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Research Article

One-Kind Hybrid Power Means of the Two-Term Exponential Sums and Quartic Gauss Sums

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The main purpose of this article is using the analytic methods and the properties of the classical Gauss sums to study the calculating problem of the hybrid power mean of the two-term exponential sums and quartic Gauss sums and then prove two interesting linear recurrence formulas. As applications, some asymptotic formulas are obtained.

1. Introduction

Let $q \geq 3$ be a fixed integer. For any integers $k \geq 2$ and m with $(m, q) = 1$, the two-term exponential sums $G(m, k; q)$ and quartic Gauss sums $G(m, q)$ are defined by

$$G(m,k;q) = \sum_{a=0}^{q-1} e\left(\frac{ma^k + a}{q}\right), \qquad G(m,q) = \sum_{a=0}^{q-1} e\left(\frac{ma^4}{q}\right), \tag{1}$$

where, as usual, $e(y) = e^{2\pi i y}$ and $i^2 = -1$.

These sums play a significant role in the research of analytic number theory, and many number theory problems are closely related to them. Therefore, for the sake of promoting the development of research work in related fields, it is necessary to study the various properties of $G(m, k; q)$ and $G(m, q)$. Some research results in these fields can be found in references [1–12]. We will not list all of them. For example, Zhang and Zhang [1] proved the identity

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3 + na}{p}\right)\right|^4 = \begin{cases} 2p^3 - p^2, & \text{if } 3 \nmid p-1, \\ 2p^3 - 7p^2, & \text{if } 3 \mid p-1, \end{cases} \tag{2}$$

where p be an odd prime and n denotes any integer with $(n, p) = 1$.

Zhang and Han [2] obtained the identity:

$$\sum_{a=1}^{p-1}\left|\sum_{n=0}^{p-1} e\left(\frac{n^3 + an}{p}\right)\right|^6 = 5p^4 - 8p^3 - p^2, \tag{3}$$

where p denotes an odd prime with $3 \nmid (p-1)$.

Zhang and Zhang [3] derived that, for any prime p, one has the identity:

$$\sum_{m=1}^{p-1} \chi_2(m)\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right)\right|^4 = \begin{cases} p^2(\delta - 3), & \text{if } p \equiv 1 \bmod 6, \\ p^2(\delta + 3), & \text{if } p \equiv -1 \bmod 6, \end{cases} \tag{4}$$

where, as usual, $(*/p) = \chi_2$ denotes the Legendre's symbol modulo p, $d \cdot \overline{d} \equiv 1 \bmod p$, and $\delta = \sum_{d=1}^{p-1}((d - 1 + \overline{d})/p)$ is an integer which satisfies the estimate $|\delta| \leq 2\sqrt{p}$.

The author [4] studied the following hybrid power mean:

$$M_k(p) = \sum_{m=1}^{p-1}\left(\sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right)\right)^k \cdot \left|\sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p}\right)\right|^2, \tag{5}$$

and obtained two interesting fourth-order linear recurrence formula as follows:

(1) $M_k(p) = -2pM_{k-2}(p) + 8p\alpha M_{k-3}(p) - p(9p - 4\alpha^2)M_{k-4}(p)$, where the first four item in sequence $\{M_k(p)\}$ is $M_0(p) = p(p-3)$; $M_1(p) = 2p\alpha$; $M_2(p) = -p(p^2 - 3p - 4\alpha^2)$; and $M_3(p) = 2p^2\alpha(3p - 14)$

(2) $M_k(p) = 6pM_{k-2}(p) + 8p\alpha M_{k-3}(p) - p(p - 4\alpha^2) M_{k-4}(p)$

The first four items in the sequence $\{M_k(p)\}$ is $M_0(p) = p(p-3)$; $M_1(p) = -6p\alpha$; $M_2(p) = p(3p^2 - 17p - 4\alpha^2)$; and $M_3(p) = 6p^2\alpha(p-8)$.

Inspired by references [1–4], in this paper, we will consider the following $2k$-th hybrid power mean:

$$A_k(p) = \sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right)\right|^{2k} \cdot \left|\sum_{b=0}^{p-1} e\left(\frac{mb^3+b}{p}\right)\right|^4, \tag{6}$$

and k-th hybrid power mean

$$B_k(p) = \sum_{m=1}^{p-1}\left(\sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right)\right)^{k} \cdot \left|\sum_{b=0}^{p-1} e\left(\frac{mb^3+b}{p}\right)\right|^4. \tag{7}$$

Of course, the work in this paper looks a little imaginative with that in [4], but they have different essence, and the main difference lies in the power of the two-term exponential sums. In fact, it is a lot easier that we are dealing with the quadratic power mean of the two-term exponential sum in [4]. In this paper, we are dealing with the fourth power of the two-term exponential sums, and it is very difficult.

In this paper, we will give a second-order linear recurrence formula for $A_k(p)$ and a fourth-order linear recurrence formula for $B_k(p)$ by using the properties of Legendre's symbol and the classical Gauss sums. That is, we will prove the following results.

Theorem 1. *If $p > 3$ is an odd prime with $p \equiv 1 \bmod 4$, for any four-order character $\chi_4 \bmod p$, we have*

$$\sum_{m=1}^{p-1}\chi_4(m)\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^4 = \tau^2(\chi_4) + \chi_4(4)\cdot p\cdot\tau^2(\chi_4)\cdot\sum_{u=1}^{p-1}\overline{\chi}_4\left(u^2+u+\overline{u}+\overline{u}^2\right) + 2\chi_4(-36)\cdot p^{5/2}, \tag{8}$$

where $\tau(\chi) = \sum_{a=1}^{p-1}\chi(a)e(a/p)$ denotes the classical Gauss sums.

