Research Article

Number of Distinct Homomorphic Images in Coset Diagrams

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The representation of the action of $\text{PGL}(2, \mathbb{Z})$ on $F_t \cup \{\infty\}$ in a graphical format is labeled as coset diagram. These finite graphs are acquired by the contraction of the circuits in infinite coset diagrams. A circuit in a coset diagram is a closed path of edges and triangles. If one vertex of the circuit is fixed by $(pq)^{\Delta_1} (pq^{-1})^{\Delta_2} (pq)^{\Delta_3} \dots (pq^{-1})^{\Delta_m} \in \text{PSL}(2, \mathbb{Z})$, then this circuit is titled to be a length- m circuit, denoted by $(\Delta_1, \Delta_2, \Delta_3, \dots, \Delta_m)$. In this manuscript, we consider a circuit Δ of length 6 as $(\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5, \Delta_6)$ with vertical axis of symmetry, that is, $\Delta_2 = \Delta_6, \Delta_3 = \Delta_5$. Let Γ_1 and Γ_2 be the homomorphic images of Δ acquired by contracting the vertices a, u and b, v , respectively, then it is not necessary that Γ_1 and Γ_2 are different. In this study, we will find the total number of distinct homomorphic images of Δ by contracting its all pairs of vertices with the condition $\Delta_1 > \Delta_2 > \Delta_3 > \Delta_4$. The homomorphic images are obtained in this way having versatile applications in coding theory and cryptography. One can attain maximum nonlinearity factor using this in the encryption process.

1. Introduction

It is prominent that the finite presentation $\langle p, q; p^2 = q^3 = 1 \rangle$ is known as the modular group $\text{PSL}(2, \mathbb{Z})$ generated by the linear fractional transformations $p: \chi \longrightarrow (-1/\chi)$ and $q: \chi \longrightarrow ((\chi - 1)/\chi)$. In [1], Akbas discussed sub-orbitalgraphs for the modular group by showing that these graphs contains no circuit if and only if it contains no triangles. If we insert an extra generator $r = (1/\chi)$ with p and q , another group is emerged, denoted as $\text{PGL}(2, \mathbb{Z})$ [2], an extension of $\text{PSL}(2, \mathbb{Z})$ with the finite presentation as

$$\langle p, q, r; p^2 = q^3 = r^2 = (pr)^2 = (qr)^2 = 1 \rangle. \quad (1)$$

In 1978, Professor Graham Higman propounds an unfamiliar type of a graph, titled as coset diagram, which presents the action of $\text{PGL}(2, \mathbb{Z})$ on $\text{PL}(F_t)$, where F_t is a finite field and t shows a prime power. In 1983, this foundation is laid by Qaisar Mushtaq [3]. Small triangles are proposed for the cycle q^3 , such that q permutes the vertices of triangles in the opposite direction of rotation of clock and an edge is attached to any two vertices that are interchanged by p . Heavy dots represent the

fixed points of p and q . Note that $(pr)^2 = 1$ equals $rqr = q^{-1}$, which means r reverses the triangle orientation proposed for the cycle q^3 . For that reason, the diagram need not to be made more perplexing by interjecting r edges.

A coset diagram (subdiagram) Γ_1 is said to be a homomorphic image of the coset diagram (subdiagram) Γ_2 if and only if $|V(\Gamma_1)| < |V(\Gamma_2)| \forall a \in V(\Gamma_2)$ with $(a)h = a$, where $h \in \text{PSL}(2, \mathbb{Z})$, there exist a vertex u in $V(\Gamma_1)$ such that $(u)h = u$.

Coset diagrams obtained from the action of $\text{PSL}(2, \mathbb{Z})$ over Q_ϵ are infinite graphs [4], where $Q_\epsilon = \{a_1 + a_2\sqrt{\epsilon}; a_1, a_2 \in \mathbb{Q} \text{ and } \epsilon \in \mathbb{Z}^+ \text{ is a square free}\}$. These diagrams are not easy to study because they are infinite. Thence, coset diagrams are considered as important for the action of $\text{PSL}(2, \mathbb{Z})$ on $\text{PL}(F_q)$ because this action presents finite graphs. The number $((x_1 + \sqrt{\epsilon})/x_2)$ is an expression of the number $a_1 + a_2\sqrt{\epsilon} \in Q_\epsilon$, where $(x_1, x_2, ((x_1^2 - \epsilon)/x_2)) = 1$. These finite coset diagrams are the homomorphic images of the coset diagrams for $((x_1 + \sqrt{\epsilon})/x_2)$, where $\epsilon \equiv z^2 \pmod{p}$ for some $z \in N$.

To explain more, coset diagram in Figure 1 illustrates the action on $\text{PL}(F_{17})$ by $\text{PGL}(2, \mathbb{Z})$ with permutation

FIGURE 1: Coset diagram for the action of $\text{PGL}(2, \mathbb{Z})$ on $\text{PL}(F_{17})$.

representations p, q , and r by $(\chi)p = (-1/\chi)$, $(\chi)q = ((\chi - 1)/\chi)$, and $(\chi)r = (1/\chi)$, respectively, as

$$p: (0 \infty), (1 \ 16), (2 \ 8), (3 \ 11), (4), (5 \ 10), (6 \ 14), (7 \ 12), (9 \ 15), (13),$$

$$q: (0 \infty \ 1), (2 \ 9 \ 16), (3 \ 12 \ 8), (4 \ 5 \ 11), (6 \ 15 \ 10), (7 \ 13 \ 14), \quad (2)$$

$$r: (0 \infty), (1), (2 \ 9), (3 \ 6), (4 \ 13), (5 \ 7), (8 \ 15), (10 \ 12), (11 \ 14), (16).$$

Thus, $13 = 8^2 \bmod 17$ gives that the coset diagram $\text{PGL}(2, 17)$ is a homomorphic image of the coset diagram for $((x_1 + \sqrt{13})/x_2)$.

Coset diagrams obtained from the action of $\text{PSL}(2, \mathbb{Z})$ on $\mathbb{Q}_\varepsilon$ have some attractive narrative. In [5], the quadratic irrational numbers are classified by taking prime modulus that proved helpful in investigating the modular group action on the real quadratic field. The number $\bar{B} = ((x_1 - \sqrt{\varepsilon})/x_2)$ is called the conjugate of $B = ((x_1 + \sqrt{\varepsilon})/x_2)$, where x_1 and x_2 are integers and ε is a fixed number from $\mathbb{Z}^+$, which is not a perfect square. B is said to be an ambiguous number [6], if the sign of B is different from the sign of $\bar{B}$. B is said to be a totally negative (positive) if B and $\bar{B}$ both have the same signs. For a fixed ε , the number of ambiguous numbers of the form $B = ((x_1 + \sqrt{\varepsilon})/x_2)$ is finite and that segment of the coset diagram attained by the ambiguous numbers forms a closed path (circuit) and it is the only closed path in B -orbit [4]. With the help of coset diagram, Anna Torstensson not only described the applications to study the finitely presented group but also discussed the one-relator quotients of the modular group [7].

A closed path of triangles and edges in a coset diagram is called a circuit. In a coset diagram, a circuit is said to be a length- k circuit, denoted by $(\Delta_1, \Delta_2, \Delta_3, \dots, \Delta_k)$, if its one vertex is fixed by

$$(ax)^{\Delta_1} (ax^{-1})^{\Delta_2} (ax)^{\Delta_3} \dots (ax^{-1})^{\Delta_k} \in \text{PSL}(2, \mathbb{Z}). \quad (3)$$

Alternatively, it means that one vertex of the Δ_1 triangles lies outside of the circuit and one vertex of the Δ_2 triangles lies inside of the circuit and likewise. Since $(\Delta_1, \Delta_2, \Delta_3, \dots, \Delta_k)$ is a cycle, so it does not matter if one

vertex of the Δ_1 triangles lies inside of the circuit and one vertex of the Δ_2 triangles lies outside of the circuit and likewise. Note that k is always even.

The circuit of the type $(\Delta_1, \Delta_2, \Delta_3, \dots, \Delta_l, \Delta_1, \Delta_2, \Delta_3, \dots, \Delta_l, \dots, \Delta_1, \Delta_2, \Delta_3, \dots, \Delta_l)$ is termed as a periodic circuit with period of length l .

Note 1. By $V(\Delta)$, we mean the collection of vertices lies on the circuit $\Delta = (\Delta_1, \Delta_2, \Delta_3, \dots, \Delta_k)$.

Let $a, u \in V(\Delta)$ be any two vertices fixed by the words h_1 and h_2 , that is, $(a)h_1 = a$ and $(u)h_2 = u$. Suppose h_3 is the word that maps a to u , then $(a)h_1^{-1}h_3 = u$. Note that h_3 and $h_1^{-1}h_3$ are the only two paths that assign a to u . Now, by contraction of the pair of vertices a and u , we mean that a and u melt together to become one node $s = a = u$ such that $(s)h_3 = (s)h_1^{-1}h_3 = s$. As a result of this contraction, a closed path Γ is created that contains the vertex s fixed by h_3 and $h_1^{-1}h_3$. This closed path Γ is the homomorphic image of the circuit Δ . It is important to note that a and u is not the only pair of contraction in Δ that creates homomorphic image Γ . There are also many pairs of contraction other than a and u that create the same homomorphic image Γ . The following theorems proved in [8] will help us to find the total number of such contracted pairs that produce the same homomorphic image Γ of Δ .

Theorem 1. *Let the vertices a and u in Δ are contracted and a homomorphic image Γ of Δ is evolved, then Γ is also obtainable if the vertices $(a)h$ and $(u)h$ in Δ are contracted.*

Theorem 2. *If a and u are contracted to obtain Γ , then during this process $|E|$ number of pairs are contracted all together, where E is the collection of words such that $\forall h \in E, (a)h$ and $(u)h$ are contained by Δ .*

Example 1. Let us contract the vertices a and u from the circuit $(3, 4, 2, 4, 2, 4)$ (Figure 2) and acquire a homomorphic Γ (Figure 3) of the circuit $(3, 4, 2, 4, 2, 4)$. Thus, $h_1 = q(pq)$ and $h_2 = (pq)(pq^{-1})^4(pq)^2(pq^{-1})^4(pq)^2(pq^{-1})^4p$ are the two possible paths between a and u that are fixing the vertex $s = a = u$ in Γ .

Let E be the family of words such that for all $h \in E$ implies $(a)h$ and $(u)h$ lie on the circuit $(3, 4, 2, 4, 2, 4)$, then $E = \{e, q, q^{-1}, p, pq, pq^{-1}, (pq)p, (pq)^2, (pq)pq^{-1}\}$. Then by Theorem 2, the cardinality of E implies that there are 9 pairs of vertices contracted to generate the homomorphic image Γ .

Note that the cardinality of E does not give the total number of contracted pairs to generate the homomorphic image Γ . In the following, we will discuss the process to find the total number of contracted pairs to generate Γ .

Let Γ^* denote itself as the mirror image of Γ . Thus, the permutation r ensures that the coset diagram is symmetric along the vertical axis. This implies Γ^* will assuredly occur.

If $u = pq^{\pi_1}pq^{\pi_2} \dots pq^{\pi_n}(\pi_i = 1 \text{ or } -1)$ is a word, then $u^* = pq^{-\pi_1}pq^{-\pi_2} \dots pq^{-\pi_n}$. If the word h fixes the vertex s , then the vertex s^* is fixed by u^* .

FIGURE 2: Circuit (3, 4, 2, 4, 2, 4).

FIGURE 3: Homomorphic image Γ .

A homomorphic image Γ has a symmetry with respect to vertical axis if and only if by contracting a and u , the vertices a^* and u^* are also contracted.

Remark 1. In coset diagrams, r reverses the orientation of the triangles representing the three cycles of q (as reflection does). So corresponding to each vertex s fixed by the pair h_1, h_2 , there is a vertex s^* in Γ^* (mirror image of Γ) such that s^* is a fixed point of h_1^*, h_2^* . In other words, it is created by contracting a^* and u^* . There are certain Γ 's which have a vertical symmetry and so have the same orientations as those of their mirror images. The homomorphic image Γ of a circuit $\Delta = (\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5, \Delta_6)$, which has a vertex s fixed

by the pair h_1, h_2 , has the same orientation as that of its mirror image if and only if there is a vertex s^* in Γ such that $(s^*)h_1 = (s^*)h_2 = s^*$.

1.1. Counting the Number of Pairs of Contracting Vertices of a Homomorphic Image. Let Γ be a homomorphic image of the circuit Δ acquired by the contraction of pair of vertices a and u of Δ . Then by Theorem 2, Γ has $|E|$ number of pairs of vertices. To find the total number of pairs of vertices, one should follow the following steps.

To know how many total pairs of contracting vertices are there, special precaution must be taken.

- (1) If by contracting a and u to create Γ , the pair of vertices a^* and u^* are not contracted, then Γ has different orientation from its mirror image Γ^* . So, there are $|E|$ number of more pairs of vertices for the mirror image of Γ .
- (2) If by contracting a and u to create Γ , the pair of vertices a^* and u^* are also contracted, then Γ has the same orientation as that of its mirror image Γ^* . So, in this case, Γ has $|E|$ number of pairs of contracted vertices.

Consider a circuit of length 6 as $\Delta = (\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5, \Delta_6)$ (Figure 4) with vertical axis of symmetry, that is, $\Delta_2 = \Delta_6, \Delta_3 = \Delta_5$. Suppose $\Delta_1 > \Delta_2 > \Delta_3 > \Delta_4$. The coset diagrams are composed of circuits. The vertices of the circuits in infinite diagrams are contracted in a certain way, and a finite coset diagram evolves. It is therefore necessary to ask how many distinct homomorphic images are obtained if we contract all the pairs of vertices of the circuit Δ ? We not only give the answer to this question for a circuit Δ but also mention those pairs of vertices which are “important”. There is no need to contract the pairs which are not mentioned as “important”. If we contract those, we obtain a homomorphic image, which we have already obtained by contracting “important” pairs.

Note 2. It is clear from Figure 4 that

- (1) The mirror image of the vertex e_l^k is $e_{3\Delta_k - (l-1)}^{2-k}$, that is, $e_l^{k*} = e_{3\Delta_k - (l-1)}^{2-k}$
- (2) The vertex a_{3m+1}^k is fixed by the word $(pq^{-1})^m (pq)^{\frac{\Delta_{k+5}}{\Delta_{k+4}} (pq^{-1})^{\frac{\Delta_{k+3}}{\Delta_{k+2}} (pq)^{\frac{\Delta_{k+1}}{\Delta_{k-1}} (pq^{-1})^{\Delta_{k-m}}}}$
- (3) The vertex x_{3m}^k is fixed by the word $(pq)^{\frac{\Delta_{k+1}}{\Delta_{k-m}} (pq)^{\frac{\Delta_{k+2}}{\Delta_{k+3}} (pq^{-1})^{\frac{\Delta_{k+4}}{\Delta_{k+5}} (pq)^m}}$, where

$$k \in \{1, 2, 3, 4, 5, 6\}; l = 1, 2, 3, \dots, \Delta_k; m =$$

$$1, 2, 3, \dots, \Delta_k - 1, \underline{n} = \begin{cases} 6, & \text{if } n = 0, 6 \\ n \pmod{6} & \text{otherwise} \end{cases}$$

$$\text{and } \pm = \begin{cases} + & \text{if } k \text{ is odd} \\ - & \text{otherwise} \end{cases}.$$

FIGURE 4: The circuit $\Delta = (\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5, \Delta_6)$.

Lemma 1. *If we contract the vertices $e_{3l_1+1}^1$: $l_1 = 0, 1, 2, \dots, \Delta_2 - 1$, with the vertex $e_{3\Delta_1}^1$ in the circuit Δ , then there arise Δ_2 distinct homomorphic images of Δ and $3(l_1 + 2)$ pairs of contracted vertices create each homomorphic image. Moreover, the number of total pairs of contracted vertices to generate all Δ_2 and their mirror homomorphic images of Δ is $3(\Delta_2^2 + 3\Delta_2 - 2)$.*

Proof. Let $H_1 = \{\Omega_{l_1}^1; l_1 = 0, 1, 2, \dots, \Delta_2 - 1\}$ (Figure 5) be the collection of homomorphic images of Δ acquired by the contraction of the vertices $e_{3l_1+1}^1$: $l_1 = 0, 1, 2, \dots, \Delta_2 - 1$ with the vertex $e_{3\Delta_1}^1$ in Δ , where $e_{3l_1+1}^1$ and $e_{3\Delta_1}^1$ are fixed by the words $(pq)^{l_1}(pq^{-1})^{\Delta_6}(pq)^{\Delta_5}(pq^{-1})^{\Delta_4}(pq)^{\Delta_3}(pq^{-1})^{\Delta_2}(pq)^{\Delta_1-l_1}$ and $(pq)^{\Delta_2}(pq^{-1})^{\Delta_3}(pq)^{\Delta_4}(pq^{-1})^{\Delta_5}(pq)^{\Delta_6}(pq^{-1})^{\Delta_1}$. It is easy to verify that $P_1 = q^{-1}(pq^{-1})^{\Delta_1-l_1-1}$ and $P_2 = (pq)^{\Delta_2}(pq^{-1})^{\Delta_3}(pq)^{\Delta_4}(pq^{-1})^{\Delta_5}(pq)^{\Delta_6}(pq^{-1})^{l_1}p$ are the possible paths between $e_{3l_1+1}^1$ and $e_{3\Delta_1}^1$ (Figure 4). This implies that for each l_1 , the homomorphic image $\Omega_{l_1}^1$ has a vertex s fixed by P_1 and P_2 .

