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Research Article

A New Study on Halpern and Nonconvex Combination Algorithm for Nonlinear Mappings in Banach Spaces with Applications

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In this paper, we introduce a Halpern algorithm and a nonconvex combination algorithm to approximate a solution of the split common fixed problem of quasi-ϕ-nonexpansive mappings in Banach space. In our algorithms, the norm of linear bounded operator does not need to be known in advance. As the application, we solve a split equilibrium problem in Banach space. Finally, some numerical examples are given to illustrate the main results in this paper and compare the computed results with other ones in the literature. Our results extend and improve some recent ones in the literature.

1. Introduction

Let H_1 be a Hilbert space, and let C be the nonempty closed convex subset of H_1. Let H_2 be a real Hilbert space, and let Q be the nonempty closed convex subset of H_2. Let $A\colon H_1 \longrightarrow H_2$ be a linear bounded operator. In 1994, Censor and Elfving [1] introduced the split feasibility problem (SFP) as a generalization of convex feasibility problem as follows:

$$\text{find a point } x^* \in C \text{ such that } Ax^* \in Q. \tag{1}$$

Recntly, the SFP and its variants have been investigated by many authors due to its real applications such as medical imaging, radiation therapy, and treatment planning; see, e.g., [2–5]. For solving SFP (1), it needs to get the inverse A^{-1} (assuming the existence of A^{-1}) in algorithm of Censor and Elfving [1]. However, few authors continue to study the algorithm of Censor and Elfving since the difficulty of computing A^{-1}, even if it exists. In fact, another algorithm solving SFP (1) is more popular which is called CQ algorithm given by Byrne [6, 7]. The CQ algorithm of Byrne is a gradient projection method in convex minimization. Since the CQ algorithm does need to compute A^{-1} and only involves the projections P_C and P_Q, it is easy to implement when P_C and P_Q have the closed-form expressions. However, the computations of P_C and P_Q are also difficult if these projections did not have the closed-form expressions which is such that the CQ algorithm of Byrne [6, 7] is not easy to implement in this case. In 2010, Xu [8] investigated the CQ algorithm from the ways of optimization and fixed point, proposed Mann's algorithm, and relaxed CQ algorithm to solve SFP (1). In the relaxed CQ algorithm, the sets C and Q are level sets of convex functions so that the projections involved in the CQ algorithm are onto half-spaces, which makes the algorithm implementable. Also, in 2010, Moudafi [9] proposed an iterative method to solve a split common fixed point problem for quasi-nonexpansive mappings in which the projection is not involved which is such that the algorithm is easy to implement. In 2014, Kraikaew and Saejung [10] combined the Moudafi method and the Halpern algorithm to propose a new iteration in which the projection is not involved for solving the SFP. In the recent years, many algorithms have been given to solve the SFP in Hilbert spaces; see, for instance, [11–15] and the references therein.

However, because of the complexity of properties in Banach space, it is very difficulty to solve SFP (and fixed point problem) in Banach spaces. Until now, only limited works on SFP (and fixed point problem) in Banach spaces have been reported in the literature. For instance, the authors in [16] gave an algorithm to solve SFP in Banach space. In [17], Tang et al. introduced some iterative algorithms to solve a split common fixed point problem for a quasi-strict pseudocontractive mapping and an asymptotically nonexpansive mapping in two Banach spaces and obtained the weak and strong convergence for the proposed algorithms. In [18], Chen et al. proposed a new hybrid projection method for solving split feasibility and fixed point problems involved in Bregman quasi-strictly pseudocontractive mapping in p-uniformly convex and uniformly smooth real Banach spaces. They proved the strong convergence for the proposed algorithm using the Bregman projection method. On the feasible and common fixed point problem, the authors also refer to [19–21].

Let E_1 be a 2-uniformly convex and 2-uniformly smooth real Banach space with the best smoothness constant $k > 0$ and E_2 be a uniformly smooth, strictly convex, and reflexive Banach space. Let S: $E_1 \longrightarrow E_1$ be a closed quasi-ϕ-nonexpansive mapping and A: $E_1 \longrightarrow E_2$ be a linear bounded operator. Very recently, Ma et al. [22] proposed a hybrid projection algorithm to solve the following split feasibility problem and fixed point problem:

$$\text{find } x^* \in C \text{ such that } Ax^* \in Q, \tag{2}$$

where $C = \{x \in E_1 : x = Sx\}$ and $Q \subset E_2$ is a nonempty closed convex subset. Precisely, their algorithm to solve (2) is as follows:

$$\begin{cases} x_1 \in E_1, C_1 = E_1, \\ z_n = J^{-1}\left(J_1 x_n + \gamma A^* J_2\left(P_Q - I\right)A x_n\right), \\ y_n = J^{-1}\left[\alpha_n J_1 z_n + (1-\alpha_n) J_1 S z_n\right], \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \le \phi(v, x_n), \phi(v, z_n) \le \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad n \ge 1, \end{cases} \tag{3}$$

where $\{\alpha\} \subset [\delta, 1)$ with $\delta > 0$, $\gamma \in (0, (1/\|A\|^2 k^2))$, P_Q is the metric projection of E_2 onto Q, and $\Pi_{C_{n+1}}$ is the generalized projection of E_1 in C_{n+1}. The authors proved that the sequence generated by (3) strongly converges to a point which solves (2).

On the contrary, the most algorithms of approximating the fixed points of quasi-ϕ-nonexpansive mappings in Banach spaces are constructed by the hybrid or shrinking projection methods, see [23–25]. However, in 2018, Hieu and Strodiot [26] introduced a new iterative algorithm for solving pseudomonotone equilibrium problem involving the fixed point problem for quasi-ϕ-nonexpansive mapping in Banach space without using the hybrid or shrinking projection methods. More precisely, their algorithm is

$$\begin{cases} y_n = \operatorname{argmin}\left\{\lambda_n f(x_n, y) + \dfrac{1}{2}\phi(y, x_n) : y \in C\right\}, \\ z_n = \operatorname{argmin}\left\{\lambda_n f(y_n, y) + \dfrac{1}{2}\phi(y, x_n) : y \in C\right\}, \\ x_{n+1} = \Pi_C\left(J^{-1}\left(\alpha_n J u + (1-\alpha_n)\left(\beta_n J z_n + (1-\beta_n) J S z_n\right)\right)\right), \end{cases} \tag{4}$$

where $f : C \times C \longrightarrow \mathbb{R}$ is a pseudomonotone bifunction and $S: C \longrightarrow C$ is a quasi-ϕ-nonexpansive mapping. The authors proved that the sequence generated by (4) strongly converges to a common point that solves the pseudomonotone equilibrium problem on f and is a fixed point of S.

In general, there are three kinds of iterations of strong convergence that are used to approximate the fixed point of the nonlinear operator. The iterations are the Halpern iteration, the viscosity iteration, and the hybrid projection iteration. Recently, Hussain et al. [27] proposed a new surprising iteration that strongly converges to a fixed point of a nonexpansive mapping in Hilbert space. More precisely, the iteration is

$$x_1 \in H, \quad x_{n+1} = \alpha_n (1-\mu_n) x_n + (1-\alpha_n) T x_n, \ n \ge 1, \tag{5}$$

where H is a Hilbert space, $T : H \longrightarrow H$ is a nonexpansive mapping, and $\{\alpha_n\}, \{\mu_n\} \subset (0, 1]$ are the control sequences. The authors proved that $\{x_n\}$ generated by (5) strongly converges to a fixed point of T under some certain conditions on $\{\alpha_n\}$ and $\{\mu_n\}$. Later on, Marino et al. [28] extended (5) to strict pseudocontraction.

In this paper, motivated by the work of [22, 26, 27], we introduce some algorithms to solve a split common fixed point problem for two families of quasi-ϕ-nonexpansive mappings in Banach spaces and prove the strong convergence for the proposed algorithms. As the application, we solve a split equilibrium problem in Banach space. Finally, we give a numerical example in infinite dimension Banach space to illustrate the main result of this paper. Our results extend the one of Ma et al. [22] from one quasi-nonexpansive mapping to two quasi-nonexpansive mappings and [27] from Hilbert space to Banach space.

2. Preliminaries

Let E be a Banach space, and let E^* be the dual space of E. For all $x \in E$ and $x^* \in E^*$, we denote the value of x^* at x by $\langle x, x^* \rangle$. The duality mapping J on E is defined by

$$J(x) = \left\{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \left\|x^*\right\|^2\right\}, \quad \forall x \in E. \tag{6}$$

It is known that $J(x)$ is nonempty for all $x \in E$. A Banach space E is said to be smooth if the limit

$$\lim_{n\longrightarrow\infty}\frac{\|x+ty\|-\|x\|}{t} \tag{7}$$

exists for all $x, y \in S(E) = \{z \in E : \|z\| = 1\}$. The space E is smooth if and only if the duality mapping J is single-valued.

A Banach space E is said to be strictly convex if $(\|x+y\|/2)<1$ for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$ and uniformly convex if for each $\epsilon \in (0,2]$, there exists $\delta > 0$ such that $(\|x+y\|/2) \le 1-\delta$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $\|x-y\| \ge \epsilon$. It is known that if E is smooth, strictly convex, and reflexive, then the duality mapping J is single-valued, one-to-one, and onto. Let E be a smooth Banach space. The function ϕ: $E\times E \longrightarrow \mathbb{R}$ is defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \tag{8}$$

for all $x, y \in E$. From the definition of ϕ, it is easy to see that, for all $x, y, z \in E$, the following hold:

$$\begin{aligned}(\|x\|-\|y\|)^2 &\le \phi(x,y) \le (\|x\|+\|y\|)^2,\\ \phi\left(x, J^{-1}(\lambda Jy) + (1-\lambda)Jz\right) &\le \lambda\phi(x,y) + (1-\lambda)\phi(x,z), \quad \lambda \in (0,1).\end{aligned} \tag{9}$$

The following is an important property for the function ϕ:

$$\phi(x,y) = \phi(z,y) + \phi(x,z) + 2\langle z-x, Jy-Jz\rangle, \tag{10}$$

for all $x, y, z \in E$.

