Research Article

The Net in a Fuzzy Soft Topological Space and Its Applications

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In this paper, the fuzzy soft net is explored to study the properties of a fuzzy soft topological space. Some important results about the closure, separation, and compactness are obtained. It is demonstrated that the net is a powerful tool for studying fuzzy soft topological spaces.

1. Introduction

In 2001, Maji et al. [1] combined fuzzy sets [2] with soft sets [3] and proposed the concept of fuzzy soft sets. After that, the fuzzy soft set was applied to group theory, decision-making, medical diagnosis, and other fields (see [4–13]). Meanwhile, the theory of fuzzy soft set has been developed rapidly. In particular, the research on fuzzy soft topology has made a lot of achievements (see [14–24]).

Noting the contribution of the pointed approach in fuzzy topology, Roy and Samanta [21] defined a fuzzy soft point on a fuzzy soft topological space. In 2018, Ibedou and Abbas [17] redefined this concept. Recently, Gao and Wu [7] studied the properties of fuzzy soft points introduced in [17] deeply and pointed that the fuzzy soft point given in [17] was more effective than that given in [21]. They also gave the definitions of a fuzzy soft net consisting of fuzzy soft points and its convergence. On these bases, they characterized the continuity of fuzzy soft mappings by the net approach.

This paper aims to further the study of [7]. In Section 2, some preliminaries will be recalled. In Section 3, the characterizations of some important results involving closure, separation, and compactness will be obtained by means of fuzzy soft nets.

2. Preliminaries

Throughout this paper, U refers to an initial universe and E is the set of all parameters for U . In this case, U is also denoted by (U, E) . I^U is the set of all fuzzy subsets over U , where $I = [0, 1]$. The elements $\bar{0}, \bar{1} \in I^U$, respectively, refer to the functions $\bar{0}(x) = 0$ and $\bar{1}(x) = 1$ for all $x \in U$. For an element $A \in I^U$, if there exists an $x \in U$ such that $A(x) = \lambda > 0$ and $A(y) = 0, \forall y \in (U/\{x\})$, then A is called a fuzzy point over U and is denoted by x_λ , x and λ are its support and height, respectively.

The definitions in this section are all sourced from the existing literature [7, 17, 21, 22].

Definition 1. Let $A \subseteq E$. A mapping $F_A: E \longrightarrow I^U$, is called a fuzzy soft set over (U, E) , where $F_A(e) = \bar{0}$ if $e \in (E/A)$ and $F_A(e) \neq \bar{0}$ if $e \in A$.

The set of all fuzzy soft sets over (U, E) is denoted by $FS(U, E)$.

The fuzzy soft set $F_\phi \in FS(U, E)$ is called the null fuzzy soft set and is denoted by $\bar{\Phi}$. Here, $F_\phi(e) = \bar{0}$ for every $e \in E$.

For $F_E \in FS(U, E)$, if $F_E(e) = \bar{1}$ for all $e \in E$, then F_E is called the absolute fuzzy soft set and is denoted by $\bar{E}$.

Let $F_A, F_B \in \text{FS}(U, E)$. If $F_A(e) \subseteq F_B(e)$ for all $e \in E$, then F_A is said to be a fuzzy soft subset of F_B and is denoted by $F_A \subseteq F_B$ or $F_B \supseteq F_A$. In addition, $F_A \subsetneq F_B$ means that F_A is not a fuzzy soft subset of F_B . If $F_A \subseteq F_B$ and $F_B \subseteq F_A$, then F_A and F_B are said to be equivalent, which is denoted by $F_A = F_B$.

Remark 1. If $F_A \subseteq F_B$, then $A \subseteq B$.

Definition 2. Let $F_A, F_B \in \text{FS}(U, E)$.

- (1) The complement of F_A , denoted by F_A^c , is then defined as

$$F_A^c(e) = \begin{cases} \bar{1} - F_A(e) & \text{for } e \in A, \\ \bar{1}, & \text{otherwise.} \end{cases} \quad (1)$$

- (2) The union of F_A and F_B is also a fuzzy soft set F_C defined by $F_C(e) = F_A(e) \cup F_B(e)$ for all $e \in E$, where $C = A \cup B$, and is denoted by $F_C = F_A \cup F_B$.
- (3) The intersection of F_A and F_B is also a fuzzy soft set F_C defined by $F_C(e) = F_A(e) \cap F_B(e)$ for all $e \in E$, where $C = A \cap B$, and is denoted by $F_C = F_A \cap F_B$.