Thus,

$$E_1 = \{e, q, q^{-1}, p, pq, pq^{-1}, pqp, (pq)^2, pqpq^{-1}, \dots,$$

$$(pq)^{l_1}p, (pq)^{l_1+1}, (pq)^{l_1}pq^{-1}\} \quad (4)$$

is the family of elements in $\text{PSL}(2, \mathbb{Z})$ such that $\forall x \in E, \Delta$ contains the vertices $(e_{3l_1+1}^1)x$ and $(e_{3\Delta_1}^1)x$. This gives that the cardinality of E_1 , that is, $3(l_1 + 2)$ is the number of contracted pairs of vertices to produce the homomorphic image $\Omega_{l_1}^1$ (Theorem 2). For $m \neq n$, let Ω_m^1 and Ω_n^1 be any two elements of H_1 , then the number of triangles in Ω_m^1 and Ω_n^1 are not equal (Figure 5). This implies that all the elements in H_1 are different and no one is the mirror image of the other. This

FIGURE 5: Homomorphic images $\Omega_{l_1}^1$.

further forms the result as $|H_1| = \Delta_2$. Thence, the number of contracted pairs of vertices of Δ to create all the elements of H_1 is $\sum_{l_1=0}^{\Delta_2-1} 3(l_1 + 2)$.

From Figure 5, it is also clear that no element of H_1 except Ω_0^1 has vertical axis of symmetry. So, Ω_0^1 is the only homomorphic image whose orientation is not different from its mirror image Ω_0^{1*} and all the remaining $\Delta_2 - 1$ elements of H_1 have different orientations from their mirror images. Hence, there are

$$6 \sum_{l_1=1}^{\Delta_2-1} (l_1 + 2) + 6 = 3(\Delta_2^2 + 3\Delta_2 - 2) \quad (5)$$

pairs of contracted vertices to produce all the homomorphic images in H_1 . $\square$

Lemma 2. *If we contract the vertices $e_{3l_1+1}^2$: $l_1 = 0, 1, 2, \dots, \Delta_2 - 1$, with the vertex $e_{3\Delta_1}^1$ in the circuit Δ , then there arise Δ_2 distinct homomorphic images of Δ and $3(l_1 + 2)$ pairs of contracted vertices create each homomorphic image. Moreover, the number of total pairs of contracted vertices to generate all Δ_2 and their mirror homomorphic images of Δ is $3(\Delta_2^2 + 3\Delta_2)$.*

Let $H_2 = \{\Omega_{l_1}^2; l_1 = 0, 1, 2, \dots, \Delta_2 - 1\}$ be the collection of homomorphic images of Δ acquired by the contraction of the vertices $e_{3l_1+1}^2; l_1 = 0, 1, 2, \dots, \Delta_2 - 1$ with the vertex $e_{3\Delta_1}^1$ in Δ . Suppose r shows itself as the remainder of $(l_1/(\Delta_2 - l_1))$. Then, graphically we make four partitions of H_2 as follows:

- (i) $\Omega_{l_1}^2: \Delta_2 - 2l_1 > 1$ (Figure 6(a))
- (ii) $\Omega_{l_1}^2: \Delta_2 - 2l_1 = 1$ (Figure 6(b))
- (iii) $\Omega_{l_1}^2: \Delta_2 - 2l_1 < 1$ and $\Delta_2 - l_1 > r + 1$ (Figure 6(c))
- (iv) $\Omega_{l_1}^2: \Delta_2 - 2l_1 < 1$ and $\Delta_2 - l_1 = r + 1$ (Figure 6(d))

From all the homomorphic images presented in these figures, it is not intricated to check that no one is the mirror image of itself. This lemma can be proved by using the same procedure as that for Lemma 1.

Let $k_1 \in \{3, 4, 5\}$.

Lemma 3. *If we contract the vertices $e_{3l_{(2,k_1)}+1}^{k_1}; l_{(2,k_1)} = 0, 1, 2, \dots, \Delta_{k_1} - 1$, with the vertex $e_{3\Delta_1}^1$ in the circuit Δ , then for each k_1 , there arise Δ_{k_1} distinct homomorphic images of Δ and $3(l_{(2,k_1)} + 2)$ pairs of contracted vertices that create each homomorphic image. Moreover, the number of total pairs of contracted vertices to generate all Δ_{k_1} and their mirror homomorphic images of Δ is $3(\Delta_{k_1}^2 + 3\Delta_{k_1})$.*

For a fix value of k_1 , let $H_3^{k_1} = \{\Omega_{l_{(2,k_1)}}^3; l_{(2,k_1)} = 0, 1, 2, \dots, \Delta_{k_1} - 1\}$ be the collection of homomorphic images of Δ acquired by the contraction of the vertices $e_{3l_{(2,k_1)}+1}^{k_1}; l_{(2,k_1)} = 0, 1, 2, \dots, \Delta_{k_1} - 1$, with the vertex $e_{3\Delta_1}^1$ in Δ . Figures 7–9 present $H_3^{k_1}$ graphically. From all the homomorphic images presented in these figures, it is not intricated to check that no one is the mirror image of itself.

This lemma can be proved by using the same procedure as that for Lemma 1.

Lemma 4. *If we contract the vertices $e_{3l_3+1}^6; l_3 = 0, 1, 2, \dots, \Delta_6 - 1$, with the vertex $e_{3\Delta_1}^1$ in the circuit Δ , then there arise Δ_6 distinct homomorphic images of Δ and $3(l_3 + 2)$ pairs of contracted vertices create each homomorphic image. Moreover, the number of total pairs of contracted vertices to generate all Δ_6 and their mirror homomorphic images of Δ is $(3/2)(\Delta_6^2 + 3\Delta_6)$.*

Let $H_4 = \{\Omega_{l_3}^4; l_3 = 0, 1, 2, \dots, \Delta_6 - 1\}$ be the collection of homomorphic images of Δ acquired by the contraction of the vertices $e_{3l_3+1}^1; l_3 = 0, 1, 2, \dots, \Delta_6 - 1$ with the vertex $e_{3\Delta_1}^1$ in Δ . Figure 10 presents H_4 graphically. From all the homomorphic images presented in Figure 10, it is not intricated to check that every one is the mirror image of itself.

This lemma can be proved by using the same procedure as that for Lemma 1.

Lemma 5. *If we contract the vertices $e_{3l_4+1}^1; l_4 = 1, 2, \dots, \Delta_1 - 1$, with the vertex $e_{3\Delta_2}^2$ in the circuit Δ , then there arise $\Delta_1 - 1$ distinct homomorphic images of Δ and $3(l_4 + 2)$ pairs of contracted vertices create each homomorphic image. Moreover, the number of total pairs of contracted vertices to*

generate all $\Delta_1 - 1$ and their mirror homomorphic images of Δ is $(3/2)(\Delta_1^2 + 3\Delta_1 - 4)$.

Let $H_5 = \{\Omega_{l_4}^5; l_4 = 1, 2, \dots, \Delta_1 - 1\}$ be the collection of homomorphic images of Δ acquired by the contraction of the vertices $e_{3l_4+1}^1; l_4 = 1, 2, \dots, \Delta_1 - 1$, with the vertex $e_{3\Delta_2}^2$ in Δ . Suppose r shows itself as the remainder of $(l_4/(\Delta_1 - l_4))$, then graphically, we make four partitions of H_5 as follows:

- (i) $\Omega_{l_4}^5: \Delta_1 - 2l_4 > 1$ (Figure 11(a))
- (ii) $\Omega_{l_4}^5: \Delta_1 - 2l_4 = 1$ (Figure 11(b))
- (iii) $\Omega_{l_4}^5: \Delta_1 - 2l_4 < 1$ and $\Delta_1 - l_4 > r + 1$ (Figure 11(c))
- (iv) $\Omega_{l_4}^5: \Delta_1 - 2l_4 < 1$ and $\Delta_1 - l_4 = r + 1$ (Figure 11(d))

From all the homomorphic images presented in these figures, it is not intricated to check that every one is the mirror image of itself. This lemma can be proved by using the same procedure as that for Lemma 1.

Lemma 6. *If we contract the vertices $e_{3l_5+1}^2; l_5 = 1, 2, \dots, \Delta_2 - 1$, with the vertex $e_{3\Delta_2}^2$ in the circuit Δ , then there arise $\Delta_2 - 1$ distinct homomorphic images of Δ and $3(l_5 + 2)$ pairs of contracted vertices create each homomorphic image. Moreover, the number of total pairs of contracted vertices to generate all $\Delta_2 - 1$ and their mirror homomorphic images of Δ is $3(\Delta_2^2 + 3\Delta_2 - 4)$.*

Let $H_6 = \{\Omega_{l_5}^6; l_5 = 1, 2, \dots, \Delta_2 - 1\}$ be the collection of homomorphic images of Δ acquired by the contraction of the vertices $e_{3l_5+1}^2; l_5 = 1, 2, \dots, \Delta_2 - 1$, with the vertex $e_{3\Delta_2}^2$ in Δ . Graphically, we make two partitions of H_5 as follows:

- (i) $\Omega_{l_5}^6: l_5 < \Delta_2 - 1$ (Figure 12(a))
- (ii) $\Omega_{l_5}^6: l_5 = \Delta_2 - 1$ (Figure 12(b))

From all the homomorphic images presented in Figures 12(a) and 12(b), it is not intricated to check that no one is the mirror image of itself. This lemma can be proved by using the same procedure as that for Lemma 1.

Let $k_1 \in \{3, 4, 5\}$.

Lemma 7. *If we contract the vertices $e_{3l_{(6,k_1)}+1}^{k_1}; l_{(6,k_1)} = 1, 2, \dots, \Delta_{k_1} - 1$, with the vertex $e_{3\Delta_2}^2$ in the circuit Δ , then for each k_1 , there arise $\Delta_{k_1} - 1$ distinct homomorphic images of Δ and $3(l_{(6,k_1)} + 2)$ pairs of contracted vertices create each homomorphic image. Moreover, the number of total pairs of contracted vertices to generate all $\Delta_{k_1} - 1$ and their mirror homomorphic images of Δ is $3(\Delta_{k_1}^2 + 3\Delta_{k_1} - 4)$.*

For a fix value of k_1 , let $H_7^{k_1} = \{\Omega_{l_{(6,k_1)}}^7; l_{(6,k_1)} = 1, 2, \dots, \Delta_{k_1} - 1\}$ be the collection of homomorphic images of Δ acquired by the contraction of the vertices $e_{3l_{(6,k_1)}+1}^{k_1}; l_{(6,k_1)} = 1, 2, \dots, \Delta_{k_1} - 1$, with the vertex $e_{3\Delta_2}^2$ in Δ . Figures 13–15 present $H_7^{k_1}$ graphically. From all the homomorphic images presented in these figures, it is not intricated to check that no one is the mirror image of itself. This lemma can be proved by using the same procedure as that for Lemma 1.

Lemma 8. *If we contract the vertices $e_{3l_7+1}^3; l_7 = 1, 2, \dots, \Delta_4 - 1$, with the vertex $e_{3\Delta_3}^3$ in the circuit Δ , then there arise*

FIGURE 6: (a) Homomorphic images $\Omega_{l_1}^2$: $\Delta_2 - 2l_1 > 1$. (b) Homomorphic images $\Omega_{l_1}^2$: $\Delta_2 - 2l_1 = 1$. (c) Homomorphic images $\Omega_{l_1}^2$: $\Delta_2 - 2l_1 < 1$ and $\Delta_2 - l_1 > r + 1$. (d) Homomorphic images $\Omega_{l_1}^2$: $\Delta_2 - 2l_1 < 1$ and $\Delta_2 - l_1 = r + 1$.

FIGURE 7: Homomorphic images $\Omega_{l_{(2,3)}}^3$.

$\Delta_4 - 1$ distinct homomorphic images of Δ and $3(l_7 + 2)$ pairs of contracted vertices create each homomorphic image. Moreover, the number of total pairs of contracted vertices to generate all $\Delta_4 - 1$ and their mirror homomorphic images of Δ is $3(\Delta_4^2 + 3\Delta_4 - 4)$.

Let $H_8 = \{\Omega_{l_7}^8; l_7 = 1, 2, \dots, \Delta_4 - 1\}$ be the collection of homomorphic images of Δ acquired by the contraction of the vertices $e_{3l_7+1}^3; l_7 = 1, 2, \dots, \Delta_4 - 1$, with the vertex $e_{3\Delta_4}^3$ in Δ . Figure 16 presents H_8 graphically. From all the homomorphic images presented in Figure 16, it is not intricate to check that no one is the mirror image of itself. This lemma can be proved by using the same procedure as that for Lemma 1.

Lemma 9. If we contract the vertices $e_{3l_7+1}^4; l_7 = 1, 2, \dots, \Delta_4 - 1$, with the vertex $e_{3\Delta_4}^4$ in the circuit Δ , then there arise $\Delta_4 - 1$ distinct homomorphic images of Δ and $3(l_7 + 2)$ pairs of contracted vertices create each homomorphic image. Moreover, the number of total pairs of contracted vertices to generate all $\Delta_4 - 1$ and their mirror homomorphic images of Δ is $(3/2)(\Delta_4^2 + 3\Delta_4 - 4)$.

Let $H_9 = \{\Omega_{l_7}^9; l_7 = 1, 2, \dots, \Delta_4 - 1\}$ be the collection of homomorphic images of Δ acquired by the contraction of the

vertices $e_{3l_7+1}^4; l_7 = 1, 2, \dots, \Delta_4 - 1$, with the vertex $e_{3\Delta_4}^4$ in Δ . Suppose r shows itself as the remainder of $(l_7/\Delta_4 - l_7)$, then graphically we make four partitions of H_9 as follows:

- (i) $\Omega_{l_7}^9: \Delta_4 - 2l_7 > 1$ (Figure 17(a))
- (ii) $\Omega_{l_7}^9: \Delta_4 - 2l_7 = 1$ (Figure 17(b))
- (iii) $\Omega_{l_7}^9: \Delta_4 - 2l_7 < 1$ and $\Delta_4 - l_7 > r + 1$ (Figure 17(c))
- (iv) $\Omega_{l_7}^9: \Delta_4 - 2l_7 < 1$ and $\Delta_4 - l_7 = r + 1$ (Figure 17(d))

From all the homomorphic images presented in these figures, it is not intricate to check that every one is the mirror image of itself. This lemma can be proved by using the same procedure as that for Lemma 1.

Lemma 10. If we contract the vertices $e_{3l_8+1}^4; l_8 = 0, 1, 2, \dots, \Delta_4 - 1$, with the vertex $e_{3\Delta_4}^4$ in the circuit Δ , then there arise Δ_4 distinct homomorphic images of Δ and $3(l_8 + 2)$ pairs of contracted vertices create each homomorphic image. Moreover, the number of total pairs of contracted vertices to generate all Δ_4 and their mirror homomorphic images of Δ is $3(\Delta_4^2 + 3\Delta_4 - 2)$.

Let $H_{10} = \{\Omega_{l_8}^{10}; l_8 = 0, 1, 2, \dots, \Delta_4 - 1\}$ be the collection of homomorphic images of Δ acquired by the contraction of the vertices $e_{3l_8+1}^4; l_8 = 0, 1, 2, \dots, \Delta_4 - 1$, with the vertex $e_{3\Delta_4}^4$ in Δ . Graphically, we make two partitions of H_{10} as follows:

- (i) $\Omega_{l_8}^{10}: l_8 < \Delta_4 - 1$ (Figure 18(a))
- (ii) $\Omega_{l_8}^{10}: l_8 = \Delta_4 - 1$ (Figure 18(b))

From all the homomorphic images presented in these figures, it is not intricate to check that no one is the mirror image of itself except Ω_0^{10} . This lemma can be proved by using the same procedure as that for Lemma 1.