Theorem 2. *If $p > 3$ is an odd prime with $p \equiv 5 \bmod 8$, then we have the second-order linear recurrence formula:*

$$A_k(p) = 6p\cdot A_{k-1}(p) - \left(9p^2 - 4p\alpha^2\right)\cdot A_{k-2}(p), \quad k \geq 2, \tag{9}$$

with the initial values

$$A_0(p) = \sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+na}{p}\right)\right|^4 = \begin{cases} 2p^3 - p^2, & \text{if } 3\nmid p-1, \\ 2p^3 - 7p^2, & \text{if } 3\mid p-1, \end{cases}$$

$$A_1(p) = 3p\cdot A_0(p) - 2\sqrt{p}\cdot\alpha\cdot\sum_{m=1}^{p-1}\chi_2(m)\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+na}{p}\right)\right|^4, \tag{10}$$

$$\sum_{m=1}^{p-1}\chi_2(m)\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^4 = \begin{cases} p^2(\delta-3), & \text{if } p \equiv 1 \bmod 6, \\ p^2(\delta+3), & \text{if } p \equiv -1 \bmod 6, \end{cases}$$

where $(*/p) = \chi_2$ denotes Legendre's symbol modulo p. $\alpha = \alpha(p) = \sum_{a=1}^{(p-1)/2}((a+\overline{a})/p)$ and $\delta = \delta(p) = \sum_{d=1}^{p-1}((d-1+\overline{d})/p)$ are two integers, and they satisfy the estimates $|\alpha| \leq \sqrt{p}$ and $|\delta| \leq 2\sqrt{p}$, $d\cdot\overline{d} \equiv 1 \bmod p$.

Theorem 3. *If $p > 3$ is an odd prime with $p \equiv 1 \bmod 8$, then we have the fourth-order linear recurrence formula:*

$$B_k(p) = 6p\cdot B_{k-2}(p) + 8p\alpha\cdot B_{k-3}(p) + \left(4p\alpha^2 - p^2\right)\cdot B_{k-4}(p), \tag{11}$$

with the initial values

$$B_0(p)=\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1}e\left(\frac{ma^3+na}{p}\right)\right|^4=\begin{cases}2p^3-p^2, & \text{if } 3\nmid p-1,\\ 2p^3-7p^2, & \text{if } 3\mid p-1,\end{cases}$$

$$\begin{aligned}B_1(p)&=\sqrt{p}\cdot\sum_{m=1}^{p-1}\chi_2(m)\left|\sum_{a=0}^{p-1}e\left(\frac{ma^3+a}{p}\right)\right|^4\\&\quad+\tau(\overline{\chi}_4)\cdot\sum_{m=1}^{p-1}\chi_4(m)\left|\sum_{a=0}^{p-1}e\left(\frac{ma^3+a}{p}\right)\right|^4\\&\quad+\tau(\chi_4)\cdot\sum_{m=1}^{p-1}\overline{\chi}_4(m)\left|\sum_{a=0}^{p-1}e\left(\frac{ma^3+a}{p}\right)\right|^4;\end{aligned}$$

$$\begin{aligned}B_2(p)&=3p\cdot B_0(p)+2\sqrt{p}\alpha\cdot\sum_{m=1}^{p-1}\chi_2(m)\left|\sum_{a=0}^{p-1}e\left(\frac{ma^3+a}{p}\right)\right|^4\\&\quad+2\sqrt{p}\cdot\tau(\overline{\chi}_4)\cdot\sum_{m=1}^{p-1}\overline{\chi}_4(m)\left|\sum_{a=0}^{p-1}e\left(\frac{ma^3+a}{p}\right)\right|^4\\&\quad+2\sqrt{p}\cdot\tau(\chi_4)\cdot\sum_{m=1}^{p-1}\chi_4(m)\left|\sum_{a=0}^{p-1}e\left(\frac{ma^3+a}{p}\right)\right|^4,\end{aligned}$$

$$\begin{aligned}B_3(p)&=\alpha\cdot B_2(p)+5p\cdot B(p)+3p\alpha\cdot B_0(p)\\&\quad+2\sqrt{p}\cdot\left(p-\alpha^2\right)\cdot\sum_{m=1}^{p-1}\chi_2(m)\left|\sum_{a=0}^{p-1}e\left(\frac{ma^3+a}{p}\right)\right|^4.\end{aligned}\tag{12}$$

From (4), Theorem 1, and Weil's works [13,14], we have the estimates:

$$\begin{aligned}&\left|\sum_{m=1}^{p-1}\chi_2(m)\left|\sum_{a=0}^{p-1}e\left(\frac{ma^3+a}{p}\right)\right|^4\right|=O\left(p^{5/2}\right),\\&\left|\sum_{m=1}^{p-1}\chi_4(m)\left|\sum_{a=0}^{p-1}e\left(\frac{ma^3+a}{p}\right)\right|^4\right|=O\left(p^{5/2}\right),\\&\left|\sum_{m=1}^{p-1}\left(\sum_{a=0}^{p-1}e\left(\frac{ma^4}{p}\right)\right)^{2k-1}\cdot\left|\sum_{a=0}^{p-1}e\left(\frac{ma^3+a}{p}\right)\right|^4\right|=O_k\left(p^{(2k+3)/2}\right).\end{aligned}\tag{13}$$

Applying (2), Theorems 2 and 3, the properties of the linear recursive sequences, and these three estimates, we can deduce the following three corollaries.

