presented some examples to illustrate the feasibility of the proposed methods and obtained results. We have also enriched this work with an application. Since our framework is more general than the class of fuzzy and fuzzy $b$-metric spaces, our results extend and generalize several existing ones in the literature.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

## Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and agreed to the published version of the manuscript.

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