Lemma 11. If we contract the vertices $e_{3l_9+1}^5; l_9 = 0, 1, 2, \dots, \Delta_5 - 1$, with the vertex $e_{3\Delta_4}^5$ in the circuit Δ , then there arise Δ_5 distinct homomorphic images of Δ and $3(l_9 + 2)$ pairs of contracted vertices create each homomorphic image. Moreover, the number of total pairs of contracted vertices to generate all Δ_5 and their mirror homomorphic images of Δ is $(3/2)(\Delta_5^2 + 3\Delta_5)$.

Let $H_{11} = \{\Omega_{l_9}^{11}; l_9 = 0, 1, 2, \dots, \Delta_5 - 1\}$ be the collection of homomorphic images of Δ acquired by the contraction of the vertices $e_{3l_9+1}^5; l_9 = 0, 1, 2, \dots, \Delta_5 - 1$, with the vertex $e_{3\Delta_4}^5$ in Δ . Figure 19 presents H_{11} graphically. From all the homomorphic images presented in Figure 19, it is not intricate to check that every one is the mirror image of itself. This lemma can be proved by using the same procedure as that for Lemma 1.

Lemma 12. If we contract the vertices $e_{3l_{10}+1}^3; l_{10} = 0, 1, 2, \dots, \Delta_3 - 1$, with the vertex $e_{3\Delta_4}^3$ in the circuit Δ , then there arise Δ_3 distinct homomorphic images of Δ and $3(l_{10} + 2)$ pairs of contracted vertices create each homomorphic image. Moreover, the number of total pairs of contracted vertices to generate all Δ_3 and their mirror homomorphic images of Δ is $3(\Delta_3^2 + 3\Delta_3)$.

Let $H_{12} = \{\Omega_{l_{10}}^{12}; l_{10} = 0, 1, 2, \dots, \Delta_3 - 1\}$ be the collection of homomorphic images of Δ acquired by the contraction of the vertices $e_{3l_{10}+1}^3; l_{10} = 0, 1, 2, \dots, \Delta_3 - 1$, with the vertex $e_{3\Delta_4}^3$ in Δ .

FIGURE 8: Homomorphic images $\Omega_{l_{(2,5)}}^3$. FIGURE 9: Homomorphic images $\Omega_{l_{(2,4)}}^3$.

in Δ . Suppose r shows itself as the reminder of $(l_{10}/(\Delta_3 - l_{10}))$, then graphically we make four partitions of H_{12} as follows:

- (i) $\Omega_{l_{10}}^{12}$: $\Delta_3 - 2l_{10} > 1$ (Figure 20(a))
- (ii) $\Omega_{l_{10}}^{12}$: $\Delta_3 - 2l_{10} = 1$ (Figure 20(b))
- (iii) $\Omega_{l_{10}}^{12}$: $\Delta_3 - 2l_{10} < 1$ and $\Delta_3 - l_{10} > r + 1$ (Figure 20(c))
- (iv) $\Omega_{l_{10}}^{12}$: $\Delta_3 - 2l_{10} < 1$ and $\Delta_3 - l_{10} = r + 1$ (Figure 20(d))

From all the homomorphic images presented in these figures, it is not intricate to check that no one is the mirror

image of itself. This lemma can be proved by using the same procedure as that for Lemma 1.

Lemma 13. If we contract the vertices $e_{3l_1+1}^2$: $l_1 = 0, 1, 2, \dots, \Delta_2 - 1$, with the vertex $e_{3\Delta_5}^5$ in the circuit Δ , then there arise Δ_2 distinct homomorphic images of Δ and $3(l_1 + 2)$ pairs of contracted vertices create each homomorphic image. Moreover, the number of total pairs of contracted vertices to generate all Δ_2 and their mirror homomorphic images of Δ is $(3/2)(\Delta_2^2 + 3\Delta_2)$.

Let $H_{13} = \{\Omega_{l_1}^{13}; l_1 = 0, 1, 2, \dots, \Delta_2 - 1\}$ be the collection of homomorphic images of Δ acquired by the contraction of the vertices $e_{3l_1+1}^2$: $l_1 = 0, 1, 2, \dots, \Delta_2 - 1$, with the vertex $e_{3\Delta_5}^5$ in Δ . Figure 21 presents H_{13} graphically. From all the homomorphic images presented in Figure 21, it is not intricate to check that every one is the mirror image of itself. This lemma can be proved by using the same procedure as that for Lemma 1.

Lemma 14. If we contract the vertices $e_{3l_{11}+1}^3$: $l_{11} = 0, 1, 2, \dots, \Delta_3 - 1$, with the vertex $e_{3\Delta_5}^5$ in the circuit Δ , then there arise Δ_3 distinct homomorphic images of Δ and $3(l_{11} + 2)$ pairs of contracted vertices create each homomorphic image. Moreover, the number of total pairs of

FIGURE 10: Homomorphic images $\Omega_{l_3}^4$.

FIGURE 11: Continued.

FIGURE 11: (a) Homomorphic images $\Omega_i^5: \Delta_1 - 2l_4 > 1$. (b) Homomorphic images $\Omega_i^5: \Delta_1 - 2l_4 = 1$. (c) Homomorphic images $\Omega_i^5: \Delta_1 - 2l_4 < 1$ and $\Delta_1 - l_4 > r + 1$. (d) Homomorphic images $\Omega_i^5: \Delta_1 - 2l_4 < 1$ and $\Delta_1 - l_4 = r + 1$

contracted vertices to generate all Δ_3 and their mirror homomorphic images of Δ is $3(\Delta_3^2 + 3\Delta_3 - 2)$.

Let $H_{14} = \{\Omega_{l_{11}}^{14}; l_{11} = 0, 1, 2, \dots, \Delta_3 - 1\}$ be the collection of homomorphic images of Δ acquired by the contraction of the vertices $e_{3l_{11}+1}^3: l_{11} = 0, 1, 2, \dots, \Delta_3 - 1$, with the vertex $e_{3\Delta_3}^5$ in Δ . Figure 22 presents H_{14} graphically. From all the homomorphic images presented in Figure 22, it is not intricate to check that no one is the mirror image of itself except Ω_0^{14} . This lemma can be proved by using the same procedure as that for Lemma 1.

Lemma 15. If we contract the vertices $e_{3l_7+1}^4: l_7 = 1, 2, \dots, \Delta_4 - 1$, with the vertex $e_{3\Delta_5}^5$ in the circuit Δ , then there arise $\Delta_4 - 1$ distinct homomorphic images of Δ and $3(l_7 + 2)$ pairs of contracted vertices create each homomorphic image. Moreover, the number of total pairs of contracted vertices to generate all $\Delta_4 - 1$ and their mirror homomorphic images of Δ is $3(\Delta_4^2 + 3\Delta_4 - 4)$.

Let $H_{15} = \{\Omega_{l_7}^{15}; l_7 = 1, 2, \dots, \Delta_4 - 1\}$ be the collection of homomorphic images of Δ acquired by the contraction of the vertices $e_{3l_7+1}^4: l_7 = 1, 2, \dots, \Delta_4 - 1$, with the vertex $e_{3\Delta_5}^5$ in Δ . Figure 23 presents H_{15} graphically. From all the homomorphic images presented in Figure 23, it is not intricate to check that no one is the mirror image of itself.

This lemma can be proved by using the same procedure as that for Lemma 1.

Lemma 16. If we contract the vertices $e_{3l_{12}+1}^5: l_{12} = 0, 1, 2, \dots, \Delta_5 - 1$, with the vertex $e_{3\Delta_5}^5$ in the circuit Δ , then there arise Δ_5 distinct homomorphic images of Δ and $3(l_{12} + 2)$ pairs of contracted vertices create each homomorphic image. Moreover, the number of total pairs of contracted vertices to generate all Δ_5 and their mirror homomorphic images of Δ is $3(\Delta_5^2 + 3\Delta_5)$.

Let $H_{16} = \{\Omega_{l_{12}}^{16}; l_{12} = 0, 1, 2, \dots, \Delta_5 - 1\}$ be the collection of homomorphic images of Δ acquired by the contraction of the vertices $e_{3l_{12}+1}^5: l_{12} = 0, 1, 2, \dots, \Delta_5 - 1$, with the vertex $e_{3\Delta_5}^5$ in Δ . Graphically, we make two partitions of H_{16} as follows:

(i) $\Omega_{l_{12}}^{16}: l_{12} < \Delta_5 - 1$ (Figure 24(a))

(ii) $\Omega_{l_{12}}^{16}: l_{12} = \Delta_5 - 1$ (Figure 24(b))

From all the homomorphic images presented in these figures, it is not intricate to check that no one is the mirror image of itself. This lemma can be proved by using the same procedure as that for Lemma 1.

Lemma 17. If we contract the vertices $e_{3l_{13}+1}^3: l_{13} = 1, 2, \dots, \Delta_3 - 1$, with the vertex $e_{3\Delta_6}^6$ in the circuit Δ , then there arise $\Delta_3 - 1$ distinct homomorphic images of Δ and $3(l_{13} + 2)$ pairs of contracted vertices create each homomorphic image. Moreover, the number of total pairs of contracted vertices to generate all $\Delta_3 - 1$ and their mirror homomorphic images of Δ is $(3/2)(\Delta_3^2 + 3\Delta_3 - 4)$.

FIGURE 12: (a) Homomorphic images $\Omega_{l_5}^6: l_5 < \Delta_2 - 1$. (b) Homomorphic images $\Omega_{l_5}^6: l_5 = \Delta_2 - 1$.

Let $H_{17} = \{\Omega_{l_{13}}^{17}; l_{13} = 1, 2, \dots, \Delta_3 - 1\}$ be the collection of homomorphic images of Δ acquired by the contraction of the vertices $e_{3l_{13}+1}^3$: $l_{13} = 1, 2, \dots, \Delta_3 - 1$, with the vertex $e_{3\Delta_6}^6$ in Δ . Figure 25 presents H_{17} graphically. From all the homomorphic images presented in Figure 25, it is not intricate to check that every one is the mirror image of itself. This lemma can be proved by using the same procedure as that for Lemma 1.

Lemma 18. If we contract the vertices $e_{3l_{14}+1}^1$: $l_{14} = \Delta_2 + 1, \Delta_2 + 2, \dots, \Delta_1 - 1$, with the vertex $e_{3\Delta_1}^1$ in the circuit Δ , then there arise $\Delta_1 - \Delta_2 - 1$ distinct homomorphic images of Δ and $3(\Delta_2 + 2)$ pairs of contracted vertices create each homomorphic image. Moreover, the number of total pairs of contracted vertices to generate all $\Delta_1 - \Delta_2 - 1$ and their mirror homomorphic images of Δ is $6(\Delta_2 + 2)(\Delta_1 - \Delta_2 - 1)$.

Proof. Let $H_{18} = \{\Omega_{l_{14}}^{18}; l_{14} = \Delta_2 + 1, \Delta_2 + 2, \dots, \Delta_1 - 1\}$ (Figure 26) be the collection of homomorphic images of Δ acquired by the contraction of the vertices $e_{3l_{14}+1}^1$: $l_{14} = \Delta_2 + 1, \Delta_2 + 2, \dots, \Delta_1 - 1$, with the vertex $e_{3\Delta_1}^1$ in Δ , where $e_{3l_{14}+1}^1$ and $e_{3\Delta_1}^1$ are fixed by the words $(pq)^{\Delta_1 - l_{14} - 1} (pq^{-1})^{\Delta_6} (pq)^{\Delta_5} (pq^{-1})^{\Delta_4} (pq)^{\Delta_3} (pq^{-1})^{\Delta_2} (pq)^{l_{14}}$ and

$(pq)^{\Delta_2} (pq^{-1})^{\Delta_3} (pq)^{\Delta_4} (pq^{-1})^{\Delta_5} (pq)^{\Delta_2} (pq^{-1})^{\Delta_3} (pq)^{\Delta_4} (pq^{-1})^{\Delta_5} (pq)^{\Delta_6} (pq^{-1})^{\Delta_1}$. It is easy to verify that $P_1 = q^{-1} (pq^{-1})^{\Delta_1 - l_{14} - 1}$ and $P_2 = (pq)^{\Delta_2} (pq^{-1})^{\Delta_3} (pq)^{\Delta_4} (pq^{-1})^{\Delta_5} (pq)^{\Delta_6} (pq^{-1})^{l_{14}} p$ are the possible paths between $e_{3l_{14}+1}^1$ and $e_{3\Delta_1}^1$ (Figure 4). This implies that, for each l_{14} , the homomorphic image $\Omega_{l_{14}}^{18}$ has a vertex s fixed by P_1 and P_2 . Thus,

$$E_{18} = \{e, q, q^{-1}, p, pq, pq^{-1}, pqp, (pq)^2, pqpq^{-1}, \dots, (pq)^{\Delta_2} p, (pq)^{\Delta_2+1}, (pq)^{\Delta_2} pq^{-1}\} \quad (6)$$

is the family of elements in $\text{PSL}(2, \mathbb{Z})$ such that $\forall x \in E_{18}, \Delta$ contains the vertices $(e_{3l_{14}+1}^1)x$ and $(e_{3\Delta_1}^1)x$. This gives that the cardinality of E_{18} , that is, $3(\Delta_2 + 2)$ is the number of contracted pairs of vertices to produce the homomorphic image $\Omega_{l_{14}}^{18}$ (Theorem 2).

Now, we will prove that all the homomorphic images in H_{18} are distinct and no one is the mirror image of itself.

Let Ω_m^{18} and Ω_n^{18} be any two elements of H_{18} , then Ω_m^{18} is produced by contraction of e_{3m+1}^1 and $e_{3\Delta_1}^1$ and Ω_n^{18} is produced by contraction of e_{3n+1}^1 and $e_{3\Delta_1}^1$. Suppose Ω_m^{18} and Ω_n^{18} are the same. This concludes that Ω_m^{18} is procurable also

FIGURE 13: Homomorphic images $\Omega_{l_{(6,3)}}^7$. FIGURE 14: Homomorphic images $\Omega_{l_{(6,4)}}^7$. FIGURE 15: Homomorphic images $\Omega_{l_{(6,5)}}^7$. FIGURE 16: Homomorphic images $\Omega_{l_7}^8$.

by contracting e_{3n+1}^1 and $e_{3\Delta_1}^1$, implying that $e_{3n+1}^1 \leftrightarrow e_{3\Delta_1}^1$ is one of the pairs of contracted vertices for Ω_m^{18} . Then there must exist an element $w \in E_{18}$ such that $(e_{3n+1}^1)w = e_{3n+1}^1$ and $(e_{3\Delta_1}^1)w = e_{3\Delta_1}^1$. $e \in E_{18}$ is the only element such that $(e_{3\Delta_1}^1)e = e_{3\Delta_1}^1$ but $(e_{3n+1}^1)e \neq e_{3n+1}^1$. Thus, Ω_m^{18} and Ω_n^{18} are not the same, that is, by contracting e_{3n+1}^1 and $e_{3\Delta_1}^1$ to produce Ω_m^{18} , e_{3n+1}^1 and $e_{3\Delta_1}^1$ are not contracted. Now, if Ω_m^{18} and Ω_n^{18}

FIGURE 17: (a) Homomorphic images $\Omega_{l_7}^9$: $\Delta_4 - 2l_7 > 1$. (b) Homomorphic images $\Omega_{l_7}^9$: $\Delta_4 - 2l_7 = 1$. (c) Homomorphic images $\Omega_{l_7}^9$: $\Delta_4 - 2l_7 < 1$ and $\Delta_4 - l_7 > r + 1$. (d) Homomorphic images $\Omega_{l_7}^9$: $\Delta_4 - 2l_7 < 1$ and $\Delta_4 - l_7 = r + 1$.

FIGURE 18: (a) Homomorphic images $\Omega_{l_8}^{10}$; $l_8 < \Delta_4 - 1$. (b) Homomorphic images $\Omega_{l_8}^{10}$; $l_8 = \Delta_4 - 1$.

are the same, then there must exist an element $w \in E_{18}$ such that $(e_{3m+1}^1)w = e_{3m+1}^{1*}$ and $(e_{3\Delta_1}^1)w = e_{3\Delta_1}^{1*}$. But there does not such an element exist in E_{18} . Thus, Ω_m^{18} is not the same as the mirror image of Ω_m^{18} , that is, by contracting e_{3m+1}^1 and $e_{3\Delta_1}^1$ to create Ω_m^{18} , e_{3m+1}^{1*} and $e_{3\Delta_1}^{1*}$ are not contracted. This implies that all the elements in H_{18} are distinct. Thus, $H_{18} = \{\Omega_{l_{14}}^{18}; l_{14} = \Delta_2 + 1, \Delta_2 + 2, \dots, \Delta_1 - 1\}$ gives $|H_{18}| = \Delta_1 - \Delta_2 - 1$.