Corollary 1. *If p is an odd prime with $p\equiv 5\bmod 8$, for any positive integer k, we have the asymptotic formula:*

$$\begin{aligned}&\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1}e\left(\frac{ma^4}{p}\right)\right|^{2k}\cdot\left|\sum_{b=0}^{p-1}e\left(\frac{mb^3+b}{p}\right)\right|^4\\&=p^3\cdot\left[(3p+2\sqrt{p}|\alpha|)^k+(3p-2\sqrt{p}|\alpha|)^k\right]\\&\quad+O_k\left(p^{(2k+5)/2}\right),\end{aligned}\tag{14}$$

where O_k denotes the big -O constant depending only on the positive integer k.

Especially for $k=2$, we have the asymptotic formula:

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1}e\left(\frac{ma^4}{p}\right)\right|^4\cdot\left|\sum_{b=0}^{p-1}e\left(\frac{mb^3+b}{p}\right)\right|^4=2p^4\cdot\left(9p+4\alpha^2\right)+O\left(p^{9/2}\right).\tag{15}$$

Corollary 2. *If p is an odd prime with $p\equiv 1\bmod 8$, for any positive integer k, we have the asymptotic formula:*

$$\begin{aligned}&\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1}e\left(\frac{ma^4}{p}\right)\right|^{2k}\cdot\left|\sum_{b=0}^{p-1}e\left(\frac{mb^3+b}{p}\right)\right|^4\\&=p^3\cdot\left[\left(3p+2\sqrt{2p^2+p\alpha^2}\right)^k+\left(3p-2\sqrt{2p^2+p\alpha^2}\right)^k\right]+O_k\left(p^{(2k+5)/2}\right),\end{aligned}\tag{16}$$

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1}e\left(\frac{ma^4}{p}\right)\right|^4\cdot\left|\sum_{b=0}^{p-1}e\left(\frac{mb^3+b}{p}\right)\right|^4=2p^4\cdot\left(17p+4\alpha^2\right)+O\left(p^{9/2}\right).$$

Corollary 3. *If p is an odd prime with $p \equiv 1 \bmod 4$, then we have the asymptotic formula:*

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right)\right|^2 \cdot \left|\sum_{b=0}^{p-1} e\left(\frac{mb^3+b}{p}\right)\right|^4 = 6p^4 + O\left(p^{7/2}\right). \tag{17}$$

2. Several Lemmas

In this section, we will give four basic lemmas that they are all necessary in the proofs of the theorems. Certainly, the proofs of these lemmas need some theoretical knowledge of elementary and analytic number theory. They can be found in references [15–17]. Firstly, we have the following:

Lemma 1. *If $p > 3$ is an odd prime with $p \equiv 1 \bmod 4$, for any four-order character $\chi_4 \bmod p$, we have*

$$\sum_{m=1}^{p-1} \chi_4(m)\left(\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right)^2 = \chi_4(-36)\cdot p^{3/2}. \tag{18}$$

Proof. Let χ_2 denotes the Legendre's symbol modulo p. Then, for any integer b, from the properties of the Legendre's symbol modulo p, we have

$$\sum_{a=0}^{p-1} e\left(\frac{ba^2}{p}\right) = \begin{cases} \chi_2(b)\cdot\tau(\chi_2), & \text{if } p\nmid b, \\ p, & \text{if } p\mid b. \end{cases} \tag{19}$$

Note that $\chi_4^3 = \overline{\chi}_4$ and $\tau(\chi_2) = \sqrt{p}$, and using the definition and properties of Gauss sums, reduced residue system modulo p, and formula (19), we have

$$\begin{aligned}
&\sum_{m=1}^{p-1} \chi_4(m)\left(\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right)^2 \\
&= \sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{m=1}^{p-1} \chi_4(m) e\left(\frac{m\left(a^3+b^3\right)+a+b}{p}\right) \\
&= \tau(\chi_4)\sum_{a=0}^{p-1}\sum_{b=0}^{p-1} \overline{\chi}_4\left(a^3+b^3\right) e\left(\frac{a+b}{p}\right) \\
&= \tau(\chi_4)\sum_{a=0}^{p-1} \overline{\chi}_4\left(a^3\right) e\left(\frac{a}{p}\right) + \tau(\chi_4)\sum_{a=0}^{p-1}\sum_{b=1}^{p-1} \overline{\chi}_4\left(a^3+b^3\right) e\left(\frac{a+b}{p}\right) \\
&= \tau^2(\chi_4) + \tau^2(\chi_4)\sum_{a=0}^{p-1} \overline{\chi}_4\left(a^3+1\right)\overline{\chi}_4(a+1) \\
&= \tau^2(\chi_4) + \tau^2(\chi_4)\sum_{b=0}^{p-1} \overline{\chi}_4\left(b^3-3b^2+3b\right)\overline{\chi}_4(b) \\
&= \tau^2(\chi_4) + \tau^2(\chi_4)\sum_{b=1}^{p-1} \overline{\chi}_4\left(1-3\overline{b}+3\overline{b}^2\right) \\
&= \tau^2(\chi_4)\sum_{b=0}^{p-1} \overline{\chi}_4\left(3b^2-3b+1\right) \\
&= \chi_2(2)\tau^2(\chi_4)\sum_{b=0}^{p-1} \overline{\chi}_4\left(3(2b-1)^2+1\right) \\
&= \overline{\chi}_4(12)\tau^2(\chi_4)\sum_{b=0}^{p-1} \overline{\chi}_4\left(b^2+\overline{3}\right).
\end{aligned} \tag{20}$$