Lemma 1 (see [29]). *Let E be a uniformly convex and smooth Banach space, and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \longrightarrow 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \longrightarrow 0$.*

For any bounded sequences $\{x_n\}$ and $\{y_n\}$ in a uniformly convex and uniformly smooth Banach space, the following hold:

$$\phi(x_n, y_n) \longrightarrow 0 \Leftrightarrow \|x_n - y_n\| \longrightarrow 0 \Leftrightarrow \|Jx_n - Jy_n\| \longrightarrow 0. \tag{11}$$

Let $\Pi_C: E \longrightarrow C$ be mapping called the generalized projection [30] that assigns to an arbitrary element $x \in E$ the minimum point of the functional $\phi(x,y)$; that is, $\Pi_C x = \operatorname{argmin}_{y\in C}\phi(y,x)$.

Lemma 2 (see [30]). *Let E be a smooth, strictly convex, and reflexive Banach space and C be a nonempty closed convex subset of E. Then, the following conclusions hold:*

(a) $\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \le \phi(x,y)$, $\forall x \in C$, $\forall y \in E$

(b) For $x \in E$, $z = \Pi_C x$ if and only if $\langle z-y, Jx-Jz\rangle \ge 0$, $\forall y \in C$

(c) For $x, y \in E$, $\phi(x,y) = 0$ if and only if $x = y$

Let E be a strictly convex and reflexive Banach space and C be a nonempty closed and convex subset. The metric projection

$$P_C x = \operatorname{argmin}_{y\in C}\|y-x\|, \quad \forall x \in E. \tag{12}$$

Lemma 3 (see [31]). *Let E be a smooth, strictly convex, and reflexive Banach space and C be a nonempty, closed convex subset of E. Let $x \in E$. Then,*

$$z = P_C x \text{ if and only if } \langle z-y, J(x-z)\rangle \ge 0, \quad \forall y \in C. \tag{13}$$

Let E be a strictly convex, smooth, and reflexive Banach space. The duality mapping J^* from E^* onto $E^{**} = E$ coincides with the inverse of the duality mapping J from E onto E^*, that is, $J^* = J^{-1}$. Define a mapping V: $E\times E^* \longrightarrow \mathbb{R}$ [32] by

$$V(x,x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2, \quad \forall (x,x^*) \in E\times E^*. \tag{14}$$

Lemma 4 (see [32]). *Let E be a reflexive, smooth, and strictly convex Banach space. Then,*

$$V(x,x^*) \le V(x, x^* + y^*) - 2\langle J^{-1}x^* - x, y^*\rangle, \tag{15}$$

for all $x \in E$ and $x^*, y^* \in E^*$. Obviously, $V(x,x^*) = \phi(x, J^{-1}x^*)$ for all $x \in E$ and $x^* \in E$.

Let E be a smooth Banach space. A mapping T: $E \longrightarrow E$ is said to be closed if for any sequence $\{x_n\} \subset E$ with $x_n \longrightarrow x$ and $Tx_n \longrightarrow y$, then $Tx_n = y$. T is said to be quasi-ϕ-nonexpansive mapping if $\operatorname{Fix}(T) \neq \emptyset$ and

$$\phi(p, Tx) \le \phi(p,x), \tag{16}$$

for all $p \in \operatorname{Fix}(T)$ and $x \in E$. For a quasi-ϕ-nonexpansive mapping T, $\operatorname{Fix}(T)$ is convex. If T is closed, then $\operatorname{Fix}(T)$ is closed, see [24].

Lemma 5 (see [33]). *Let $r > 0$. A real Banach space E is uniformly convex if and only if there exists a continuous strictly increasing function $g: [0,\infty) \longrightarrow [0,\infty)$ with $g(0) = 0$ such that*

$$\|tx + (1-t)y\|^2 \le t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g(\|x-y\|), \tag{17}$$

for all $t \in [0,1]$ and $x, y \in B_r$, where $B_r = \{x \in E: \|x\| \le r\}$.

Lemma 6 (see [33]). *Let $r > 0$. Let E be a 2-uniformly smooth Banach space with the best smoothness constants $k > 0$. Then,*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, Jx\rangle + 2k^2\|y\|^2, \tag{18}$$

for all $x, y \in E$.

Lemma 7 (see [34]). *Let* $\{a_n\}$ *be a sequence of nonnegative real numbers. Suppose that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n\delta_n, \quad \forall n \in \mathbb{N}, \tag{19}$$

where $\{\gamma_n\} \subset (0,1)$ *and* $\{\delta_n\} \subset \mathbb{R}$ *satisfy the conditions:*

$$\begin{aligned} &\lim_{n\longrightarrow\infty} \gamma_n = 0, \\ &\sum_{n=1}^{\infty} \gamma_n = \infty, \text{ and } \limsup_{n\longrightarrow\infty} \delta_n \leq 0. \end{aligned} \tag{20}$$

Then, $\lim_{n\longrightarrow\infty} a_n = 0$.

Lemma 8 (see [35]). *Let* $\{a_n\}$ *be a sequence of real numbers such that there exists a subsequence* $\{n_i\}$ *of* $\{n\}$ *such that* $a_{n_i} < a_{n_i+1}$ *for all* $i \in \mathbb{N}$. *Then, there exists a nondecreasing sequence* $\{m_k\} \subset \mathbb{N}$ *such that* $m_k \longrightarrow \infty$ *as* $k \longrightarrow \infty$, *and the following properties are satisfied by all (sufficiently large) numbers* $k \in \mathbb{N}$:

$$a_{m_k} \leq a_{m_k+1} \text{ and } a_k \leq a_{m_k+1}. \tag{21}$$

In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.

Lemma 9 (see [36]). *Suppose that* $\{a_n\}$ *and* $\{b_n\}$ *are sequences of nonnegative real numbers such that*

$$a_{n+1} \leq a_n + b_n, \quad n \geq 1. \tag{22}$$

If $\sum_{n=1}^{\infty} b_n < \infty$, *then* $\lim_{n\longrightarrow\infty} a_n$ *exists.*

3. Main Results

In this section, let E_1 be a 2-uniformly convex and 2-uniformly smooth real Banach space with the best smoothness constant $k > 0$ and E_2 be a uniformly smooth, strictly convex, and reflexive Banach space. Define the functions ϕ_1 and ϕ_2 by

$$\begin{aligned} \phi_1(x,y) &= \|x\|_1^2 - 2\langle x, J_1 y\rangle_1 + \|y\|_1^2, \quad \forall x, y \in E_1, \\ \phi_2(u,v) &= \|u\|_2^2 - 2\langle u, J_2 v\rangle_2 + \|v\|_2^2, \quad \forall u, v \in E_2, \end{aligned} \tag{23}$$

where $\langle x, J_1 y\rangle_1$ (resp., $\langle u, J_1 v\rangle_2$) and $\|x\|_1$ (resp., $\|u\|_2$) denote the value of $J_1 y$ at x and norm of x (resp., the value of $J_1 v$ at u and norm of u) in E_1 (resp. E_2), respectively. However, for convenience, we use the same symbols $\langle \cdot, \cdot \rangle$, $\|\cdot\|$, and ϕ in E_1 and E_2 without the confusion.

Let $A: E_1 \longrightarrow E_2$ be a linear bounded operator with adjoint A^*. Let $S: E_1 \longrightarrow E_1$ and $T: E_2 \longrightarrow E_2$ be the quasi-ϕ-nonexpansive mappings. Consider the following split common fixed point problem:

$$\text{find } x \in \text{Fix}(S) \text{ such that } Ax \in \text{Fix}(T). \tag{24}$$

Denote the set of solutions of the above split common fixed point problem by Ω. In this section, assume that S and T are closed and $I-S$ and $I-T$ are demiclosed at zeros in E_1 and E_2. Note that, from the closedness of S and T, it follows that $\text{Fix}(S)$ and $\text{Fix}(T)$ are closed [24], which implies that Ω is closed. The convexity of Ω is from the convexity of $\text{Fix}(S)$. Assume that Ω is nonempty.

Let $x^* = \Pi_\Omega \theta$, where θ is the zero element in E_1. We will prove that sequence $\{x_n\}$ generated by the following algorithm converges strongly to x^*.

Algorithm 1. Take $x_1 \in E_1$, and define a sequence $\{x_n\}$ by

$$\begin{cases} w_n = TAx_n, \\ Q_n = \{w \in E_2 : \phi(w, w_n) \leq \phi(w, Ax_n)\}, \\ z_n = J_1^{-1}\left(J_1 x_n - \gamma_n A^* J_2\left(I - P_{Q_n}\right)Ax_n\right), \\ y_n = J_1^{-1}(\beta_n J_1 z_n + (1-\beta_n) J_1 S z_n), \\ x_{n+1} = J_1^{-1}(\alpha_n(1-\tau_n) J_1 x_n + (1-\alpha_n) J_1 y_n), \quad n \geq 1, \end{cases} \tag{25}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0,1), \{\tau_n\} \subset (\tau, 1)$ with $\tau \in (0,1)$ and

$$\gamma_n = \begin{cases} \dfrac{\left\|\left(P_{Q_n} - I\right)Ax_n\right\|^2}{2k^2\left\|A^* J_2\left(I - P_{Q_n}\right)Ax_n\right\|^2}, & \text{if } \left\|\left(P_{Q_n} - I\right)Ax_n\right\| \neq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{26}$$

Lemma 10. *The sequence* $\{x_n\}$ *is well-defined and bounded.*

Proof. Since $\phi(w, w_n) \leq \phi(w, Ax_n)$ is equivalent to $2\langle w, J_2 Ax_n - J_2 w_n\rangle \leq \|Ax_n\|^2 - \|w_n\|^2$, it follows that Q_n is closed and convex for each $n \geq 1$. For any $p \in \Omega$, it follows that $Ap \in Q_n$ for all $n \geq 1$. Hence, each Q_n is nonempty closed convex, which implies that $\{P_{Q_n} Ax_n\}$ is well-defined. Now, we show that $\|(P_{Q_n} - I)Ax_n\| \neq 0$ implies that $\|A^* J_2 (P_{Q_n} - I)Ax_n\| \neq 0$. Assume that $\|A^* J_2 (P_{Q_n} - I)Ax_n\| = 0$. We have $\langle Ap - P_{Q_n} Ax_n, J_2 (P_{Q_n} - I)Ax_n\rangle \geq 0$ by Lemma 3 and hence

$$\begin{aligned} 0 &= \langle p - x_n, A^* J_2\left(P_{Q_n} - I\right)Ax_n\rangle = \langle Ap - Ax_n, J_2\left(P_{Q_n} - I\right)Ax_n\rangle \\ &= \langle Ap - P_{Q_n} Ax_n, J_2\left(P_{Q_n} - I\right)Ax_n\rangle + \langle P_{Q_n} Ax_n - Ax_n, J_2\left(P_{Q_n} - I\right)Ax_n\rangle \\ &= \langle Ap - P_{Q_n} Ax_n, J_2\left(P_{Q_n} - I\right)Ax_n\rangle + \left\|\left(P_{Q_n} - I\right)Ax_n\right\|^2 \geq \left\|\left(P_{Q_n} - I\right)Ax_n\right\|^2. \end{aligned} \tag{27}$$

It is a contradiction. It follows that $\|(P_{Q_n} - I)Ax_n\| \neq 0$ implies that $\|A^* J_2 (P_{Q_n} - I)Ax_n\| \neq 0$. Hence, $\{z_n\}$ is well-defined. Furthermore, $\{x_n\}$ is well-defined.