Similarly, the union (intersection) of a family of fuzzy soft sets may be defined as $\{F_{C_\alpha} : \alpha \in \Lambda\}$ and denoted by $\bigcup_{\alpha \in \Lambda} F_{C_\alpha}$ ($\bigcap_{\alpha \in \Lambda} F_{C_\alpha}$), where Λ is an arbitrary index set.

Remark 2. The following is, therefore, clear:

- (1) $\tilde{\Phi}^c = \tilde{E}$, $\tilde{E}^c = \tilde{\Phi}$
- (2) $(\bigcup_{\alpha \in \Lambda} F_{A_\alpha})^c = \bigcap_{\alpha \in \Lambda} F_{A_\alpha}^c$
- (3) $(\bigcap_{\alpha \in \Lambda} F_{A_\alpha})^c = \bigcup_{\alpha \in \Lambda} F_{A_\alpha}^c$

Definition 3. A fuzzy soft topology τ over (U, E) is a family of fuzzy soft sets over (U, E) satisfying the following properties:

- (1) $\tilde{\Phi}, \tilde{E} \in \tau$
- (2) If $F_A, F_B \in \tau$, then $F_A \cap F_B \in \tau$
- (3) If $F_{A_\alpha} \in \tau$ for all $\alpha \in \Lambda$ (an index set), then $\bigcup_{\alpha \in \Lambda} F_{A_\alpha} \in \tau$

If τ is a fuzzy soft topology over (U, E) , the triple (U, E, τ) is said to be a fuzzy soft topological space. Each element of τ is called an open set. If F_A^c is an open set, then F_A is called a closed set.

Definition 4. Let (U, E, τ) be a fuzzy soft topological space, $F_A \in \text{FS}(U, E)$.

- (1) The intersection of all closed sets $F_B \supseteq F_A$ is called the closure of F_A and is denoted by $\overline{F_A}$
- (2) The union of all open subsets of F_A over (U, E, τ) is called the interior of F_A and is denoted by $\text{int}F_A$

Remark 3. Let $F_A \in \text{FS}(U, E)$. The following is evident:

- (1) $\overline{F_A}$ is closed and $\text{int}F_A$ is open
- (2) F_A is closed if and only if $F_A = \overline{F_A}$

- (3) F_A is open if and only if $F_A = \text{int}F_A$

Definition 5. A mapping $\xi: E \rightarrow I^U$ is called a fuzzy soft point over (U, E) if there is an $e \in E$ such that $\xi(e) = x_\lambda \in \text{FP}(U)$, and $\xi(a) = \bar{0}$ when $a \in (E/\{e\})$.

In this case, ξ is also denoted by $P_e^{x_\lambda}$, and e is called its parameter support. The set of all fuzzy soft points over (U, E) is denoted by $\text{FSP}(U, E)$.

In the remainder of this paper, a fuzzy soft point is always referred to as given by Definition 5 and is called a point for short.

For $P_e^{x_\lambda}, P_f^{y_\beta} \in \text{FSP}(U, E)$, if $x \neq y$ or $\lambda \neq \beta$ or $e \neq f$, then it is said that $P_e^{x_\lambda}$ and $P_f^{y_\beta}$ are different, which is written as $P_e^{x_\lambda} \neq P_f^{y_\beta}$.

Theorem 1. Let $P_e^{x_\lambda} \in \text{FSP}(U, E)$, $F_A, F_B \in \text{FS}(U, E)$, and $F_{A_\alpha} \in \text{FS}(U, E)$ ($\forall \alpha \in \Lambda$).

- (1) $P_e^{x_\lambda} \in F_A \cup F_B$ if and only if $P_e^{x_\lambda} \in F_A$ or $P_e^{x_\lambda} \in F_B$
- (2) $P_e^{x_\lambda} \in F_{A_\alpha}$ ($\forall \alpha \in \Lambda$) if and only if $P_e^{x_\lambda} \in \bigcap_{\alpha \in \Lambda} F_{A_\alpha}$

Definition 6. A point $P_e^{x_\lambda}$ is said to be quasi-coincident with $F_A \in \text{FS}(U, E)$, which is denoted by $P_e^{x_\lambda} \tilde{\in} F_A$, if $\lambda + F_A(e)(x) > 1$.

That $P_e^{x_\lambda}$ is not quasi-coincident with F_A is denoted by $P_e^{x_\lambda} \not\tilde{\in} F_A$.

Remark 4. It is obvious that $P_e^{x_\lambda} \tilde{\in} F_A$ if and only if $P_e^{x_\lambda} \notin F_A^c$. Additionally, if $P_e^{x_\lambda} \tilde{\in} F_A$, then there exists $0 < \mu < \lambda$ such that $P_e^{x_{1-\mu}} \in F_A$.