Note that $\tau(\chi_4)\tau(\overline{\chi}_4) = \chi_4(-1)\cdot p$, and from the properties of Gauss sums, we have

$$\sum_{a=0}^{p-1}\chi_4\left(a^2+r\right) = \frac{1}{\tau(\overline{\chi}_4)}\sum_{b=1}^{p-1}\overline{\chi}_4(b)\sum_{a=0}^{p-1}e\left(\frac{b\left(a^2+r\right)}{p}\right) = \begin{cases}\dfrac{\overline{\chi}_4(-r)\cdot\tau^2(\chi_4)}{\sqrt{p}}, & \text{if } p\nmid r,\\ 0, & \text{if } p\mid r.\end{cases} \tag{21}$$

From (20) and (21), we have

$$\sum_{m=1}^{p-1}\chi_4(m)\left(\sum_{a=0}^{p-1}e\left(\frac{ma^3+a}{p}\right)\right)^2 = \chi_4(-36)\cdot p^{3/2}. \tag{22}$$

This proves Lemma 1. □

Lemma 2. *If $p>3$ is an odd prime with $p\equiv 1\bmod 4$, for any four-order character $\chi_4 \bmod p$, we have*

$$\sum_{m=1}^{p-1}\chi_4(m)\left(\sum_{a=0}^{p-1}e\left(\frac{ma^3+a}{p}\right)\right)^2\left(\sum_{b=0}^{p-1}e\left(\frac{-mb^3-b}{p}\right)\right) = (p+1)\cdot\tau^2(\chi_4)+\chi_4(-36)\cdot p^{3/2}\cdot\sum_{a=0}^{p-1}\overline{\chi}_4(a+2)\chi_4\left(a^3+2\right). \tag{23}$$

Proof. From Lemma 1, we have

$$\begin{aligned}&\sum_{m=1}^{p-1}\chi_4(m)\left(\sum_{a=0}^{p-1}e\left(\frac{ma^3+a}{p}\right)\right)^2\left(\sum_{b=0}^{p-1}e\left(\frac{-mb^3-b}{p}\right)\right)\\ &= \sum_{m=1}^{p-1}\chi_4(m)\left(\sum_{a=0}^{p-1}e\left(\frac{ma^3+a}{p}\right)\right)^2\left(\sum_{b=1}^{p-1}e\left(\frac{-mb^3-b}{p}\right)\right)\\ &\quad+\sum_{m=1}^{p-1}\chi_4(m)\left(\sum_{a=0}^{p-1}e\left(\frac{ma^3+a}{p}\right)\right)^2\\ &= \tau^2(\chi_4)\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\overline{\chi}_4\left(a^3+b^3-1\right)\overline{\chi}_4(a+b-1)\\ &\quad+\chi_4(-36)\cdot p^{3/2}.\end{aligned} \tag{24}$$

Let $c=b-1$, then from the properties of the complete residue system modulo p, we have

$$\begin{aligned}&\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\overline{\chi}_4\left(a^3+b^3-1\right)\overline{\chi}_4(a+b-1)\\ &= \sum_{a=0}^{p-1}\sum_{c=0}^{p-1}\overline{\chi}_4\left(a^3+(c+1)^3-1\right)\overline{\chi}_4(a+c)\\ &= \sum_{a=0}^{p-1}\sum_{c=0}^{p-1}\overline{\chi}_4\left(a^3+c^3+3c^2+3c\right)\overline{\chi}_4(a+c)\\ &= p+\sum_{a=0}^{p-1}\sum_{c=1}^{p-1}\overline{\chi}_4\left(a^3c^3+c^3+3c^2+3c\right)\overline{\chi}_4(ac+c)\\ &= p+\sum_{a=0}^{p-1}\sum_{c=1}^{p-1}\overline{\chi}_4\left(a^3+1+3c+3c^2\right)\overline{\chi}_4(a+1)\\ &= p+\chi_4(4)\sum_{a=0}^{p-1}\sum_{c=1}^{p-1}\overline{\chi}_4\left(4a^3+1+3(2c+1)^2\right)\overline{\chi}_4(a+1)\\ &= p+\chi_4(4)\sum_{a=0}^{p-1}\sum_{c=0}^{p-1}\overline{\chi}_4\left(4a^3+1+3c^2\right)\overline{\chi}_4(a+1)\\ &\quad-\sum_{a=0}^{p-1}\overline{\chi}_4\left(a^3+1\right)\overline{\chi}_4(a+1).\end{aligned} \tag{25}$$

From (19) and the methods of proving (20), we have

$$\begin{aligned}\sum_{a=0}^{p-1}\overline{\chi}_4\left(a^3+1\right)\overline{\chi}_4(a+1) &= \sum_{b=1}^{p-1}\overline{\chi}_4\left(3b^2-3b+1\right)\\ &= \chi_4(4)\sum_{c=0}^{p-1}\overline{\chi}_4\left(3c^2+1\right)-1 = \frac{\chi_4(4)}{\tau(\chi_4)}\sum_{b=1}^{p-1}\chi_4(b)\sum_{c=0}^{p-1}e\left(\frac{b\left(3c^2+1\right)}{p}\right)-1\\ &= \frac{\chi_4(-36)\cdot\tau^2\left(\overline{\chi}_4\right)}{\sqrt{p}}-1.\end{aligned} \tag{26}$$

