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Research Article

On Distance in Some Finite Planes and Graphs Arising from Those Planes

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In this paper, affine and projective graphs are obtained from affine and projective planes of order p^r by accepting a line as a path. Some properties of these affine and projective graphs are investigated. Moreover, a definition of distance is given in the affine and projective planes of order p^r and, with the help of this distance definition, the point or points having the most advantageous (central) position in the corresponding graphs are determined, with some examples being given. In addition, the concepts of a circle, ellipse, hyperbola, and parabola, which are well known for the Euclidean plane, are carried over to these finite planes. Finally, the roles of finite affine and projective Klingenberg planes in all the results obtained are considered and their equivalences in graph applications are discussed.

1. Introduction

E. Galois (1811–1832) established Galois fields $\mathscr{F} = \mathrm{GF}(p^r)$, where p is a prime number and r is a positive integer [1]. A projective plane $P_2\mathscr{F}$ can be constructed over $\mathscr{F}$, and the order of this plane is also p^r. Such a plane is called a field plane. Field planes are a class of projective planes, and the majority of projective planes are not in this class. The known finite projective planes are of the order p^r. It is not known whether there are finite projective planes that are not of this order.

So far, it has been proved that there are unique projective planes of orders 2, 3, 4, 5, 7, and 8 (see [2] for proofs of this). Four different projective planes of order 9 have been obtained. There is controversy over whether there are any other projective planes of order 9. Following a computer-based study, Lam and colleagues, in [3], claimed that there was no other projective plane of order 9. In another computer-based article [4], Lam came to the conclusion that there is no projective plane of order 10.

The projective planes of orders 2, 3, 4, 5, and 7 were constructed by hand. However, a projective plane of order 8 can be constructed with the help of a computer, and its uniqueness can be proved by using Latin squares of order 7. The theoretical source of these accounts is based on the fact, quoted in [5], that "a projective plane of order k exists if and only if the set of $k-1$ mutually orthogonal Latin squares (MOLS) of order k exists." For the proof of this fact, see Theorem 2.2.10 in [6] or pp. 136–140 in [7].

Once geometry came to be expressed with coordinates, which are algebraic concepts, very close relationships arose between the two areas and this led to the discovery of many innovations in both. For example, a projective plane is a Pappian plane, a Desarguesian plane, or a Moufang plane if and only if the coordinatizing ring is a field, a division ring (skew field), or an alternative ring, respectively ([8], p. 154). Moreover, a geometric structure is a Moufang–Klingenberg (MK) plane if and only if the coordinatizing ring is a local alternative ring (Theorems 3.10 and 4.1 in [9]). Since an MK plane is a projective Klingenberg (PK) plane, PK planes are coordinatized as MK planes. More specifically, projective Klingenberg planes can be considered as a generalization of projective planes.

Like finite affine, projective, and PK planes, graphs consist of two sets, called points and lines, and are an important branch of mathematics that emerges from a certain set of axioms.

Graph theory, where many fields come together, is an area that gives different and useful results, and relationships have been discovered and transferred to technological developments by the combination of methods and results that seem unconnected.

The reason for the rapid increase of interest in graph theory and its applications in recent years is that many problems encountered in daily life can be solved with graph theory. Graphs are actually mathematical models of real-life events. By using these models, and with the help of the theories that exist in various fields of mathematics, the mathematical values and results obtained can be used to get an idea about the events represented by the graphs. As an example of this, in [10] Hosseini et al. reported that the analysis of neuroimaging information using graph theory contributes to an understanding of the organization of large-scale structural and functional brain networks.

Using the concepts of graph theory, an analysis of magnetoencephalographic functional connectivity in the brains of Alzheimer's patients was performed, and the relationship between the neurons in the related areas of the patients' brains was investigated in [11].

In their recent use in the visualization of social networks [12], in particular, graphs have contributed to the formation of new and useful technologies in the field of communication technologies.

Graph theory is applied in many different areas, such as route policies in air, land, and sea transportation, traffic flow control, routing packages in a computer network, and relationships between staff and managers. These applications cover independent fields such as economics, management science, marketing, information transmission, transportation planning, genetics, environmental science, archaeology, and music. Examples of such applications can be seen in [13, 14].

Graph theory, which has been linked to many fields from sciences to the social sciences, has led to the strengthening of the existing relationship between geometry and algebra and the establishment of connections with new fields.

The taxicab plane used in the planning of an ideal city is a non-Euclidean (infinite) plane because of the absence of the well-known side-angle-side relationship of triangles in the Euclidean plane. Many concepts of Euclidean plane geometry in the taxicab plane have been studied with the taxicab metric, and interesting results have been obtained. For example, it has been observed that a taxicab circle may have a shape similar to a rhombus (see Figure 23 on p. 79 in [15]). Similarly, answers have been sought to questions such as the locus of equidistant points to two points (see Figure 25 on p. 80 in [15]), and concepts such as the taxicab parabola, taxicab ellipse (see Figure 26 on p. 80 in [15]), and taxicab hyperbola (see Figure 28 on p. 81 in [15]) in these planes have been examined.

In this study, we want to examine the correspondences of these concepts in finite PK planes that are not Euclidean but are finite, like the taxicab plane, or the projective plane lying under a finite PK plane as an epimorphic image. In the literature, the concept of distance has been given in some cases in PK planes. However, the literature does not give any information about the definition of distance in finite PK planes or therefore in the finite projective planes underlying them. It is indeed well known that, in finite planes, no nontrivial algebraic distance can be defined because the triangle inequality can never be satisfied over finite fields (since there is no natural ordering relation in these fields).

The study has two main objectives. The first is to propose a concept of distance on PK planes, with a mathematical basis rather than an assumption. To achieve this aim, the conjugation of two points in the PK plane will be defined with the help of rank in the matrix representation of points, and then the definition of distance will be obtained using "conjugate" points on a line joining any two points of the plane. The correspondence of such a line in graph language can be considered as a path. In this case, finite PK planes (and therefore finite projective planes and finite affine planes) become appropriate geometric models for solving some important problems in graph theory. Therefore, this study is expected to bring innovation to both geometry and the areas of application of graph theory. Our second aim is to examine the geometric correspondences of circles, ellipses, hyperbolas, and parabolas, which are basic geometric concepts whose properties are well known from Euclidean geometry, in the finite planes. The aim of this study is to obtain some numerical results, such as how many different or identical circles there are with a radius of 1, 2, 3, and so on. This numerical examination will be limited to the projective planes of orders 2, 3, 4, 5, 7, and 8, which are known to be unique, to the field projective plane of order 9, and, by a generalization from this point, to the projective planes of order p^r.

The remainder of this paper is structured as follows. In Section 2, some definitions, results, and notation, a construction of a complete set of mutually orthogonal Latin squares, and two theorems are presented from the literature. In Section 3, the definition of distance for projective Klingenberg planes is introduced. In Section 4, the main results are given. First, the definition of distance is carried over to affine and projective planes of order p^r. Second, affine and projective graphs are obtained from the affine and projective planes of order p^r by taking a line as a path. Third, some properties of these affine and projective graphs are investigated, and then, the circles of the affine and projective plane of order 3 are examined. Next, with the help of this definition, the point or points having the most advantageous (central) position in the corresponding graph are determined. The other concepts of ellipse, hyperbola, and parabola, which are well known from the Euclidean plane, are then carried over to these finite planes, and so a process for finding all ellipses and hyperbolas of the affine plane of order p^r is given. Finally, the roles of finite affine and projective Klingenberg planes for all the results obtained are considered and their equivalences in graph applications are discussed.

2. Preliminaries

In this section, we will remind readers of some information that will be needed in this study. First, let us present some information, compiled from [16–20], related to graphs.

A graph is a pair $G = (V, E)$ where $V\colon = V(G)$ is a nonempty finite set of vertices and $E\colon = E(G)$ is a finite set of edges. A simple graph is an unweighted and undirected graph, containing no loops or multiple edges.

The order (the size) of a graph G is the number of elements in $V(G)$ $(E(G))$ and is denoted by $n = |V(G)|$ $(m = |E(G)|$, respectively).

Let r and s be two vertices in a graph G. If r and s are connected to each other by an edge e, this situation will be denoted by $e = rs$. The vertices r and s are called adjacent vertices and the edge e is said to be incident with r and s.

A walk in a graph G is an alternating sequence $r_0, e_0, r_1, e_1, r_2, \ldots, r_{j-1}, e_{j-1}, r_j$ of vertices and edges beginning and ending at vertices, where e_i is incident with r_i and r_{i+1} for $i = 0, 1, 2, \ldots, j-1$.

A walk is called a path if no vertex in it is visited more than once.

A graph is connected when there is a path between every pair of vertices. Otherwise, it is called disconnected.

The degree of a vertex $r \in V(G)$ is denoted by $d_G(r)$ or $d(r)$ and is defined as the number of edges in G incident to r.

A graph G is said to be k-regular if the degree of every vertex in G is k. Otherwise, it is called nonregular.

The biggest and smallest vertex degrees of G are denoted by Δ and δ, respectively, and they are defined as follows:

$$\begin{aligned} \Delta(G) &= \max\{d(r) | rt \in nVq(G)\}, \\ \delta(G) &= \min\{d(r) | rt \in nVq(G)\}. \end{aligned} \tag{1}$$

Let $V(G)$ be $\{r_1, r_2, r_3, \ldots, r_n\}$. The nonincreasing sequence is as follows:

$$(d(r_1), d(r_2), d(r_3), \ldots, d(r_n)), \tag{2}$$

where $d(r_1)\colon = \Delta(G)$ and $d(r_n)\colon = \delta(G)$ is called the degree sequence of G.

We complete this section on graphs by giving a result known as the handshaking lemma in graph theory. This result, for any graph $G = (V, E)$, states that the sum of the degrees in the degree sequence of G is equal to half the number of edges; that is

$$\sum_{r \in V(G)} d(r) = 2|E(G)|. \tag{3}$$

Like graphs, projective planes consist of two sets. When N and D are two distinct sets whose elements are called points and lines, respectively, and $\in$ is the incidence relation between N and D, then the triple $(N, D, \in)$ satisfying the following three axioms is called a projective plane and is denoted by $\mathbf{P}$. If N is finite, the projective plane $\mathbf{P}$ is called a finite projective plane.

P1: any two distinct points are incident to just one line

P2: any two lines are incident to at least one point

P3: there exist four points of which no three are collinear

The order of $\mathbf{P}$ is defined to be the number of points on any line (or, equivalently, the number of lines on any point) of the projective plane $\mathbf{P} = (N, D, \in)$ minus 1. If the order of a finite projective plane is p^r, then every point lies on $p^r + 1$ lines, every line contains $p^r + 1$ points, and the total number of its points (or lines) is $(p^r)^2 + p^r + 1$. The smallest order of a projective plane is 2, and this projective plane is called a Fano plane. If any line of a projective plane of order p^r is omitted, then the resulting plane is an affine plane of order p^r. Conversely, any affine plane of order p^r can be completed to a projective plane of the same order by adding a line.