Theorem 2. Let $F_A, F_B, F_{A_\alpha} \in \text{FS}(U, E)$ ($\forall \alpha \in \Lambda$), and $P_e^{x_\lambda} \in \text{FSP}(U, E)$.

- (1) $F_A \subseteq F_B$ if and only if $P_e^{x_\lambda} \tilde{\in} F_B$ for any $P_e^{x_\lambda} \tilde{\in} F_A$
- (2) $P_e^{x_\lambda} \tilde{\in} \bigcap_{\alpha \in \Lambda} F_{A_\alpha}$ implies $P_e^{x_\lambda} \tilde{\in} F_{A_\alpha}$ ($\forall \alpha \in \Lambda$)
- (3) $P_e^{x_\lambda} \tilde{\in} F_A \cap F_B$ if and only if $P_e^{x_\lambda} \tilde{\in} F_A$ and $P_e^{x_\lambda} \tilde{\in} F_B$
- (4) $P_e^{x_\lambda} \tilde{\in} \bigcup_{\alpha \in \Lambda} F_{A_\alpha}$ if and only if there exists an $\alpha_0 \in \Lambda$ such that $P_e^{x_\lambda} \tilde{\in} F_{A_{\alpha_0}}$

Definition 7. Let $F_A, F_B \in \text{FS}(U, E)$ and $\xi \in \text{FSP}(U, E)$. If $\xi \tilde{\in} F_A \cap F_B$, then it is said that F_A and F_B are quasi-coincident at ξ .

Definition 8. A fuzzy soft set F_A is said to be quasi-coincident with F_B , which is denoted by $F_A q F_B$, if $F_A(e)(x) + F_B(e)(x) > 1$ for some $x \in U$ and $e \in A \cap B$.

That F_A is not quasi-coincident with F_B is denoted by $F_A \bar{q} F_B$.

Theorem 3. Let $F_A, F_B \in \text{FS}(U, E)$. If $F_A q F_B$, then $F_A \cap F_B \neq \tilde{\Phi}$.

Definition 9. Let $\xi \in \text{FSP}(U, E)$ and $F_A, F_B \in \text{FS}(U, E)$.

- (1) F_A is said to be a neighborhood of ξ if there exists $F_B \in \tau$ such that $\xi \in F_B \subseteq F_A$

- (2) F_A is called a Q-neighborhood of ξ if there exists $F_B \in \tau$ such that $\xi \in F_B \subseteq F_A$.

The set of all Q-neighborhoods of ξ is denoted by $A(\xi)$.

In the remainder of this paper, Δ is a directed set with the partial order " $<$ ".

Definition 10. The mapping $S: \Delta \longrightarrow \text{FSP}(U, E)$ is called a fuzzy soft net over (U, E) and is denoted by $\{S(\delta), \delta \in \Delta\}$ or S for simplicity.

In particular, if there exists $F_A \in \text{FS}(U, E)$ such that $S(\delta) \in F_A$ for any $\delta \in \Delta$, then S is said to be a fuzzy soft net in F_A , or a net for simplicity.

Definition 11. Let $F_A \in \text{FS}(U, E)$ and $S = \{S(\delta), \delta \in \Delta\}$ be a net over (U, E) . If there exists $\delta_0 \in \Delta$ such that $S(\delta) \in F_A$ whenever $\delta_0 < \delta$, then S is said to be eventually quasi-coincident with F_A . If for each $\delta \in \Delta$ there exists $\delta_0 \in \Delta$ with $\delta < \delta_0$ such that $S(\delta_0) \in F_A$, then S is said to be frequently quasi-coincident with F_A .

Definition 12. A net $\{S(\delta), \delta \in \Delta\}$ over (U, E, τ) is said to be convergent to a point ξ if S is eventually quasi-coincident with each Q-neighborhood of ξ . In this case, ξ is called the limit of S and is denoted by $\lim S(\delta)$.

3. Main Results

In this section, nets are applied to characterize the closure of a fuzzy soft set, T_2 separation, and compactness of a fuzzy soft topological space.

Obviously, if $P_e^{x_\lambda} \in \text{FSP}(U, E)$, then $A(P_e^{x_\lambda})$ forms a directed set under the relation $\supseteq$.

Theorem 4. Let (U, E, τ) be a fuzzy soft topological space. A point $\xi \in \text{int} F_A$ if and only if there exists an $F_B \in A(\xi)$ such that $F_B \subseteq F_A$.