From (19), we have

$$\begin{aligned}&\sum_{a=0}^{p-1}\sum_{c=0}^{p-1}\overline{\chi}_4\left(4a^3+1+3c^2\right)\overline{\chi}_4(a+1)\\ &= \frac{1}{\tau(\chi_4)}\sum_{a=0}^{p-1}\overline{\chi}_4(a+1)\sum_{b=1}^{p-1}\chi_4(b)\sum_{c=0}^{p-1}e\left(\frac{b\left(4a^3+1+3c^2\right)}{p}\right)\\ &= \frac{\chi_2(3)\cdot\sqrt{p}\cdot\tau\left(\overline{\chi}_4\right)}{\tau(\chi_4)}\sum_{a=0}^{p-1}\overline{\chi}_4(a+1)\chi_4\left(4a^3+1\right)\\ &= \frac{\chi_4(-9)\cdot\tau^2\left(\overline{\chi}_4\right)}{\sqrt{p}}\cdot\sum_{a=0}^{p-1}\overline{\chi}_4(a+2)\chi_4\left(a^3+2\right).\end{aligned} \tag{27}$$

Combining (24)–(27), we have the identity

$$\begin{aligned}&\sum_{m=1}^{p-1}\chi_4(m)\left(\sum_{a=0}^{p-1}e\left(\frac{ma^3+a}{p}\right)\right)^2\left(\sum_{b=0}^{p-1}e\left(\frac{-mb^3-b}{p}\right)\right)\\ &= (p+1)\cdot\tau^2(\chi_4)+\chi_4(-36)\cdot p^{3/2}\\ &\quad\cdot\sum_{a=0}^{p-1}\overline{\chi}_4(a+2)\chi_4\left(a^3+2\right).\end{aligned} \tag{28}$$

This proves Lemma 2. □

Lemma 3. *If $p>3$ is an odd prime with $p\equiv 1\bmod 4$, then we have*

$$\begin{aligned}&\sum_{m=1}^{p-1}\chi_4(m)\left(\sum_{a=0}^{p-1}e\left(\frac{ma^3+a}{p}\right)\right)^2\sum_{b=0}^{p-1}e\left(\frac{-mb^3-b}{p}\right)\sum_{c=1}^{p-1}e\left(\frac{-mc^3-c}{p}\right)\\ &= \chi_4(4)\cdot p\cdot\tau^2(\chi_4)\cdot\sum_{u=1}^{p-1}\overline{\chi}_4\left(u^2+u+\overline{u}+\overline{u}^2\right)+2\chi_4(-36)\cdot p^{5/2}\\ &\quad - p\cdot\tau^2(\chi_4)-\chi_4(-36)\cdot p^{3/2}\cdot\sum_{a=0}^{p-1}\overline{\chi}_4(a+2)\chi_4\left(a^3+2\right).\end{aligned} \tag{29}$$

Proof. From the properties of the classical Gauss sums and the methods of proving Lemmas 1 and 2, we have

$$\begin{aligned}&\sum_{m=1}^{p-1}\chi_4(m)\left(\sum_{a=0}^{p-1}e\left(\frac{ma^3+a}{p}\right)\right)^2\sum_{b=0}^{p-1}e\left(\frac{-mb^3-b}{p}\right)\sum_{c=1}^{p-1}e\left(\frac{-mc^3-c}{p}\right)\\ &= \tau^2(\chi_4)\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\overline{\chi}_4\left(a^3+b^3-c^3-1\right)\overline{\chi}_4(a+b-c-1).\end{aligned} \tag{30}$$

Let $u=a-1$ and $v=b-c$, then from the properties of the complete residue system modulo p, we have

$$\begin{aligned}
&\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\overline{\chi}_4\left(a^3+b^3-c^3-1\right)\overline{\chi}_4(a+b-c-1)\\
&\quad=\sum_{u=0}^{p-1}\sum_{v=0}^{p-1}\sum_{c=0}^{p-1}\overline{\chi}_4\left((u+1)^3+(v+c)^3-c^3-1\right)\overline{\chi}_4(u+v)\\
&\quad=\sum_{u=0}^{p-1}\sum_{v=0}^{p-1}\sum_{c=0}^{p-1}\overline{\chi}_4\left(u^3+3u^2+3u+v^3+3v^2c+3vc^2\right)\overline{\chi}_4(u+v)\\
&\quad=\sum_{u=0}^{p-1}\sum_{v=1}^{p-1}\sum_{c=0}^{p-1}\overline{\chi}_4\left(u^3+3u^2v+3uv^2+1+3c+3c^2\right)\overline{\chi}_4(u+1)+p\cdot\sum_{u=1}^{p-1}\overline{\chi}_4\left(3u^2+3u+1\right)\\
&\quad=\chi_4(4)\sum_{u=0}^{p-1}\sum_{v=1}^{p-1}\sum_{c=0}^{p-1}\overline{\chi}_4\left(u^3+3u(2v+u)^2+1+3(2c+1)^2\right)\overline{\chi}_4(u+1)\\
&\qquad+\chi_4(4)\cdot p\cdot\sum_{u=0}^{p-1}\overline{\chi}_4\left(3(2u+1)^2+1\right)-p\\
&\quad=\chi_4(4)\sum_{u=0}^{p-1}\sum_{v=0}^{p-1}\sum_{c=0}^{p-1}\overline{\chi}_4\left(u^3+3uv^2+1+3c^2\right)\overline{\chi}_4(u+1)\\
&\qquad-\chi_4(4)\sum_{u=0}^{p-1}\sum_{c=0}^{p-1}\overline{\chi}_4\left(4u^3+1+3c^2\right)\overline{\chi}_4(u+1)+\chi_4(4)\cdot p\cdot\sum_{u=0}^{p-1}\overline{\chi}_4\left(3u^2+1\right)-p.
\end{aligned}\tag{31}$$