To check does any element of H_{18} is the mirror image of itself, we suppose Ω_m^{18} is the same to its mirror image Ω_m^{18*} . Then by definition, there must exist an element $w \in E_{18}$ such that $(e_{3m+1}^1)w = e_{3m+1}^{1*}$ and $(e_{3\Delta_1}^1)w = e_{3\Delta_1}^{1*}$. But there does not such an element exist in E_{18} . Thus, Ω_m^{18} is not the same as its mirror image, that is, by contracting e_{3m+1}^1 and $e_{3\Delta_1}^1$ to create Ω_m^{18} , e_{3m+1}^{1*} and $e_{3\Delta_1}^{1*}$ are not contracted. Hence, there are

$$2 \times |E_{18}| \times |H_{18}| = 6(\Delta_2 + 2)(\Delta_1 - \Delta_2 - 1) \quad (7)$$

pairs of contracting vertices to produce all the homomorphic images in H_{18} .

$$\text{Let } \eta_1 = \begin{cases} 0, & \text{if } \Delta_1 + \Delta_4 = 0 \pmod{2} \\ 1, & \text{otherwise.} \end{cases} \quad \square$$

Lemma 19. *If we contract the vertices $e_{3l_{15}+1}^1$: $l_{15} = \Delta_4 + 1, \Delta_4 + 2, \dots, ((\Delta_1 + \Delta_4 - \eta_1)/2)$, with the vertex $e_{3\Delta_3}^3$ in the circuit Δ , then there arise $((\Delta_1 - \Delta_4 - \eta_1)/2)$ distinct homomorphic images of Δ and $3(\Delta_4 + 2)$ pairs of contracted vertices create each homomorphic image. Moreover, the number of total pairs of contracted vertices to generate all $((\Delta_1 - \Delta_4 - \eta_1)/2)$ and their mirror homomorphic images of Δ is $3(\Delta_4 + 2)(\Delta_1 - \Delta_4 - 1)$.*

Proof. Let

$H_{19} = \{\Omega_{l_{15}}^{19}; l_{15} = \Delta_4 + 1, \Delta_4 + 2, \dots, ((\Delta_1 + \Delta_4 - \eta_1)/2)\}$ (Figure 27) be the collection of homomorphic images of Δ acquired by the contraction of the vertices $e_{3l_{15}+1}^1$: $l_{15} = \Delta_4 + 1, \Delta_4 + 2, \dots, ((\Delta_1 + \Delta_4 - \eta_1)/2)$, with the vertex $e_{3\Delta_3}^3$ in Δ , where $e_{3l_{15}+1}^1$ and $e_{3\Delta_3}^3$ are fixed by the words $(pq)^{\Delta_1 - l_{15}}(pq^{-1})^{\Delta_6}(pq)^{\Delta_5 + 1}(pq)^{\Delta_4}(pq)^{\Delta_3}(pq^{-1})^{\Delta_2}(pq)^{l_{15}}$ and $(pq)^{\Delta_4}(pq^{-1})^{\Delta_5}(pq)^{\Delta_6}(pq^{-1})^{\Delta_1}(pq)^{\Delta_2}(pq^{-1})^{\Delta_3}$. It is easy to verify that $P_1 = q^{-1}(pq^{-1})^{\Delta_1 - l_{15} - 1}(pq)^{\Delta_2}(pq^{-1})^{\Delta_3}$ and $P_2 = (pq)^{\Delta_4}(pq^{-1})^{\Delta_5}(pq)^{\Delta_6}(pq^{-1})^{l_{15}}p$ are the possible paths between $e_{3l_{15}+1}^1$ and $e_{3\Delta_3}^3$ (Figure 4). This implies that,

FIGURE 19: Homomorphic images $\Omega_{l_9}^{11}$.

FIGURE 20: Continued.

FIGURE 20: (a) Homomorphic images $\Omega_{l_{10}}^{12}$: $\Delta_3 - 2l_{10} > 1$. (b) Homomorphic images $\Omega_{l_{10}}^{12}$: $\Delta_3 - 2l_{10} = 1$. (c) Homomorphic images $\Omega_{l_{10}}^{12}$: $\Delta_3 - 2l_{10} < 1$ and $\Delta_3 - l_{10} > r + 1$. (d) Homomorphic images $\Omega_{l_{10}}^{12}$: $\Delta_3 - 2l_{10} < 1$ and $\Delta_3 - l_{10} = r + 1$.

FIGURE 21: Homomorphic images $\Omega_{l_1}^{13}$.

for each l_{15} , the homomorphic image $\Omega_{l_{15}}^{19}$ has a vertex s fixed by P_1 and P_2 . Thus,

$$E_{19} = \{e, q, q^{-1}, p, pq, pq^{-1}, pqp, (pq)^2, pqpq^{-1}, \dots, (pq)^{\Delta_4} p, (pq)^{\Delta_4+1}, (pq)^{\Delta_4} pq^{-1}\} \quad (8)$$

is the family of elements in $\text{PSL}(2, \mathbb{Z})$ such that $\forall x \in E_{19}, \Delta$ contains the vertices $(e_{3l_{15}+1}^1)x$ and $(e_{3\Delta_3}^3)x$. This gives that the cardinality of E_{19} , that is, $3(\Delta_4 + 2)$ is the number of contracted pairs of vertices to produce the homomorphic image $\Omega_{l_{15}}^{19}$ (Theorem 2).

Now, we will prove that (i) for $\Delta_1 + \Delta_4 = 1 \pmod{2}$, all the homomorphic images in H_{19} are distinct and no one is

the mirror image of itself, and (ii) for $\Delta_1 + \Delta_4 = 0 \pmod{2}$, all the homomorphic images in H_{19} are distinct and only $\Omega_{l_{15}}^{19}$ is the mirror image of itself.

Let Ω_m^{19} and Ω_n^{19} be any two elements of H_{19} , then Ω_m^{19} is produced by contraction of e_{3m+1}^1 and $e_{3\Delta_3}^3$ and Ω_n^{19} is produced by contraction of e_{3n+1}^1 and $e_{3\Delta_3}^3$.

Suppose Ω_m^{19} and Ω_n^{19} are the same. This concludes that Ω_m^{19} is procurable also by contracting e_{3n+1}^1 and $e_{3\Delta_3}^3$, implying that $e_{3n+1}^1 \leftrightarrow e_{3\Delta_3}^3$ is one of the pairs of contracted vertices for Ω_m^{19} . Then there must exist an element $w \in E_{19}$ such that $(e_{3m+1}^1)w = e_{3n+1}^1$ and $(e_{3\Delta_3}^3)w = e_{3\Delta_3}^3$. $e \in E_{19}$ is the only element such that $(e_{3\Delta_3}^3)e = e_{3\Delta_3}^3$ but $(e_{3m+1}^1)e \neq e_{3m+1}^1$. Thus, Ω_m^{19} and Ω_n^{19} are not the same, that is, by contracting e_{3m+1}^1 and $e_{3\Delta_3}^3$ to produce Ω_m^{19} , e_{3n+1}^1 and $e_{3\Delta_3}^3$ are not contracted.

FIGURE 22: Homomorphic images $\Omega_{l_{11}}^{14}$. FIGURE 23: Homomorphic images $\Omega_{l_7}^{15}$.

Now, if Ω_m^{19} and Ω_n^{19*} are the same, then there must exist an element $w \in E_{19}$ such that $(e_{3m+1}^1)w = e_{3n+1}^{1*}$ and $(e_{3\Delta_3}^3)w = e_{3\Delta_3}^{3*} \cdot (pq)^{\Delta_4} p \in E_{19}$ is the only element such that $(e_{3\Delta_3}^3)(pq)^{\Delta_4} p = e_{3\Delta_3}^{3*}$ and $(e_{3m+1}^1)(pq)^{\Delta_4} p = e_{3(\Delta_1+\Delta_4-m)+1}^{1*}$. This implies that Ω_m^{19} and Ω_n^{19} are the mirror images of each other if and only if $n = \Delta_1 + \Delta_4 - m$ and Ω_m^{19} is the mirror image of itself if and only if $m = \Delta_1 + \Delta_4 - m$, that is, $m = ((\Delta_1 + \Delta_4)/2)$. Now,

- (i) If $\Delta_1 + \Delta_4 = 1 \pmod{2}$, then $\forall m \in l_{15}$, $\Delta_1 + \Delta_4 - m > ((\Delta_1 + \Delta_4 - 1)/2)$, implying that $\Omega_{\Delta_1+\Delta_4-m}^{19} \notin H_{19}$. This states that no homomorphic image in H_{19} is the mirror image of other. Hence, all the homomorphic images in H_{19} are distinct. Thus, $H_{19} = \{\Omega_{l_{15}}^{19}; l_{15} = \Delta_4 + 1, \Delta_4 + 2, \dots, ((\Delta_1 + \Delta_4 - 1)/2)\}$ gives $|H_{19}| = ((\Delta_1 - \Delta_4 - 1)/2)$.

Also, $((\Delta_1 + \Delta_4)/2) > ((\Delta_1 + \Delta_4 - 1)/2)$ gives that no homomorphic image in H_{19} is the mirror image of itself. Hence, there are

$$2 \times |E_{19}| \times |H_{19}| = 6(\Delta_4 + 2) \left(\frac{\Delta_1 - \Delta_4 - 1}{2} \right) \quad (9)$$

$$= 3(\Delta_4 + 2)(\Delta_3 - \Delta_4 - 1)$$

pairs of contracting vertices to produce all the homomorphic images in H_{19} .

- (ii) If $\Delta_1 + \Delta_4 = 0 \pmod{2}$, then $\forall m \in (l_{15}/\{(\Delta_1 + \Delta_4)/2\})$, $\Delta_1 + \Delta_4 - m > ((\Delta_1 + \Delta_4)/2)$, implying that $\Omega_{\Delta_1+\Delta_4-m}^{19} \notin H_{19}$. This states that no homomorphic image in H_{19} is the mirror image of other. Hence, all the homomorphic images in H_{19} are distinct. Thus, $H_{19} = \{\Omega_{l_{15}}^{19}; l_{15} = \Delta_4 + 1, \Delta_4 + 2, \dots, ((\Delta_1 + \Delta_4)/2)\}$ gives $|H_{19}| = ((\Delta_1 - \Delta_4)/2)$.

For $m = ((\Delta_1 + \Delta_4)/2)$, we have $\Delta_1 + \Delta_4 - m = ((\Delta_1 + \Delta_4)/2) \in l_{15}$ gives $\Omega_{(\Delta_1+\Delta_4)/2}^{19} \in H_{19}$ is the mirror image of itself. Hence, there are

$$2 \times |E_{19}| \times (|H_{19}| - 1) + |E_{19}| = 6(\Delta_4 + 2) \left(\frac{\Delta_1 - \Delta_4}{2} - 1 \right) \quad (10)$$

$$+ 3(\Delta_4 + 2) = 3(\Delta_4 + 2)(\Delta_3 - \Delta_4 - 1)$$

pairs of contracting vertices to produce all the homomorphic images in H_{19} .

Let $k_2 \in \{2, 3\}$. $\square$

Lemma 20. If we contract the vertices $e_{3l_{(3,k_2)}+1}^{k_2} \cdot l_{(3,k_2)} = \Delta_4 + 1, \Delta_4 + 2, \dots, \Delta_{k_2} - 1$, with the vertex $e_{3\Delta_3}^3$ in the circuit Δ , then for each k_2 , there arise $\Delta_{k_2} - \Delta_4 - 1$ distinct homomorphic images of Δ and $3(\Delta_4 + 2)$ pairs of contracted vertices create each homomorphic image. Moreover, the number of total pairs of contracted vertices to generate all $\Delta_{k_2} - \Delta_4 - 1$ and their mirror homomorphic images of Δ is $6(\Delta_4 + 2)(\Delta_{k_2} - \Delta_4 - 1)$.

For a fix value of k_2 , let $H_{20}^{k_2} = \{\Omega_{l_{(3,k_2)}}^{20}; l_{(3,k_2)} = \Delta_4 + 1, \Delta_4 + 2, \dots, \Delta_{k_2} - 1\}$ be the collection of homomorphic images of Δ acquired by the contraction of the vertices $e_{3l_{(3,k_2)}}^{k_2}$

FIGURE 24: (a) Homomorphic images $\Omega_{l_{12}}^{16}$: $l_{12} < \Delta_5 - 1$. (b) Homomorphic images $\Omega_{l_{12}}^{16}$: $l_{12} = \Delta_5 - 1$.

$+1^{k_2}$: $l_{(3,k_2)} = \Delta_4 + 1, \Delta_4 + 2, \dots, \Delta_{k_2} - 1$, with the vertex $e_{3\Delta_3}^3$ in Δ . Figures 28 and 29 present $H_3^{k_2}$ graphically. From all the homomorphic images presented in these figures, it is not intricate to check that no one is the mirror image of itself.

This lemma can be proved by using the same procedure as that for Lemma 18.

Lemma 21. If we contract the vertices $e_{3l_{16}+1}^2$: $l_{16} = \Delta_3 + 1, \Delta_3 + 2, \dots, \Delta_2 - 1$, with the vertex $e_{3\Delta_4}^4$ in the circuit Δ , then there arise $\Delta_2 - \Delta_3 - 1$ distinct homomorphic images of Δ and $3(\Delta_3 + 2)$ pairs of contracted vertices create each homomorphic image. Moreover, the number of total pairs of contracted vertices to generate all $\Delta_2 - \Delta_3 - 1$ and their mirror homomorphic images of Δ is $6(\Delta_3 + 2)(\Delta_2 - \Delta_3 - 1)$.

Let $H_{21} = \{\Omega_{l_{16}}^{21}; l_{16} = \Delta_3 + 1, \Delta_3 + 2, \dots, \Delta_2 - 1\}$ be the collection of homomorphic images of Δ acquired by the contraction of the vertices $e_{3l_{16}+1}^2$: $l_{16} = \Delta_3 + 1, \Delta_3 + 2, \dots, \Delta_2 - 1$, with the vertex $e_{3\Delta_4}^4$ in Δ . Figure 30 presents H_{21} graphically. From all the homomorphic images presented in Figure 30, it is not intricate to check

that no one is the mirror image of itself. This lemma can be proved by using the same procedure as that for Lemma 18.

Let $k_3 \in \{1, 2\}$.

Lemma 22. If we contract the vertices $e_{3l_{(4,k_3)}+1}^{k_3}$: $l_{(4,k_3)} = \Delta_5 + 1, \Delta_5 + 2, \dots, \Delta_{k_3} - 1$, with the vertex $e_{3\Delta_6}^6$ in the circuit Δ , then for each k_3 , there arise $\Delta_{k_3} - \Delta_5 - 1$ distinct homomorphic images of Δ and $3(\Delta_5 + 2)$ pairs of contracted vertices create each homomorphic image. Moreover, the number of total pairs of contracted vertices to generate all $\Delta_{k_3} - \Delta_5 - 1$ and their mirror homomorphic images of Δ is $6(\Delta_5 + 2)(\Delta_{k_3} - \Delta_5 - 1)$.

For a fix value of k_3 , let $H_{22}^{k_3} = \{\Omega_{l_{(4,k_3)}}^{22}; l_{(4,k_3)} = \Delta_5 + 1, \Delta_5 + 2, \dots, \Delta_{k_3} - 1\}$ be the collection of homomorphic images of Δ acquired by the contraction of the vertices $e_{3l_{(4,k_3)}+1}^{k_3}$: $l_{(4,k_3)} = \Delta_5 + 1, \Delta_5 + 2, \dots, \Delta_{k_3} - 1$, with the vertex $e_{3\Delta_6}^6$ in Δ . Figures 31 and 32 present $H_{22}^{k_3}$ graphically. From all the homomorphic images presented in these figures, it is not intricate to check that no

FIGURE 25: Homomorphic images $\Omega_{l_{13}}^{17}$. FIGURE 26: Homomorphic images $\Omega_{l_{14}}^{18}$.

one is the mirror image of itself. This lemma can be proved by using the same procedure as that for Lemma 18.

Let $k_2 \in \{2, 3\}$.

Lemma 23. If we contract the vertices $e_{3l_4+1}^1$; $l_4 = 1, 2, \dots, \Delta_1 - 1$, with the vertices $e_{3l_{(5,k_2)}}^{k_2}$; $l_{(5,k_2)} = 1, 2, \dots, \Delta_{k_2} - 1$, in the circuit Δ , then for each k_2 , there arise $(\Delta_1 - 1)(\Delta_{k_2} - 1)$

distinct homomorphic images of Δ and 6 pairs of contracted vertices create each homomorphic image. Moreover, the number of total pairs of contracted vertices to generate all $(\Delta_1 - 1)(\Delta_{k_2} - 1)$ and their mirror homomorphic images of Δ is $12(\Delta_1 - 1)(\Delta_{k_2} - 1)$.