Proof (Necessity). Let $\xi \in \text{int} F_A$. From Definition 4 and Theorem 2 (4), there exists an open set $F_B \subseteq F_A$ such that $\xi \in F_B$. Because F_B is open, $F_B \in A(\xi)$.

(Sufficiency) Let $F_B \in A(\xi)$ and $F_B \subseteq F_A$. From Definition 9, there is $F_C \in \tau$ such that $\xi \in F_C \subseteq F_B$. It is obvious that $F_C \subseteq \text{int} F_B \subseteq \text{int} F_A$. Then, $\xi \in \text{int} F_A$. $\square$

Theorem 5. A point $P_e^{x_\lambda} \in \overline{F_A}$ if and only if each Q-neighborhood of $P_e^{x_\lambda}$ is quasi-coincident with F_A .

Proof (Necessity). Let $P_e^{x_\lambda} \in \overline{F_A}$, $F_B \in A(P_e^{x_\lambda})$. Without loss of generality, it is supposed that F_B is open. To complete the proof of necessity, it is sufficient to show $F_B q F_A$. In fact, if $F_B \not q F_A$, then $F_A(d)(y) + F_B(d)(y) \leq 1$ for any $y \in U$ and $d \in E$. Thus, $F_A \subseteq F_B^c$. Note that F_B^c is closed and $\overline{F_A} \subseteq F_B^c$. So, $P_e^{x_\lambda} \in F_B^c$, which together with Remark 4 conflicts with $P_e^{x_\lambda} \in F_B$. Hence, $F_B q F_A$.

(Sufficiency) Suppose each Q-neighborhood of $P_e^{x_\lambda}$ is quasi-coincident with F_A . If $P_e^{x_\lambda} \notin \overline{F_A}$, then $P_e^{x_\lambda} \in \overline{F_A^c}$. Thus, $F_A^c \in A(P_e^{x_\lambda})$. It follows from the supposition that $\overline{F_A} q F_A^c$. So, there exist $y \in U$ and $d \in E$ such that

$$\overline{F_A^c}(d)(y) + F_A(d)(y) > 1. \quad (2)$$

Hence, $\overline{F_A}(d)(y) < F_A(d)(y)$, which is a contradiction. $\square$

Theorem 6. Let $F_A \in \text{FS}(U, E)$. Then, a point $P_e^{x_\lambda} \in \overline{F_A}$ if and only if there is a net $S = \{S(\delta): \delta \in \Delta\}$ in F_A such that $\lim S(\delta) = P_e^{x_\lambda}$.

Proof (Necessity). Let $P_e^{x_\lambda} \in \overline{F_A}$. Take $\{A(P_e^{x_\lambda}), \supseteq\}$ as a directed set. For any $F_B \in A(P_e^{x_\lambda})$, by Theorem 5, $F_B q F_A$. Then, there exist $d_B \in E$ and $y_B \in U$ such that

$$F_B(d_B)(y_B) + F_A(d_B)(y_B) > 1. \quad (3)$$

Let $\lambda_B = F_A(d_B)(y_B)$. Then, the points $P_{d_B}^{(y_B)\lambda_B} \in F_A$ and $P_{d_B}^{(y_B)\lambda_B} \in F_B$ from (1). Thus $\{P_{d_B}^{(y_B)\lambda_B}: F_B \in A(P_e^{x_\lambda})\}$ is a net in F_A .

It is now proven that $\{P_{d_B}^{(y_B)\lambda_B}: F_B \in A(P_e^{x_\lambda})\}$ converges to $P_e^{x_\lambda}$. In fact, for any $F_C \in A(P_e^{x_\lambda})$, let $F_{B_0} = F_C$. When $F_{B_0} \supseteq F_B \in A(P_e^{x_\lambda})$, by (3), the following is obtained:

$$\begin{aligned} 1 &< F_B(d_B)(y_B) + F_A(d_B)(y_B) \\ &\leq F_{B_0}(d_B)(y_B) + \lambda_B. \end{aligned} \quad (4)$$

So, $P_{d_B}^{(y_B)\lambda_B} \in F_{B_0}$, that is, $P_{d_B}^{(y_B)\lambda_B} \in F_C$. From the arbitrariness of F_C , $\{P_{d_B}^{(y_B)\lambda_B}: F_B \in A(P_e^{x_\lambda})\}$ converges to $P_e^{x_\lambda}$.