From (19), we have

$$\begin{aligned}
\sum_{u=0}^{p-1}\overline{\chi}_4\left(3u^2+1\right)&=\frac{1}{\tau(\chi_4)}\cdot\sum_{b=1}^{p-1}\chi_4(b)\sum_{u=0}^{p-1}e\left(\frac{b\left(3u^2+1\right)}{p}\right)\\
&=\frac{\chi_2(3)\cdot\sqrt{p}}{\tau(\chi_4)}\cdot\sum_{b=1}^{p-1}\overline{\chi}_4(b)e\left(\frac{b}{p}\right)=\frac{\chi_4(-9)\cdot\tau^2\left(\overline{\chi_4}\right)}{\sqrt{p}}.
\end{aligned}\tag{32}$$

From (19), we also have

$$\begin{aligned}
&\sum_{u=0}^{p-1}\sum_{v=0}^{p-1}\sum_{c=0}^{p-1}\overline{\chi}_4\left(u^3+3uv^2+1+3c^2\right)\overline{\chi}_4(u+1)\\
&\quad=\frac{1}{\tau(\chi_4)}\sum_{u=0}^{p-1}\overline{\chi}_4(u+1)\sum_{b=1}^{p-1}\chi_4(b)\sum_{v=0}^{p-1}\sum_{c=0}^{p-1}e\left(\frac{b\left(u^3+3uv^2+1+3c^2\right)}{p}\right)\\
&\quad=\frac{p}{\tau(\chi_4)}\sum_{u=1}^{p-1}\overline{\chi}_4\left(u^3+u^2\right)\sum_{b=1}^{p-1}\chi_4(b)e\left(\frac{b\left(u^3+1\right)}{p}\right)+\chi_4(-9)\cdot\sqrt{p}\cdot\tau^2\left(\overline{\chi}_4\right)\\
&\quad=p\cdot\sum_{u=1}^{p-1}\overline{\chi}_4\left(u^3+u^2\right)\overline{\chi}_4\left(u^3+1\right)+\chi_4(-9)\cdot\sqrt{p}\cdot\tau^2\left(\overline{\chi}_4\right)\\
&\quad=p\cdot\sum_{u=1}^{p-1}\overline{\chi}_4\left(u^2+u+\overline{u}+\overline{u}^2\right)+\chi_4(-9)\cdot\sqrt{p}\cdot\tau^2\left(\overline{\chi}_4\right).
\end{aligned}\tag{33}$$

Now, combining (27)–(33), we have the identity:

$$\sum_{m=1}^{p-1}\chi_4(m)\left(\sum_{a=0}^{p-1}e\left(\frac{ma^3+a}{p}\right)\right)^2\sum_{b=0}^{p-1}e\left(\frac{-mb^3-b}{p}\right)\sum_{c=1}^{p-1}e\left(\frac{-mc^3-c}{p}\right)$$

$$=\chi_4(4)\cdot p\cdot\tau^2(\chi_4)\cdot\sum_{u=1}^{p-1}\overline{\chi}_4\left(u^2+u+\overline{u}+\overline{u}^2\right)+2\chi_4(-36)\cdot p^{5/2} \tag{34}$$

$$-p\cdot\tau^2(\chi_4)-\chi_4(-36)\cdot p^{3/2}\cdot\sum_{a=0}^{p-1}\overline{\chi}_4(a+2)\chi_4\left(a^3+2\right).$$

This proves Lemma 3. □

Lemma 4. *Let $p>3$ is an odd prime with $p\equiv 1\bmod 4$, then for any four-order character $\chi_4 \bmod p$, we have the identity:*

$$\tau^2(\chi_4)+\tau^2(\overline{\chi}_4)=2\sqrt{p}\cdot\alpha, \tag{35}$$

where $\alpha=\alpha(p)=\sum_{a=1}^{(p-1)/2}((a+\overline{a})/p)$ is an integer.

Proof. See Lemma 2.2 in Han [5]. □

3. Proofs of the Theorems

In this section, we will complete the proofs of our theorems. In fact, for any prime p with $p\equiv 1\bmod 4$, from Lemmas 2 and 3, we have

$$\sum_{m=1}^{p-1}\chi_4(m)\left|\sum_{a=0}^{p-1}e\left(\frac{ma^3+a}{p}\right)\right|^4$$

$$=\sum_{m=1}^{p-1}\chi_4(m)\left(\sum_{a=0}^{p-1}e\left(\frac{ma^3+a}{p}\right)\right)^2\left(\sum_{b=0}^{p-1}e\left(\frac{-mb^3-b}{p}\right)\right)$$