Proof. For $k_2 = 2$, let $H_{23}^2 = \left\{ \Omega_{(l_4, l_{(5,2)})}^{23}; l_4 = 1, 2, \dots, \Delta_1 - 1; l_{(5,2)} = 1, 2, \dots, \Delta_2 - 1 \right\}$ (Figure 33) be the collection of homomorphic images of Δ acquired by the contraction of the vertices $e_{3l_4+1}^1$ with the vertex $e_{3l_{(5,2)}}^2$ in Δ , where $e_{3l_4+1}^1$ and $e_{3l_{(5,2)}}^2$ are fixed by the words $(pq)^{l_{(5,2)}} (pq^{-1})^{\Delta_6} (pq)^{\Delta_5} (pq^{-1})^{\Delta_4} (pq)^{\Delta_3} (pq^{-1})^{\Delta_2} (pq)^{\Delta_1 - l_4}$ and $(pq^{-1})^{\Delta_2 - l_{(5,2)}} (pq)^{\Delta_1} (pq^{-1})^{\Delta_6} (pq)^{\Delta_5} (pq^{-1})^{\Delta_4} (pq)^{\Delta_3} (pq^{-1})^{l_{(5,2)}}$. It is easy to verify that $P_1 = (pq^{-1})^{\Delta_2 - l_{(5,2)}} (pq)^{\Delta_1 - l_4}$ and $P_2 = (pq)^{l_4} (pq^{-1})^{\Delta_6} (pq)^{\Delta_5} (pq^{-1})^{\Delta_4} (pq)^{\Delta_3} (pq^{-1})^{l_{(5,2)}}$ are the possible paths between $e_{3l_4+1}^1$ and $e_{3l_{(5,2)}}^2$ (Figure 4). This implies that the homomorphic image $\Omega_{(l_4, l_{(5,2)})}^{23}$ has a vertex s fixed by P_1 and P_2 . Thus,

$$E_{23}^2 = \{e, q, q^{-1}, p, pq, pq^{-1}\} \quad (11)$$

is the family of elements in $\text{PSL}(2, \mathbb{Z})$ such that $\forall x \in E_{23}^2, \Delta$ contains the vertices $(e_{3l_4+1}^1)x$ and $(e_{3l_{(5,2)}}^2)x$. This gives that the cardinality of E_{23}^2 , that is, 6 is the number of contracted pairs of vertices to produce the homomorphic image $\Omega_{(l_4, l_{(5,2)})}^{23}$ (Theorem 2).

Now, we will prove that all the homomorphic images in H_{23}^2 are distinct and no one is the mirror image of itself.

Let $\Omega_{(m_1, m_2)}^{23}$ and $\Omega_{(n_1, n_2)}^{23}$ be any two elements of H_{23}^2 , then $\Omega_{(m_1, m_2)}^{23}$ is produced by contraction of $e_{3m_1+1}^1$ and $e_{3m_2}^2$ and $\Omega_{(n_1, n_2)}^{23}$ is produced by contraction of $e_{3n_1+1}^1$ and $e_{3n_2}^2$.

Suppose $\Omega_{(m_1, m_2)}^{23}$ and $\Omega_{(n_1, n_2)}^{23}$ are the same. This concludes that $\Omega_{(m_1, m_2)}^{23}$ is procurable also by contracting $e_{3n_1+1}^1$ and $e_{3n_2}^2$, implying that $e_{3n_1+1}^1 \leftrightarrow e_{3n_2}^2$ is one of the pairs of contracted vertices for $\Omega_{(m_1, m_2)}^{23}$. Then there must exist an element $w \in E_{23}^2$ such that $(e_{3m_1+1}^1)w = e_{3n_1+1}^1$ and $(e_{3m_2}^2)w = e_{3n_2}^2$. $pq^{-1} \in E_5^2$ is the only element such that $(e_{3m_2}^2)pq^{-1} = e_{3(m_2+1)}^2$ but $(e_{3m_1+1}^1)pq^{-1} \neq e_{3n_1+1}^1$. Thus, $\Omega_{(m_1, m_2)}^{23}$ and $\Omega_{(n_1, n_2)}^{23}$ are not the same, that is, by contracting $e_{3m_1+1}^1$ and $e_{3m_2}^2$ to produce $\Omega_{(m_1, m_2)}^{23}$, $e_{3n_1+1}^1$ and $e_{3n_2}^2$ are not contracted. Now, if $\Omega_{(m_1, m_2)}^{23}$ and $\Omega_{(n_1, n_2)}^{23*}$ are the same, then there must exist an element $w \in E_{23}^2$ such that $(e_{3m_1+1}^1)w = e_{3n_1+1}^{1*}$ and $(e_{3m_2}^2)w = e_{3n_2}^{2*}$. $p \in E_{23}^2$ is the only element such that $(e_{3m_1+1}^1)p = e_{3(\Delta_1 - m_1) + 1}^{1*}$ but $(e_{3m_2}^2)p \neq e_{3n_2}^{2*}$. This implies that $\Omega_{(m_1, m_2)}^{23}$ and $\Omega_{(n_1, n_2)}^{23}$ are not the mirror images of each other, that is, by contracting $e_{3m_1+1}^1$ and $e_{3m_2}^2$ to produce $\Omega_{(m_1, m_2)}^{23}$, $e_{3n_1+1}^{1*}$ and $e_{3n_2}^{2*}$ are not contracted. Hence, all the homomorphic images in H_{23}^2 are distinct. Thus, $H_{23}^2 = \left\{ \Omega_{(l_4, l_{(5,2)})}^{23}; l_4 = 1, 2, \dots, \Delta_1 - 1; l_{(5,2)} = 1, 2, \dots, \Delta_2 - 1 \right\}$ gives $|H_{23}^2| = (\Delta_1 - 1)(\Delta_2 - 1)$.

Suppose $\Omega_{(m_1, m_2)}^{23}$ is the mirror image of itself, then there must exist an element $w \in E_{23}^2$ such that $(e_{3m_1+1}^1)w = e_{3m_1+1}^{1*}$ and $(e_{3m_2}^2)w = e_{3m_2}^{2*}$. $p \in E_{23}^2$ is the only element such that

FIGURE 27: Homomorphic images $\Omega_{l_{15}}^{19}$. FIGURE 28: Homomorphic images $\Omega_{l_{(3,2)}}^{20}$.

$(e_{3m_1+1}^1)p = e_{3(\Delta_1-m_1)+1}^{1*}$ but $(e_{3m_2}^2)p \neq e_{3m_2}^{2*}$. This implies that $\Omega_{(m_1, m_2)}^{23}$ is not the same as $\Omega_{(m_1, m_2)}^{23*}$, that is, by contracting $e_{3m_1+1}^1$ and $e_{3m_2}^2$ to produce $\Omega_{(m_1, m_2)}^{23}$, $e_{3m_1+1}^{1*}$ and $e_{3m_2}^{2*}$ are not contracted.

Hence, there are

$$2 \times |E_{23}^2| \times |H_{23}^2| = 12(\Delta_1 - 1)(\Delta_2 - 1) \quad (12)$$

pairs of contracting vertices to produce all the homomorphic images in H_{23}^2 .

We can prove this lemma for $k_2 = 3$ (Figure 34) in similar way as that for $k_2 = 2$.

Let $\eta_2 = \begin{cases} 0 & \text{if } \Delta_4 = 0 \pmod{2} \\ 1 & \text{otherwise} \end{cases}$ and $\eta_3 = \begin{cases} 0 & \text{if } \Delta_1 = 0 \pmod{2} \\ 1 & \text{otherwise} \end{cases}$. $\square$

Lemma 24. If we contract the vertices $e_{3l_{17}+1}^1: l_{17} = \begin{cases} 1, 2, \dots, ((\Delta_1 - \eta_3)/2), & \text{if } l_{18} = (\Delta_4/2) \\ 1, 2, \dots, \Delta_1 - 1 & \text{otherwise} \end{cases}$, with the vertices $e_{3l_{18}}^4: l_{18} = 1, 2, \dots, ((\Delta_4 - \eta_2)/2)$, in the circuit Δ , then there arise distinct homomorphic images of Δ and 6 pairs of contracted vertices create each homomorphic image. Moreover, the number of total pairs of contracted vertices to generate all these homomorphic images and their mirror homomorphic images of Δ is $6(\Delta_1 - 1)(\Delta_4 - 1)$.

Proof. Let $H_{24} = \{\Omega_{(l_{17}, l_{18})}^{24}; l_{17} = \begin{cases} 1, 2, \dots, ((\Delta_1 - \eta_3)/2), & \text{if } l_{18} = (\Delta_4/2) \\ 1, 2, \dots, \Delta_1 - 1 & \text{otherwise} \end{cases} \}$ (Figure 35) be the collection of homomorphic images of Δ acquired by the contraction of the vertices $e_{3l_{17}+1}^1$ with the vertices $e_{3l_{18}}^4$ in Δ , where $e_{3l_{17}+1}^1$ and $e_{3l_{18}}^4$ are fixed by the words $(pq)^{l_{17}}(pq^{-1})^{\Delta_6}(pq)^{\Delta_5}(pq^{-1})^{\Delta_4}(pq)^{\Delta_3}(pq^{-1})^{\Delta_2}(pq)^{\Delta_1-l_{18}}$ and $(pq^{-1})^{\Delta_4-l_{18}}(pq)^{\Delta_3}(pq^{-1})^{\Delta_2}(pq)^{\Delta_1-l_{17}}$. It is easy to verify that $P_1 = (pq)^{l_{17}}(pq^{-1})^{\Delta_6}(pq)^{\Delta_5}(pq^{-1})^{l_{18}}$ and $P_2 = (pq^{-1})^{\Delta_4-l_{18}}(pq)^{\Delta_3}(pq^{-1})^{\Delta_2}(pq)^{\Delta_1-l_{17}}$ are the possible paths between $e_{3l_{17}+1}^1$ and $e_{3l_{18}}^4$ (Figure 4). This implies that the homomorphic image $\Omega_{(l_{17}, l_{18})}^{24}$ has a vertex s fixed by P_1 and P_2 . Thus,

$$E_{24} = \{e, q, q^{-1}, p, pq, pq^{-1}\} \quad (13)$$

is the family of elements in $\text{PSL}(2, \mathbb{Z})$ such that $\forall x \in E_{24}, \Delta$ contains the vertices $(e_{3l_{17}+1}^1)x$ and $(e_{3l_{18}}^4)x$. This gives that the cardinality of E_{24} , that is, 6 is the number of contracted pairs of vertices to produce the homomorphic image $\Omega_{(l_{17}, l_{18})}^{24}$ (Theorem 2).

Now, we prove that (i) for $\Delta_4 = 0 \pmod{2}$, all the elements in H_{24} are distinct and no one is the mirror image of itself, and (ii) for $\Delta_4 = 1 \pmod{2}$, all the elements in H_{24} are distinct and the homomorphic image $\Omega_{((\Delta_1/2), (\Delta_4/2))}^{24}$ is the mirror image of itself.

FIGURE 29: Homomorphic images $\Omega_{l_{(3,3)}}^{20}$. FIGURE 30: Homomorphic images $\Omega_{l_{16}}^{21}$.

Suppose $\Omega_{(m_1, m_2)}^{24}$ and $\Omega_{(n_1, n_2)}^{24}$ are the same. This concludes that $\Omega_{(m_1, m_2)}^{24}$ is procurable also by contracting $e_{3m_1+1}^1$ and $e_{3m_2}^4$, implying that $e_{3m_1+1}^1 \leftrightarrow e_{3m_2}^4$ is one of the pairs of contracted vertices for $\Omega_{(m_1, m_2)}^{24}$. Then there must exist an element $w \in E_{24}$ such that $(e_{3m_1+1}^1)w = e_{3n_1+1}^1$ and $(e_{3m_2}^4)w = e_{3n_2}^4$. $pq^{-1} \in E_{24}$ is the only element such that $(e_{3m_1+1}^1)pq^{-1} = e_{3m_2+1}^4$ but $(e_{3m_1+1}^1)pq^{-1} \neq e_{3n_1+1}^1$. Thus, $\Omega_{(m_1, m_2)}^{24}$ and $\Omega_{(n_1, n_2)}^{24}$ are not the same, that is, by contracting $e_{3m_1+1}^1$ and $e_{3m_2}^4$ to produce $\Omega_{(m_1, m_2)}^{24}$, $e_{3m_1+1}^1$ and $e_{3m_2}^4$ are not contracted. Now, if $\Omega_{(m_1, m_2)}^{24}$ and $\Omega_{(n_1, n_2)}^{24}$ are the same, then there must exist an element $w \in E_{24}$ such that $(e_{3m_1+1}^1)w =$

$e_{3n_1+1}^{1*}$ and $(e_{3m_2}^4)w = e_{3n_2}^{4*}$. $p \in E_{24}$ is the only element such that $(e_{3m_1+1}^1)p = e_{3(\Delta_1-m_1)+1}^{1*}$ and $(e_{3m_2}^4)p = e_{3(\Delta_4-m_2)}^{4*}$. This implies that $\Omega_{(m_1, m_2)}^{24}$ and $\Omega_{(n_1, n_2)}^{24}$ are the mirror images of each other if and only if $n_1 = \Delta_1 - m_1$ and $n_2 = \Delta_4 - m_2$, that is, by contracting $e_{3m_1+1}^1$ and $e_{3m_2}^4$ to produce $\Omega_{(m_1, m_2)}^{24}$, $e_{3n_1+1}^{1*}$ and $e_{3n_2}^{4*}$ are also contracted and the homomorphic image $\Omega_{(m_1, m_2)}^{24}$ is the mirror image of itself if and only if $m_1 = \Delta_1 - m_1$ and $m_2 = \Delta_4 - m_2$ that is $m_1 = (\Delta_1/2)$ and $m_2 = (\Delta_4/2)$. Now,

- (1) If $\Delta_4 = 1 \pmod{2}$ then $\forall m_2 \in l_{18}$, we get $\Delta_4 - m_2 > ((\Delta_4 - 1)/2)$ implies $\Omega_{(\Delta_1-m_1, \Delta_4-m_2)}^{24} \notin H_{24}$. This indicates that no homomorphic image in H_{24} is the mirror image of other. Hence, all the elements in H_{24} are distinct. Thus, $H_{24} = \{\Omega_{(l_{17}, l_{18})}^{24}; l_{17} = \{1, 2, \dots, ((\Delta_1 - \eta_3)/2) \text{ if } l_{18} = (\Delta_4/2), l_{18} = 1, 2, \dots, ((\Delta_4 - \eta_2)/2)1, 2, \dots, \Delta_1 - 1 \text{ otherwise}\}$, implying that

$$|H_{24}| = \frac{1}{2} (\Delta_1 - 1)(\Delta_4 - 1). \quad (14)$$

Also, $(\Delta_4/2) > ((\Delta_4 - 1)/2)$ implies no homomorphic image in H_{24} is the mirror image of itself. Hence, there are

$$2 \times |E_{24}| \times |H_{24}| = 6(\Delta_1 - 1)(\Delta_4 - 1) \quad (15)$$

pairs of contracting vertices to produce all the homomorphic images in H_{24} .

- (2) If $\Delta_4 = 0 \pmod{2}$, then only for $m_2 = (\Delta_4/2)$, $\Delta_4 - m_2 = (\Delta_4/2) \in l_{18}$ and $\forall m_1 \in l_{17} \setminus \{\Delta_1/2\}$, $\Delta_1 - m_1 \notin l_{17}$, implying that $\Omega_{(\Delta_1-m_1, (\Delta_4/2))}^{24} \notin H_{24}$. Hence, all the elements in H_{24} are distinct. Thus, $H_{24} = \{\Omega_{(l_{17}, l_{18})}^{24}; l_{17} = \{1, 2, \dots, ((\Delta_1 - \eta_3)/2) \text{ if } l_{18} = (\Delta_4/2); l_{18} = 1, 2, \dots, ((\Delta_4 - \eta_2)/2)1, 2, \dots, \Delta_1 - 1 \text{ otherwise}\}$, implying that

$$\begin{aligned} |H_{24}| &= \frac{1}{2} (\Delta_1 - 1)(\Delta_4 - 2) + \frac{1}{2} (\Delta_1 - \eta_3) \\ &= \frac{1}{2} [(\Delta_1 - 1)(\Delta_4 - 1) + 1 - \eta_3]. \end{aligned} \quad (16)$$

Now,

- (a). If Δ_1 is odd, then $(\Delta_1/2) \notin l_{17}$, implying that $\Omega_{((\Delta_1/2), (4/2))}^{24} \notin H_{24}$. So, no homomorphic image in H_{24} is the mirror image of itself. Hence, there are

$$2 \times |E_{24}| \times |H_{24}| = 6(\Delta_1 - 1)(\Delta_4 - 1) \quad (17)$$

pairs of contracting vertices to produce all the homomorphic images in H_{24} .