Since the construction of a projective plane with the help of Latin squares will be taken into consideration in this study, some basic information about Latin squares will first be shared and then a summary of this construction will be given. Those who want to find more information about this can refer to [21] and the master's thesis [22].

The matrix of dimension $n \times n$ whose components are taken from a set with n elements and in which these elements appear exactly once in each row and column is called a Latin square of order n. A Latin square is usually constituted using elements of the set $\{0, 1, 2, \ldots, n-1\}$. If the elements of the first row of a Latin square are in the order given in the set, this Latin square is called a reduced Latin square.

If the resulting n^2 element pairs differ from each other when two Latin squares of order n constituted by taking the inputs from the set $\{0, 1, 2, \ldots, n-1\}$ are superimposed, these two Latin squares are called orthogonal. If every pair in a set of $n-1$ Latin squares of order n is orthogonal, then this set is called a complete set of mutually orthogonal Latin squares (MOLS for short). If $N(n)$ represents the maximum number of elements of such a complete set, there is an upper limit for this number.

Theorem 1. *$N(p^r) = p^r - 1$ where p is a prime number and r is a positive integer (Theorem 2.19 in [21]).*

Construction 1. A complete set of MOLS of order p^r is built from the following two steps (Construction 2.20 in [21]):

(1) The Latin square $L_\alpha(i, j) = i + \alpha j$ is defined for each $\alpha \in \mathcal{F}_{p^r} - \{0\}$ where $i, j \in \mathcal{F}_{p^r}$ and its numbers are $p^r - 1$

(2) $\{L_\alpha \mid \alpha \in \mathcal{F}_{p^r} - \{0\}\}$ is a complete set of MOLS of order p^r

The complete sets constructed as above for $p^r = 2, 3, 4, 5, 7, 8, 9$ are given in pp. 114-115 in [21].

Theorem 2. *There are $(p^r - 2)!/r$ different reduced complete sets where $N(p^r) = p^r - 1$ (Theorem 5.60 in [22]).*

With the help of a reduced complete set of MOLS of order p^r with $p^r - 1$ elements, a projective plane of order p^r is established as follows.

Let us assume that $(p^r)^2$ points of a finite affine plane of order p^r are given as an ordered pair $(a, b)\colon = ab$ (see Figure 1) and the lexicographical order of the points of all lines of the plane is defined.

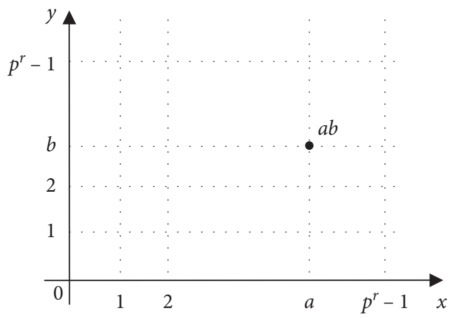

FIGURE 1: A representation of the points of an affine plane of order p^r.

Since the total number of points of the plane is $(p^r)^2 + p^r + 1$, the remaining $p^r + 1$ points of the plane must be determined. Add the following two matrices to complete the set of Latin squares:

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 \\ 2 & 2 & 2 & \cdots & 2 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ p^r-1 & p^r-1 & p^r-1 & \cdots & p^r-1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 2 & \cdots & p^r-1 \\ 0 & 1 & 2 & \cdots & p^r-1 \\ 0 & 1 & 2 & \cdots & p^r-1 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 1 & 2 & \cdots & p^r-1 \end{bmatrix}. \tag{4}$$

Thus, a set with $p^r + 1$ elements is obtained. The elements of this set are matrices of order p^r. Let one of these matrices be placed on each of the $(p^r)^2$ points represented above of the plane, and assume that a line of the plane is determined by the points where the components of the matrix are the same. With such a process, a parallel line class with p^r elements of the plane is obtained from a matrix, in which case the finite affine planes of order p^r in Appendix A in [23] are constituted. Since the points of each line are written here according to their lexicographical order, we ensure that the elements of the line are expressed in one way.

Select a new point from each parallel line class so that these parallel classes intersect at a single point. Let us assume that these points are represented by the (X) at point $1X$ on the first line of each parallel class, except for the last parallel class (for which the representation (∞) will be used here). A point obtained in this way is called the ideal (or infinite) point and its numbers are $p^r + 1$. In this case, the total number of points of the plane is $(p^r)^2 + p^r + 1$ and the total number of lines of the plane is $(p^r + 1)p^r$. The line $\{(0), (1), (2), \ldots, (p^r - 1), (\infty)\}$ on which lie all the ideal points is the ideal (or infinite) line of the plane, and thus, the total number of lines of the plane is also $(p^r)^2 + p^r + 1$. It is clear that there are $p^r + 1$ lines on every point and $p^r + 1$ points on every line. It is a known fact that the plane obtained in this way is a projective plane of order p^r.

Now, we can give the definition of a projective Klingenberg plane.

Let $\mathbf{M} = (\mathbf{P}, \mathbf{L}, \in, \sim)$ consist of an incidence structure $(\mathbf{P}, \mathbf{L}, \in)$ (points, lines, and incidence) and an equivalence relation " $\sim$ " (neighbour relation) on $\mathbf{P}$ and on $\mathbf{L}$. Then $\mathbf{M}$ is called a projective Klingenberg plane (PK plane), if it satisfies the following axioms:

(PK1) If P, Q are nonneighbour points, then there is a unique line PQ through P and Q

(PK2) If g, h are nonneighbour lines, then there is a unique point $g \wedge h$ on both g and h

(PK3) There are a projective plane $\mathbf{M}^* = (\mathbf{P}^*, \mathbf{L}^*, \in)$ and an incidence structure epimorphism $\Psi\colon \mathbf{M} \longrightarrow \mathbf{M}^*$, such that the conditions

$$\Psi(P) = \Psi(Q) \Longleftrightarrow P \sim Q, \Psi(g) = \Psi(h) \Longleftrightarrow g \sim h \tag{5}$$

hold for all $P, Q \in \mathbf{P}$, $g, h \in \mathbf{L}$.

A point $P \in \mathbf{P}$ is said to be near a line $g \in \mathbf{L}$ iff there exists a line h such that $P \in h$ for some line $h \sim g$.

An incidence structure automorphism preserving and reflecting the neighbour relation is called a collineation of $\mathbf{M}$. The notion of a centre and an axis of a collineation is the same as in an ordinary projective plane. A (C, a)-collineation φ of $\mathbf{M}$ is a collineation with centre C and axis a. If $C \in a$, then φ is a (C, a)-elation, and if $C \notin a$, then φ is a (C, a)-homology.

A PK plane $\mathbf{M}$ is called a Moufang–Klingenberg plane (MK plane) if $\mathbf{M}$ is (C, a)-transitive for all (C, a) with $C \in a$. In this case, the canonical image $\mathbf{M}^*$ is a Moufang plane (for details, see [9, 24]).

For each local alternating ring, there is an MK plane that can be coordinated with it, and, conversely, for each MK plane, there is a local alternating ring to which this plane can be coordinatized [9].

Based on the following algebraic structure and some information to be given about the class of octonion planes constructed with this algebraic structure, we will give a definition of distance that is not encountered in the literature on finite projective planes or finite projective Klingenberg planes (which can be viewed as their more general form). What makes this study important is not only this definition of distance, but also the fact that concepts related to distance have not been studied on finite projective Klingenberg planes or finite projective planes.

We consider the local ring (or the local ring $\mathcal{O}[\varepsilon]/\langle \varepsilon^m \rangle$ or the algebra of order m) $\mathbf{A}\colon = \mathcal{O} + \mathcal{O}\varepsilon + \mathcal{O}\varepsilon^2 + \cdots + \mathcal{O}\varepsilon^{m-1} = \{a_0 + a_1\varepsilon + a_2\varepsilon^2 + \cdots + a_{m-1}\varepsilon^{m-1} | a_0, a_1, a_2, \ldots, a_{m-1} \in \mathcal{O}\}$ with the set of $\mathbf{I} = \mathbf{A}\varepsilon = \{a_0\varepsilon + a_1\varepsilon^2 + \cdots + a_{m-2}\varepsilon^{m-1} \; | a_0, a_1, \ldots, a_{m-2} \in \mathcal{O}\}$ as an ideal of nonunits, where $\mathcal{O}$ is a Cayley division ring, $\varepsilon \notin \mathcal{O}$, and $\varepsilon^m = 0$. Then, $\mathbf{A}/\mathbf{I}$ is a division ring and also x or $1 - x$ is a unit. In this case, with the help of the conjugate transformation

$$\mathbf{j}\colon a \longrightarrow \overline{a} = \overline{a}_0 + \overline{a}_1\varepsilon + \overline{a}_2\varepsilon^2 + \cdots + \overline{a_{m-1}}\varepsilon^{m-1}, \tag{6}$$

on $\mathbf{A}$ for $a = a_0 + a_1\varepsilon + a_2\varepsilon^2 + \cdots + a_{m-1}\varepsilon^{m-1} \in \mathbf{A}$, the trace form $\mathbf{t}(a)$ and norm form $\mathbf{n}(a)$ can be defined (see [25]).

Thus, we can give some results based on the definition of the octonion plane given in [26].

It is possible to divide the sets of points and lines of the octonion plane $\mathbb{P}(\prod)$ consisting of the elements which are taken from the octonion algebra $\mathbf{A}$ into three equivalence classes as follows:

$$
\begin{aligned}
N = &\left\{ X = \begin{pmatrix} 1 & \gamma_1^{-1}\gamma_2\overline{y} & \gamma_1^{-1}\gamma_3\overline{z} \\ y & \gamma_1^{-1}\gamma_2\mathbf{n}(y) & \gamma_1^{-1}\gamma_3 y\overline{z} \\ z & \gamma_1^{-1}\gamma_2 z\overline{y} & \gamma_1^{-1}\gamma_3\mathbf{n}(z) \end{pmatrix} =: (1, y, z) \mid y \in \mathbf{A}, z \in \mathbf{I} \right\} \\
&\cup \left\{ Y = \begin{pmatrix} \gamma_1\gamma_2^{-1}\mathbf{n}(w) & w & \gamma_3\gamma_2^{-1}(w\overline{z}) \\ \gamma_1\gamma_2^{-1}\overline{w} & 1 & \gamma_3\gamma_2^{-1}\overline{z} \\ \gamma_1\gamma_2^{-1}(z\overline{w}) & z & \gamma_3\gamma_2^{-1}\mathbf{n}(z) \end{pmatrix} =: (w, 1, z) \mid w, z \in \mathbf{I} \right\} \\
&\cup \left\{ Z = \begin{pmatrix} \gamma_3^{-1}\gamma_1\mathbf{n}(x) & \gamma_3^{-1}\gamma_2(x\overline{y}) & x \\ \gamma_3^{-1}\gamma_1(y\overline{x}) & \gamma_3^{-1}\gamma_2\mathbf{n}(y) & y \\ \gamma_3^{-1}\gamma_1\overline{x} & \gamma_3^{-1}\gamma_2\overline{y} & 1 \end{pmatrix} =: (x, y, 1) \mid x, y \in \mathbf{A} \right\},
\end{aligned} \tag{7}
$$