Since E_1 is a 2-uniformly convex and 2-uniformly smooth real Banach space, E_1^* is 2-uniformly smooth real Banach space, and $J_1 = (J_1^*)^{-1}$. From (25) and Lemma 6, we have

$$\begin{aligned}\phi(x^*, z_n) &= \|x^*\|^2 - 2\langle x^*, J_1 x_n + \gamma_n A^* J_2(P_{Q_n} - I)Ax_n\rangle + \left\|J_1 x_n + \gamma_n A^* J_2(P_{Q_n} - I)Ax_n\right\|^2 \\ &\leq \|x^*\|^2 - 2\langle x^*, J_1 x_n\rangle - 2\gamma_n\langle x^*, A^* J_2(P_{Q_n} - I)Ax_n\rangle + \|x_n\|^2 \\ &\quad + 2\gamma_n\langle x_n, A^* J_2(P_{Q_n} - I)Ax_n\rangle + 2\gamma_n^2 k^2\left\|A^* J_2(P_{Q_n} - I)Ax_n\right\|^2 \\ &= \phi(x^*, x_n) - 2\gamma_n\langle x^* - x_n, A^* J_2(P_{Q_n} - I)Ax_n\rangle + 2\gamma_n^2 k^2\left\|A^* J_2(P_{Q_n} - I)Ax_n\right\|^2.\end{aligned} \tag{28}$$

Since $Ax^* \in Q_n$, $\langle Ax^* - P_{Q_n}Ax_n, J_2(P_{Q_n} - I)Ax_n\rangle \geq 0$. Hence, we have

$$\begin{aligned}2\langle x^* - x_n, A^* J_2(P_{Q_n} - I)Ax_n\rangle &= 2\langle Ax^* - Ax_n, J_2(P_{Q_n} - I)Ax_n\rangle \\ &= 2\langle Ax^* - P_{Q_n}Ax_n, J_2(P_{Q_n} - I)Ax_n\rangle + 2\left\|(P_{Q_n} - I)Ax_n\right\|^2 \geq 2\left\|(P_{Q_n} - I)Ax_n\right\|^2.\end{aligned} \tag{29}$$

Combining (28) with (29), we obtain

$$\begin{aligned}\phi(x^*, z_n) &\leq \phi(x^*, x_n) - 2\gamma_n\left\|(P_{Q_n} - I)Ax_n\right\|^2 + 2\kappa^2\gamma_n^2\left\|A^* J_2(P_{Q_n} - I)Ax_n\right\|^2 \\ &= \phi(x^*, x_n) - \frac{\left\|(P_{Q_n} - I)Ax_n\right\|^4}{2k^2\left\|A^* J_2(P_{Q_n} - I)Ax_n\right\|^2} \leq \phi(x^*, x_n).\end{aligned} \tag{30}$$

Furthermore, by Lemma 5, (25), and (30) we obtain

$$\begin{aligned}\phi(x^*, y_n) &= \|x^*\|^2 - 2\langle x^*, \beta_n J_1 z_n + (1 - \beta_n)J_1 S z_n\rangle + \|\beta_n J_1 z_n + (1 - \beta_n)J_1 S z_n\|^2 \\ &\leq \|x^*\|^2 - 2\langle x^*, \beta_n J_1 z_n + (1 - \beta_n)J_1 S z_n\rangle + \beta_n\|z_n\|^2 + (1 - \beta_n)\|Sz_n\|^2 - \beta_n(1 - \beta_n)g\left(\|J_1 z_n - J_1 S z_n\|\right) \\ &= \beta_n\phi(x^*, z_n) + (1 - \beta_n)\phi(x^*, Sz_n) - \beta_n(1 - \beta_n)g\left(\|J_1 z_n - J_1 S z_n\|\right) \\ &\leq \phi(x^*, x_n) - \frac{\left\|(P_{Q_n} - I)Ax_n\right\|^4}{2k^2\left\|A^* J_2(P_{Q_n} - I)Ax_n\right\|^2} - \beta_n(1 - \beta_n)g\left(\|J_1 z_n - J_1 S z_n\|\right).\end{aligned} \tag{31}$$

It follows from (25), (31), and Lemma 5 that

$$\begin{aligned}
\phi(x^*, x_{n+1}) &= \phi\left(x^*, J_1^{-1}(\alpha_n(1-\tau_n)J_1x_n + (1-\alpha_n)J_1y_n)\right) = \|x^*\|^2 - 2\alpha_n(1-\tau_n)\langle x^*, J_1x_n\rangle - 2(1-\alpha_n)\langle x^*, J_1y_n\rangle \\
&\quad + \|\alpha_n(1-\tau_n)J_1x_n + (1-\alpha_n)J_1y_n\|^2 \le \|x^*\|^2 - 2\alpha_n(1-\tau_n)\langle x^*, J_1x_n\rangle - 2(1-\alpha_n)\langle x^*, J_1y_n\rangle \\
&\quad + \alpha_n\|(1-\tau_n)J_1x_n\|^2 + (1-\alpha_n)\|J_1y_n\|^2 \le \|x^*\|^2 - 2\alpha_n(1-\tau_n)\langle x^*, J_1x_n\rangle - 2(1-\alpha_n)\langle x^*, J_1y_n\rangle \\
&\quad + \alpha_n(1-\tau_n)\|x_n\|^2 + (1-\alpha_n)\|y_n\|^2 = \alpha_n(1-\tau_n)\phi(x^*, x_n) + (1-\alpha_n)\phi(x^*, y_n) + \alpha_n\tau_n\|x^*\|^2 \\
&\le \alpha_n(1-\tau_n)\phi(x^*, x_n) + (1-\alpha_n)\left(\phi(x^*, x_n) - \frac{\|(P_{Q_n} - I)Ax_n\|^4}{2k^2\|A^*J_2(P_{Q_n} - I)Ax_n\|^2} - \beta_n(1-\beta_n)g(\|J_1z_n - J_1Sz_n\|)\right) \\
&\quad + \alpha_n\tau_n\|x^*\|^2 \\
&= (1-\alpha_n\tau_n)\phi(x^*, x_n) + \alpha_n\tau_n\|x^*\|^2 - (1-\alpha_n)\left(\frac{\|(P_{Q_n} - I)Ax_n\|^4}{2k^2\|A^*J_2(P_{Q_n} - I)Ax_n\|^2} + \beta_n(1-\beta_n)g(\|J_1z_n - J_1Sz_n\|)\right) \\
&\le \max\left\{\phi(x^*, x_n), \|x^*\|^2\right\} \le \cdots \le \max\left\{\phi(x^*, x_1), \|x^*\|^2\right\}, \quad n \ge 1.
\end{aligned} \tag{32}$$

So, $\{\phi(x^*, x_n)\}$ is bounded. □

Lemma 11. *Let $\{x_n\}$ be the sequence generated by Algorithm 1. Then,*

$$\phi(x^*, x_{n+1}) \le (1-\alpha_n\tau_n)\phi(x^*, x_n) + 2\alpha_n\tau_n\langle x^* - x_{n+1}, J_1x^* + (1-\alpha_n)(J_1x_n - J_1y_n)\rangle. \tag{33}$$

Proof. Let $h_n = \alpha_n J_1x_n + (1-\alpha_n)J_1y_n$. Then, by (31), we have

$$\begin{aligned}
\phi(x^*, J_1^{-1}h_n) &\le \alpha_n\phi(x^*, x_n) + (1-\alpha_n)\phi(x^*, y_n) \\
&\le \alpha_n\phi(x^*, x_n) + (1-\alpha_n)\phi(x^*, x_n) \\
&= \phi(x^*, x_n).
\end{aligned} \tag{34}$$

Note that

$$x_{n+1} = J_1^{-1}((1-\alpha_n\tau_n)h_n + \alpha_n\tau_n(1-\alpha_n)(J_1y_n - J_1x_n)). \tag{35}$$

By (34) and (35) and Lemma 4, we have

$$\begin{aligned}
\phi(x^*, x_{n+1}) &= \phi\left(x^*, J_1^{-1}((1-\alpha_n\tau_n)h_n + \alpha_n\tau_n(1-\alpha_n)(J_1y_n - J_1x_n))\right) \\
&= V(x^*, (1-\alpha_n\tau_n)h_n + \alpha_n\tau_n(1-\alpha_n)(J_1y_n - J_1x_n)) \\
&\le V(x^*, (1-\alpha_n\tau_n)h_n + \alpha_n\tau_n(1-\alpha_n)(J_1y_n - J_1x_n) + \alpha_n\tau_n(J_1x^* - (1-\alpha_n)(J_1y_n - J_1x_n))) \\
&\quad - 2\langle x_{n+1} - x^*, \alpha_n\tau_n(J_1x^* - (1-\alpha_n)(J_1y_n - J_1x_n))\rangle = V(x^*, (1-\alpha_n\tau_n)h_n + \alpha_n\tau_n J_1x^*) \\
&\quad - 2\langle x_{n+1} - x^*, \alpha_n\tau_n(J_1x^* - (1-\alpha_n)(J_1y_n - J_1x_n))\rangle \le (1-\alpha_n\tau_n)\phi\left(x^*, J_1^{-1}h_n\right) + \alpha_n\tau_n\phi(x^*, x^*) \\
&\quad - 2\langle x_{n+1} - x^*, \alpha_n\tau_n(J_1x^* - (1-\alpha_n)(J_1y_n - J_1x_n))\rangle \\
&\le (1-\alpha_n\tau_n)\phi(x^*, x_n) + 2\alpha_n\tau_n\langle x^* - x_{n+1}, J_1x^* - (1-\alpha_n)(J_1y_n - J_1x_n)\rangle.
\end{aligned} \tag{36}$$

□

Theorem 1 *If the following conditions hold:*

$$\lim_{n\longrightarrow\infty}\alpha_n = 0, \quad \sum_{n=1}^{\infty}\alpha_n = \infty \text{ and } \liminf_{n\longrightarrow\infty}\beta_n(1-\beta_n) > 0, \tag{37}$$

then the sequence $\{x_n\}$ generated by Algorithm 1 converges strongly to the element x^*.