Sufficiency: it is supposed that $\lim S(\delta) = P_e^{x_\lambda}$, where $S = \{S(\delta): \delta \in \Delta\}$ is in F_A . Then, for any $F_B \in A(P_e^{x_\lambda})$, there is a $\delta_0 \in \Delta$ such that $S(\delta_0) \in F_A$ and $S(\delta_0) \in F_B$. $S(\delta_0)$ is written as $P_d^{y_\beta}$, where $d \in E$, $y \in U$, and $\beta \in (0, 1]$. Thus $F_A(d)(y) \geq \beta$ and $\beta + F_B(d)(y) > 1$. So, $F_A(d)(y) + F_B(d)(y) > 1$, which means that $F_B q F_A$. From the arbitrariness of F_B and Theorem 5, $P_e^{x_\lambda} \in \overline{F_A}$ is obtained. $\square$

Definition 13. Let (U, E, τ) be a fuzzy soft topological space. If for any two different points $P_e^{x_\lambda}$ and $P_f^{y_\beta}$ there exist $F_A \in A(P_e^{x_\lambda})$ and $F_B \in A(P_f^{y_\beta})$ such that $F_A \cap F_B = \emptyset$, then (U, E, τ) is said to be T_2 separated.

Theorem 7. A fuzzy soft topological space (U, E, τ) is T_2 separated if and only if the limit of any net is unique.

Proof (Necessity). Suppose that a net $S = \{S(\lambda): \lambda \in \Lambda\}$ converges to two different points $P_e^{x_\alpha}$ and $P_f^{y_\beta}$. Because (U, E, τ) is T_2 separated, there exist $F_A \in A(P_e^{x_\alpha})$ and $F_B \in A(P_f^{y_\beta})$ such that $F_A \cap F_B = \emptyset$. By Definition 12, there exists $S(\lambda)$ such that $S(\lambda) \in F_A$ and $S(\lambda) \in F_B$. Then, $S(\lambda) \in F_A \cap F_B$. There is a contradiction with the fact that $F_A \cap F_B = \emptyset$.

Sufficiency: suppose that (U, E, τ) is not T_2 separated, then there are two different points $P_e^{x_\alpha}$ and $P_f^{y_\beta}$, and for any $F_A \in A(P_e^{x_\alpha})$ and $F_B \in A(P_f^{y_\beta})$, F_A and F_B are quasi-coincident at some points. Thus, $S(A, B) \in \text{FSP}(U, E)$ such that $S(A, B) \in F_A$ and $S(A, B) \in F_B$. Let $\Delta = \{(F_A, F_B): F_A \in A(P_e^{x_\alpha}), F_B \in A(P_f^{y_\beta})\}$. Then, Δ forms a directed set

under the order relation " $\prec$ " defined as $(F_A, F_B) \prec (F_C, F_D)$ if and only if $F_A \supsetneq F_C$, $F_B \supsetneq F_D$. For any $\delta = (F_A, F_B) \in \Delta$, let $S(\delta) = S(A, B)$. So, a net $S = \{S(\delta): \delta \in \Delta\}$ is obtained.

It is now shown that S converges to $P_e^{x_a}$ and $P_f^{y_\beta}$. In fact, for any $F_A \in A(P_e^{x_a})$ and $F_B \in A(P_f^{y_\beta})$, $(F_A, F_B) \in \Delta$. When $(F_A, F_B) \prec (F_C, F_D) \in \Delta$, $S(C, D) \supsetneq F_C$, $S(C, D) \supsetneq F_D$, $F_A \supsetneq F_C$, and $F_B \supsetneq F_D$. Then, $S(C, D) \supsetneq F_A$ and $S(C, D) \supsetneq F_B$. From the arbitrariness of F_A and F_B , $\lim S(\delta) = P_e^{x_a}$ and $\lim S(\delta) = P_f^{y_\beta}$ are obtained, which is a contradiction with the supposition.

A fuzzy soft net $\{T(\omega): \omega \in \Omega\}$ is called a subnet of a fuzzy soft net $\{S(\lambda): \lambda \in \Lambda\}$ if there exists a mapping $\varphi: \Omega \longrightarrow \Lambda$ such that

- (i) $\forall p \in \Lambda$, there exists $\omega \in \Omega$ such that $p < \varphi(\omega)$
- (ii) For any $\omega_1, \omega_2 \in \Omega$ with $\omega_1 < \omega_2$, then $\varphi(\omega_1) < \varphi(\omega_2)$
- (iii) $T(\omega) = S(\varphi(\omega))$

Let Σ be a family of fuzzy soft sets, and $F_B \in FS(U, E)$. If $F_B \subsetneq \bigcup_{F_A \in \Sigma} F_A$, then Σ is called a cover of F_B . If a cover contains finite elements, then it is called a finite cover. If a subset Σ_1 of Σ is also a cover of F_B , then Σ_1 is called a subcover of Σ . If Σ is a cover of F_B , every element of Σ being open (closed), then Σ is said to be an open (closed) cover of F_B . $\square$

Definition 14. Let $F_A \in FS(U, E)$. If every open cover of F_A contains a finite subcover, then F_A is said to be compact.