and dually

$$
\begin{aligned}
D = &\left\{ S = \begin{bmatrix} 1 & -q & -p \\ -\gamma_2^{-1}\gamma_1\overline{q} & \gamma_2^{-1}\gamma_1\mathbf{n}(q) & \gamma_2^{-1}\gamma_1(\overline{q}p) \\ -\gamma_3^{-1}\gamma_1\overline{p} & \gamma_3^{-1}\gamma_1(\overline{p}q) & \gamma_3^{-1}\gamma_1\mathbf{n}(p) \end{bmatrix} =: [1, q, p] \mid p \in \mathbf{A}, q \in \mathbf{I} \right\} \\
&\cup \left\{ Q = \begin{bmatrix} \gamma_1^{-1}\gamma_2\mathbf{n}(m) & -\gamma_1^{-1}\gamma_2\overline{m} & \gamma_1^{-1}\gamma_2\overline{m}k \\ -m & 1 & -k \\ \gamma_3^{-1}\gamma_2\overline{k}m & -\gamma_3^{-1}\gamma_2\overline{k} & \gamma_3^{-1}\gamma_2\mathbf{n}(k) \end{bmatrix} =: [m, 1, k] \mid m, k \in \mathrm{A} \right\} \\
&\cup \left\{ W = \begin{bmatrix} \gamma_1^{-1}\gamma_3\mathbf{n}(q) & \gamma_1^{-1}\gamma_3\overline{q}r & -\gamma_1^{-1}\gamma_3\overline{q} \\ \gamma_2^{-1}\gamma_3\overline{r}q & \gamma_2^{-1}\gamma_3\mathbf{n}(r) & -\gamma_2^{-1}\gamma_3\overline{r} \\ -q & -r & 1 \end{bmatrix} =: [q, r, 1] \mid q, r \in \mathbf{I} \right\}.
\end{aligned} \tag{8}
$$

It can also be shown that it is possible to express the incidence relation and the neighbourhood relations on the octonion plane $\mathbb{P}(\prod)$ in a shorter form, which is different from the original definition. Hence, we can obtain the result that the octonion plane $\mathbb{P}(\prod)$ is an MK plane.

Now, we are ready to give the definition of distance.

3. The Definition of Distance in a PK Plane

It is clear that each MK plane is a PK plane and that the coordinatizing for the MK plane can also be used for the coordinatizing of the PK plane. Thus, if matrix representations in an octonion plane $\mathbb{P}(\prod)$ are used to represent the points and lines of a PK plane, a definition of distance can be given in the PK plane and hence in the underlying projective plane as a canonical image. With the help of this definition of distance, the correspondence of geometric concepts and shapes such as circles, ellipses, parabolas, and hyperbolas, whose properties are well known in relation to distance in Euclidean planes, will be examined for finite projective planes and finite projective Klingenberg planes. Some numerical results will also be obtained as part of this examination. This numerical analysis will be limited to the projective planes of orders 2, 3, 4, 5, 7, and 8, to one of the four different projective planes of order 9 that are known to be unique, and, by making a generalization, to the projective planes of order p^r. Moreover, the results are enriched by carrying out this analysis on some finite Klingenberg planes. For examples of such finite Klingenberg planes, refs. [27, 28] can be consulted. Moreover, more detailed information about such finite structures can be obtained from [29–31].

The distance definition we propose emerges from the definitions given for some special matrix spaces in [32]. Although octonion planes are studied with a special subset of

Hermitian matrix spaces, here it would be more appropriate to use the definitions of the arithmetic distance given for alternate matrix spaces (the only difference between the two matrix spaces is the factor 1/2 in the following arithmetic distance definition) and the definition of conjugation (this definition is the same in the two matrix spaces). Otherwise, there would be no conjugate point pair in the space we are studying, and this would prevent us from creating a definition of distance.

Definition 1. Let X_1 and X_2 be any two points of a PK plane $\mathbf{M}$. The arithmetic distance between X_1 and X_2 is denoted by $\mathrm{ad}(X_1, X_2)$ and is defined as $\mathrm{ad}(X_1, X_2) = (\mathrm{rank}(X_1 - X_2)/2)$. If $\mathrm{ad}(X_1, X_2) = 1$, that is, $\mathrm{rank}(X_1 - X_2) = 2$, then X_1 and X_2 are called conjugates.

This arithmetic distance has the following properties.

Theorem 3. *Let* $\mathbf{M}$ *be a PK plane and any three points of* $\mathbf{M}$ *be* X_1, X_2, *and* X_3. *Then,*

(1) $ad(X_1, X_2) = 1 \Longleftrightarrow X_1 \nsim X_2$ *and* $ad(X_1, X_2) = 0 \Longleftrightarrow X_1 \sim X_2$. *So,* $ad(X_1, X_2) \geq 0$

(2) $ad(X_1, X_2) = ad(X_2, X_1)$

(3) $ad(X_1, X_2) + ad(X_2, X_3) \geq ad(X_1, X_3)$

Proof

(1) First, an examination will be made for nonneighbour point pairs. Since the PK plane has three distinct classes of points, and nonneighbour point pairs can be found in the same classes, there are six cases:

Case 1: let

$$X_1 = \begin{pmatrix} 1 & \gamma_1^{-1}\gamma_2\overline{y} & \gamma_1^{-1}\gamma_3\overline{z} \\ y & \gamma_1^{-1}\gamma_2\mathbf{n}(y) & \gamma_1^{-1}\gamma_3 y\overline{z} \\ z & \gamma_1^{-1}\gamma_2 z\overline{y} & \gamma_1^{-1}\gamma_3\mathbf{n}(z) \end{pmatrix} \nsim \begin{pmatrix} \gamma_1\gamma_2^{-1}\mathbf{n}(u) & u & \gamma_3\gamma_2^{-1}(u\overline{v}) \\ \gamma_1\gamma_2^{-1}\overline{u} & 1 & \gamma_3\gamma_2^{-1}\overline{v} \\ \gamma_1\gamma_2^{-1}(v\overline{u}) & v & \gamma_3\gamma_2^{-1}\mathbf{n}(v) \end{pmatrix} = X_2. \tag{9}$$

In this case, we must find the rank of the matrix $X_1 - X_2$. For this, the number of linear independent rows or columns must be determined. Since the algebra we are working with is not Abelian, the left product will be used for multiplying over rows and the right product for multiplying over columns, as in [33]. For $1 \leq i, j \leq 3$, the row i. and column j. of the matrix $X_1 - X_2$are denoted by S_i and K_j, respectively. Some elementary row and column operations will be applied to this matrix and thus the rank will be calculated. First, if the elementary operations $S_2 \longrightarrow S_2 - yS_1$ and $S_3 \longrightarrow S_3 - zS_1$ are applied, then the matrix

$$\begin{pmatrix} 1 - \gamma_1\gamma_2^{-1}\mathbf{n}(u) & \gamma_1^{-1}\gamma_2\overline{y} - u & \gamma_1^{-1}\gamma_3\overline{z} - \gamma_3\gamma_2^{-1}(u\overline{v}) \\ \gamma_1\gamma_2^{-1}[y\mathbf{n}(u) - \overline{u}] & yu - 1 & \gamma_3\gamma_2^{-1}[y(u\overline{v}) - \overline{v}] \\ \gamma_1\gamma_2^{-1}[z\mathbf{n}(u) - (v\overline{u})] & zu - v & \gamma_3\gamma_2^{-1}[z(u\overline{v}) - \mathbf{n}(v)] \end{pmatrix} \tag{10}$$

is obtained. If the elementary operations $K_1 \longrightarrow K_1 - K_2(\gamma_1\gamma_2^{-1}\overline{u})$ and $K_3 \longrightarrow K_3 - K_2(\gamma_3\gamma_2^{-1}\overline{v})$ are applied to the last matrix, then the matrix

$$\begin{pmatrix} 1 - \overline{yu} & \gamma_1^{-1}\gamma_2\overline{y} - u & \gamma_1^{-1}\gamma_3(\overline{z} - \overline{yv}) \\ 0 & yu - 1 & \gamma_3\gamma_2^{-1}[y(u\overline{v}) - (yu)\overline{v}] \\ 0 & zu - v & \gamma_3\gamma_2^{-1}[z(u\overline{v}) - (zu)\overline{v}] \end{pmatrix} \tag{11}$$

is found. If the elementary operations $K_2 \longrightarrow K_2 - K_1[(1 - \overline{yu})^{-1}(u - \gamma_1^{-1}\gamma_2\overline{y})]$ and $K_3 \longrightarrow K_3 - K_1[(1 - \overline{yu})^{-1}\gamma_1^{-1}\gamma_3(\overline{yv} - \overline{z})]$ where $1 - \overline{yu} \notin \mathbf{I}$ are applied to the resulting matrix, then the matrix

$$\begin{pmatrix} 1 - \overline{yu} & 0 & 0 \\ 0 & yu - 1 & \gamma_3\gamma_2^{-1}[y(u\overline{v}) - (yu)\overline{v}] \\ 0 & zu - v & \gamma_3\gamma_2^{-1}[z(u\overline{v}) - (zu)\overline{v}] \end{pmatrix} \tag{12}$$

is obtained, and by applying the elementary operations $S_3 \longrightarrow S_3 - (v - zu)(yu - 1)^{-1}S_2$ where $yu - 1 \notin \mathbf{I}$ we have the following matrix:

$$\begin{pmatrix} 1 - \overline{yu} & 0 & 0 \\ 0 & yu - 1 & \gamma_3\gamma_2^{-1}[y(u\overline{v}) - (yu)\overline{v}] \\ 0 & 0 & \gamma_3\gamma_2^{-1}\left([z(u\overline{v}) - (zu)\overline{v}] - (v - zu)(yu - 1)^{-1}[y(u\overline{v}) - (yu)\overline{v}]\right) \end{pmatrix}. \tag{13}$$

If the elementary operation $K_2 \longrightarrow K_3 - K_2[(yu - 1)^{-1}\gamma_3\gamma_2^{-1}[(yu)\overline{v} - y(u\overline{v})]]$ is applied to this matrix, then we obtain

$$\begin{pmatrix} 1-\overline{yu} & 0 & 0 \\ 0 & yu-1 & 0 \\ 0 & 0 & \gamma_3\gamma_2^{-1}\left(\left[z\left(u\overline{v}\right)-(zu)\overline{v}\right]-(v-zu)\left(yu-1\right)^{-1}\left[y\left(u\overline{v}\right)-(yu)\overline{v}\right]\right) \end{pmatrix}. \tag{14}$$

If the component (3,3) of this matrix is not in $\mathbf{I}$, it is possible that the rank of the matrix is 3. The only way to eliminate this situation is by showing that the algebra we are working with is associative. If the algebra is associative, the component (3,3) is zero and the rank of the matrix is 2. In [33], some examples are given in which the rank changes depending on whether or not the algebra is commutative. Moreover, in such a case, the definitions of commutative rank (:=crk) and noncommutative rank (:=ncrk) are used. Here, since the rank calculation changes according to whether or not the algebra is associative, it is possible to distinguish between associative rank (:=ark) and nonassociative rank (:=nark). Since the octonion division ring with which we are working is not associative, we may prefer to work with the elements of the quaternion division ring, which is an associative subset of the octonion division ring. If we continue to work with the elements of the octonion division ring, the rank of a matrix must be determined by evaluating any row or column consisting of the elements of the ideal $\mathbf{I}$ as a row or column with zeros. In such a case, the rank of the above matrix would be 2. Since $z, u, v \in \mathbf{I}$, the component (3,3) is an element of $\mathbf{I}$ and so the linearly independent row or column number is also 2.