- (b) If Δ_1 is even, then $(\Delta_1/2) \in l_{17}$, implying that $\Omega_{((\Delta_1/2), (\Delta_4/2))}^{24} \in H_{24}$ is the mirror image of itself. Hence, there are

$$2 \times |E_{24}| \times (|H_{24}| - 1) + |E_{24}| = 6(\Delta_1 - 1)(\Delta_4 - 1) \quad (18)$$

pairs of contracting vertices to produce all the homomorphic images in H_{24} .

FIGURE 31: Homomorphic images $\Omega_{l_{(4,1)}}^{22}$. FIGURE 32: Homomorphic images $\Omega_{l_{(4,2)}}^{22}$.

Let $k_1 \in \{3, 4, 5\}$.

□

Lemma 25. If we contract the vertices $e_{3l_5+1}^2: l_5 = 1, 2, \dots, \Delta_2 - 1$, with the vertices $e_{3l_{(6,k_1)}}^{k_1}: l_{(6,k_1)} = 1, 2, \dots, \Delta_{k_1} - 1$, in the circuit Δ , then for each k_5 , there arise $(\Delta_2 - 1)(\Delta_{k_1} - 1)$ distinct homomorphic images of Δ and 6 pairs of contracted vertices create each homomorphic image. Moreover, the

number of total pairs of contracted vertices to generate all $(\Delta_2 - 1)(\Delta_{k_1} - 1)$ and their mirror homomorphic images of Δ is $12(\Delta_2 - 1)(\Delta_{k_1} - 1)$.

For a fix value of k_1 , let $H_{25}^{k_1} = \left\{ \Omega_{(l_5, l_{(6,k_1)})}^{25}; l_5 = 1, 2, \dots, \Delta_2 - 1; l_{(6,k_1)} = 1, 2, \dots, \Delta_{k_1} - 1 \right\}$ be the collection of homomorphic images of Δ acquired by the contraction of the vertices $e_{3l_5+1}^2$ with the vertices $e_{3l_{(6,k_1)}}^{k_1}$ in Δ . Figures 36, 37, and

FIGURE 33: Homomorphic images $\Omega^{23}_{(l_4)J_{(5,2)}}$.

FIGURE 34: Homomorphic images $\Omega^{23}_{(l_4)J_{(5,3)}}$.

FIGURE 35: Homomorphic images $\Omega_{(l_{17}, l_{18})}^{24}$. FIGURE 36: Homomorphic images $\Omega_{(l_5, l_{(6,3)})}^{25}$.

38 present $H_{25}^{k_1}$ graphically. From all the homomorphic images presented in these figures, it is not intricated to check that no one is the mirror image of itself.

This lemma can be proved by using the same procedure as that for Lemma 23.

Lemma 26. If we contract the vertices $e_{3l_{13}+1}^3: l_{13} = 1, 2, \dots, \Delta_3 - 1$, with the vertices $e_{3l_7}^4: l_7 = 1, 2, \dots, \Delta_4 - 1$, in the circuit Δ , then there arise $(\Delta_3 - 1)(\Delta_4 - 1)$ distinct homomorphic images of Δ and 6 pairs of contracted vertices create each homomorphic image. Moreover, the number of total pairs of contracted vertices to generate all $(\Delta_3 - 1)(\Delta_4 - 1)$ and their mirror homomorphic images of Δ is $12(\Delta_3 - 1)(\Delta_4 - 1)$.

Let $H_{26} = \{\Omega_{(l_{13}, l_7)}^{26}; l_{13} = 1, 2, \dots, \Delta_3 - 1; l_7 = 1, 2, \dots, \Delta_4 - 1\}$ be the collection of homomorphic images of Δ acquired by the contraction of the vertices $e_{3l_{13}+1}^3$ with the vertices $e_{3l_7}^4$ in Δ . Figure 39 presents H_{26} graphically. From all the homomorphic images presented in Figure 39, it is not intricated to check that no

one is the mirror image of itself. This lemma can be proved by using the same procedure as that for Lemma 23.

Let $\bar{k}_2 = 8 - k_2$.

Lemma 27. If we contract the vertices $e_{3l_{(7,k_2)}+1}^{k_2}: l_{(7,k_2)} = 1, 2, \dots, \Delta_{k_2} - 1$, with the vertices $e_{3l_{(8,k_2)}}^{k_2}: l_{(8,k_2)} = 1, 2, \dots, \Delta_{k_2} - l_{(7,k_2)}$, in the circuit Δ , then for each k_3 , there arise $(1/2)\Delta_{k_2}(\Delta_{k_2} - 1)$ distinct homomorphic images of Δ and 6 pairs of contracted vertices create each homomorphic image. Moreover, the number of total pairs of contracted vertices to generate all $(1/2)\Delta_{k_2}(\Delta_{k_2} - 1)$ and their mirror homomorphic images of Δ is $6(\Delta_{k_2} - 1)^2$.

Proof. For $k_2 = 2$, let $H_{27}^2 = \{\Omega_{(l_{(7,2)}, l_{(8,2)})}^{27}; l_{(7,2)} = 1, 2, \dots, \Delta_2 - 1; l_{(8,2)} = 1, 2, \dots, \Delta_2 - l_{(7,2)}\}$ (Figure 40) be the collection of homomorphic images of Δ acquired by the contraction of the vertices $e_{3l_{(7,2)}+1}^2$ with the vertex $e_{3l_{(8,2)}}^6$ in Δ , where $e_{3l_{(7,2)}+1}^2$ and $e_{3l_{(8,2)}}^6$ are fixed by the words

FIGURE 37: Homomorphic images $\Omega^{25}_{(l_5, l_{(6,4)})}$. FIGURE 38: Homomorphic images $\Omega^{25}_{(l_5, l_{(6,5)})}$.

$(pq)^{l_{(7,2)}} (pq^{-1})^{\Delta_3} (pq)^{\Delta_4} (pq^{-1})^{\Delta_5} (pq)^{\Delta_6} (pq^{-1})^{\Delta_1} (pq)^{\Delta_2 - l_{(7,2)}}$ and $(pq^{-1})^{\Delta_6 - l_{(8,2)}} (pq)^{\Delta_5} (pq^{-1})^{\Delta_4} (pq)^{\Delta_3} (pq^{-1})^{\Delta_2} (pq)^{\Delta_1} (pq^{-1})^{l_{(8,2)}}$. It is easy to verify that $P_1 = (pq)^{l_{(7,2)}} (pq^{-1})^{\Delta_3} (pq)^{\Delta_4} (pq^{-1})^{\Delta_5} (pq)^{\Delta_6 - l_{(8,2)}} p$ and $P_2 = q (pq)^{l_{(8,2)} - 1} (pq^{-1})^{\Delta_1} (pq)^{\Delta_2 - l_{(7,2)}}$ are the possible paths between $e_{3l_{(7,2)}+1}^2$ and $e_{3l_{(8,2)}}^6$ (Figure 4). This implies that the homomorphic image $\Omega^{27}_{(l_{(7,2)}, l_{(8,2)})}$ has a vertex s fixed by P_1 and P_2 . Thus,

$$E_{27}^2 = \{e, q, q^{-1}, p, pq, pq^{-1}\} \quad (19)$$

is the family of elements in $\text{PSL}(2, \mathbb{Z})$ such that $\forall w \in E_{27}^2, \Delta$ contains the vertices $(e_{3l_{(7,2)}+1}^2)w$ and $(e_{3l_{(8,2)}}^6)w$. This gives that the cardinality of E_{27}^2 , that is, 6 is the number of contracted pairs of vertices to produce each homomorphic image $\Omega^{27}_{(l_{(7,2)}, l_{(8,2)})}$ (Theorem 2).

Suppose $\Omega^{27}_{(m_1, m_2)}$ and $\Omega^{27}_{(n_1, n_2)}$ are the same. This concludes that $\Omega^{27}_{(m_1, m_2)}$ is procurable also by contracting $e_{3n_1+1}^2$ and $e_{3n_2}^6$, implying that $e_{3n_1+1}^2 \leftrightarrow e_{3n_2}^6$ is one of the pairs of contracted vertices for $\Omega^{27}_{(m_1, m_2)}$. Then there must exist an element $w \in E_{27}^2$ such that $(e_{3m_1+1}^2)w = e_{3n_1+1}^2$ and $(e_{3m_2}^6)w = e_{3n_2}^6$. $pq^{-1} \in E_{27}^2$ is the only element such that $(e_{3m_2}^6)pq^{-1} = e_{3(m_2+1)}^6$ but $(e_{3m_1+1}^2)pq^{-1} \neq e_{3n_1+1}^2$. Thus, $\Omega^{27}_{(m_1, m_2)}$ and $\Omega^{27}_{(n_1, n_2)}$ are not the same, that is, by contracting $e_{3m_1+1}^2$ and $e_{3m_2}^6$ to produce $\Omega^{27}_{(m_1, m_2)}$, $e_{3n_1+1}^2$ and $e_{3n_2}^6$ are not contracted. Now, if $\Omega^{27}_{(m_1, m_2)}$ and $\Omega^{27}_{(n_1, n_2)}$ are the same, then there must exist an element $w \in E_{27}^2$ such that $(e_{3m_1+1}^2)w = e_{3n_1+1}^{2*}$ and $(e_{3m_2}^6)w = e_{3n_2}^{6*}$ or $(e_{3m_1+1}^2)w = e_{3n_1+1}^{6*}$ and $(e_{3m_2}^6)w = e_{3n_2}^{2*}$. $p \in E_{27}^2$ is the only element such that $(e_{3m_1+1}^2)p = e_{3m_1+1}^{2*} = e_{3(\Delta_2 - m_1) + 1}^{6*}$ and $(e_{3m_2}^6)p = e_{3m_2+1}^{6*} = e_{3(\Delta_2 - m_2)}^{2*}$. This implies that $\Omega^{27}_{(m_1, m_2)}$ and

FIGURE 39: Homomorphic images $\Omega^{26}_{(l_{13}, l_7)}$. FIGURE 40: Homomorphic images $\Omega^{27}_{(l_{(7,2)}, l_{(8,2)})}$.

FIGURE 41: Homomorphic images $\Omega^{27}_{(l_{(7,3)}, l_{(8,3)})}$.

FIGURE 42: Homomorphic images $\Omega^{28}_{(l_{(9,1)}, l_{(10,1)})}$.

$\Omega_{(n_1, n_2)}^{27}$ are the mirror images of each other if and only if $n_1 = \Delta_2 - m_2$ and $n_2 = \Delta_2 - m_1$, that is, by contracting $e_{3m_1+1}^2$ and $e_{3m_2}^6$ to produce $\Omega_{(m_1, m_2)}^{27}$, $e_{3m_1+1}^{2*}$ and $e_{3m_2}^{6*}$ are also contracted. For a fix $m_1 \in l_{(7,2)}$ and for all $m_2 \in l_{(8,2)} \setminus \{\Delta_2 - m_1\}$, $\Omega_{(\Delta_2 - m_2, \Delta_2 - m_1)}^{22} \notin H_{27}^2$, implying that $\Omega_{(m_1, m_2)}^{27}$ and $\Omega_{(n_1, n_2)}^{27}$ are not the mirror images of each other. This implies that all the homomorphic images in H_{27}^2 are distinct. Thus, $H_{27}^2 = \left\{ \Omega_{(l_{(7,2)}, l_{(8,2)})}^{27}; l_{(7,2)} = 1, 2, \dots, \Delta_2 - 1; l_{(8,2)} = 1, 2, \dots, \Delta_2 - l_{(7,2)} \right\}$ implies

$$|H_{27}^2| = \frac{1}{2} \Delta_2 (\Delta_2 - 1). \quad (20)$$

Let $\Omega_{(m_1, m_2)}^{27}$ be the mirror image of itself, then from Figure 40, we have $m_1 - 1 = \Delta_2 - m_2 - 1$ and $m_2 - 1 = \Delta_2 - m_1 - 1$, implying $m_2 = \Delta_2 - m_1$. Now, $\forall m_1 \in l_{(7,2)}, \Delta_2 - m_1 \in l_{(8,2)}$, implying that $\Omega_{(m_1, \Delta_2 - m_1)}^{27} \in H_{27}^2$ is the mirror image of itself. So, out of $(1/2)\Delta_2(\Delta_2 - 1)$ homomorphic images in H_{27}^2 , $\Delta_2 - 1$ are the mirror images of itself. Hence, there are

$$2 \times |E_{27}^2| \times [|H_{27}^2| - (\Delta_2 - 1)] + |E_{27}^2| \times (\Delta_2 - 1) = 6(\Delta_2 - 1)^2 \quad (21)$$

pairs of contracting vertices to produce all the homomorphic images in H_{27}^2 . We can prove this lemma for $k_2 = 3$ (Figure 41) in similar way as that for $k_2 = 2$.

$$\text{Let } k_4 = \{1, 4\} \text{ and } \eta_{(4, k_4)} = \begin{cases} 0 & \text{if } \Delta_{k_4} = 0 \pmod{2} \\ 1 & \text{otherwise} \end{cases}. \quad \square$$

Lemma 28. *If we contract the vertices $e_{3l_{(7, k_4)}+1}^{k_4} : l_{(9, k_4)} = 1, 2, \dots, ((\Delta_{k_4} - 2 - \eta_{(4, k_4)})/2)$, with the vertices $e_{3l_{(10, k_4)}}^{k_4} : l_{(10, k_4)} = l_{(9, k_4)} + 1, l_{(9, k_4)} + 2, \dots, \Delta_{k_4} - l_{(9, k_4)}$, in the circuit Δ , then for each k_4 , there arise $(1/4)(\Delta_{k_4}^2 - 2\Delta_{k_4} + \eta_{(4, k_4)})$ distinct homomorphic images of Δ and 6 pairs of contracted vertices create each homomorphic image. Moreover, the number of total pairs of contracted vertices to generate all $(1/4)(\Delta_{k_4}^2 - 2\Delta_{k_4} + \eta_{(4, k_4)})$ and their mirror homomorphic images of Δ is $3(\Delta_{k_4}^2 - 3\Delta_{k_4} + 2)$.*

Proof. For $k_4 = 1$, let $H_{28}^1 = \left\{ \Omega_{(l_{(9,1)}, l_{(10,1)})}^{28}; l_{(9,1)} = 1, 2, \dots, ((\Delta_1 - 2 - \eta_{(4,1)})/2); l_{(10,1)} = l_{(9,1)} + 1, l_{(9,1)} + 2, \dots, \Delta_1 - l_{(9,1)} \right\}$ (Figure 42) be the

collection of homomorphic images of Δ acquired by the contraction of the vertices $e_{3l_{(9,1)}+1}^1$ with the vertices $e_{3l_{(10,1)}}^1$ in Δ , where $e_{3l_{(9,1)}+1}^1$ and $e_{3l_{(10,1)}}^1$ are fixed by the words $(pq)^{l_{(9,1)}}(pq^{-1})^{\Delta_6}(pq)^{\Delta_5}(pq^{-1})^{\Delta_4}(pq)^{\Delta_3}(pq^{-1})^{\Delta_2}(pq)^{\Delta_1-l_{(9,1)}}$ and $(pq^{-1})^{\Delta_1-l_{(10,1)}}$. It is easy to verify that $P_1 = (pq)^{l_{(9,1)}}(pq^{-1})^{\Delta_6}(pq)^{\Delta_5}(pq^{-1})^{\Delta_4}(pq)^{\Delta_3}(pq^{-1})^{\Delta_2}(pq)^{\Delta_1-l_{(10,1)}}p$ and $P_2 = q(pq)^{l_{(10,1)}-l_{(9,1)}-1}$ are the possible paths between $e_{3l_{(9,1)}+1}^1$ and $e_{3l_{(10,1)}}^1$ (Figure 4). This implies that the homomorphic image $\Omega_{(l_{(9,1)}, l_{(10,1)})}^{28}$ has a vertex s fixed by P_1 and P_2 . Thus,

$$E_{28}^1 = \{e, q, q^{-1}, p, pq, pq^{-1}\} \quad (22)$$

is the family of elements in $\text{PSL}(2, \mathbb{Z})$ such that $\forall w \in E_{28}^1, \Delta$ contains the vertices $(e_{3l_{(9,1)}+1}^1)w$ and $(e_{3l_{(10,1)}}^1)w$. This gives that the cardinality of E_{28}^1 , that is, 6 is the number of contracted pairs of vertices to produce each homomorphic image $\Omega_{(l_{(9,1)}, l_{(10,1)})}^{28}$ (Theorem 2).