Proof. By (32), we have

$$(1-\alpha_n)\left(\frac{\left\|\left(P_{Q_n}-I\right)Ax_n\right\|^4}{2k^2\left\|A^*J_2\left(P_{Q_n}-I\right)Ax_n\right\|^2}+\beta_n(1-\beta_n)g\left(\left\|J_1z_n-J_1Sz_n\right\|\right)\right)$$
$$\leq(1-\alpha_n\tau_n)\phi(x^*,x_n)-\phi(x^*,x_{n+1})+\alpha_n\tau_n\|x^*\|^2\leq\phi(x^*,x_n)-\phi(x^*,x_{n+1})+\alpha_n\tau_n\|x^*\|^2. \tag{38}$$

Now, we show that $\|x_n-x^*\|\longrightarrow 0$ by the following two cases:

Case 1. Suppose that there exists $n_0\in\mathbb{N}$ such that $\{\phi(x^*,x_n)\}_{n=n_0}^{\infty}$ is nonincreasing. In this situation, $\{\phi(x^*,x_n)\}$ is convergent. By (37) and (38), we have

$$\lim_{n\longrightarrow\infty}\frac{\left\|\left(P_{Q_n}-I\right)Ax_n\right\|^4}{\left\|A^*J_2\left(P_{Q_n}-I\right)Ax_n\right\|^2}=\lim_{n\longrightarrow\infty}g\left(\left\|J_1z_n-J_1Sz_n\right\|\right)=0, \tag{39}$$

which implies that

$$\lim_{n\longrightarrow\infty}\left\|J_1z_n-J_1Sz_n\right\|=0. \tag{40}$$

Since $\left\{\|A^*J_2(P_{Q_n}-I)Ax_n\|\right\}$ is bounded, we have

$$\lim_{n\longrightarrow\infty}\left\|\left(P_{Q_n}-I\right)Ax_n\right\|=0. \tag{41}$$

By (40), we have

$$\left\|J_1y_n-J_1z_n\right\|=(1-\beta_n)\left\|J_1z_n-J_1Sz_n\right\|\longrightarrow 0. \tag{42}$$

Combining (39) with (42), we obtain

$$\left\|J_1y_n-J_1x_n\right\|\leq\left\|J_1y_n-J_1z_n\right\|+\left\|J_1z_n-J_1x_n\right\|=\left\|J_1y_n-J_1z_n\right\|+\gamma_n\left\|A^*J_2\left(P_{Q_n}-I\right)Ax_n\right\|$$
$$=\left\|J_1y_n-J_1z_n\right\|+\frac{\left\|\left(P_{Q_n}-I\right)Ax_n\right\|^2}{2\kappa^2\left\|A^*J_2\left(P_{Q_n}-I\right)Ax_n\right\|}\longrightarrow 0. \tag{43}$$

On the contrary, from (25) and (43), it follows that

$$\left\|J_1z_n-J_1x_{n+1}\right\|\leq\left\|J_1z_n-J_1y_n\right\|+\left\|J_1y_n-J_1x_{n+1}\right\|=\left\|J_1z_n-J_1y_n\right\|+\alpha_n\left\|(1-\tau_n)J_1x_n-J_1y_n\right\|\longrightarrow 0. \tag{44}$$

Since E_1 is a 2-uniformly convex and 2-uniformly smooth real Banach space, J_1 is uniformly norm-to-norm continuous. From (40), (42), and (44), it follows that

$$\lim_{n\longrightarrow\infty}\left\|z_n-Sz_n\right\|=\lim_{n\longrightarrow\infty}\left\|y_n-z_n\right\|=\lim_{n\longrightarrow\infty}\left\|z_n-x_{n+1}\right\|=0. \tag{45}$$

Since $\{z_n\}$ is bounded, there exist a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ converging weakly to $p\in E_1$ such that

$$\limsup_{n\longrightarrow\infty}\langle x^*-z_n,J_1x^*\rangle=\lim_{k\longrightarrow\infty}\langle x^*-z_{n_k},J_1x^*\rangle$$
$$=\langle x^*-p,J_1x^*\rangle. \tag{46}$$

Now, we show that $p\in\Omega$. First, by (45) and demicloseness principle at zero of S, we have $p\in\mathrm{Fix}(S)$. On the contrary, since $P_{Q_n}Ax_n\in Q_n$ and $\|P_{Q_n}Ax_n-Ax_n\|\longrightarrow 0$, we have

$$\phi\left(P_{Q_n}Ax_n,w_n\right)\leq\phi\left(P_{Q_n}Ax_n,Ax_n\right)\longrightarrow 0. \tag{47}$$

By Lemma 1, it follows that

$$\left\|P_{Q_n}Ax_n-w_n\right\|=\left\|P_{Q_n}Ax_n-TAx_n\right\|\longrightarrow 0. \tag{48}$$

Hence,

$$\left\|Ax_n-TAx_n\right\|\leq\left\|Ax_n-P_{Q_n}Ax_n\right\|+\left\|P_{Q_n}Ax_n-TAx_n\right\|\longrightarrow 0. \tag{49}$$

Since A is bounded and linear, by (45), we can conclude that $\left\{Ax_{n_k+1}\right\}$ converges weakly to $Ap\in E_2$. By (49) and demi-closedness principle of T, we obtain that $Ap\in\mathrm{Fix}(T)$. Hence, $p\in\Omega$. Therefore, by (45) and Lemma 3,

$$\limsup_{n\longrightarrow\infty}\langle x^*-x_{n+1},J_1x^*\rangle=\limsup_{n\longrightarrow\infty}\langle x^*-z_n,J_1x^*\rangle$$
$$=\langle x^*-p,J_1x^*\rangle\leq 0. \tag{50}$$

Finally, the conclusion$\|x_n-x^*\|\longrightarrow 0$ follows from the hypothesis on $\{\alpha_n\}$, (33), (43), (50), and Lemma 4.

Case 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\phi\left(x^*, x_{n_i}\right) < \phi\left(x^*, x_{n_i+1}\right), \tag{51}$$

for all $i \in \mathbb{N}$.

Then, by Lemma 5, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \longrightarrow \infty$:

$$\begin{aligned}\phi\left(x^*, x_{m_k}\right) &\leq \phi\left(x^*, x_{m_k+1}\right) \text{ and } \phi\left(x^*, x_k\right) \\ &\leq \phi\left(x^*, x_{m_k+1}\right), \quad \forall k \geq 1.\end{aligned} \tag{52}$$

Replacing n with m_k in (38), by (52), we have

$$\begin{aligned}&\left(1-\alpha_{m_k}\right)\left(\frac{\left\|\left(P_{Q_{m_k+1}}-I\right)Ax_{m_k}\right\|^4}{2k^2\left\|A^*J_2\left(P_{Q_{m_k+1}}-I\right)Ax_{m_k}\right\|^2}+\beta_{m_k}\left(1-\beta_{m_k}\right)g\left(\left\|J_1z_{m_k}-J_1Sz_{m_k}\right\|\right)\right) \\ &\quad \leq \phi\left(x^*, x_{m_k}\right)-\phi\left(x^*, x_{m_k+1}\right)+\alpha_{m_k}\tau_{m_k}\left\|x^*\right\|^2 \leq \alpha_{m_k}\tau_{m_k}\left\|x^*\right\|^2.\end{aligned} \tag{53}$$

Then, by a similar process with proving (43)–(50), we can obtain that

$$\lim_{k\longrightarrow\infty}\left\|J_1x_{m_k}-J_1y_{m_k}\right\| = 0 \text{ and } \limsup_{n\longrightarrow\infty}\langle x^*-x_{m_k+1}, J_1x^*\rangle \leq 0. \tag{54}$$

Replacing n with m_k in (33), we have

$$\begin{aligned}\phi\left(x^*, x_{m_k+1}\right) \leq{}& \left(1-\alpha_{m_k}\tau_{m_k}\right)\phi\left(x^*, x_{m_k}\right) \\ &+ 2\alpha_{m_k}\tau_{m_k}\langle x^*-x_{m_k+1}, J_1x^* \\ &+\left(1-\alpha_{m_k}\right)\left(J_1y_{m_k}-J_1x_{m_k}\right)\rangle,\end{aligned} \tag{55}$$

from which we obtain

$$\begin{aligned}\alpha_{m_k}\tau_{m_k}\phi\left(x^*, x_{m_k}\right) &\leq \phi\left(x^*, x_{m_k}\right)-\phi\left(x^*, x_{m_k+1}\right)+2\alpha_{m_k}\tau_{m_k}\langle x^*-x_{m_k+1}, J_1x^*+\left(1-\alpha_{m_k}\right)\left(J_1y_{m_k}-J_1x_{m_k}\right)\rangle \\ &\leq 2\alpha_{m_k}\tau_{m_k}\langle x^*-x_{m_k+1}, J_1x^*+\left(1-\alpha_{m_k}\right)\left(J_1y_{m_k}-J_1x_{m_k}\right)\rangle.\end{aligned} \tag{56}$$

Since $\alpha_{m_k}\tau_{m_k} > 0$, by (54) and (56), we have

$$\begin{aligned}\phi\left(x^*, x_{m_k}\right) \leq{}& 2\langle x^*-x_{m_k+1}, J_1x^* \\ &+\left(1-\alpha_{m_k}\right)\left(J_1y_{m_k}-J_1x_{m_k}\right)\rangle \longrightarrow 0.\end{aligned} \tag{57}$$

Furthermore, by (54), (55), and (57), it follows that

$$\lim_{k\longrightarrow\infty}\phi\left(x^*, x_{m_k+1}\right) = 0. \tag{58}$$

However, $\phi(x^*, x_k) \leq \|x_{m_k+1}-x^*\|$ for all $k \geq 1$. So, we conclude that $\phi(x^*, x_k) \longrightarrow 0$ as $k \longrightarrow \infty$ and hence $\|x_k - x^*\| \longrightarrow 0$ as $k \longrightarrow \infty$ by Lemma 1. The proof is complete. □