Case 2: let

$$X_1 = \begin{pmatrix} 1 & \gamma_1^{-1}\gamma_2\overline{y} & \gamma_1^{-1}\gamma_3\overline{z} \\ y & \gamma_1^{-1}\gamma_2\mathbf{n}(y) & \gamma_1^{-1}\gamma_3 y\overline{z} \\ z & \gamma_1^{-1}\gamma_2 z\overline{y} & \gamma_1^{-1}\gamma_3\mathbf{n}(z) \end{pmatrix} \nsim \begin{pmatrix} \gamma_3^{-1}\gamma_1\mathbf{n}(u) & \gamma_3^{-1}\gamma_2\left(u\overline{v}\right) & u \\ \gamma_3^{-1}\gamma_1\left(v\overline{u}\right) & \gamma_3^{-1}\gamma_2\mathbf{n}(v) & v \\ \gamma_3^{-1}\gamma_1\overline{u} & \gamma_3^{-1}\gamma_2\overline{v} & 1 \end{pmatrix} = X_2. \tag{15}$$

A proof similar to the proof in Case 1 can be given.

Case 3: let

$$\begin{aligned} X_1 &= \begin{pmatrix} \gamma_1\gamma_2^{-1}\mathbf{n}(w) & w & \gamma_3\gamma_2^{-1}\left(w\overline{z}\right) \\ \gamma_1\gamma_2^{-1}\overline{w} & 1 & \gamma_3\gamma_2^{-1}\overline{z} \\ \gamma_1\gamma_2^{-1}\left(z\overline{w}\right) & z & \gamma_3\gamma_2^{-1}\mathbf{n}(z) \end{pmatrix} \\ &\nsim \begin{pmatrix} \gamma_3^{-1}\gamma_1\mathbf{n}(u) & \gamma_3^{-1}\gamma_2\left(u\overline{v}\right) & u \\ \gamma_3^{-1}\gamma_1\left(v\overline{u}\right) & \gamma_3^{-1}\gamma_2\mathbf{n}(v) & v \\ \gamma_3^{-1}\gamma_1\overline{u} & \gamma_3^{-1}\gamma_2\overline{v} & 1 \end{pmatrix} = X_2. \end{aligned} \tag{16}$$

A proof similar to the proof in Case 1 can be given.

Case 4: let

$$\begin{aligned} X_1 &= \begin{pmatrix} 1 & \gamma_1^{-1}\gamma_2\overline{y} & \gamma_1^{-1}\gamma_3\overline{z} \\ y & \gamma_1^{-1}\gamma_2\mathbf{n}(y) & \gamma_1^{-1}\gamma_3 y\overline{z} \\ z & \gamma_1^{-1}\gamma_2 z\overline{y} & \gamma_1^{-1}\gamma_3\mathbf{n}(z) \end{pmatrix} \\ &\nsim \begin{pmatrix} 1 & \gamma_1^{-1}\gamma_2\overline{u} & \gamma_1^{-1}\gamma_3\overline{v} \\ u & \gamma_1^{-1}\gamma_2\mathbf{n}(u) & \gamma_1^{-1}\gamma_3 u\overline{v} \\ v & \gamma_1^{-1}\gamma_2 v\overline{u} & \gamma_1^{-1}\gamma_3\mathbf{n}(v) \end{pmatrix} = X_2. \end{aligned} \tag{17}$$

Unlike the first three cases, a proof will be given for this situation. Therefore, we must calculate the rank of the matrix $X_1 - X_2$ where $y - u \notin \mathbf{I}$. After applying the elementary operations $S_2 \longrightarrow S_2 + (-y)S_1$ and $S_3 \longrightarrow S_3 + (-z)S_1$ to this matrix, the elementary operations $K_2 \longrightarrow K_2 + K_1\left(-\gamma_1^{-1}\gamma_2\overline{u}\right)$ and $K_3 \longrightarrow K_3 + K_1\left(-\gamma_1^{-1}\gamma_3\overline{v}\right)$ are applied to the resulting matrix, and then we have the following matrix:

$$\begin{pmatrix} 0 & \gamma_1^{-1}\gamma_2\left(\overline{y}-\overline{u}\right) & \gamma_1^{-1}\gamma_3\left(\overline{z}-\overline{v}\right) \\ y-u & 0 & 0 \\ z-v & 0 & 0 \end{pmatrix}. \tag{18}$$

Since $z - v \in \mathbf{I}$ and $\overline{z} - \overline{v} \in \mathbf{I}$, if the last row and last column of this matrix are evaluated as a zero row or zero column, the rank of this matrix is 2. Moreover, if the elementary operations $S_3 \longrightarrow S_3 + (v-z)(y-u)^{-1}S_2$ and $K_3 \longrightarrow K_3 + K_2\gamma_2^{-1}\gamma_3\left(\overline{y}-\overline{u}\right)^{-1}\left(\overline{v}-\overline{z}\right)$ are applied to the last matrix, then we obtain the following matrix:

$$\begin{pmatrix} 0 & \gamma_1^{-1}\gamma_2\left(\overline{y}-\overline{u}\right) & 0 \\ y-u & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{19}$$

and the rank of this matrix is also 2.

Case 5: let

$$Y = \begin{pmatrix} \gamma_1\gamma_2^{-1}\mathbf{n}(w) & w & \gamma_3\gamma_2^{-1}(w\overline{z}) \\ \gamma_1\gamma_2^{-1}\overline{w} & 1 & \gamma_3\gamma_2^{-1}\overline{z} \\ \gamma_1\gamma_2^{-1}(z\overline{w}) & z & \gamma_3\gamma_2^{-1}\mathbf{n}(z) \end{pmatrix} \nsim \begin{pmatrix} \gamma_1\gamma_2^{-1}\mathbf{n}(u) & u & \gamma_3\gamma_2^{-1}(u\overline{v}) \\ \gamma_1\gamma_2^{-1}\overline{u} & 1 & \gamma_3\gamma_2^{-1}\overline{v} \\ \gamma_1\gamma_2^{-1}(v\overline{u}) & v & \gamma_3\gamma_2^{-1}\mathbf{n}(v) \end{pmatrix} = Y'. \tag{20}$$

The proof of this case is similar to the proof in Case 4.

Case 6: let

$$\begin{aligned} Z &= \begin{pmatrix} \gamma_3^{-1}\gamma_1\mathbf{n}(x) & \gamma_3^{-1}\gamma_2(x\overline{y}) & x \\ \gamma_3^{-1}\gamma_1(y\overline{x}) & \gamma_3^{-1}\gamma_2\mathbf{n}(y) & y \\ \gamma_3^{-1}\gamma_1\overline{x} & \gamma_3^{-1}\gamma_2\overline{y} & 1 \end{pmatrix} \\ &\nsim \begin{pmatrix} \gamma_3^{-1}\gamma_1\mathbf{n}(u) & \gamma_3^{-1}\gamma_2(u\overline{v}) & u \\ \gamma_3^{-1}\gamma_1(v\overline{u}) & \gamma_3^{-1}\gamma_2\mathbf{n}(v) & v \\ \gamma_3^{-1}\gamma_1\overline{u} & \gamma_3^{-1}\gamma_2\overline{v} & 1 \end{pmatrix} = Z'. \end{aligned} \tag{21}$$

The proof of this case is similar to the proof in Case 4.

Thus, it is concluded that the arithmetic distance is 1 for all nonneighbour point pairs.

This investigation will now be conducted for neighbour point pairs. Since only points of the same type can be neighbours, there are three cases here:

Case (i): let

$$\begin{aligned} X &= \begin{pmatrix} 1 & \gamma_1^{-1}\gamma_2\overline{y} & \gamma_1^{-1}\gamma_3\overline{z} \\ y & \gamma_1^{-1}\gamma_2\mathbf{n}(y) & \gamma_1^{-1}\gamma_3 y\overline{z} \\ z & \gamma_1^{-1}\gamma_2 z\overline{y} & \gamma_1^{-1}\gamma_3\mathbf{n}(z) \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & \gamma_1^{-1}\gamma_2\overline{u} & \gamma_1^{-1}\gamma_3\overline{v} \\ u & \gamma_1^{-1}\gamma_2\mathbf{n}(u) & \gamma_1^{-1}\gamma_3 u\overline{v} \\ v & \gamma_1^{-1}\gamma_2 v\overline{u} & \gamma_1^{-1}\gamma_3\mathbf{n}(v) \end{pmatrix} = X'. \end{aligned} \tag{22}$$

We will give the proof for this case. For this, we must calculate the rank of the matrix $X - X'$ where $y - u \in \mathbf{I}$ and $z - v \in \mathbf{I}$. If the first four elementary operations as in Case 4 are applied to this matrix, then we find the matrix

$$\begin{pmatrix} 0 & \gamma_1^{-1}\gamma_2(\overline{y} - \overline{u}) & \gamma_1^{-1}\gamma_3(\overline{z} - \overline{v}) \\ y - u & 0 & 0 \\ z - v & 0 & 0 \end{pmatrix}. \tag{23}$$

All entries of the matrix therefore become elements of $\mathbf{I}$. In this case, the rank of this matrix can be evaluated as zero.

Case (ii): let

$$\begin{aligned} Y &= \begin{pmatrix} \gamma_1\gamma_2^{-1}\mathbf{n}(w) & w & \gamma_3\gamma_2^{-1}(w\overline{z}) \\ \gamma_1\gamma_2^{-1}\overline{w} & 1 & \gamma_3\gamma_2^{-1}\overline{z} \\ \gamma_1\gamma_2^{-1}(z\overline{w}) & z & \gamma_3\gamma_2^{-1}\mathbf{n}(z) \end{pmatrix} \\ &\sim \begin{pmatrix} \gamma_1\gamma_2^{-1}\mathbf{n}(u) & u & \gamma_3\gamma_2^{-1}(u\overline{v}) \\ \gamma_1\gamma_2^{-1}\overline{u} & 1 & \gamma_3\gamma_2^{-1}\overline{v} \\ \gamma_1\gamma_2^{-1}(v\overline{u}) & v & \gamma_3\gamma_2^{-1}\mathbf{n}(v) \end{pmatrix} = Y'. \end{aligned} \tag{24}$$

The proof of this case is similar to the proof in Case (i).