Lemma 1. Let $\xi \in FSP(U, E)$, $F_A \in FS(U, E)$, and $S = \{S(\lambda): \lambda \in \Lambda\}$ be a net. If any subnet of S does not take ξ as its limit, then there exists $O_\xi \in A(\xi)$ and $\lambda_\xi \in \Lambda$ such that $S(\lambda) \not\supsetneq O_\xi$ whenever $\lambda_\xi < \lambda \in \Lambda$.

Proof. Conversely, it is supposed that, for any $O \in A(\xi)$ and any $\lambda \in \Lambda$, there exists $\lambda_O \in \Lambda$ with $\lambda < \lambda_O$ such that $S(\lambda_O) \supsetneq O$. Let $\bar{\Lambda}$ be a set of the corresponding λ_O with respect to all $O \in A(\xi)$ and all $\lambda \in \Lambda$, that is, $\bar{\Lambda} = \{\lambda_O: \lambda < \lambda_O \text{ and } S(\lambda_O) \supsetneq O, O \in A(\xi), \lambda \in \Lambda\}$. Then, $\bar{\Lambda}$ is a directed set under the same order relation " $\prec$ " as in Λ .

To complete the proof, it must only be shown that there is a subnet of S converging to ξ . Set $\Omega = A(\xi)$. Under the relation $\supsetneq$, Ω is a directed set. Let $\Omega \times \bar{\Lambda}$ be the usual product directed set of Ω and $\bar{\Lambda}$. The mapping φ from $\Omega \times \bar{\Lambda}$ to Λ is defined as $\varphi(O, \lambda_O) = \lambda_O$, $\forall (O, \lambda_O) \in \Omega \times \bar{\Lambda}$. Then, for any $\lambda' \in \Lambda$, there always exists $(O', \lambda_{O'}) \in \Omega \times \bar{\Lambda}$ such that $\lambda' < \lambda_{O'}$ and $S(\lambda_{O'}) \supsetneq O'$. When $(O', \lambda_{O'}) < (O, \lambda_O) \in \Omega \times \bar{\Lambda}$, $O' \supsetneq O$, $\lambda_{O'} < \lambda_O$. Thus $\lambda' < \lambda_{O'} < \lambda_O = \varphi(O, \lambda_O)$. So, $\{S(\lambda_O): (O, \lambda_O) \in \Omega \times \bar{\Lambda}\}$ is a subnet of $S = \{S(\lambda): \lambda \in \Lambda\}$. Let $O' \in A(\xi)$ arbitrarily. Then, $(O, \lambda_{O'}) \in \Omega \times \bar{\Lambda}$. When $(O', \lambda_{O'}) < (O, \lambda_O) \in \Omega \times \bar{\Lambda}$, $O' \supsetneq O$, $\lambda_{O'} < \lambda_O$, and $S(\lambda_O) \supsetneq O$.

Thus, $S(\lambda_O) \supsetneq O'$. So, $\{S(\lambda_O): (O, \lambda_O) \in \Omega \times \bar{\Lambda}\}$ converges to ξ . That is, there is a subnet of S converging to ξ . $\square$

Theorem 8. Let $F_A \in FS(U, E)$. Then, F_A is compact if and only if any net whose element is quasi-coincident with F_A has a subnet whose limit is quasi-coincident with F_A .

Proof (Necessity). Let F_A be a compact fuzzy soft set and $S = \{S(\lambda): \lambda \in \Lambda\}$ a net whose element is quasi-coincident with F_A . To prove the Necessity, it must only be shown that S has a subnet whose limit is quasi-coincident with F_A . Suppose it is not true, then for any point ξ which is quasi-coincident with F_A , any subnet of S does not take ξ as its limit. Therefore, by Lemma 1, there exists $O_\xi \in A(\xi)$ and $\lambda_\xi \in \Lambda$ such that when $\lambda_\xi < \lambda \in \Lambda$, and