Remark 1. If $\|(P_{Q_n}-I)Ax_n\| = 0$ for all $n \geq 1$, then $\gamma_n = 0$ and $z_n = x_n$ for all $n \geq 1$. In this case, $Ax_n = P_{Q_n}Ax_n$ and $\phi(Ax_n, w_n) = \phi(Ax_n, TAx_n) \leq \phi(Ax_n, Ax_n) = 0$, which implies that $Ax_n = TAx_n$ for all $n \geq 1$. The iterative scheme (25) becomes

$$\begin{cases} y_n = J_1^{-1}\left(\beta_nJ_1x_n+(1-\beta_n)J_1Sx_n\right), \\ x_{n+1} = J_1^{-1}\left(\alpha_n(1-\tau_n)J_1x_n+(1-\alpha_n)J_1y_n\right), \quad n \geq 1. \end{cases} \tag{59}$$

By the proof process above, we still can see that $\{x_n\}$ converges strongly to $x^* = P_{\mathrm{Fix}(S)}\theta$. Since A is linear and bounded, $Ax_n \longrightarrow Ax^*$, which implies that $Ax_n \rightharpoonup x^*$. Note that $Ax_n = TAx_n$, for all $n \geq 1$, and $Ax_n - TAx_n \longrightarrow 0$ as $n \longrightarrow \infty$. By the hypothesis that $I - T$ is demi-closedness at zero, we get $Ax^* = TAx^*$. Hence, $x^* \in \Omega$. Hence, without loss generality, we assume that $\gamma_n \neq 0$ for all $n \geq 1$ in the proof process.

Algorithm 2. Take $u = x_1 \in E_1$, and define a sequence $\{x_n\}$ by

$$\begin{cases} w_n = TAx_n, \\ Q_n = \{w \in E_2\colon \phi(w, w_n) \leq \phi(w, Ax_n)\}, \\ z_n = J_1^{-1}\left(J_1x_n - \gamma_nA^*J_2\left(I-P_{Q_n}\right)Ax_n\right), \\ y_n = J_1^{-1}\left(\beta_nJ_1z_n+(1-\beta_n)J_1Sz_n\right), \\ x_{n+1} = J_1^{-1}\left(\alpha_nJ_1u+(1-\alpha_n)J_1y_n\right), \quad n \geq 1, \end{cases} \tag{60}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ and

$$\gamma_n = \begin{cases} \dfrac{\left\|\left(P_{Q_n}-I\right)Ax_n\right\|^2}{2k^2\left\|A^*J_2\left(I-P_{Q_n}\right)Ax_n\right\|^2}, & \left\|\left(P_{Q_n}-I\right)Ax_n\right\| \neq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{61}$$

Lemma 12. *$\{x_n\}$ is well-defined and bounded.*

Proof. By a similar proof lines of Lemma 10, we can show that $\{x_n\}$ is well-defined. Now, we prove that $\{x_n\}$ is bounded. By (29)–(31), (60), and Lemma 5, for any $\widehat{x} \in \Omega$, we have

$$\begin{aligned}\phi(\widehat{x}, x_{n+1}) &= \phi\left(\widehat{x}, J_1^{-1}(\alpha_n J_1 u + (1-\alpha_n) J_1 y_n)\right) = \|\widehat{x}\|^2 - 2\alpha_n \langle \widehat{x}, J_1 u\rangle - 2(1-\alpha_n)\langle \widehat{x}, J_1 y_n\rangle \\ &\quad + \left\|\alpha_n J_1 u + (1-\alpha_n) J_1 y_n\right\|^2 \leq \|\widehat{x}\|^2 - 2\alpha_n \langle \widehat{x}, J_1 u\rangle - 2(1-\alpha_n)\langle \widehat{x}, J_1 y_n\rangle \\ &\quad + \alpha_n \|u\|^2 + (1-\alpha_n)\left\|y_n\right\|^2 = \alpha_n \phi(\widehat{x}, u) + (1-\alpha_n)\phi(\widehat{x}, y_n) \leq \alpha_n \phi(\widehat{x}, u) + (1-\alpha_n) \\ &\quad \cdot \left(\phi(\widehat{x}, x_n) - \frac{\left\|\left(P_{Q_n} - I\right)Ax_n\right\|^4}{2k^2\left\|A^* J_2\left(P_{Q_n} - I\right)Ax_n\right\|^2} - \beta_n(1-\beta_n) g\left(\left\|J_1 z_n - J_1 S z_n\right\|\right)\right) \\ &\leq \alpha_n \phi(\widehat{x}, u) + (1-\alpha_n)\phi(\widehat{x}, x_n) \leq \alpha_n \phi(\widehat{x}, u) + \phi(\widehat{x}, x_n), \quad n \geq 1.\end{aligned} \tag{62}$$

By the hypothesis on $\{\alpha_n\}$ and Lemma 9, it follows that the limit of $\{\phi(\widehat{x}, x_n)\}$ exists. Hence, $\{x_n\}$ is bounded. □

Theorem 2. *Assume that S and T are closed. If the interior of Ω is nonempty and $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following conditions*

$$\sum_{n=1}^{\infty} \alpha_n < \infty \text{ and } \liminf_{n\longrightarrow\infty} \beta_n(1-\beta_n) > 0, \tag{63}$$

then $\{x_n\}$ generated by Algorithm 2 converges strongly to the element $x^* = lim_{n\longrightarrow\infty}\Pi_\Omega x_n$.

Proof. We first show that $\{x_n\}$ is a Cauchy sequence and hence converges strongly to some point $x^* \in E_1$. Since the interior of Ω is nonempty, there exist $p \in \Omega$ and $r > 0$ such that

$$p + rh \in \Omega, \tag{64}$$

whenever $\|h\| \leq 1$. By (10), we have

$$\begin{aligned}\phi(p, x_n) &= \phi(x_{n+1}, x_n) + \phi(p, x_{n+1}) + 2\langle x_{n+1} - p, J_1 x_n - J_1 x_{n+1}\rangle \\ &= \phi(x_{n+1}, x_n) + \phi(p, x_{n+1}) + 2\langle x_{n+1} - (p + rh), J_1 x_n - J_1 x_{n+1}\rangle + 2r\langle h, J_1 x_n - J_1 x_{n+1}\rangle.\end{aligned} \tag{65}$$

On the contrary, by (10), again we have

$$\begin{aligned}\phi(p + rh, x_n) &= \phi(x_{n+1}, x_n) + \phi(p + rh, x_{n+1}) \\ &\quad + 2\langle x_{n+1} - (p + rh), J_1 x_n - J_1 x_{n+1}\rangle.\end{aligned} \tag{66}$$

Combining (65) with (66), we obtain

$$\begin{aligned}2r\langle h, J_1 x_n - J_1 x_{n+1}\rangle &= \phi(p, x_n) - (\phi(x_{n+1}, x_n) + \phi(p, x_{n+1}) + 2\langle x_{n+1} - (p + rh), J_1 x_n - J_1 x_{n+1}\rangle) \\ &= \phi(p, x_n) - \phi(x_{n+1}, x_n) - \phi(p, x_{n+1}) - \phi(p + rh, x_n) + \phi(x_{n+1}, x_n) + \phi(p + rh, x_{n+1}) \\ &= \frac{1}{2r}(\phi(p, x_n) - \phi(p, x_{n+1})) + \phi(p + rh, x_{n+1}) - \phi(p + rh, x_n).\end{aligned} \tag{67}$$

Since $p + rh \in \Omega$, from (62) and (67), it follows that

$$2r\langle h, J_1x_n - J_1x_{n+1}\rangle \le \phi(p, x_n) - \phi(p, x_{n+1}) + \alpha_n(\phi(p+rh, u) - \phi(p+rh, x_n)) \\ \le \phi(p, x_n) - \phi(p, x_{n+1}) + \alpha_n\phi(p+rh, u). \tag{68}$$

Since h with $\|h\| \le 1$ is arbitrary, we have

$$\|J_1x_n - J_1x_{n+1}\| \le \frac{1}{2r}(\phi(p, x_n) - \phi(p, x_{n+1}) \\ + \alpha_n\phi(p+rh, u)). \tag{69}$$

So, for all $m > n$,

$$\|J_1x_n - J_1x_m\| = \|J_1x_n - J_1x_{n+1} + J_1x_{n+1} - \cdots - J_1x_{m-1} + J_1x_{m-1} + J_1x_m\| \\ \le \sum_{i=n}^{m-1}\|J_1x_i - J_1x_{i+1}\| \le \frac{1}{2r}\sum_{i=n}^{m-1}(\phi(p, x_i) - \phi(p, x_{i+1}) + \alpha_i\phi(p+rh, u)) \\ = \frac{1}{2r}\sum_{i=n}^{m-1}(\phi(p, x_i) - \phi(p, x_{i+1})) + \frac{\phi(p+rh, u)}{2r}\sum_{i=n}^{m-1}\alpha_i = \phi(p, x_n) - \phi(p, x_m) + \frac{\phi(p+rh, u)}{2r}\sum_{i=n}^{m-1}\alpha_i. \tag{70}$$

Since the limit of $\{\phi(p, x_n)\}$ exists and $\sum_{n=1}^{\infty}\alpha_n < \infty$, from (70), we see

$$\lim_{m,n\longrightarrow\infty}\|J_1x_n - J_1x_m\| = 0, \tag{71}$$

which implies that $\{J_1x_n\}$ is a Cauchy sequence in E_1^*. Hence, $\{J_1x_n\}$ converges strongly to some point in E_1^*. Since E_1^* has a Fréchet differentiable norm, then J_1^{-1} is continuous on E_1^*. Hence, x_n converges strongly to some point x^* in E_1.

