

Research Article

On the Change of Distance Energy of Complete Bipartite Graph due to Edge Deletion

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The distance energy of a graph is defined as the sum of absolute values of distance eigenvalues of the graph. The distance energy of a graph plays an important role in many fields. By constructing a new polynomial, we transform a problem on the sum of the absolute values of the roots of a quadratic polynomial into a problem on the largest root of a cubic polynomial. Hence, we give a new and shorter proof on the change of distance energy of a complete bipartite graph due to edge deletion, which was given by Varghese et al.

1. Introduction

Let $G = (V(G), E(G))$ be a simple connected graph, whose vertex set is $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set is $E(G)$. For any pair of vertices (v_i, v_j) , the distance $d(v_i, v_j)$ is defined as the length of the shortest path between v_i and v_j . In particular, $d(v_i, v_i) = 0$ for any $v_i \in V(G)$. The distance matrix of G , denoted by $D(G)$, is the $n \times n$ matrix with (i, j) -entry being $d(v_i, v_j)$. The distance energy $E_D(G)$ of G is defined by the sum of absolute values of distance eigenvalues of G . The distance energy of G has been studied for several years in the literature, see [1–5] and the references therein.

It is not easy to compute the distance energy of general graphs. Thus, we are interested in finding the distance energy of some special graphs. For example, it has been proved in [6–8] that

$$E_D(K_{n_1, n_2, \dots, n_k}) = 2 \left(\sum_{i=1}^k n_i - k \right), \quad (1)$$

where K_{n_1, n_2, \dots, n_k} is a complete k -partite graph with the size of each partition being at least 2. The distance energy of some graphs with diameter 2 is also determined in [9]. Another interesting problem is to study how the distance energy of a

graph changes by deleting an edge. Zhou and Ilić [10] showed that the deletion of any edge increases the distance energy of a connected graph with unique positive distance eigenvalue whenever the resulting graph is still connected. Varghese et al. [11] proved that $E_D(K_{p,q}) < E_D(K_{p,q} - e)$ for any edge $e \in E(K_{p,q})$, where $K_{p,q}$ is a complete bipartite graph with $p, q \geq 2$. Recently, Tian et al. [12] showed that the deletion of any edge increases the distance energy of some special complete multipartite graphs.

In this paper, we will give a new and shorter proof of the main result in [11] by transforming a problem on the sum of the absolute values of the roots of a quadratic polynomial into a problem on the largest root of a cubic polynomial.

2. Main Result

In this section, we will give a new and shorter proof on $p, q \geq 2$, $E_D(K_{p,q}) < E_D(K_{p,q} - e)$, which has been proved in [11]. Before giving our main proof, we need a result on distance eigenvalues of a graph.

Lemma 1 (see [13]). *Let G be a graph of order n . If there exist k vertices having the same neighborhood, then G has distance eigenvalue -2 with multiplicity at least $k - 1$.*

Next, we introduce a concept about the equitable quotient matrix. Let M be a symmetric matrix of order n whose block form is as follows:

$$M = \begin{bmatrix} M_{11} & \cdots & M_{1t} \\ \vdots & \ddots & \vdots \\ M_{t1} & \cdots & M_{tt} \end{bmatrix}, \tag{2}$$

where the blocks M_{ij} are $n_i \times n_j$ matrices for any $1 \leq i, j \leq t$ and $n = n_1 + \cdots + n_t$. Let b_{ij} denote the sum of all entries in M_{ij} divided by the number of rows. Then, $B(M) = (b_{ij})$ is called the quotient matrix of M . If M_{ij} has a constant row sum for each pair (i, j) , then B is called the equitable quotient matrix of M . Haemers [14] obtained the relation between the spectra of M and $B(M)$ as follows.

Lemma 2 (see [14]). *Let B be the equitable quotient matrix of M as defined above. Then, any eigenvalue of B is a eigenvalue of M .*

Now, we are ready to give the main result of this paper.

Theorem 1. *For integers $p, q \geq 2$ and $n = p + q$,*

$$E_D(K_{p,q}) < E_D(K_{p,q} - e), \tag{3}$$

where e is any edge of $K_{p,q}$.

Proof. If $n = 4$ or 5 , one can easily confirm that $E_D(K_{p,q}) < E_D(K_{p,q} - e)$ holds; otherwise, $n \geq 6$. From the structure of the graph $K_{p,q} - e$, by Lemma 1, we found that the distance eigenvalues of $K_{p,q} - e$ are -2 with multiplicity $n - 4$. Note that, after relabeling the matrix $D(K_{p,q} - e)$, we rewrite $D(K_{p,q} - e)$ as a 4×4 block matrix in the following form:

$$M = \begin{bmatrix} 0 & 3 & 2J_{1,p-1} & J_{1,q-1} \\ 3 & 0 & J_{1,p-1} & 2J_{1,q-1} \\ 2J_{p-1,1} & J_{p-1,1} & 2J_{p-1,p-1} - 2I_{p-1} & J_{p-1,q-1} \\ J_{q-1,1} & 2J_{q-1,1} & J_{q-1,p-1} & 2J_{q-1,q-1} - 2I_{q-1} \end{bmatrix}, \tag{4}$$

where $J_{i \times j}$ be a $i \times j$ matrix with all entries being 1. It is clear that each block of M has a constant row sum. Thus, the equitable quotient matrix of M is

$$B = \begin{bmatrix} 0 & 3 & 2(p-2) & q-1 \\ 3 & 0 & p-1 & 2(q-1) \\ 2 & 1 & 2(p-2) & q-1 \\ 1 & 2 & p-1 & 2(q-2) \end{bmatrix}. \tag{5}$$