Suppose $\Omega_{(m_1, m_2)}^{28}$ and $\Omega_{(n_1, n_2)}^{28}$ are the same. This concludes that $\Omega_{(m_1, m_2)}^{28}$ is procurable also by contracting $e_{3m_1+1}^1$ and $e_{3m_2}^1$, implying that $e_{3m_1+1}^1 \leftrightarrow e_{3m_2}^1$ is one of the pairs of contracted vertices for $\Omega_{(m_1, m_2)}^{28}$. Then there must exist an element $w \in E_{28}^1$ such that $(e_{3m_1+1}^1)w = e_{3m_1+1}^1$ and $(e_{3m_2}^1)w = e_{3m_2}^1$. $pq, pq^{-1} \in E_{28}^1$ are the only element such that $(e_{3m_1+1}^1)pq = e_{3(m_1-1)+1}^1$ and $(e_{3m_2}^1)pq^{-1} = e_{3(m_2+1)}^1$ but $(e_{3m_2}^1)pq \neq e_{3m_2}^1$ and $(e_{3m_1+1}^1)pq^{-1} \neq e_{3m_1+1}^1$. $\Omega_{(m_1, m_2)}^{28}$ and $\Omega_{(n_1, n_2)}^{28}$ are not the same, that is, by contracting $e_{3m_1+1}^1$ and $e_{3m_2}^1$ to produce $\Omega_{(m_1, m_2)}^{28}$, $e_{3m_1+1}^1$ and $e_{3m_2}^1$ are not contracted. Now, if $\Omega_{(m_1, m_2)}^{28}$ and $\Omega_{(n_1, n_2)}^{28*}$ are same, then there must exist an element $w \in E_{28}^1$ such that $(e_{3m_1+1}^1)w = e_{3n_1+1}^{1*}$ and $(e_{3m_2}^1)w = e_{3n_2}^{1*}$. $p \in E_{28}^1$ is the only element such that $(e_{3m_1+1}^1)p = e_{3m_1}^{1*} = e_{3(\Delta_1-m_1)+1}^{1*}$ and $(e_{3m_2}^1)p = e_{3m_2+1}^{1*} = e_{3(\Delta_1-m_2)}^{1*}$. This implies that $\Omega_{(m_1, m_2)}^{28}$ and $\Omega_{(n_1, n_2)}^{28}$ are the mirror images of each other if and only if $n_1 = \Delta_1 - m_1$ and $n_2 = \Delta_1 - m_2$, that is, by contracting $e_{3m_1+1}^1$ and $e_{3m_2}^1$ to produce $\Omega_{(m_1, m_2)}^{28}$, $e_{3m_1+1}^{1*}$ and $e_{3m_2}^{1*}$ are also contracted. For all $m_1 \in l_{(9,1)}, \Delta_1 - m_1 > ((\Delta_1 - 2 - \eta_{(4,1)})/2)$, implying that $\Omega_{(n_1, n_2)}^{28} = \Omega_{(\Delta_2 - m_2, \Delta_2 - m_1)}^{28} \notin H_{28}^1$ and that $\Omega_{(m_1, m_2)}^{28}$ and $\Omega_{(n_1, n_2)}^{28}$ are not the mirror images of each other. This implies that all the homomorphic images in H_{28}^1 are distinct. Thus,

$$H_{28}^1 = \left\{ \Omega_{(l_{(9,1)}, l_{(10,1)})}^{28}; l_{(9,1)} = 1, 2, \dots, \frac{\Delta_1 - 2 - \eta_{(4,1)}}{2}; l_{(10,1)} = l_{(9,1)} + 1, l_{(9,1)} + 2, \dots, \Delta_1 - l_{(9,1)} \right\} \quad (23)$$

implies

$$\begin{aligned} |H_{28}^1| &= (\Delta_1 - 2) + (\Delta_2 - 4) + (\Delta_2 - 6) + \dots + (4 - \eta_{(4,1)}) \\ &\quad + (2 - \eta_{(4,1)}), \\ &= \frac{1}{4}(\Delta_1^2 - 2\Delta_1 + \eta_{(4,1)}). \end{aligned} \quad (24)$$

Let $\Omega_{(m_1, m_2)}^{28}$ be the mirror image of itself, then from Figure 42, we have $m_1 - 1 = \Delta_1 - m_2 - 1$, implying $m_2 = \Delta_1 - m_1$. Now, $\forall m_1 \in l_{(9,1)}, \Delta_1 - m_1 \in l_{(10,1)}$, implying that $\Omega_{(m_1, \Delta_1 - m_1)}^{28} \in H_{28}^1$ is the mirror image of itself. So, out of $(1/4)(\Delta_1^2 - 2\Delta_1 + \eta_{(4,1)})$ homomorphic images in H_{28}^1 , $((\Delta_1 - 2 - \eta_{(4,1)})/2)$ are the mirror images of itself. Hence, there are

FIGURE 43: Homomorphic images $\Omega_{(l(9,4)l(10,4))}^{28}$.

$$\begin{aligned}
 & 2 \times |E_{28}^2| \times \left[|H_{28}^2| - \frac{\Delta_1 - 2 - \eta_{(4,1)}}{2} \right] + |E_{28}^2| \times \frac{\Delta_1 - 2 - \eta_{(4,1)}}{2} \\
 &= 3(\Delta_1^2 - 3\Delta_1 + 2)
 \end{aligned} \tag{25}$$

pairs of contracting vertices to produce all the homomorphic images in H_{28}^1 . We can prove this lemma for $k_4 = 4$ (Figure 43) in similar way as that for $k_4 = 1$. $\square$

Lemma 29. If we contract the vertices $e_{3l(11,k_2)+1}^{k_2} : l_{(11,k_2)} = 1, 2, \dots, \Delta_{k_2} - 1$, with the vertices $e_{3l(12,k_2)}^{k_2} : l_{(12,k_2)} = 1, 2, \dots, \Delta_{k_2} - l_{(11,k_2)}$, in the circuit Δ , then for each k_2 , there arise $(1/6)(\Delta_{k_2} - 1)(\Delta_{k_2} - 2)$ distinct homomorphic images of Δ and 6 pairs of contracted vertices create each homomorphic image. Moreover, the number of total pairs of contracted vertices to generate all $(1/2)(\Delta_{k_2} - 1)(\Delta_{k_2} - 2)$ and their mirror homomorphic images of Δ is $6(\Delta_{k_2} - 1)(\Delta_{k_2} - 2)$.

For a fix value of k_2 , let $H_{29}^{k_2} = \left\{ \Omega_{(l(11,k_2)l(12,k_2))}^{29} ; l_{(11,k_2)} = 1, 2, \dots, \Delta_{k_2} - 1 ; l_{(12,k_2)} = 1, 2, \dots, \Delta_{k_2} - l_{(11,k_2)} \right\}$ be the collection of homomorphic images of Δ acquired by the contraction of the vertices $e_{3l(11,k_2)+1}^{k_2}$ with the vertices $e_{3l(12,k_2)}^{k_2}$ in Δ . Figures 44 and 45 present $H_{29}^{k_2}$ graphically. From all the homomorphic images presented in these figures, it is not intricated to check that no one is the mirror image of itself. This lemma can be proved by using the same procedure as that for Lemma 23.

$$\text{Let } k_4 = \{1, 4\} \text{ and } \eta_{(4,k_4)} = \begin{cases} 0 & \text{if } \Delta_{k_4} = 0 \pmod{2} \\ 1 & \text{otherwise} \end{cases}$$

Lemma 30. If we contract the vertices $e_{3l(13,k_4)+1}^{k_4} : l_{(13,k_4)} = 1, 2, \dots, ((\Delta_{k_4} - \eta_{(4,k_4)})/2)$, with the vertices $e_{3l(13,k_4)}^{k_4}$ in the circuit Δ , then for each k_4 , there arise $(1/2)(\Delta_{k_4} - \eta_{(4,k_4)})$ distinct homomorphic images of Δ and 3 pairs of contracted vertices create each homomorphic image. Moreover, the number of total pairs of contracted vertices to generate all $(1/2)(\Delta_{k_4} - \eta_{(4,k_4)})$ and their mirror homomorphic images of Δ is $3(\Delta_{k_4} - 1)$.

For a fix value of k_4 , let $H_{30}^{k_4} = \left\{ \Omega_{(l(13,k_4)l(13,k_4))}^{30} ; l_{(13,k_4)} = 1, 2, \dots, ((\Delta_{k_4} - \eta_{(4,k_4)})/2) \right\}$ be the collection of homomorphic images of Δ acquired by the contraction of the vertices $e_{3l(13,k_4)+1}^{k_4}$ with the vertices $e_{3l(13,k_4)}^{k_4}$ in Δ . Figures 46 and 47 present $H_{30}^{k_4}$ graphically. From all the homomorphic images presented in these figures, it is not intricated to check that if $\Delta_{k_2} = 0 \pmod{2}$, then $\Omega_{((\Delta_{k_2}/2), (\Delta_{k_2}/2))}^{30}$ is the only homomorphic image which is the mirror image of itself; otherwise, no one is the mirror image of itself.

This lemma can be proved by using the same procedure as that for Lemma 28.

Lemma 31. If we contract the vertices $e_{3l(11,k_2)+1}^{k_2} : l_{(11,k_2)} = 1, 2, \dots, \Delta_{k_2} - 1$, with the vertices $e_{3l(11,k_2)}^{k_2}$ in the circuit Δ , then for each k_2 , there arise $\Delta_{k_2} - 1$ distinct homomorphic images of Δ and 3 pairs of contracted vertices create each homomorphic image. Moreover, the number of total pairs of contracted vertices to generate all $\Delta_{k_2} - 1$ and their mirror homomorphic images of Δ is $6(\Delta_{k_2} - 1)$.

For a fix value of k_2 , let $H_{31}^{k_2} = \left\{ \Omega_{(l(11,k_2)l(11,k_2))}^{31} ; l_{(11,k_2)} = 1, 2, \dots, \Delta_{k_2} - 1 \right\}$ be the collection of homomorphic images of Δ acquired by the

FIGURE 44: Homomorphic images $\Omega_{(l_{(11,2)}, l_{(12,2)})}^{29}$.

contraction of the vertices $e_{3l_{(11,k_2)}}^{k_2}$ with the vertices $e_{3l_{(11,k_2)}}^{k_2}$ in Δ . Figures 48 and 49 present $H_{3l_{(11,k_2)}}^{k_2}$ graphically. From all the homomorphic images presented in these figures, it is not intricate to check that no one is the mirror image of itself. This lemma can be proved by using same procedure as that for Lemma 23.

Let $k_5 = \{2, 4\}$.

Lemma 32. For each k_5 , there are $6(\Delta_{k_5} + \Delta_{k_5+1} + 2)$ pairs of contracting vertices to produce the homomorphic image of Δ by contracting the vertex e_1^1 with the vertex $e_1^{k_5}$.

For $k_5 = 2$, let Ω_2^{32} be the homomorphic image of Δ acquired by contracting the vertex e_1^1 with the vertex e_1^2 in Δ , where e_1^1 and e_1^2 are fixed by the words $(pq^{-1})^{\Delta_6}(pq)^{\Delta_5}(pq^{-1})^{\Delta_4}(pq)^{\Delta_3}(pq^{-1})^{\Delta_2}(pq)^{\Delta_1}$ and $(pq^{-1})^{\Delta_3}(pq)^{\Delta_4}(pq^{-1})^{\Delta_5}(pq)^{\Delta_6}(pq^{-1})^{\Delta_1}(pq)^{\Delta_2}$. It is easy to verify that $P_1 = (pq^{-1})^{\Delta_6}(pq)^{\Delta_5}(pq^{-1})^{\Delta_4}(pq)^{\Delta_3}p$ and $P_2 = q^{-1}(pq^{-1})^{\Delta_2-1}(pq)^{\Delta_1}$ are the possible paths between e_1^1 and e_1^2 (Figure 4). This implies that the homomorphic image Ω_2^{32} has a vertex s fixed by P_1 and P_2 . Thus,

$$E_2^{32} = \left\{ \begin{array}{l} e, q, q^{-1}, p, pq, (pq^{-1})^1, (pq^{-1})^1 p, (pq^{-1})^1 pq, (pq^{-1})^2, \dots, (pq^{-1})^{\Delta_3} p, (pq^{-1})^{\Delta_3} pq, \\ (pq^{-1})^{\Delta_3+1}, (q^{-1}p)^1, (q^{-1}p)^1 q, (q^{-1}p)^1 q^{-1}, \dots, (q^{-1}p)^{\Delta_2}, (q^{-1}p)^{\Delta_2} q, (q^{-1}p)^{\Delta_2} q^{-1} \end{array} \right\} \quad (26)$$

is the family of elements in $\text{PSL}(2, \mathbb{Z})$ such that $\forall w \in E_2^{32}, \Delta$ contains the vertices $(e_1^1)w$ and $(e_1^2)w$. This gives that the cardinality of E_2^{32} , that is, $3(\Delta_2 + \Delta_3 + 2)$ is the number of contracted pairs of vertices to produce the homomorphic image Ω_2^{32} (Theorem 2). From Figure 50, it is not intricate to

check that Ω_2^{32} and its mirror image Ω_2^{32*} are not the same. Hence, there are

$$2 \times |E_2^{32}| = 3(\Delta_2 + \Delta_3 + 2), \quad (27)$$

FIGURE 45: Homomorphic images $\Omega_{(l_{(11,3)}^4, l_{(12,3)}^2)}^{29}$.

pairs of contracting vertices to produce the homomorphic images Ω_2^{32} .

We can prove this lemma for $k_5 = 4$ (Figure 51) in similar way as that for $k_5 = 2$.

Lemma 33. For each k_3 , there are 6 $(\Delta_4 + \Delta_5 + 2)$ pairs of contracting vertices to produce the homomorphic image of Δ by contracting the vertex $e_1^{k_3}$ with the vertex $e_1^{k_3}$.

For a fix value of k_3 , let $\Omega_{k_3}^{33}$ be the homomorphic image of Δ acquired by the contraction of the vertex $e_1^{k_3}$ with the vertex $e_1^{k_3}$ in Δ . Figures 52 and 53 present $\Omega_{k_3}^{33}$ graphically. From these figures, it is not intricated to check that $\Omega_{k_3}^{33}$ is not the mirror image of itself.

This lemma can be proved by using the same procedure as that for Lemma 32.

Let $k_6 = \{1, 2, 5\}$ and $\overline{k_6} = \begin{cases} 6 & \text{if } k_6 = 1, \\ 3 & \text{if } k_6 = 2, \\ 4 & \text{if } k_6 = 5. \end{cases}$

Lemma 34. For each k_6 , there are 6 $(\Delta_{\overline{k_6}} + 1)$ pairs of contracting vertices to produce the homomorphic image of Δ by contracting the vertex $e_1^{k_6}$ with the vertex $e_1^{k_6}$.

For a fix value of k_6 , let $\Omega_{k_6}^{34}$ be the homomorphic image of Δ acquired by the contraction of the vertex $e_1^{k_6}$ with the vertex $e_1^{k_6}$ in Δ . Figures 54, 55, and 56 present $\Omega_{k_6}^{34}$ graphically. From these figures, it is not intricated to check that $\Omega_{k_6}^{34}$ is not the mirror image of itself. This lemma can be proved by using the same procedure as that for Lemma 32.

FIGURE 46: Homomorphic images $\Omega_{(l_{(13,1)}^4, l_{(13,1)}^2)}^{30}$.

Lemma 35. For each k_3 , there are 3 $(\Delta_{k_3+1} + 2\Delta_{k_3+2} + 2)$ pairs of contracting vertices to produce the homomorphic image of Δ by contracting the vertex $e_1^{k_3}$ with the vertex $e_1^{k_3+2}$.

For a fix value of k_3 , let $\Omega_{k_3}^{35}$ be the homomorphic image of Δ acquired by the contraction of the vertex $e_1^{k_3}$ with the vertex $e_1^{k_3+2}$ in Δ . Figures 57(a), 57(b), and 57(c) present $\Omega_{k_3}^{35}$ and Figure 58 presents $\Omega_{k_3}^{35}$ graphically. From these figures, it is not intricated to check that $\Omega_{k_3}^{35}$ is the mirror image of itself. This lemma can be proved by using the same procedure as that for Lemma 32.

Let $V(\Delta \times \Delta) = \{(a, b); a, b \in V(\Delta) \text{ and } a \neq b\}$, then

$$|V(\Delta \times \Delta)| = \frac{1}{2} [9(\Delta_1 + 2\Delta_2 + 2\Delta_3 + \Delta_4) - 3(\Delta_1 + 2\Delta_2 + 2\Delta_3 + \Delta_4)]. \quad (28)$$

We are now able to prove our primary outcome. Let

$$\eta = \begin{cases} 4, & \text{if } (\Delta_1, \Delta_4) = (E, E), \\ 0, & \text{if } (\Delta_1, \Delta_4) = (O, O), \\ -1, & \text{otherwise.} \end{cases} \quad (29)$$

where E and O stand for even and odd, respectively.