For any $\hat{x} \in \Omega$, by (62), we have

$$(1-\alpha_n)\left(\frac{\|(P_{Q_n} - I)Ax_n\|^4}{2k^2\|A^*J_2(P_{Q_n} - I)Ax_n\|^2} + \beta_n(1-\beta_n)g(\|J_1z_n - J_1Sz_n\|)\right) \\ \le \alpha_n\phi(\hat{x}, u) + (1-\alpha_n)\phi(\hat{x}, x_n) - \phi(\hat{x}, x_{n+1}) \le \alpha_n\phi(\hat{x}, u) + \phi(\hat{x}, x_n) - \phi(\hat{x}, x_{n+1}). \tag{72}$$

Since the limit of $\{\phi(\hat{x}, x_n)\}$ exists, by the hypothesis on $\{\alpha_n\}$ and $\{\beta_n\}$, it follows that

$$\lim_{n\longrightarrow\infty}\frac{\|(P_{Q_n} - I)Ax_n\|^4}{\|A^*J_2(P_{Q_n} - I)Ax_n\|^2} = \lim_{n\longrightarrow\infty}g(\|J_1z_n - J_1Sz_n\|) = 0, \tag{73}$$

which implies that

$$\lim_{n\longrightarrow\infty}\|(P_{Q_n} - I)Ax_n\| = 0 \text{ and } \lim_{n\longrightarrow\infty}\|J_1z_n - J_1Sz_n\| = 0, \tag{74}$$

and hence

$$\|z_n - Sz_n\| \longrightarrow 0. \tag{75}$$

On the contrary, by (60) and (73), we have

$$\|J_1z_n - J_1x_n\| = \gamma_n\|A^*J_2(I - P_{Q_n})Ax_n\| \\ = \frac{\|(I - P_{Q_n})Ax_n\|^2}{\|A^*J_2(I - P_{Q_n})Ax_n\|} \longrightarrow 0. \tag{76}$$

It follows that

$$\|z_n - x_n\| \longrightarrow 0. \tag{77}$$

Hence, $\{z_n\}$ converges strongly to $x^* \in E_1$. Since S is closed, by (75), we get $x^* = Sx^*$.

Now, we show that $Ax^* = TAx^*$. From (49), it follows that $\|Ax_n - TAx_n\| \longrightarrow 0$. Since A is linear bounded, $Ax_n \longrightarrow Ax^*$. From the closedness of T, we get $Ax^* = TAx^*$. Therefore, $x^* \in \Omega$. Finally, we show that $x^* = \lim_{n\longrightarrow\infty}\Pi_\Omega x_n$. In fact, since $x^* \in \Omega$, by Lemma 2, we have

$$\phi(x^*, \Pi_\Omega x_n) \le \phi(x^*, x_n) \longrightarrow 0. \tag{78}$$

It follows that $x^* = \lim_{n\longrightarrow\infty}\Pi_{\Omega}x_n$. The proof is complete.

Let Q be a nonempty closed convex subset of E_2. In Algorithms 1 and 2, if putting $T = I$ and $Q_1 = Q$, we have $w_n = Ax_n$ and $Q_n = Q$ for all $n \geq 1$. Then, we have the following results. □

Corollary 1. *Let E_1 be a 2-uniformly convex and 2-uniformly smooth real Banach space with the best smoothness constant $k > 0$ and E_2 be a uniformly smooth, strictly convex, and reflexive Banach space with a nonempty closed convex subset $Q \subset E_2$. Let A: $E_1 \longrightarrow E_2$ be a linear bounded operator with adjoint A^*. Let S: $E_1 \longrightarrow E_1$ and $Q \subset E_2$ be a nonempty subset. Assume that $I - S$ is demi-closedness at zero and $\Gamma \neq \varnothing$, where $\Gamma = \{x \in E_1\colon x \in Fix(S),\ Ax \in Q\}$. Let $x_1 \in E_1$ and define a sequence $\{x_n\}$ by*

$$\begin{cases} z_n = J_1^{-1}\left(J_1x_n - \gamma_n A^* J_2\left(I - P_Q\right)Ax_n\right), \\ y_n = J_1^{-1}\left(\beta_n J_1 z_n + (1-\beta_n)J_1 S z_n\right), \\ x_{n+1} = J_1^{-1}\left(\alpha_n(1-\tau_n)J_1x_n + (1-\alpha_n)J_1y_n\right), \quad n \geq 1, \end{cases} \tag{79}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0,1), \{\tau_n\} \subset (\tau, 1)$ with $\tau \in (0,1)$ and

$$\gamma_n = \begin{cases} \dfrac{\left\|\left(P_{Q_n} - I\right)Ax_n\right\|^2}{2k^2\left\|A^*J_2\left(I - P_{Q_n}\right)Ax_n\right\|^2}, & \left\|\left(P_{Q_n} - I\right)Ax_n\right\| \neq 0, \\ 0, & \text{otherwise}. \end{cases} \tag{80}$$

If the following conditions hold,

$$\lim_{n\longrightarrow\infty}\alpha_n = 0, \qquad \sum_{n=1}^{\infty}\alpha_n = \infty \text{ and } \liminf_{n\longrightarrow\infty}\beta_n\left(1-\beta_n\right) > 0, \tag{81}$$

then the sequence $\{x_n\}$ generated by (60) converges strongly to the element $x^* = \Pi_{\Gamma}\theta$, where θ is the zero element in E_1.

Corollary 2. *Let E_1 be a 2-uniformly convex and 2-uniformly smooth real Banach space with the best smoothness constant $k > 0$ and E_2 be a uniformly smooth, strictly convex, and reflexive Banach space with a nonempty closed convex subset $Q \subset E_2$. Let A: $E_1 \longrightarrow E_2$ be a linear bounded operator with adjoint A^*. Let S: $E_1 \longrightarrow E_1$ and $Q \subset E_2$ be a nonempty subset. Assume that S is closed and the interior of Γ is nonempty, where $\Gamma = \{x \in E_1\colon x \in Fix(S),\ Ax \in Q\}$. Let $u = x_1 \in E_1$ and define a sequence $\{x_n\}$ by*

$$\begin{cases} z_n = J_1^{-1}\left(J_1x_n - \gamma_n A^* J_2\left(I - P_Q\right)Ax_n\right), \\ y_n = J_1^{-1}\left(\beta_n J_1 z_n + (1-\beta_n)J_1 S z_n\right), \\ x_{n+1} = J_1^{-1}\left(\alpha_n J_1 u + (1-\alpha_n)J_1y_n\right),\ n \geq 1, \end{cases} \tag{82}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ and $\gamma_n = \{\|(P_{Q_n} - I)Ax_n\|^2/\ 2k^2\|A^* J_2(I - P_{Q_n})Ax_n\|^2, \|(P_{Q_n} - I)Ax_n\| \neq 0, 0, otherwise.$

If the following conditions hold

$$\lim_{n\longrightarrow\infty}\alpha_n = 0, \qquad \sum_{n=1}^{\infty}\alpha_n < \infty \text{ and } \liminf_{n\longrightarrow\infty}\beta_n\left(1-\beta_n\right) > 0, \tag{83}$$

then the sequence $\{x_n\}$ generated by (82) converges strongly to some element $x^* = lim_{n\longrightarrow\infty}\Pi_{\Gamma}x_n$.

4. Application

Let E_1 and E_2 be two Banach spaces and f_1: $E_1 \times E_1 \longrightarrow \mathbb{R}$ and f_2: $E_2 \times E_2 \longrightarrow \mathbb{R}$ be the bifunctions. Let A: $E_1 \longrightarrow E_2$ be a linear bounded operator. In this section, we consider a split equilibrium problem: find a point $x^* \in E_1$ such that

$$x^* \in \mathrm{EP}\left(f_1\right) \text{ and } Ax^* \in \mathrm{EP}\left(f_2\right), \tag{84}$$

where $\mathrm{EP}(f_1) = \{x \in E_1\colon f_1(x,y) \geq 0,\ \forall y \in E_1\}$ and $\mathrm{EP}(f_2) = \{u \in E_2\colon f_2(u,v) \geq 0,\ \forall v \in E_2\}$. We denote the set of solution of problem (84) by Λ. That is, $\Lambda = \{x \in \mathrm{EP}(f_1)\colon Ax \in \mathrm{EP}(f_2)\}$.

The split equilibrium problem has been studied by many authors in Hilbert space, see [37–41]. However, few results on the split equilibrium problem in Banach space is reported by far.

Lemma 13 (see [24]). *Let E be a strictly convex, reflexive, and uniform smooth Banach space and f: $E \times E \longrightarrow \mathbb{R}$ be a bifunction satisfying the following conditions:*

(A1) $f(x,x) = 0$ for all $x \in E$.

(A2) f is monotone, i.e., $f(x,y) + f(y,x) \leq 0$ for all $x, y \in E$.

(A3) For all $x, y, z \in E$,

$$\limsup_{t\longrightarrow 0^+} f\left(tz + (1-t)x, y\right) \leq f(x,y). \tag{85}$$

(A4) For all $x \in E$, $f(x,\cdot)$ is convex and lower semicontinuous.

For $r > 0$ and $x \in E$, define a mapping T_r: $E \longrightarrow E$ as follows:

$$\mathrm{T}_r^f x = \left\{z \in E\colon f(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0 \text{ for all } y \in E\right\}, \tag{86}$$

for all $x \in E$. Then, the following hold:

(1) T_r^f is single-valued

(2) $Fix(\mathrm{T}_r^f) = EP(f)$

(3) $EP(f)$ is closed and convex

(4) $\phi(q, T_r^f x) + \phi(T_r^f x, x) \leq \phi(q, x)$ for all $x \in E$ and $q \in EP(f)$, which shows that T_r^f is a quasi-ϕ-nonexpansive mapping

Now, we show that the mapping $I - T_r^f$ is demi-closedness at zero on a bounded subset of E.