$$S(\lambda) \not\supsetneq O_\xi. \quad (5)$$

By Theorem 2 (1), $F_A \subsetneq \bigcup \{O_\xi: O_\xi \in A(\xi), \xi \in F_A\}$. Because F_A is compact, there exists $\Psi = \{O_{\xi_i}: i = 1, 2, \dots, k\} \subseteq \Omega$ such that $F_A \subsetneq \bigcup_{i=1}^k O_{\xi_i}$. For each $i = 1, 2, \dots, k$, by (5), there exists $\lambda_{\xi_i} \in \Lambda$, $S(\lambda) \not\supsetneq O_{\xi_i}$ whenever $\lambda_{\xi_i} < \lambda \in \Lambda$. Because Λ is a directed set, there exists $\lambda_0 \in \Lambda$ such that $\lambda_{\xi_i} < \lambda_0$ ($i = 1, 2, \dots, k$). It follows that $S(\lambda_0) \not\supsetneq O_{\xi_i}$ ($i = 1, 2, \dots, k$). Then, $S(\lambda_0) \not\supsetneq F_A$. This is in contradiction with the supposition that $S = \{S(\lambda): \lambda \in \Lambda\}$ is a net whose element is quasi-coincident with F_A . Thus, S has a subnet whose limit is quasi-coincident with F_A .

(Sufficiency) Let $\Theta = \{O_i: i \in \Gamma\}$ be an open cover of F_A , $\Lambda = \{\lambda: \lambda \text{ is a finite subset of } \Gamma\}$. Under the usual inclusion " $\subseteq$ " relation, Λ forms a directed set. Suppose that F_A is not compact, then for any $\lambda \in \Lambda$, $F_A \not\supsetneq \bigcup_{i \in \lambda} O_i$. Then, by Theorem 2 (1), there is a point $S(\lambda) \not\supsetneq F_A$; however,

$$S(\lambda) \not\supsetneq \bigcup_{i \in \lambda} O_i, \quad \forall \lambda \in \Lambda. \quad (6)$$

Thus, a net $S = \{S(\lambda): \lambda \in \Lambda\}$ is obtained whose element is always quasi-coincident with F_A ; however, $S(\lambda) \not\supsetneq \bigcup_{i \in \lambda} O_i$, $\forall \lambda \in \Lambda$.

Under the assumption, there is a subnet $T = \{T(\omega): \omega \in \Omega\}$ of S such that $\lim T(\omega) = \xi_0 \in F_A$. Because $F_A \subsetneq \bigcup_{i \in \Gamma} O_i$ and due to Theorem 2 (3), there exists $i_0 \in \Gamma$ such that $\xi_0 \in O_{i_0}$. So, there exists $\omega_1 \in \Omega$ when $\omega_1 < \omega \in \Omega$:

$$T(\omega) \supsetneq O_{i_0}. \quad (7)$$

Because $T = \{T(\omega): \omega \in \Omega\}$ is a subnet of $S = \{S(\lambda): \lambda \in \Lambda\}$, there exists a mapping φ from Ω to Λ . For the above i_0 , $\{i_0\} \in \Lambda$. Hence, there exists $\omega_2 \in \Omega$ such that $\{i_0\} < \varphi(\omega_2)$. Subsequently, when $\omega_2 < \omega \in \Omega$, the following is obtained: $\{i_0\} < \varphi(\omega)$, that is, $i_0 \in \varphi(\omega)$. From (6), $S(\varphi(\omega)) \not\supsetneq O_{i_0}$, that is,

$$T(\omega) \not\supsetneq O_{i_0}. \quad (8)$$

Let $\omega \in \Omega$ with $\omega_1 < \omega$ and $\omega_2 < \omega$; then, both (7) and (8) have a contradiction. Therefore, F_A is compact. $\square$

4. Conclusions

In this paper, we have obtained some important theorems by means of the convergence of a fuzzy soft net. The obtained results demonstrate that the existing net is a powerful tool for studying fuzzy soft topological spaces. It is prospective that fuzzy soft nets will play important roles in characterizing some other properties of fuzzy soft topological spaces.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Jianrong Wu conceptualized the study. Rui Gao was responsible for formal analysis. Jianrong Wu acquired funding. Rui Gao investigated the study. Jianrong Wu was responsible for methodology. Rui Gao wrote the original draft. Jianrong Wu wrote, reviewed, and edited the manuscript. All authors have read and agreed to the published version of the manuscript.