$$+\sum_{m=1}^{p-1}\chi_4(m)\left(\sum_{a=0}^{p-1}e\left(\frac{ma^3+a}{p}\right)\right)^2\sum_{b=0}^{p-1}e\left(\frac{-mb^3-b}{p}\right)\sum_{c=1}^{p-1}e\left(\frac{-mc^3-c}{p}\right)$$

$$=(p+1)\cdot\tau^2(\chi_4)+\chi_4(-36)\cdot p^{3/2}\cdot\sum_{a=0}^{p-1}\overline{\chi}_4(a+2)\chi_4\left(a^3+2\right) \tag{36}$$

$$+\chi_4(4)\cdot p\cdot\tau^2(\chi_4)\cdot\sum_{u=1}^{p-1}\overline{\chi}_4\left(u^2+u+\overline{u}+\overline{u}^2\right)+2\chi_4(-36)\cdot p^{5/2}$$

$$-p\cdot\tau^2(\chi_4)-\chi_4(-36)\cdot p^{3/2}\cdot\sum_{a=0}^{p-1}\overline{\chi}_4(a+2)\chi_4\left(a^3+2\right)$$

$$=\tau^2(\chi_4)+\chi_4(4)\cdot p\cdot\tau^2(\chi_4)\cdot\sum_{u=1}^{p-1}\overline{\chi}_4\left(u^2+u+\overline{u}+\overline{u}^2\right)+2\chi_4(-36)\cdot p^{5/2}.$$

This proves Theorem 1.

Now, we prove Theorem 2. If $p \equiv 5 \bmod 8$, then for any four-order character $\chi_4 \bmod p$, we have $\chi_4(-1) = -1$. From the properties of the Gauss sums and the four-order character modulo p, we have

$$\begin{aligned} G(m,p) &= \sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right) = 1 + \sum_{a=1}^{p-1} \left(1 + \chi_4(a) + \chi_2(a) + \overline{\chi}_4(a)\right) \cdot e\left(\frac{ma}{p}\right) \\ &= \overline{\chi}_4(m) \cdot \tau(\chi_4) + \chi_2(m) \cdot \tau(\chi_2) + \chi_4(m) \cdot \tau(\overline{\chi}_4) \\ &= \overline{\chi}_4(m) \cdot \tau(\chi_4) + \chi_2(m) \cdot \sqrt{p} + \chi_4(m) \cdot \tau(\overline{\chi}_4), \end{aligned} \tag{37}$$

$$\overline{G(m,p)} = -\chi_4(m) \cdot \tau(\overline{\chi}_4) - \overline{\chi}_4(m) \cdot \tau(\chi_4) + \chi_2(m) \cdot \sqrt{p}. \tag{38}$$

So, from (37), (38), and Lemma 4, we have

$$\begin{aligned} |G(m,p)|^2 &= p + \left|\chi_4(m) \cdot \tau(\overline{\chi}_4) + \overline{\chi}_4(m) \cdot \tau(\chi_4)\right|^2 \\ &= 3p - \chi_2(m)\left(\tau^2(\chi_4) + \tau^2(\overline{\chi}_4)\right) \\ &= 3p - 2\chi_2(m)\sqrt{p}\alpha, \end{aligned} \tag{39}$$

$$|G(m,p)|^4 = 6p|G(m,p)|^2 + 4p\alpha^2 - 9p^2. \tag{40}$$

So, for any integer $k \geq 2$, from (2), (4), (39), and (40), we have

$$\begin{aligned} A_k(p) &= \sum_{m=1}^{p-1} |G(m,p)|^{2k-4} \cdot |G(m,p)|^4 \cdot \left|\sum_{b=0}^{p-1} e\left(\frac{mb^3+b}{p}\right)\right|^4 \\ &= \sum_{m=1}^{p-1} |G(m,p)|^{2k-4} \cdot \left(6p|G(m,p)|^2 + 4p\alpha^2 - 9p^2\right) \cdot \left|\sum_{b=0}^{p-1} e\left(\frac{mb^3+b}{p}\right)\right|^4 \\ &= 6p \cdot A_{k-1}(p) - \left(9p^2 - 4p\alpha^2\right) \cdot A_{k-2}(p), \end{aligned} \tag{41}$$

with the initial values

$$\begin{aligned} A_0(p) &= \sum_{m=1}^{p-1} \left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+na}{p}\right)\right|^4 = \begin{cases} 2p^3 - p^2, & \text{if } 3 \nmid p-1, \\ 2p^3 - 7p^2, & \text{if } 3 \mid p-1, \end{cases} \\ A_1(p) &= 3p \cdot A_0(p) - 2\sqrt{p} \cdot \alpha \cdot \sum_{m=1}^{p-1} \chi_2(m) \left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+na}{p}\right)\right|^4, \end{aligned} \tag{42}$$

$$\sum_{m=1}^{p-1} \chi_2(m) \left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^4 = \begin{cases} p^2(\delta - 3), & \text{if } p \equiv 1 \bmod 6, \\ p^2(\delta + 3), & \text{if } p \equiv -1 \bmod 6, \end{cases}$$

where δ is the same as the definition in (4). This proves Theorem 2.