Lemma 14. *Let E be a strictly convex, reflexive, and uniform smooth Banach space and $f: E \times E \longrightarrow \mathbb{R}$ be a bifunction satisfying conditions (A1)–(A4). Let $r > 0$ and define the mapping T_r^f as (86). Assume that $EP(f) \neq \varnothing$. Then, $I - T_r^f$ is demi-closedness at zero on a bounded set. That is, if $\{x_n\} \subset E$ is bounded and weakly converges to $x \in E$ and $\|x_n - T_r^f x_n\| \longrightarrow 0$ as $n \longrightarrow \infty$, then $x = T_r^f x$.*

Proof. et $\{x_n\} \subset E$ be bounded and converges weakly to $x \in E$ and $\|x_n - T_r^f x_n\| \longrightarrow 0$ as $n \longrightarrow \infty$. For each $x^* \in \mathrm{EP}(f) = \mathrm{Fix}(T_r^f)$, since T_r^f is quasi-ϕ-nonexpansive, we have

$$\phi\left(x^*, T_r^f x_n\right) \leq \phi\left(x^*, x_n\right), \quad n \geq 1, \tag{87}$$

which implies that $\left\{T_r^f x_n\right\}$ is bounded. On the contrary, since J is uniformly norm-to-norm continuous on bounded sets, it follows that

$$\lim_{n \longrightarrow \infty} \left\| J T_r^f x_n - J x_n \right\| = 0. \tag{88}$$

By (A2), we have

$$\frac{1}{r}\langle y - T_r^f x_n, J T_r^f x_n - J x_n \rangle \geq -f\left(T_r^f x_n, y\right) \geq f\left(y, T_r^f x_n\right), \quad \forall y \in E. \tag{89}$$

Letting $n \longrightarrow 0$ in (89), by (A4) and (88), we obtain

$$f(y, x) \leq 0, \quad \forall y \in E. \tag{90}$$

For $0 < t \leq 1$ and $y \in E$, let $y_t = ty + (1-t)x$. Note that (90) implies that $f(y_t, x) \leq 0$. By (A1), we have

$$0 = f(y_t, y_t) \leq t f(y_t, y) + (1-t) f(y_t, x) \leq t f(y_t, y). \tag{91}$$

Dividing by t, we obtain

$$f(y_t, y) \geq 0, \quad \forall y \in E. \tag{92}$$

Let $t \longrightarrow 0^+$, by (A3), we have

$$f(x, y) \geq 0, \quad \forall y \in E. \tag{93}$$

It follows that $x \in \mathrm{EP}(f)$. That is, $x = T_r^f x$ by Lemma 13. This completes the proof.

Based on the results in Section 3, we give the following conclusion directly. □

Theorem 3. *Let E_1 be a 2-uniformly convex and 2-uniformly smooth real Banach space with the best smoothness constant $k > 0$ and E_2 be a uniformly smooth, strictly convex, and reflexive Banach space. Let $A: E_1 \longrightarrow E_2$ be a linear bounded operator with adjoint A^*. Let $f_1: E_1 \times E_1 \longrightarrow \mathbb{R}$ and $f_2: E_2 \times E_2 \longrightarrow \mathbb{R}$ be the bifunctions satisfying conditions (A1)–(A4). Assume that $\Lambda \neq \varnothing$, where $\Lambda = \{x \in E_1: x \in EP(f_1), Ax \in EP(f_2)\}$. Let $r > 0$. Take $x_1 \in E_1$ and put $Q_1 = E_2$. Define a sequence $\{x_n\}$ by*

$$\begin{cases} w_n = T_r^{f_2} A x_n, \\ Q_n = \{w \in Q_n: \phi(w, w_n) \leq \phi(w, A x_n)\}, \\ z_n = J^{-1}\left(J_1 x_n + \gamma_n A^* J_2\left(P_{Q_n} - I\right) A x_n\right), \\ y_n = J^{-1}\left[(1-\beta_n) J_1 z_n + (1-\beta_n) J_1 T_r^{f_1} z_n\right], \\ x_{n+1} = J_1^{-1}\left(\alpha_n (1-\tau_n) J_1 x_n + (1-\alpha_n) J_1 y_n\right), \quad n \geq 1, \end{cases} \tag{94}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0,1), \{\tau_n\} \subset (\tau, 1)$ with $\tau \in (0,1)$ and

$$\gamma_n = \begin{cases} \dfrac{\left\|\left(P_{Q_n} - I\right) A x_n\right\|^2}{2k^2 \left\|A^* J_2\left(I - P_{Q_n}\right) A x_n\right\|^2}, & \left\|\left(P_{Q_n} - I\right) A x_n\right\| \neq 0, \\ 0, & \text{else}. \end{cases} \tag{95}$$

If the following conditions hold

$$\lim_{n \longrightarrow \infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \text{ and } \liminf_{n \longrightarrow \infty} \beta_n (1-\beta_n) > 0, \tag{96}$$

then the sequence $\{x_n\}$ generated by (94) converges strongly to the element $x^* = \Pi_\Lambda \theta$, where θ is the zero element in E_1.

Theorem 4. *Let E_1 be a 2-uniformly convex and 2-uniformly smooth real Banach space with the best smoothness constant $k > 0$ and E_2 be a uniformly smooth, strictly convex, and reflexive Banach space. Let $A: E_1 \longrightarrow E_2$ be a linear bounded operator with adjoint A^*. Let $f_1: E_1 \times E_1 \longrightarrow \mathbb{R}$ and $f_2: E_2 \times E_2 \longrightarrow \mathbb{R}$ be the bifunctions satisfying conditions (A1)–(A4). Assume that the interior of Λ is nonempty, where $\Lambda = \{x \in E_1: x \in EP(f_1), Ax \in EP(f_2)\}$. Let $r > 0$. Take $u, x_1 \in E_1$ and put $Q_1 = E_2$. Define a sequence $\{x_n\}$ by*

$$\begin{cases} w_n = T_r^{f_2} A x_n, \\ Q_n = \{w \in Q_n: \phi(w, w_n) \leq \phi(w, A x_n)\}, \\ z_n = J^{-1}\left(J_1 x_n + \gamma_n A^* J_2\left(P_{Q_n} - I\right) A x_n\right), \\ y_n = J^{-1}\left[(1-\beta_n) J_1 z_n + (1-\beta_n) J_1 T_r^{f_1} z_n\right], \\ x_{n+1} = J_1^{-1}\left(\alpha_n J_1 u + (1-\alpha_n) J_1 y_n\right), \quad n \geq 1, \end{cases} \tag{97}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ and

$$\gamma_n = \begin{cases} \dfrac{\left\|\left(P_{Q_n} - I\right)Ax_n\right\|^2}{2k^2\left\|A^* J_2\left(I - P_{Q_n}\right)Ax_n\right\|^2}, & \left\|\left(P_{Q_n} - I\right)Ax_n\right\| \neq 0, \\ 0, & \text{else.} \end{cases} \tag{98}$$

If the following conditions hold

$$\lim_{n\longrightarrow\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n < \infty \text{ and } \liminf_{n\longrightarrow\infty} \beta_n\left(1-\beta_n\right) > 0, \tag{99}$$

then the sequence $\{x_n\}$ generated by (97) converges strongly to the element $x^* = \lim_{n\longrightarrow\infty}\Pi_\Lambda x_n$.

5. Numerical Examples

In this section, we give the following examples to illustrate the effectiveness of Algorithms 1 and 2. The program is performed by Matlab R2016b running on a PC Desktop with Core(TM) i5CPU M550 3.20 GHz with 4 GB Ram.

We first show the convergence of Algorithm 1 by the following example which has been used by Ma et al. [22]. In [22], the authors compare the computed results using their algorithm (25) with algorithm (100) in Kraikaew and Saejung [10] by the example. Here, we also compare the convergence of our Algorithm 1 with algorithm (25) in [22] and algorithm (100) in [10].

Example 1. Let $E_1 = \mathbb{R}$, $E_2 = \mathbb{R}^2$, $Q = [0,\infty] \times (-\infty, 0)$, $Sx = (x/4)$, for all $x \in E_1$, $Tx = P_Q x$ for all $x \in E_2$, where P_Q is the metric projection from E_2 onto Q, and $A\colon E_1 \longrightarrow E_2$ be a mapping defined by $Ax = (x/2, x/3)$ for all $x \in E_1$. Then, $A^*(u,v) = (u/2) + (v/3)$, for all $(u,v) \in E_2$. It is easy to see that $\Omega = \{x \in E_1\colon x \in \mathrm{Fix}(S), Ax \in \mathrm{Fix}(T)\} = \{0\}$.

Algorithm 3. Let $\{x_n\}$ be the sequence generated by (25) in this paper with $\alpha_n = 1/2n$ and $\beta_n = \tau_n = 6/7$. Then, scheme (25) can be simplified as

$$\begin{cases} x_1 \in E_1, \\ w_n = P_Q\left(\dfrac{x_n}{2}, \dfrac{x_n}{3}\right), \\ Q_n = \left\{w \in E_2\colon \left\|w_n - w\right\| \leq \left\|\left(\dfrac{x_n}{2}, \dfrac{x_n}{3}\right) - w\right\|\right\}, \\ Ax_n = \left(\dfrac{x_n}{2}, \dfrac{x_n}{3}\right), z_n = x_n + \gamma_n A^*\left(P_{Q_n} - I\right)Ax_n, \\ y_n = \dfrac{6}{7}z_n + \dfrac{1}{28}z_n, \\ x_{n+1} = \dfrac{1}{14n}x_n + \dfrac{2n-1}{2n}y_n, \quad n \geq 1, \end{cases} \tag{100}$$

where

$$\gamma_n = \begin{cases} \dfrac{\left\|\left(P_{Q_n} - I\right)Ax_n\right\|^2}{2\left\|A^*\left(I - P_{Q_n}\right)Ax_n\right\|^2}, & \text{if } \left\|\left(P_{Q_n} - I\right)Ax_n\right\| \neq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{101}$$

Algorithm 4. Let $\{x_n\}$ be the sequence generated by algorithm (100) in [10] with $\alpha_n = 1/2n$ and $\gamma = 1$. Then, scheme (100) in [10] can be simplified as

$$x_1 \in E_1, x_{n+1} = \frac{1}{2n}x_1 + \frac{2n-1}{8n}\left(x_n + A^*(T-I)Ax_n\right), \quad n \geq 1. \tag{102}$$

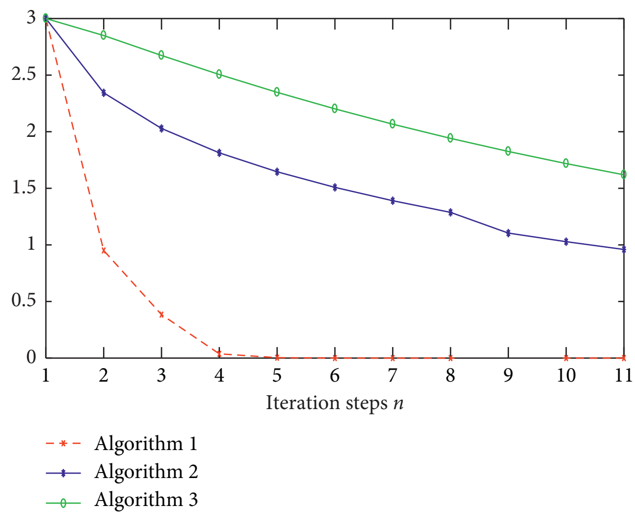

Figure 1: Convergence for Algorithms 3–5 with different initial points $x_1 = 3$.