Case (iii): let

$$\begin{aligned} Z &= \begin{pmatrix} \gamma_3^{-1}\gamma_1\mathbf{n}(x) & \gamma_3^{-1}\gamma_2(x\overline{y}) & x \\ \gamma_3^{-1}\gamma_1(y\overline{x}) & \gamma_3^{-1}\gamma_2\mathbf{n}(y) & y \\ \gamma_3^{-1}\gamma_1\overline{x} & \gamma_3^{-1}\gamma_2\overline{y} & 1 \end{pmatrix} \\ &\sim \begin{pmatrix} \gamma_3^{-1}\gamma_1\mathbf{n}(u) & \gamma_3^{-1}\gamma_2(u\overline{v}) & u \\ \gamma_3^{-1}\gamma_1(v\overline{u}) & \gamma_3^{-1}\gamma_2\mathbf{n}(v) & v \\ \gamma_3^{-1}\gamma_1\overline{u} & \gamma_3^{-1}\gamma_2\overline{v} & 1 \end{pmatrix} = Z'. \end{aligned} \tag{25}$$

The proof of this case is similar to the proof in Case (i).

Thus, it is concluded that the arithmetic distance is zero for all neighbour point pairs.

(2) The proof is obvious from the definition of arithmetic distance.

(3) The proof will be given in two cases:

Case 1: let $X_1 \sim X_3$. Since $\mathrm{ad}(X_1, X_3) = 0$, there are two subcases:

(a) If $X_2 \sim X_1$ then $X_2 \sim X_3$ and so we have $\mathrm{ad}(X_1, X_2) = 0 = \mathrm{ad}(X_2, X_3)$. Thus, the equality case of the inequality is valid.

(b) If $X_2 \nsim X_1$ then $X_2 \nsim X_3$. Since $\mathrm{ad}(X_1, X_2) = 1 = \mathrm{ad}(X_2, X_3)$, the inequality is still valid.

Case 2: let $X_1 \nsim X_3$. Then, $\mathrm{ad}(X_1, X_3) = 1$ and there are two subcases:

(a) If $X_2 \sim X_1$ then $X_2 \nsim X_3$. Thus, the equality case of the inequality is valid.

(b) If $X_2 \nsim X_1$ then either $X_2 \sim X_3$ or $X_2 \nsim X_3$. In both cases, there is no result contrary to the inequality.

Hence, the proof is complete. □

Corollary 1. *Every pair of nonneighbour points of a PK plane* **M** *is conjugate.*

Corollary 2. *In the projective plane that is the canonical image of the PK plane* **M**, *every pair of points is conjugate.*

From Corollaries 1 and 2, if we consider the distance definition given for some special matrix spaces in [32] with a small change, we can give the following definition for a PK plane and the projective plane which is the canonical image of it.

Definition 2. Let **M** be a PK plane and $X, X' \in \mathbf{M}$. If $X \nsim X'$, the distance $d(X, X')$ between X and X' is defined to be the (least) positive integer r for which there is a sequence of $r+1$ nonneighbour points $X_0, X_1, X_2, \ldots, X_r$ with $X_0 = X$ and $X_r = X'$ such that the points X_i for $1 \le i \le r-1$ are on a line joining X and X'. If $X \sim X'$, then $d(X, X') = 0$.

Proposition 1. *Let X and X' be any two points of a PK plane* **M**. *Then,*

$$\operatorname{ad}(X, X') \le d(X, X'). \tag{26}$$

Proof. If $X \sim X'$, then $\operatorname{ad}(X, X') = 0$ and $d(X, X') = 0$. If $X \nsim X'$, then $\operatorname{ad}(X, X') = 1$ and $d(X, X') \ge 1$. Thus, the given inequality is valid.

Now, we are ready to give the main results. □

4. The Main Results

By carrying Definition 2 over from finite PK planes, we would like to see a corresponding distance concept for finite projective planes and finite affine planes.

A situation that needs to be discussed here, as stated in Theorem 2, is the one in which the studied complete set is not unique. Projective planes of the same order from each of $(p^r - 2)!/r$ different complete sets found by applying certain permutations to a given complete set are obtained. In this case, the selection of these different complete sets to give projective planes whose uniqueness is known is expected to preserve the distances in isomorphism between the obtained planes. If the distances are not preserved, the permutations used in the transitions between these complete sets have an effect on the distance. In this case, it can be concluded that the definition of distance depends on the complete set selected.

As an example of the discussion in the last part of the paragraph above, there are $((5^1 - 2)!/1) = 3! = 6$ different reduced complete sets of the projective plane of order 5. On the complete set given in (Theorem 2.29 in [21]) for $n = 5$, with the help of the permutations $\sigma_1 = (34)$, $\sigma_2 = (23)$, $\sigma_3 = (234)$, $\sigma_4 = (24)$, and $\sigma_5 = (243)$, the five different complete sets are obtained (see [22]).

Since the projective plane of order 5 is unique, the six complete sets will give the same projective plane, that is, they are isomorphic. However, although these planes are isomorphic, it is not guaranteed that distances will be preserved under the isomorphism. We can give the following small example of this situation. There are two isomorphic projective planes of order 2 denoted by U and V in Figure 2.

While the points and lines of the plane $\mathbf{U} = (N, D)$ are, respectively, $N = \{A, B, C, D, E, F, G\}$ and

$$\begin{aligned} D = \{&e_1 = \{A, B, E\}, e_2 = \{A, C, F\}, e_3 = \{A, D, G\}, e_4 = \{B, C, G\}, \\ &e_5 = \{B, D, F\}, e_6 = \{C, D, E\}, e_7 = \{E, F, G\}\}, \end{aligned} \tag{27}$$

the points and lines of the plane $V = (N', D')$ are, respectively, $N' = \{1, 2, 3, 4, 5, 6, 7\}$ and

$$\begin{aligned} D' = \{&d_1 = \{1, 2, 3\}, d_2 = \{3, 6, 7\}, d_3 = \{3, 4, 5\}, d_4 = \{1, 4, 7\}, \\ &d_5 = \{1, 5, 6\}, d_6 = \{2, 5, 7\}, d_7 = \{2, 4, 6\}\}. \end{aligned} \tag{28}$$

The following isomorphism f can be defined between these two planes does not preserve the distances:

$$\begin{aligned} &f\colon N \longrightarrow N' \\ &A \longrightarrow 4 \\ &B \longrightarrow 3 \\ &C \longrightarrow 1 \\ &D \longrightarrow 6 \\ &E \longrightarrow 5 \\ &F \longrightarrow 7 \\ &G \longrightarrow 2 \\ &f\colon D \longrightarrow D' \\ &e_1 \longrightarrow d_3 \\ &e_2 \longrightarrow d_4 \\ &e_3 \longrightarrow d_7 \\ &e_4 \longrightarrow d_1 \\ &e_5 \longrightarrow d_2 \\ &e_6 \longrightarrow d_5 \\ &e_7 \longrightarrow d_6 \end{aligned} . \tag{29}$$

For example, while the distance between G and D in the plane **U** is 1, the distance between $f(G) = 2$ and $f(D) = 6$ in the plane **V** is 2. Therefore, the selection of the exact set for the projective plane we will work on will be determined by Construction 1 and the exact sets we will work on will be obtained with the help of this construction. Thus, the choice of this complete set will be determined in one way. Moreover, the lines, infinite points, and the infinite line obtained from this complete set will be determined in one way.

Now, we will consecutively give a definition, a theorem, and a result to establish the connection between the finite geometries we will study and graphs.

Definition 3. Let $\mathbf{P} = (P, L, \in)$ be a finite projective plane of order p^r. Now, consider each line

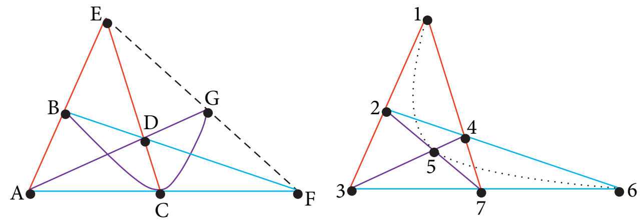

Figure 2: The projective plane of order 2. (a) $U = (N, D)$. (b)$V = (N', D')$.

$$l_i = \left\{ l_{i1}, l_{i2}, l_{i3}, \ldots, l_{ip^r}, l_{i\,(p^r+1)} \right\}, \tag{30}$$

in $\mathbf{P}$ as an ordered $p^r + 1$-tuple, and then form the set

$$c\,(l_i) = \left\{ \{l_{i1}, l_{i2}\}, \{l_{i2}, l_{i3}\}, \{l_{i3}, l_{i4}\}, \ldots, \left\{ l_{i\,(p^r-1)}, l_{ip^r} \right\}, \cdot \left\{ l_{ip^r}, l_{i\,(p^r+1)} \right\} \right\}, \tag{31}$$

where $1 \le i \le p^r + 1$.

Theorem 4. *If $\mathbf{P} = (P, L, \in)$ is a finite affine or projective plane of order p^r, then we can obtain a graph $G = (V, E)$ such that $V(G) = P$ and $E(G) = \cup_{l_i \in L} c\,(l_i)$.*

The graphs obtained from a finite affine or projective plane of order p^r as in Theorem 4 are called an affine graph and a projective graph, respectively.

Corollary 3. *Let G be the graph with the property given in Theorem 4. Then,*

(i) The order and the size of the projective graph G are $(p^r)^2 + p^r + 1$ and $((p^r)^2 + p^r + 1)p^r$, respectively

(ii) The order and the size of the affine graph G are $(p^r)^2$ and $((p^r)^2 + p^r)(p^r - 1)$, respectively

In the affine plane of order p^r, we know that a point xy is on $p^r + 1$ lines. Now, we would like to find the degree of the vertex xy in the affine graph. To find this, it is sufficient to know how many times there are intermediate points or how many times there are end points (both the starting and ending points of a line) on the $p^r + 1$ lines passing through the point xy. For the point xy of a line in the affine plane, if xy is one of the end points, this point will be on one edge in the affine graph, and if xy is one of the intermediate points, this point will be on two edges in the affine graph. In this way, it is possible to find the degree sequence of the affine graph obtained, by making this examination over the lines passing through any point of the affine plane. If we examine the tables in Appendix A in [23], the point xy is of the order of $x + 1$ for p^r times and of the order of $y + 1$ for one time. Thus, we have the following result.