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References

- [1] P. K. Maji, R. Biswas, and A. R. Roy, "Fuzzy soft sets," *Journal of Fuzzy Mathematics*, vol. 9, pp. 589–602, 2001.
- [2] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.
- [3] D. Molodtsov, "Soft set theory-First results," *Computers & Mathematics with Applications*, vol. 37, no. 4-5, pp. 19–31, 1999.
- [4] S. Alkhazaleh, A. R. Salleh, and N. Hassan, "Fuzzy parameterized interval-valued fuzzy soft set," *Applied Mathematical Sciences*, vol. 5, pp. 3335–3346, 2011.
- [5] A. Aygünöğlu and H. Aygün, "Introduction to fuzzy soft groups," *Computers & Mathematics with Applications*, vol. 58, pp. 1279–1286, 2009.
- [6] F. Feng, Y. Li, and V. Leoreanu-Fotea, "Application of level soft sets in decision making based on interval-valued fuzzy soft sets," *Computers & Mathematics with Applications*, vol. 60, no. 6, pp. 1756–1767, 2010.
- [7] R. Gao and J. R. Wu, "A net with applications for continuity in a fuzzy soft topological space," *Mathematical Problems in Engineering*, vol. 2020, Article ID 9098410, 7 pages, 2020.
- [8] Z. Kong, L. Gao, and L. Wang, "Comment on 'A fuzzy soft set theoretic approach to decision making problems'," *Journal of Computational and Applied Mathematics*, vol. 223, no. 2, pp. 540–542, 2009.
- [9] A. R. Roy and P. K. Maji, "A fuzzy soft set theoretic approach to decision making problems," *Journal of Computational and Applied Mathematics*, vol. 203, no. 2, pp. 412–418, 2007.
- [10] J. Wang, Y. Hu, F. Xiao, X. Deng, and Y. Deng, "A novel method to use fuzzy soft sets in decision making based on ambiguity measure and dempster-shafer theory of evidence: an application in medical diagnosis," *Artificial Intelligence in Medicine*, vol. 69, pp. 1–11, 2016.
- [11] Z. Xiao, K. Gong, and Y. Zou, "A combined forecasting approach based on fuzzy soft sets," *Journal of Computational and Applied Mathematics*, vol. 228, no. 1, pp. 326–333, 2009.
- [12] N. X. Xie, G. Q. Wen, and Z. W. Li, "A method for fuzzy soft sets in decision making based on grey relational analysis and d-s theory of evidence: application to medical diagnosis," *Computational and Mathematical Methods in Medicine*, vol. 2014, Article ID 581316, 12 pages, 2014.
- [13] Z. Zhang and S. Zhang, "A novel approach to multi attribute group decision making based on trapezoidal interval type-2 fuzzy soft sets," *Applied Mathematical Modelling*, vol. 37, no. 7, pp. 4948–4971, 2013.
- [14] A. Aygünöğlu, E. Aydoğdu, and H. Aygün, "Fuzzy soft metric and fuzzifying soft topology induced by fuzzy soft metric," *Filomat*, vol. 33, pp. 645–653, 2019.
- [15] S. El-Shiekh and S. El-Sayed, "γ-Operation & decomposition of some forms of fuzzy soft mappings on fuzzy soft ideal topological spaces," *Filomat*, vol. 34, 2020.
- [16] C. Gunduz (Aras) and S. Bayramov, "Some results on fuzzy soft topological spaces," *Mathematical Problems in Engineering*, vol. 2013, Article ID 835308, 10 pages, 2013.
- [17] I. Ibedou and S. E. Abbas, "Fuzzy soft filter convergence," *Filomat*, vol. 32, no. 9, pp. 3325–3336, 2018.
- [18] A. Kandil, O. A. El-Tantawy, S. A. El-Sheikh, and S. S. S. El-Sayed, "Fuzzy soft connected sets in fuzzy soft topological spaces II," *Journal of the Egyptian Mathematical Society*, vol. 25, no. 2, pp. 171–177, 2017.
- [19] J. Mahanta and P. K. Das, "Results on fuzzy soft topological spaces," 2012, <http://arxiv.org/abs/1203.0634v1>.
- [20] J. S. Ping, T. Wu, and C. Z. Yang, "Sum spaces in fuzzy soft topological spaces," *Fuzzy Systems and Mathematics*, vol. 28, pp. 69–73, 2014.
- [21] S. Roy and T. K. Samanta, "An introduction to open and closed sets on fuzzy soft topological spaces," *Annals of Fuzzy Mathematics and Informatics*, vol. 6, pp. 425–431, 2013.
- [22] S. Roy and T. K. Samanta, "A note on fuzzy soft topological spaces," *Annals of Fuzzy Mathematics and Informatics*, vol. 3, pp. 305–311, 2012.
- [23] T. Simsekler and S. Yuksel, "Fuzzy soft topological spaces," *Annals of Fuzzy Mathematics and Informatics*, vol. 5, pp. 87–96, 2013.
- [24] B. Tanay and M. B. Kandemir, "Topological structure of fuzzy soft sets," *Computers & Mathematics with Applications*, vol. 61, no. 10, pp. 2952–2957, 2011.