Theorem 5. There are $(1/4)[(\Delta_1 + 2\Delta_2 + 2\Delta_3 + \Delta_4)^2 + 4(\Delta_1 + 2\Delta_2 + 2\Delta_3 + \Delta_4) + 4\Delta_4 + \eta]$ number of distinct

FIGURE 47: Homomorphic images $\Omega_{(l(13,4), l(13,4))}^{30}$. FIGURE 48: Homomorphic images $\Omega_{(l(11,2), l(11,2))}^{31}$.

homomorphic images acquired by the contraction of all pairs of vertices of the circuit Δ .

Proof. Let us collect all the pairs of contracting vertices of Δ mentioned in Lemma 1 to Lemma 35 in the form of set S as

$$S = \left\{ \begin{aligned} &(e_{3l_1+1}^1, e_{3\Delta_1}^1), (e_{3l_1+1}^2, e_{3\Delta_1}^1), (e_{3l_{(2,k_1)}+1}^{k_1}, e_{3\Delta_1}^1), (e_{3l_5+1}^6, e_{3\Delta_1}^1), (e_{3l_4+1}^1, e_{3\Delta_2}^2), (e_{3l_5+1}^2, e_{3\Delta_2}^2), (e_{3l_{(6,k_1)}+1}^{k_1}, e_{3\Delta_2}^2), \\ &(e_{3l_7+1}^3, e_{3\Delta_3}^3), (e_{3l_7+1}^4, e_{3\Delta_3}^3), (e_{3l_8+1}^4, e_{3\Delta_4}^4), (e_{3l_9+1}^5, e_{3\Delta_4}^4), (e_{3l_{10}+1}^3, e_{3\Delta_4}^4), (e_{3l_1+1}^2, e_{3\Delta_5}^5), (e_{3l_{11}+1}^3, e_{3\Delta_5}^5), (e_{3l_7+1}^4, e_{3\Delta_5}^5), \\ &(e_{3l_{12}+1}^5, e_{3\Delta_5}^5), (e_{3l_{13}+1}^3, e_{3\Delta_6}^6), (e_{3l_{14}+1}^1, e_{3\Delta_1}^1), (e_{3l_{15}+1}^1, e_{3\Delta_3}^3), (e_{3l_{(3,k_2)}+1}^{k_2}, e_{3\Delta_3}^3), (e_{3l_{16}+1}^2, e_{3\Delta_4}^4), (e_{3l_{(4,k_3)}+1}^{k_3}, e_{3\Delta_6}^6), \\ &(e_{3l_4+1}^1, e_{3l_{(5,k_2)}}^{k_2}), (e_{3l_4+1}^1, e_{3l_{17}}^4), (e_{3l_5+1}^2, e_{3l_{(6,k_1)}}^{k_1}), (e_{3l_{13}+1}^3, e_{3l_7}^4), (e_{3l_{(7,k_2)}+1}^{k_2}, e_{3l_{(8,k_2)}}^{\bar{k}_2}), (e_{3l_{(7,k_4)}+1}^{k_4}, e_{3l_{(10,k_4)}}^{k_4}), (e_{3l_{(11,k_2)}+1}^{k_2}, e_{3l_{(12,k_2)}}^{k_2}), \\ &(e_{3l_{(13,k_4)}+1}^{k_4}, e_{3l_{(13,k_4)}}^{k_4}), (e_{3l_{(11,k_2)}+1}^{k_2}, e_{3l_{(11,k_2)}}^{k_2}), (e_1^{k_3}, e_1^1), (e_1^{k_3}, e_1^5), (e_1^{k_6}, e_1^{\bar{k}_6}), (e_1^{k_3}, e_1^{k_3+2}) \end{aligned} \right\}. \quad (30)$$

Let H be the set of homomorphic images obtained by contracting each element of S , then

$$H = \left\{ \begin{aligned} &\Omega_{l_1}^1, \Omega_{l_1}^2, \Omega_{l_{(2,k_1)}}^3, \Omega_{l_3}^4, \Omega_{l_4}^5, \Omega_{l_5}^6, \Omega_{l_{(6,k_1)}}^7, \Omega_{l_7}^8, \Omega_{l_7}^9, \Omega_{l_8}^{10}, \Omega_{l_9}^{11}, \Omega_{l_{10}}^{12}, \Omega_{l_1}^{13}, \Omega_{l_{11}}^{14}, \Omega_{l_7}^{15}, \Omega_{l_{12}}^{16}, \Omega_{l_{13}}^{17}, \\ &\Omega_{l_{14}}^{18}, \Omega_{l_{15}}^{19}, \Omega_{l_{(3,k_2)}}^{20}, \Omega_{l_{16}}^{21}, \Omega_{l_{(4,k_3)}}^{22}, \Omega_{l_{(4,l_{(5,k_2)})}}^{23}, \Omega_{l_{(4,l_{17})}}^{24}, \Omega_{l_{(5,l_{(6,k_1)})}}^{25}, \Omega_{l_{(13,l_7)}}^{26}, \Omega_{l_{(7,k_2), l_{(8,k_2)}}}^{27}, \\ &\Omega_{l_{(9,k_4), l_{(10,k_4)}}}^{28}, \Omega_{l_{(11,k_2), l_{(12,k_2)}}}^{29}, \Omega_{l_{(13,k_4), l_{(13,k_4)}}}^{30}, \Omega_{l_{(11,k_2), l_{(11,k_2)}}}^{31}, \Omega_{k_5}^{32}, \Omega_{k_3}^{33}, \Omega_{k_6}^{34}, \Omega_{k_3}^{35} \end{aligned} \right\}. \quad (31)$$

FIGURE 49: Homomorphic images $\Omega_{(l_{(11,3)}^1, l_{(11,3)})}^{31}$.

Let S' be the sum of contracted pairs obtained by contracting each element of S , then

FIGURE 50: Homomorphic image Ω_2^{32} .

$$\begin{aligned}
 S' &= 3(\Delta_2^2 + 3\Delta_2 - 2) + 3(\Delta_2^2 + 3\Delta_2) + \frac{3}{2}(\Delta_6^2 + 3\Delta_6) + \frac{3}{2}(\Delta_1^2 + 3\Delta_1 - 4) + 3(\Delta_2^2 + 3\Delta_2 - 4) + 3(\Delta_4^2 + 3\Delta_4 - 4) \\
 &+ \frac{3}{2}(\Delta_4^2 + 3\Delta_4 - 4) + 3(\Delta_4^2 + 3\Delta_4 - 2) + \frac{3}{2}(\Delta_5^2 + 3\Delta_5) + 3(\Delta_3^2 + 3\Delta_3) + \frac{3}{2}(\Delta_2^2 + 3\Delta_2) + 3(\Delta_3^2 + 3\Delta_3 - 2) \\
 &+ 3(\Delta_4^2 + 3\Delta_4 - 4) + 3(\Delta_5^2 + 3\Delta_5) + \frac{3}{2}(\Delta_3^2 + 3\Delta_3 - 4) + 6(\Delta_2 + 2)(\Delta_1 - \Delta_2 - 1) \\
 &+ 3(\Delta_4 + 2)(\Delta_1 - \Delta_4 - 1) \\
 &+ 6(\Delta_3 + 2)(\Delta_2 - \Delta_3 - 1) \\
 &+ 6(\Delta_1 - 1)(\Delta_4 - 1) + 12(\Delta_3 - 1)(\Delta_4 - 1) + \sum_{k_1=3}^5
 \end{aligned}$$

FIGURE 51: Homomorphic image Ω_4^{32} . FIGURE 52: Homomorphic image Ω_1^{33} .

$$\begin{aligned}
 & \left[3(\Delta_{k_1}^2 + 3\Delta_{k_1}) + 3(\Delta_{k_1}^2 + 3\Delta_{k_1} - 4) + 12(\Delta_2 - 1)(\Delta_{k_1} - 1) \right] \\
 & + \sum_{k_2=2}^3 \left[6(\Delta_4 + 2)(\Delta_{k_2} - \Delta_4 - 1) + 12(\Delta_1 - 1)(\Delta_{k_2} - 1) + 6(\Delta_{k_2} - 1)^2 + 6(\Delta_{k_2} - 1)(\Delta_{k_2} - 2) + 6(\Delta_{k_2} - 1) \right] \\
 & + \sum_{k_3=1}^2 \left[6(\Delta_5 + 2\Delta_{k_3} - \Delta_5 - 1) + 6(\Delta_4 + \Delta_5 + 2) + 3(\Delta_{k_3+1} + 2\Delta_{k_3+2} + 2) \right] + \sum_{k_4=1}^4 \left[3(\Delta_{k_4}^2 - 3\Delta_{k_4} + 2) + 3(\Delta_{k_4} - 1) \right] \\
 & + \sum_{k_5=2,4} \left[6(\Delta_{k_5} + \Delta_{k_5+1} + 2) \right] + \sum_{k_6=1,2,5} \left[6(\Delta_{k_6} + 1) \right] \\
 & = \frac{1}{2} [9(\Delta_1 + 2\Delta_2 + 2\Delta_3 + \Delta_4) - 3(\Delta_1 + 2\Delta_2 + 2\Delta_3 + \Delta_4)].
 \end{aligned} \tag{32}$$

shows that each element of $V(\Delta \times \Delta)$ is contracted.

Now,

$$\begin{aligned}
 |H| &= \Delta_2 + \Delta_2 + \Delta_6 + (\Delta_1 - 1) + (\Delta_2 - 1) + (\Delta_4 - 1) + (\Delta_1 - 1) + \Delta_4 + \Delta_5 + \Delta_3 + \Delta_2 + \Delta_3 + (\Delta_4 - 1) + \Delta_5 + (\Delta_3 - 1) \\
 &+ (\Delta_1 - \Delta_2 - 1) + \frac{\Delta_1 - \Delta_4 - \eta_1}{2} + (\Delta_2 - \Delta_3 - 1) + \begin{cases} \frac{1}{2}(\Delta_1 - 1)(\Delta_4 - 1) & \text{if } \Delta_4 = 1 \pmod{2} \\ \frac{1}{2}[(\Delta_1 - 1)(\Delta_4 - 1) + 1 - \eta_3] & \text{if } \Delta_4 = 0 \pmod{2} \end{cases} + (\Delta_3 - 1)(\Delta_4 - 1)
 \end{aligned}$$

FIGURE 53: Homomorphic image Ω_2^{33} . FIGURE 54: Homomorphic image Ω_1^{34} .

$$\begin{aligned}
 & + \sum_{k_1=3}^5 [\Delta_{k_1} + (\Delta_{k_1} - 1) + (\Delta_2 - 1)(\Delta_{k_1} - 1)] + \sum_{k_1=3}^5 [(\Delta_{k_2} - \Delta_4 - 1) + (\Delta_1 - 1)(\Delta_{k_2} - 1) + \frac{1}{2}\Delta_{k_2}(\Delta_{k_2} - 1) + (\Delta_{k_2} - 1) \\
 & + \frac{1}{2}(\Delta_{k_2} - 1)(\Delta_{k_2} - 2)] \\
 & + \sum_{k_3=1}^2 [\Delta_{k_3} - \Delta_5 - 1 + 1 + 1] + \sum_{k_4=1,4} \left[\frac{1}{4}(\Delta_{k_4}^2 - 2\Delta_{k_4} + \Delta_{(4,k_4)}) + \frac{1}{2}(\Delta_{k_4} - \eta_{(4,k_4)}) \right] + \sum_{k_5=2,4} [1] + \sum_{k_6=1,2,5} [1] \\
 & = \frac{1}{4} [(\Delta_1 + 2\Delta_2 + 2\Delta_3 + \Delta_4)^2 + 4(\Delta_1 + 2\Delta_2 + 2\Delta_3 + \Delta_4) + 4\Delta_4 + \eta].
 \end{aligned}$$

(33)

FIGURE 55: Homomorphic image Ω_2^{34} . FIGURE 56: Homomorphic image Ω_5^{34} .

FIGURE 57: (a) Homomorphic image Ω_1^{35} : $\Delta_1 > \Delta_3 + \Delta_5 + 1$. (b) Homomorphic image Ω_1^{35} : $\Delta_1 = \Delta_3 + \Delta_5 + 1$. (c) Homomorphic image Ω_1^{35} : $\Delta_1 < \Delta_3 + \Delta_5 + 1$.

FIGURE 58: Homomorphic image Ω_2^{35} .

The value of $|H|$ gives the total number of homomorphic images produced in the contraction of all pairs of vertices of Δ . The value of S' guaranteed that the set H contains all homomorphic images. $\square$

2. Conclusion

Thus, there are total $(1/2)[9(\Delta_1 + 2\Delta_2 + 2\Delta_3 + \Delta_4) - 3(\Delta_1 + 2\Delta_2 + 2\Delta_3 + \Delta_4)]$ number of elements in $V(\Delta \times \Delta)$. To find all distinct homomorphic images, we do not need to contract each pair of vertices of $V(\Delta \times \Delta)$. We have to contract only those pairs of vertices, which are in set S and they are $(1/4)[(\Delta_1 + 2\Delta_2 + 2\Delta_3 + \Delta_4)^2 + 4(\Delta_1 + 2\Delta_2 + 2\Delta_3 + \Delta_4) + 4\Delta_4 + \eta]$ in numbers because if we contract the pair which is not belong to set S , we attain the homomorphic image, which we already acquired by contracting the element of set S .

3. Applications of Homomorphic Copies in Lightweight Cryptography and Chemistry

The construction of any product by using minimum resources without compromising on quality is primary objective of scientists.

The coset diagrams with 256 vertices are used in the construction of strong 8×8 S-boxes [9]. In present-day block ciphers, cryptographically secure S-boxes are designed to attain the requirements of Shannon's necessity for perplexity. Substitution boxes are the fundamental segments in numerous Feistel network-based block cryptosystems or substitution-permutation (S-P) networks. The use of 8×8 S-boxes in block cipher is excessively costly. So, it is not surprising that we are seeing strong progress in the field of lightweight cryptography in recent years, for instance

PRESENT. Lightweight cryptographic calculations are utilized in business items, including DESL, PRINTCIPHER, SEA, HIGHT, PRESENT, LED, and KATAN/KTANTAN. For instance, Keeloq is a 32-bit block cipher frequently utilized in the automobile business. Digital signature transponder is a 40-bit block cipher, implemented in wireless authentication systems. Since a homomorphic copy of a graph is smaller graph with the same algebraic structure, it can be quite handy to generate small sized S-boxes by using homomorphic copies of the coset diagram having 256 vertices.

Interrelation of certain types of coset graphs and structure of carbon allotropes not only highlights the connection but also improvises applications in many fields.

Several forms of carbon can be found in nature. One of the most important allotropes of carbon is fullerene which was discovered in 1985. Fullerenes are carbon-cage-like polyhedral molecules in which a large number of carbon atoms are bonded in a nearly spherical symmetric configuration. Fullerenes C_n can be drawn for $n = 20$ and for all even $n \geq 24$. They have n carbon atoms, $(3n/2)$ bonds, 12 pentagonal faces, and $(n/2) - 10$ hexagonal faces. The most important member of the family of fullerenes is C_{60} . Fullerenes can be classified in terms of their groups of symmetries. These groups are also known as point groups. Every element of a point group is an isometry of the Euclidian space and so it is either a rotation around an axis or it is a reflection in a plane. The list of all 28 fullerene point groups is: $I_h, I, T_h, T_d, T, D_{6h}, D_{6d}, D_6, D_{5h}, D_{5d}, D_5, D_{3h}, D_{3d}, D_3, D_{2h}, D_{2d}, D_2, S_6, S_4, C_{3h}, C_{2h}, C_{3v}, C_3, C_{2v}, C_2, C_s, C_i$, and C_1 [10]. Fullerene C_{60} has icosahedral symmetry, that is, the symmetry group is isomorphic to A_5 . A constructive enumeration of fullerenes has been dealt in detail in [11]. The structure of coset graphs of fullerene C_{60} , symmetry group $PSL(2, 5)$ and adjacency matrix has been explored through the action of modular group in [12]. Another carbon allotrope with high permutational symmetry is allotrope D_{168} Schwarzite, proposed by Vanderbilt and Tersoff, which has an automorphism group of order 168. In [13], it has been shown that coset diagram for $PSL(2, 7)$ points has interesting relation with carbon allotrope with negative curvature D_{168} Schwarzite. The future studies may extend the present study by investigating the homomorphic copies of the coset graphs for $PSL(2, 5)$ and $PSL(2, 7)$ and their related chemical structures.

Data Availability

No datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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