Corollary 4. *The degree sequence of the affine graph obtained from the affine plane of order p^r is $((2\ p^r + 2)^{\{(p^r-2)}\ (p^r - 2)\}, (2p^r + 1)^{\{2\,(p^r-2)\}}, (p^r + 2)^{\{2\,(p^r-2)\}}, (p^r + 1)^{\{4\}})$ where $a^{\{b\}}$ means that b is the number of vertices of degree a.*

Let us consider the new lines formed when completing the affine plane of order p^r to the projective plane with the points (0), (1), (2), . . ., $(p^r - 1)$ and (∞). The degrees of the points $(p^r - 1)0$, $(p^r - 1)1$, $(p^r - 1)2$, . . ., $(p^r - 1)(p^r - 2)$, $(p^r - 1)(p^r - 1)$ are calculated while they are the end point in the affine graph they come to the intermediate point position p^r times, so the degree of these points increases by p^r in the projective graph, and the degree of the points $0\,(p^r - 1)$, $1\,(p^r - 1)$, $2\,(p^r - 1)$, . . ., $(p^r - 2)(p^r - 1)$, $(p^r - 1)(p^r - 1)$ increases by 1 because the points become intermediate points once. Note that the only common point in the above two groups is $(p^r - 1)(p^r - 1)$. The degree of this point will increase by $p^r + 1$ as it passes from the affine graph to the projective graph. The degrees of all the other points in the affine graph will not change in the projective graph. It is easy to find the degree of the new points (0), (1), (2),. . ., $(p^r - 1)$, (∞) of the projective graph: while each of them is located p^r times as an end point in all lines in the projective plane, it is located on the infinite line {(0), (1), (2),. . ., $(p^r - 1)$, (∞)} once. For this reason, the degrees of these points are, respectively, $p^r + 1$, $p^r + 2$, $p^r + 2$, . . ., $p^r + 2$, $p^r + 1$. A total increase in degree when switching from an affine graph to a projective graph occurs in two situations. These are the points where there is an increase from the degree in the affine graph to the degrees of the new points (0), (1), (2), . . ., $(p^r - 1)$, (∞) in the projective graph. So, the increase is $[(p^r)^2 + \ p^r] + [2\,(p^r + 1) + \ (p^r - 1)(p^r + 2)] = 2\,(p^r)^2 + 4p^r$. If Corollary 3 and the handshaking lemma are considered together, the same result can be reached again. Thus, we can obtain the following result.

Corollary 5. *The degree sequence of the projective graph obtained from the projective plane of order p^r is*

$$\left((2p^r + 2)^{\{(p^r-2)p^r+1\}}, (2p^r + 1)^{\{p^r-1\}}, (p^r + 2)^{\{2p^r-2\}}, (p^r + 1)^{\{3\}} \right). \tag{32}$$

For example, the degree sequences of the affine and projective graphs obtained from the affine and projective planes of order 2 are (3,3,3,3) and (6,5,4,4,3,3,3), respectively, while the degree sequences of the affine and projective graphs obtained from the affine and projective planes of order 3 are (8,7,7,5,5,4,4,4,4) and (8,8,8,8,7,7,5,5,5,5,4,4,4), respectively.

Now, we are ready to examine examples of the affine and projective planes of order 3 under these conditions. The three affine planes of order 3 in Figure 3 are isomorphic. For the completion to the projective planes of the same order as the planes in Figure 3, see Figure 4.

Now, let us try to find the desired results for affine and projective planes of order 3. Let us start by finding the circles of the affine plane. With the help of the definition of a circle, as is

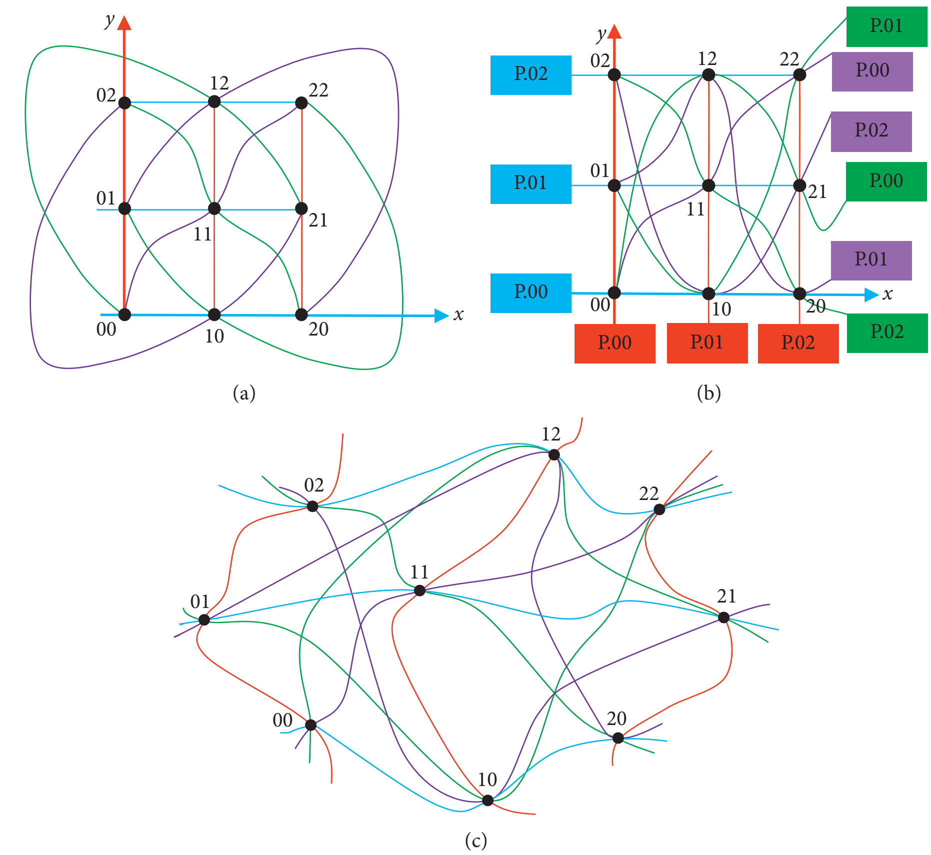

FIGURE 3: Affine planes of order 3.

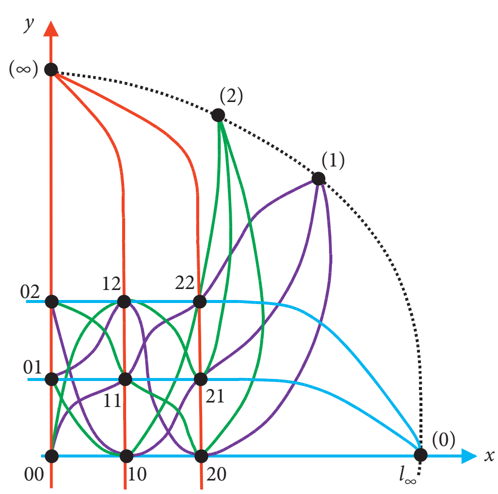

FIGURE 4: The projective plane of order 3.

TABLE 1: Circles with centre C and radius r of the affine plane of order 3.

C	$r = 1$	$r = 2$
00	{01, 10, 11, 12}	{02, 20, 21, 22}
01	{10, 11, 12, 00, 02}	{22, 21, 20}
02	{01, 12, 11, 10}	{00, 22, 21, 20}
10	{00, 01, 02, 20, 21, 22, 11}	{12}
11	{02, 01, 00, 22, 21, 20, 12, 10}	{}
12	{02, 01, 00, 22, 21, 20, 11}	{10}
20	{10, 11, 12, 21}	{00, 01, 02, 22}
21	{12, 11, 10, 22, 20}	{02, 01, 00}
22	{12, 11, 10, 21}	{02, 01, 00, 20}

known from the Euclidean plane, we can immediately obtain circles from Figure 3 (see Table 1). Note that the radius of a circle in the affine plane of order p^r can be a maximum of $p^r - 1$.

If the circles in Table 1 are examined carefully, it can be seen that all circles that can be drawn from one point (including the centre) occur as a partition of the points of the plane. Now, we are curious about the following: how many different partitions of the plane can there be in this way? The answer to this question is clear from Table 1, which shows that the number of different partitions is four: one of $C = 00$, $C = 02$, $C = 20$, and $C = 22$, one of $C = 01$ and $C = 21$, one of $C = 10$ and $C = 12$, and finally $C = 11$. For the four different partitions, see, respectively, Figures 5–8.

Corollary 6. *The most advantageous point in the problem is that the nine points should be listed in terms of the sum of their distances to the remaining points which is 11 for each of these points. Moreover, the ranking of these nine points is as follows: 11; 10 or 12; 01 or 21; any one of 00, 20, 02, and 22.*

Proof. This ranking is easily obtained if the distances to all the remaining points for each point are found and summed.

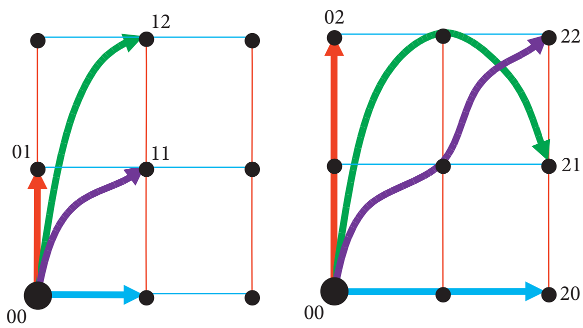

Figure 5: The points at a distance of 1 and 2 units from the point 00.

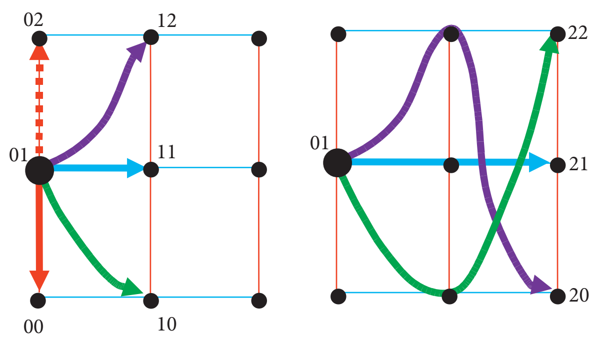

Figure 6: The points at a distance of 1 and 2 units from point 01.

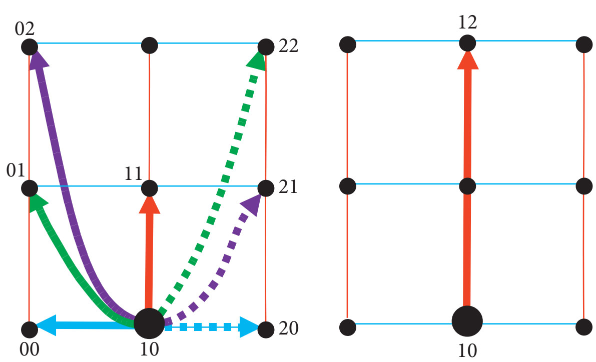

Figure 7: The points at a distance of 1 and 2 units from point 10.