Now, we prove Theorem 3. If $p \equiv 1 \bmod 8$, then note that $\chi_4(-1) = -1$, so from (37) and Lemma 4, we have

$$G^2(m,p) = 3p + \chi_2(m)\cdot 2\sqrt{p}\alpha + 2\sqrt{p}(\overline{\chi}_4(m)\tau(\overline{\chi}_4) + \chi_4(m)\cdot\tau(\chi_4)), \tag{43}$$

$$G^3(m,p) = \alpha\cdot G^2(m,p) + 5p\cdot G(m,p) + \chi_2(m)\cdot 2\sqrt{p}\cdot\left(p-\alpha^2\right) + 3p\alpha, \tag{44}$$

$$G^4(m,p) = 6p\cdot G^2(m,p) + 8p\alpha\cdot G(m,p) + \left(4p\alpha^2 - p^2\right). \tag{45}$$

So, for any integer $k \geq 4$, from (43) and (44), we have

$$B_k(p) = 6p\cdot B_{k-2}(p) + 8p\alpha\cdot B_{k-3}(p) + \left(4p\alpha^2 - p^2\right)\cdot B_{k-4}(p), \tag{46}$$

with the initial values

$$B_0(p) = \sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+na}{p}\right)\right|^4 = \begin{cases} 2p^3 - p^2, & \text{if } 3\nmid p-1, \\ 2p^3 - 7p^2, & \text{if } 3\mid p-1, \end{cases}$$

$$\begin{aligned} B_1(p) = {} & \sqrt{p}\cdot\sum_{m=1}^{p-1}\chi_2(m)\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^4 \\ & + \tau(\overline{\chi}_4)\cdot\sum_{m=1}^{p-1}\chi_4(m)\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^4 \\ & + \tau(\chi_4)\cdot\sum_{m=1}^{p-1}\overline{\chi}_4(m)\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^4, \end{aligned}$$

$$\begin{aligned} B_2(p) = {} & 3p\cdot B_0(p) + 2\sqrt{p}\alpha\cdot\sum_{m=1}^{p-1}\chi_2(m)\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^4 \\ & + 2\sqrt{p}\cdot\tau(\overline{\chi}_4)\cdot\sum_{m=1}^{p-1}\overline{\chi}_4(m)\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^4 \\ & + 2\sqrt{p}\cdot\tau(\chi_4)\cdot\sum_{m=1}^{p-1}\chi_4(m)\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^4, \end{aligned}$$

$$\begin{aligned} B_3(p) = {} & \alpha\cdot B_2(p) + 5p\cdot B_1(p) + 3p\alpha\cdot B_0(p) \\ & + 2\sqrt{p}\cdot\left(p-\alpha^2\right)\cdot\sum_{m=1}^{p-1}\chi_2(m)\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^4. \end{aligned} \tag{47}$$

This proves Theorem 3.

Now, we will give a simple proof for Corollary 2. Note the estimate

$$\left|\sum_{m=1}^{p-1}\left(\sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right)\right)^{2k-1}\cdot\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^4\right| = O_k\left(p^{(2k+3)/2}\right), \tag{48}$$

and the identity

$$\begin{aligned} B_{2k}(p) = {} & 6p\cdot B_{2k-2}(p) + 8p\alpha\cdot B_{2k-3}(p) \\ & + \left(4p\alpha^2 - p^2\right)\cdot B_{2k-4}(p). \end{aligned} \tag{49}$$

From (48) and (49), we have

$$\begin{aligned} B_{2k}(p) = {} & 6p\cdot B_{2(k-1)}(p) + \left(4p\alpha^2 - p^2\right)\cdot B_{2(k-2)}(p) \\ & + O_k\left(p^{k+2}\right). \end{aligned} \tag{50}$$

Note that the linear recurrence sequence $U_n = 6p\cdot U_{n-1} + (4p\alpha^2 - p^2)\cdot U_{n-2}$ has the general terms:

$$\begin{aligned} U_n = {} & C_1\cdot\left(3p + 2\sqrt{2p^2 + p\alpha^2}\right)^n \\ & + C_2\cdot\left(3p - 2\sqrt{2p^2 + p\alpha^2}\right)^n, \quad n \geq 0. \end{aligned} \tag{51}$$

If $U_0 = 2p^3$ and $U_1 = 6p^4$, then we have $C_1 = C_2 = p^3$ and

$$\begin{aligned} U_n = {} & p^3\cdot\left[\left(3p + 2\sqrt{2p^2 + p\alpha^2}\right)^n\right. \\ & \left. + \left(3p - 2\sqrt{2p^2 + p\alpha^2}\right)^n\right], \quad n \geq 0. \end{aligned} \tag{52}$$

Combining (50) and (52), we have the asymptotic formula:

$$\begin{aligned} B_{2k}(p) = {} & p^3\cdot\left[\left(3p + 2\sqrt{2p^2 + p\alpha^2}\right)^k\right. \\ & \left. + \left(3p - 2\sqrt{2p^2 + p\alpha^2}\right)^k\right] + O_k\left(p^{(2k+5)/2}\right). \end{aligned} \tag{53}$$

This completes the proofs of our all results.

4. Conclusion

The main results of this paper are two linear recurrence formulas for the hybrid power means of the two-term exponential sums $G(m, k; q)$ and the quartic Gauss sums $G(m, q)$. As some applications of these results, we obtained a sharp asymptotic formula for the hybrid power means:

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right)\right|^{2k}\cdot\left|\sum_{b=0}^{p-1} e\left(\frac{mb^3+b}{p}\right)\right|^4, \tag{54}$$

for all positive integer k and odd prime p with $p \equiv 1 \bmod 4$. All these results are new contributions to the related fields.

Data Availability

Our field is theoretical mathematics, and we do not use any data.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors have equally contributed to this work. All authors read and approved the final manuscript.

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