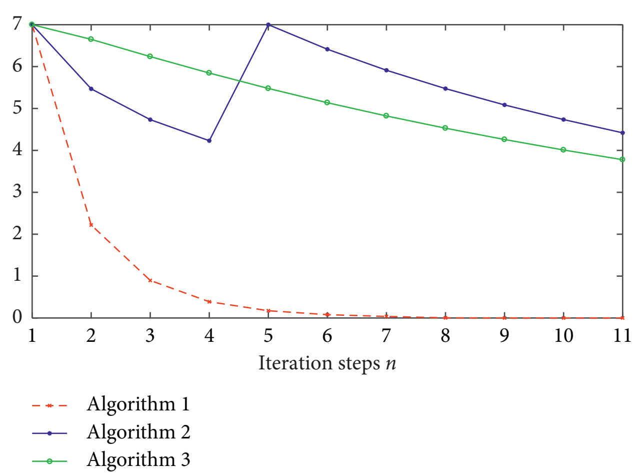

Figure 2: Convergence for Algorithms 3–5 with different initial points $x_1 = 7$.

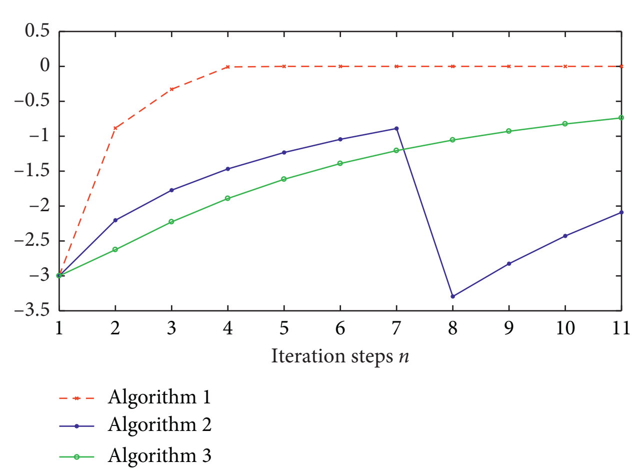

Figure 3: Convergence for Algorithms 3–5 with different initial points $x_1 = -3$.

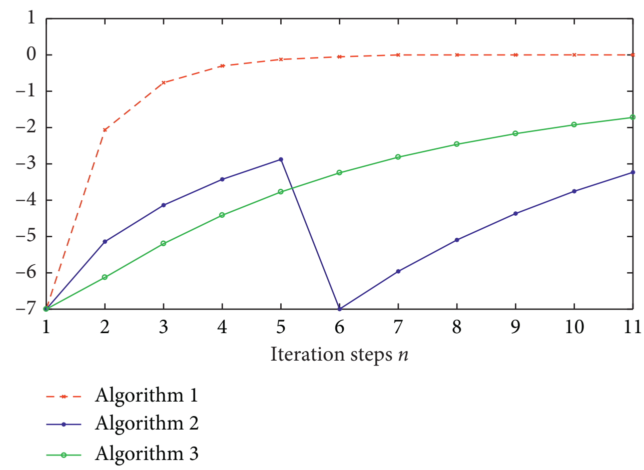

Figure 4: Convergence for Algorithms 3–5 with different initial points $x_1 = -7$.

Algorithm 5. Let $\{x_n\}$ be the sequence generated by algorithm (25) in [22] with $\alpha_n = 1/2n$ and $\gamma = 1$. Then, scheme (25) in [22] can be simplified as

$$\begin{cases} x_1 \in E_1, \\ Ax_n = \left(\dfrac{x_n}{2}, \dfrac{x_n}{3}\right), \\ z_n = x_n + A^* (T - I)Ax_n, \\ y_n = \dfrac{2n-1}{2n} z_n + \dfrac{1}{8n} z_n, \\ C_{n+1} = \left\{v_n \colon C_n \colon \left|y_n - v\right| \le \left|x_n - v\right|, \left|z_n - v\right| \le \left|x_n - v\right|\right\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad n \ge 1. \end{cases} \tag{103}$$

We perform schemes (100)–(103) with the different initial points. Figures 1–4 show that the sequence $\{x_n\}$ generated by (100)–(103) converge to 0.

Remark 2. (a) Although Theorem 1 in [22] requires that $\{\alpha_n\}$ in Algorithm 5, i.e., algorithm (25) in [22], takes values in $[\delta, 1)$ with $\delta \in (0, 1)$; here, for comparing the convergence rate of three schemes, we put the same $\alpha_n = 1/2n$. This does not affect the effectiveness of Algorithm 5 since the program stops in finite iterations. (b) Figures 1–4 above show that the convergence rate of Algorithm 3 is faster than that of Algorithms 4 and 5.

Next, we illustrate Theorem 2 by the following example.

Example 2. Let $E_1 = \mathbb{R}^2$ and $E_2 = \mathbb{R}$. Define the mappings $S\colon E_1 \longrightarrow E_1$ by $Sx = ((x_1/2), x_2)$ for all $x = (x_1, x_2) \in E_1$, and $T\colon E_2 \longrightarrow E_2$ by $Tx = x/2$ if $|x| \le 1$ and $Tx = 1$ if $|x| > 1$.

Table 1: Convergence for Algorithm 2 with initial point $x_1 = (3, 6)$.

Iteration steps	x_n
1	$(-2, -5)$
2	$(-1.48705, -5.00000)$
3	$(-1.13985, -4.99999)$
4	$(-0.89031, -4.99999)$
5	$(-0.70347, -4.99999)$
⋮	⋮
100	$(-0.00085, -4.99999)$
261	$(-0.00012, -4.99998)$
262	$(-0.00012, -4.99998)$
263	$(-0.00011, -4.99998)$
264	$(-0.00011, -4.99998)$
265	$(-0.00011, -4.99998)$
⋮	⋮
286	$(-0.00009, -4.99998)$

Table 2: Convergence for Algorithm 2 with initial point $x_1 = (3, 6)$.

Iteration steps	x_n
1	$(3, 6)$
2	$(2.23056, 0.99999)$
3	$(1.70977, 1.55555)$
4	$(1.33545, 1.83331)$
⋮	⋮
100	$(0.00128, 1.99985)$
101	$(0.00125, 2.00024)$
102	$(0.00122, 2.00022)$
300	$(0.00014, 1.99997)$
301	$(0.00013, 2.00002)$
302	$(0.00013, 2.00003)$
303	$(0.00013, 1.99997)$
⋮	⋮
349	$(0.00010, 2.00002)$
350	$(0.00010, 2.00000)$
351	$(0.00009, 2.00003)$

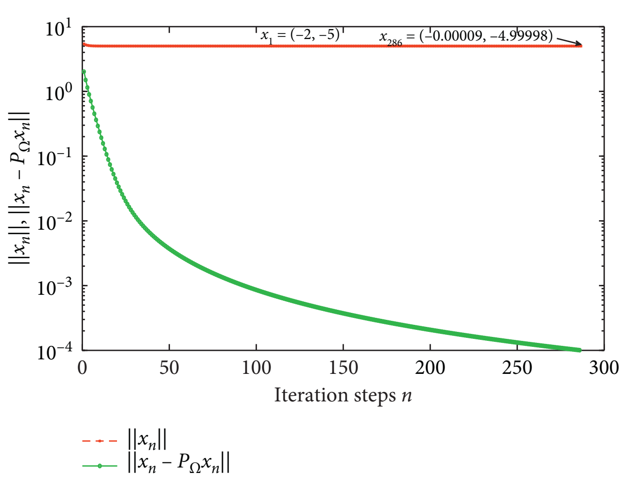

Figure 5: Convergence for Algorithm 2 with different initial points $x_1 = (-2, -2)$.

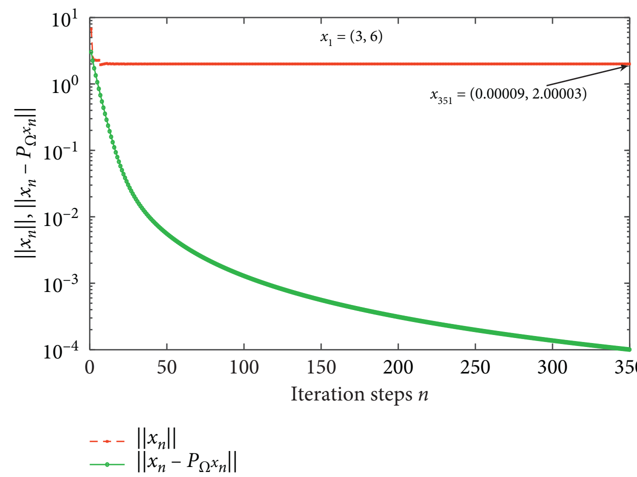

Figure 6: Convergence for Algorithm 2 with different initial points $x_1 = (3, 6)$.

Let $A: E_1 \longrightarrow E_2$ be a mapping defined by $Ax = x_1$ for all $x = (x_1, x_2) \in E_1$. Then, A is linear and bounded and $A^* y = (y, 0)$ for all $y \in E_2$. It is easy to see that $\Omega = \{(0, x_2): x_2 \in \mathbb{R}\}$. All the conditions on S, T, and Ω are satisfied for Theorem 2.

By Algorithm 2, we generate a sequence $\{x_n\}$ with $\alpha_n = 1/n^2$ and $\beta_n = 1/2(1 - e^{(n/2)})$ for all $n \geq 1$. Theorem 2 shows that $\{x_n\}$ will converge to the point $P_\Omega x_n$. We will stop the program when $\|x_n - P_\Omega x_n\| < 10^{-4}$. The computed results of the sequence $\{x_n\}$ are given in Tables 1 and 2. Figures 5 and 6 show the convergence of the sequence $\{x_n\}$.

6. Conclusion

For finding a solution of the split common fixed problem of quasi-ϕ-nonexpansive mappings in Banach space, we introduced a Halpern algorithm and a nonconvex combination algorithm where the norm of the linear bounded operator does not need to be known in advance. The convergence of the algorithms was investigated and some numerical examples were given to illustrate the convergence of the algorithms.

Data Availability

All data for our algorithms are included in this paper.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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