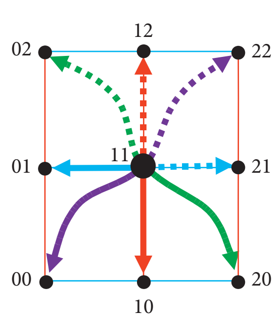

Figure 8: The points at a distance of 1 and 2 unit from the point 11.

Now we want to find the circles of the projective plane of order 3. With the help of Figure 4, we can again easily obtain these circles: see Table 2. Note that the radius of a circle in the projective plane of order p^r can be a maximum of p^r.

If the circles in Table 2 are examined carefully, it can be seen that all circles that can be drawn from one point (including the centre) occur as a partition of the points of the plane. How many different partitions of the plane can there be in this way? The answer to this question is clear from Table 2, which shows that the number of different partitions here is also four, with one of $C = 00$, $C = (0)$, and $C = (\infty)$, one of $C = 01$, $C = 02$, $C = (1)$, and $C = (2)$, one of $C = 10$ and $C = 20$, and finally one of $C = 11$, $C = 12$, $C = 21$, and $C = 22$. We leave the reader to draw the same figures of the four different partitions as in Figures 5–8. □

Corollary 7. *The most advantageous point in the problem is that the thirteen points should be listed in terms of the sum of their distances to the remaining points for each of these points are 11, 12, 21, and 22. Moreover, the ranking of the thirteen points is as follows: any one of 11, 12, 21, and 22; 10 or 20; any one of 01, 02, (1), and (2); any one of 00, (0), and (∞).*

Proof. This ranking is easily obtained if the distances to the remaining points for each point are found and summed.

Note that the projective planes we are working with have been defined uniquely because of the way in which they are constructed. If it is desired to preserve a relationship as in Corollary 6 between the points of the affine plane of order 3, which occurs when a line with all the points is thrown from the projective plane of order 3, the line thrown must be the original infinite line. Otherwise, the relationship between the points in the affine plane will be distorted. However, this new relationship between points will lead to the emergence of another problem with a different relationship in the same affine plane. Examining these relationships in detail may have important consequences for application areas.

Now, we will try to carry over the result in Corollary 6 to the affine plane of order p^r. That is, we will classify the points of the affine plane, provided that circles of the same radius pass and there are equal numbers of points on these circles. With such a classification, points will emerge for which the sums of the distances from one point of the plane to all other points differ from each other. Therefore, in such a plane, a ranking is obtained of the points that are farthest from the point located at the least distance to every place. In this ranking, we will say that the point or points that are the least distant point or points have the most advantageous location.

Now, suppose that $(p^r)^2$ points of the affine plane of order p^r where p^r is odd and $k = (p^r - 1)/2$ are given as an ordered pair (see Figure 9).

The quadrilaterals with all symmetrical enlargements according to the quadrilateral $ABCD$ where A is in region 1, B is in region 2, C is in region 3, and D is in region 4, as in Figure 9, are given in Table 3 according to the regions to which the corners belong.

All points of the affine plane are reached from the point $K = (k, k)$, which is not included in Tables 3 and 4 and is the first to have the most advantageous point position. The other rankings are given in Tables 3 and 4 and are as follows: the point pairs at $1.1, \ldots, 1.(k-1), 1.k$ after the first one; the

Table 2: Circles with centre C and radius r of the projective plane of order 3.

C	$r = 1$	$r = 2$	$r = 3$
00	{01, 10, 11, 12}	{02, 20, 21, 22}	{(0), (1), (2), (□)}
01	{10, 11, 12, 00, 02}	{22, 21, 20, (□)}	{(0), (1), (2)}
02	{01, 12, 11, 10, (□)}	{00, 22, 21, 20}	{(0), (1), (2)}
10	{00, 01, 02, 20, 21, 22, 11}	{12, (0), (1), (2)}	{(□)}
11	{02, 01, 00, 22, 21, 20, 12, 10}	{(0), (1), (2), (□)}	{}
12	{02, 01, 00, 22, 21, 20, 11, (□)}	{10, (0), (1), (2)}	{}
20	{10, 11, 12, 21, (0), (1), (2)}	{00, 01, 02, 22}	{(□)}
21	{12, 11, 10, 22, 20, (0), (1), (2)}	{02, 01, 00, (□)}	{}
22	{12, 11, 10, 21, (0), (1), (2), (□)}	{02, 01, 00, 20}	{}
(0)	{20, 21, 22, (1)}	{10, 11, 12, (2)}	{00, 01, 02, (□)}
(1)	{20, 21, 22, (0), (2)}	{10, 11, 12, (□)}	{00, 01, 02}
(2)	{20, 21, 22, (1), (□)}	{10, 11, 12, (0)}	{00, 01, 02}
(□)	{02, 12, 22, (2)}	{01, 11, 21, (1)}	{00, 10, 20, (0)}

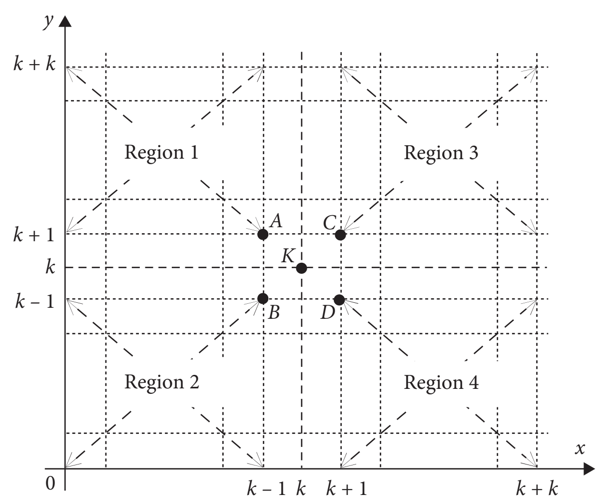

Figure 9: The points of the affine plane of p^r when p^r is odd.

Table 3: The ranking of $2(p^r - 1)$ points of the affine plane of order p^r.

Ranking	1.1		1.$(k-2)$	1.$(k-1)$	1.k
The line $x = k$	$(k, k-1)$ $(k, k+1)$	$\cdots$ $\cdots$	$(k, 2)$ $(k, k+(k-2))$	$(k, 1)$ $(k, k+(k-1))$	$(k, 0)$ $(k, k+k)$
Ranking	2		$k-1$	k	$k+1$
The line $y = k$	$(k-1, k)$ $(k+1, k)$	$\cdots$ $\cdots$	$(2, k)$ $(k+(k-2), k)$	$(1, k)$ $(k+(k-1), k)$	$(0, k)$ $(k+k, k)$

point quartets from 2.1 to the right after the point pair in 2; as in 2 for 3 and 3.1; and finally the point pair at $k+1$ and the point quartets from $(k+1)$.1 to the right.

Table 5, which is similar to Table 4, is obtained for the points of the affine plane of order p^r where p^r is even and $k = (p^r/2)$. However, in this case, the quadrilateral A', B', C', D' instead of the point K in the middle of Figure 9 will correspond (see Table 5).

The most advantageous ranking of points is given in Table 5 and is as follows: point quartets in 1 to the right, then point quartets in 2 to the right, and so on, finally the point quartets in k to the right. Thus, the points A', B', C', D' are in the most advantageous point position.

By classifying the points of the affine plane of order p^r as above, the following two results are obtained:

(1) There are $((p^r - 1)/2)^2 + (p^r - 1) + 1$ different points of the plane when p^r is odd

(2) There are $(p^r/2)^2$ different points of the plane when p^r is even

When p^r is odd and $k = (p^r - 1)/2$, all circles of radius 1, 2, 3, . . ., k that can be drawn from the point $K = (k, k)$ have $2(p^r + 1)$ points, which is the maximum number of points that can be found on the circles of radius 1, 2, 3, . . ., k that are different. If we add the points on these circles and add the point K, we reach the total number of points in the affine

Table 4: The ranking of $(p^r-1)^2$ points of the affine plane of order p^r.

Ranking	Region					
2.1	1	$A=(k-1,k+1)$	$(k-1,k+2)$	$(k-1,k+3)$	$\cdots$	$(k-1,k+k)$
	2	$B=(k-1,k-1)$	$(k-1,k-2)$	$(k-1,k-3)$	$\cdots$	$(k-1,k-k)$
	3	$C=(k+1,k+1)$	$(k+1,k+2)$	$(k+1,k+3)$	$\cdots$	$(k+1,k+k)$
	4	$D=(k+1,k-1)$	$(k+1,k-2)$	$(k+1,k-3)$	$\cdots$	$(k+1,k-k)$
3.1	1	$(k-2,k+1)$	$(k-2,k+2)$	$(k-2,k+3)$	$\cdots$	$(k-2,k+k)$
	2	$(k-2,k-1)$	$(k-2,k-2)$	$(k-2,k-3)$	$\cdots$	$(k-2,k-k)$
	3	$(k+2,k+1)$	$(k+2,k+2)$	$(k+2,k+3)$	$\cdots$	$(k+2,k+k)$
	4	$(k+2,k-1)$	$(k+2,k-2)$	$(k+2,k-3)$	$\cdots$	$(k+2,k-k)$
4.1	1	$(k-3,k+1)$	$(k-3,k+2)$	$(k-3,k+3)$	$\cdots$	$(k-3,k+k)$
	2	$(k-3,k-1)$	$(k-3,k-2)$	$(k-3,k-3)$	$\cdots$	$(k-3,k-k)$
	3	$(k+3,k+1)$	$(k+3,k+2)$	$(k+3,k+3)$	$\cdots$	$(k+3,k+k)$
	4	$(k+3,k-1)$	$(k+3,k-2)$	$(k+3,k-3)$	$\cdots$	$(k+3,k-k)$
		$\vdots$	$\vdots$	$\vdots$		$\vdots$
$(k+1).1$	1	$(k-k,k+1)$	$(k-k,k+2)$	$(k-k,k+3)$	$\cdots$	$(k-k,k+k)$
	2	$(k-k,k-1)$	$(k-k,k-2)$	$(k-k,k-3)$	$\cdots$	$(k-k,k-k)$
	3	$(k+k,k+1)$	$(k+k,k+2)$	$(k+k,k+3)$	$\cdots$	$(k+k,k+k)$
	4	$(k+k,k-1)$	$(k+k,k-2)$	$(k+k,k-3)$	$\cdots$	$(k+k,k-k)$

Table 5: The ranking of all points of the affine plane of order p^r.

Ranking	Region					
1	1	$A'=(k-1,k)$	$(k-1,k+1)$	$(k-1,k+2)$	$\cdots$	$(k-1,k+(k-1))$
	2	$B'=(k-1,k-1)$	$(k-1,k-2)$	$(k-1,k-3)$	$\cdots$	$(k-1,k-k)$
	3	$C'=(k,k)$	$(k,k+1)$	$(k,k+2)$	$\cdots$	$(k,k+(k-1))$
	4	$D'=(k,k-1)$	$(k,k-2)$	$(k,k-3)$	$\cdots$	$(k,k-k)$
2	1	$(k-2,k)$	$(k-2,k+1)$	$(k-2,k+2)$	$\cdots$	$(k-2,k+(k-1))$
	2	$(k-2,k-1)$	$(k-2,k-2)$	$(k-2,k-3)$	$\cdots$	$(k-2,k-k)$
	3	$(k+1,k)$	$(k+1,k+1)$	$(k+1,k+2)$	$\cdots$	$(k+1,k+(k-1))$
	4	$(k+1,k-1)$	$(k+1,k-2)$	$(k+1,k-3)$	$\cdots$	$(k+1,k-k)$
3	1	$(k-3,k)$	$(k-3,k+1)$	$(k-3,k+2)$	$\cdots$	$(k-3,k+(k-1))$
	2	$(k-3,k-1)$	$(k-3,k-2)$	$(k-3,k-3)$	$\cdots$	$(k-3,k-k)$
	3	$(k+2,k)$	$(k+2,k+1)$	$(k+2,k+2)$	$\cdots$	$(k+2,k+(k-1))$
	4	$(k+2,k-1)$	$(k+2,k-2)$	$(k+2,k-3)$	$\cdots$	$(k+2,k-k)$
$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
k	1	$(k-k,k)$	$(k-k,k+1)$	$(k-k,k+2)$	$\cdots$	$(k-k,k+(k-1))$
	2	$(k-k,k-1)$	$(k-k,k-2)$	$(k-k,k-3)$	$\cdots$	$(k-k,k-k)$
	3	$(k+(k-1),k)$	$(k+(k-1),k+1)$	$(k+(k-1),k+2)$	$\cdots$	$(k+(k-1),k+(k-1))$
	4	$(k+(k-1),k-1)$	$(k+(k-1),k-2)$	$(k+(k-1),k-3)$	$\cdots$	$(k+(k-1),k-k)$

plane. All circles that can be drawn from the point K give a partition of the points of the affine plane. This case is valid for all points of the plane, since all points of the plane are on lines passing through a point.

When $p^r=2$, there are 3 points on the circle of radius 1. When $p^r>2$ is even and $k=(p^r/2)$, there are a maximum of $2(p^r+1)$ points on the circle of radius 1. All circles of radius 1, 2, 3, …, k that can be drawn from each of the points A', B', C', D' have $2(p^r+1), 2(p^r+1), \ldots, 2(p^r+1), p^r+1$ points, respectively, and these points are different. If we add the points on these circles and add the centre, we reach the total number of points in the affine plane. The results we express for a partition in the previous case are then also valid in this case.

This classification obtained for affine planes can be found for projective planes of order p^r.

Now, we would like to carry over the concept of an ellipse, which is a well-known concept from the Euclidean plane, to affine planes of order p^r. In order to find the points of the ellipse, for which the sum of the distances to the foci is s for each distance a between the focal points xy and zt for $1\le a\le p^r-1$, the steps below are followed, where $u\longleftrightarrow v$ means both v to u and u to v:

(1) For $s=2$, the intersection points of circles with distances $1\longleftrightarrow 1$ to the foci are found

(2) For $s=3$, the intersection points of circles with distances $1\longleftrightarrow 2$ to the foci are found

(3) For $s=4$, the intersection points of circles with distances $1\longleftrightarrow 3$ and $2\longleftrightarrow 2$ to the foci are found

(4) For $s=p^r-1$, the intersection points of circles with distances $1\longleftrightarrow p^r-2$, $2\longleftrightarrow p^r-3$, …, and finally $((p^r-2)/2)\longleftrightarrow p^r-(p^r/2)$ if p^r is even (or $((p^r-1)/2)\longleftrightarrow p^r-((p^r+1)/2)$ if p^r is odd) to the foci are found, and so on

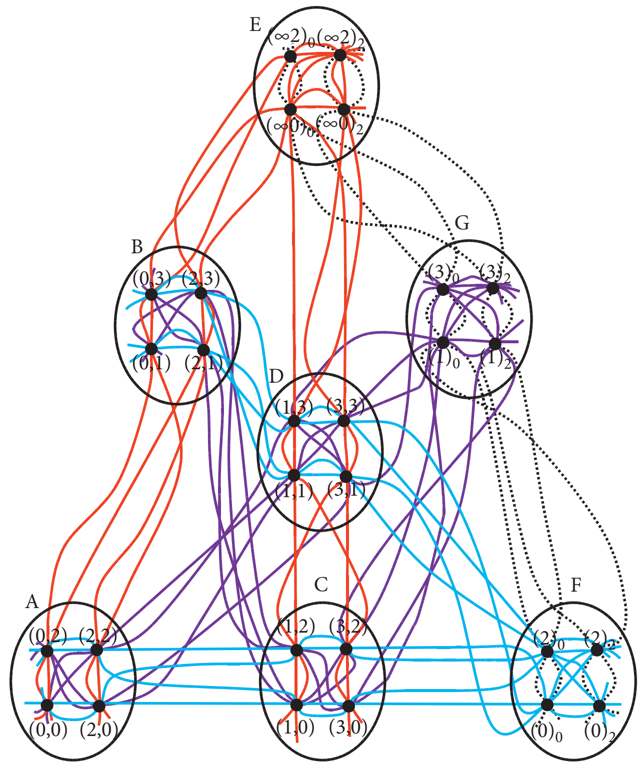

FIGURE 10: An example of a PK plane of order 2.

(5) In the final step, for $s = p^r + i$ such that $0 \le i \le p^r - 2$, the intersection points of circles with distances $i+1 \longleftrightarrow p^r - 1$, $i+2 \longleftrightarrow p^r - 2$, $\ldots$, and finally $(s/2) \longleftrightarrow s - (s/2)$ if s is even (or $((s-1)/2) \longleftrightarrow s - ((s-1)/2)$ if s is odd) to the foci are found

Now, let us carry over the concept of a hyperbola, which is a well-known concept from the Euclidean plane, to affine planes of order p^r. To find the points of the hyperbola, for which the difference of the distances to the foci is s for each distance a between the focal points xy and zt for $1 \le a \le p^r - 1$, the steps below are followed, where $u \longleftrightarrow v$ means both v to u and u to v:

(1) For $s = 0$, the intersection points of circles with distances $1 \longleftrightarrow 1$, $2 \longleftrightarrow 2$, $\ldots$, and finally $p^r - 1 \longleftrightarrow p^r - 1$ to the foci are found

(2) For $s = 1$, the intersection points of circles with distances $1 \longleftrightarrow 2$, $2 \longleftrightarrow 3$, $\ldots$, and finally $p^r - 2 \longleftrightarrow p^r - 1$ to the foci are found

(3) For $s = 2$, the intersection points of circles with distances $1 \longleftrightarrow 3$, $2 \longleftrightarrow 4$, $\ldots$, and finally $p^r - 3 \longleftrightarrow p^r - 1$ to the foci are found, and so on

(4) In the final step, for $s = p^r - 2$, the intersection points of circles with distances $1 \longleftrightarrow p^r - 1$ to the foci are found

Thus, we can state the following result. □

Corollary 8. *All ellipses (or hyperbolas) for which the distance between the focal points xy and zt is a where $1 \le a \le p^r - 1$ and the sum (or difference) of the distances to the foci is s give a partition for the points of an affine plane of order p^r.*

Note that the above processes given for finding the points of ellipses and hyperbolas and Corollary 8 are valid for the projective planes of order p^r, with small differences.

By using the processes for ellipses and hyperbolas, we leave the reader to find all ellipses and hyperbolas of the affine and projective planes of order 3 in Figures 3 and 4, respectively.

Now, we will give definition of "distance of a point to a line" which is valid in the affine planes of order p^r. Assume that we have a line l and a point P not on the line in an affine plane of order p^r. We then know that there is a unique line parallel to the line l and passing through the point P. The distance between the two parallel lines will be defined as the distance to the line l of the point P. Each line in the other parallel classes is named $0y$ since $0y$ is the point on the y-axis of this line, except for the parallel class of lines parallel to the y-axis. Each line in the remaining parallel class is named $0x$, to be compatible with the others, since $0x$ is the point on the x-axis of the line. Thus, the distance between two lines called $0z$ and $0w$ in a parallel class is defined as $|z - w|$. For such an example, see Figure 3(b). In this example, the distances of point 00 to, respectively, the purple line P.02, the red line P.00, the green line P.01, and the blue line P.01 are 2, 0, 1, and 1.

We can then carry over the concept of a parabola, which is a well-known concept from the Euclidean plane, to affine planes of order p^r.

We leave the reader to find all parabolas of the affine plane of order 3 in Figure 3.

Finally, we give an example of a finite PK plane in which the projective plane lying under its epimorphic image is a Fano plane. For the incidence table of the plane in Figure 10 coordinatized by the local ring $\mathbb{Z}_{2^2}$with the help of the representations in the coordinatization given for PK planes in [34], see [27].

We know from the geometry that we have studied that the distance between points in the same neighbourhood is zero: that is, there is no significant difference between these points, but these points are connected to each other with a certain relationship. We can think of the equivalence of points in the same neighbourhood in graph theory applications as a traffic or transaction density at these corner points. Therefore, each vertex and edge of finite projective graphs and affine graphs obtained from finite projective and affine planes, respectively, can be extended by neighbourhood classes based on a certain relationship. Thus, these projective and affine graphs can now be treated as finite projective Klingenberg graphs (PK graphs) and affine Klingenberg graphs (AK graphs), respectively. Note that if we remove the infinite line {E,G,F} from the finite PK plane in Figure 10, with all the points on it, we obtain an example of an AK plane of order 2. A finite PK graph and an AK graph, with neighbourhood classes consisting of a single point and a single line will be, respectively, finite projective and affine graphs. In finite affine and projective graphs, the results we have obtained for nonneighbour points are valid for finite PK graphs and AK graphs.

5. Conclusions

In a finite projective Klingenberg plane, or a finite projective plane lying under such a plane as an epimorphic image, or a finite affine plane obtained by throwing a line in the finite projective plane, a definition of distance has been given that is not encountered in the literature. This definition of distance was obtained with the help of the rank of a matrix in an octonion plane whose points and lines are represented by matrices. By selecting a line as a path in a finite plane of order p^r, graphs called affine and projective graphs, whose degree sequence is exactly known and connected, were obtained from this finite plane. With the help of this distance definition, the most central point problem in affine and projective graphs was discussed and the solution was obtained. Moreover, the equivalents of the concepts of circle, ellipse, hyperbola, and parabola in these finite planes were examined. Finally, the results obtained were interpreted on affine and projective Klingenberg graphs and the conclusion was reached that such graphs, obtained from finite geometries, can be used as a tool for solving some important real-life problems. For example, these graph models may be the most economical and profitable models for problems such as establishing sewage, water, train, bus, or residential networks in environmental and city planning, and also the production or distribution bands of a business.

Data Availability

No data were used to support this study.

Disclosure

A part of this study contains a result of the PhD thesis of I. Dogan.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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