After simple calculation, we obtain the characteristic polynomial of B as

$$P(x) = x^4 - 2(n-4)x^3 + (3pq - 12n + 16)x^2 + 4(3pq - 4n - 4)x + 12(n-5). \tag{6}$$

Since $P(-2) = -36pq + 108p + 108q - 140 \neq 0$ for any integer p and q , $P(x) = 0$ has four roots which are different from -2 . Combining these results with Lemma 2, we conclude that $D(K_{p,q} - e)$ has eigenvalue -2 with multiplicity $n - 4$ and the remaining four eigenvalues (say, $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$) are the roots of $P(x) = 0$. By Descartes' rule of signs with the fact that $\sum_{i=1}^4 \lambda_i = 2n - 8$, one can easily check that $\lambda_1 \geq \lambda_2 > 0 > \lambda_3 \geq \lambda_4$. Now, let $a_0, -a_1, a_2, a_3, a_4$ be the coefficients of $P(x)$. Note that $a_0 = 1$ and $a_4 = 12(n-5)$. Then, $P(x)$ can be rewritten as

$$x^4 - a_1x^3 + a_2x^2 + a_3x + a_4 = (x^2 - ax + b)(x^2 + cx + d), \tag{7}$$

where $a_1 = a - c$, $a_2 = b + d - ac$, $a_3 = bc - ad$, and $a_4 = bd$. \square

Claim 1. $(\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4)^2 - a_1^2 > 32n - 64$.

Proof of Claim. 1. From the given conditions, we have

$$\begin{aligned} (a+c)^2(b-d)^2 &= (a+c)^2((b+d)^2 - 4bd) \\ &= (a+c)^2((a_2+ac)^2 - 4a_4) \\ &= (a+c)^2\left(\left(a_2 + \frac{(a+c)^2 - a_1^2}{4}\right)^2 - 4a_4\right). \end{aligned} \tag{8}$$

On the other hand, we have

$$(a-c)(b+d) = a_1(a_2+ac) = a_1\left(a_2 + \frac{(a+c)^2 - a_1^2}{4}\right). \tag{9}$$

As $2a_3 = 2(bc - ad)$, we obtain

$$2a_3 + (a-c)(b+d) = (a+c)(b-d). \tag{10}$$

Combining (8)-(10), we can obtain that $(a+c)^2 - a_1^2$ is the root of $f(y) = 0$, where

$$f(y) = \frac{1}{16}y^3 + \frac{a_2}{2}y^2 + (a_2^2 - 4a_4 - a_1a_3)y - 4a_1^2a_4 - 4a_1a_2a_3 - 4a_3^2. \tag{11}$$

From (7), it follows that $(a+c)^2$ has three possible values:

$$\begin{aligned} &(\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4)^2, \\ &(\lambda_1 + \lambda_3 - \lambda_2 - \lambda_4)^2, \\ &(\lambda_1 + \lambda_4 - \lambda_2 - \lambda_3)^2. \end{aligned} \tag{12}$$

Since $\lambda_1 \geq \lambda_2 > 0 > \lambda_3 \geq \lambda_4$, then $(\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4)^2 - a_1^2$ is the largest root of $f(y) = 0$. Using the fact that $a_3 = 4(a_2 + 8n - 20)$ and $a_4 = 12(n-5)$, we have

$$f(y) = (y - 16a_1 - 64)a_2^2 + \left(\frac{y^2}{2} - 4a_1y - (64a_1 + 512)(2n - 5)\right)a_2 + g(n, y, a_1), \quad (13)$$

where

$$g(n, y, a_1) = \frac{y^3}{16} - 16((2n - 5)a_1 + 3n - 15)y - 64(8n - 20)^2 - 48(n - 5)a_1^2. \quad (14)$$

Again using the fact that $a_1 = 2n - 8$, we obtain

$$f(32n - 64) = 128na_2 + g(n, 32n - 64, 2n - 8) = 128na_2 - 192n^3 + 1984n^2 - 2048n - 1024. \quad (15)$$

As $p + q = n$, it follows that $a_2 \leq (3n^2/4) - 12n + 16$. Combining the above results, we have

$$f(32n - 64) \leq 128n\left(\frac{3n^2}{4} - 12n + 16\right) - 192n^3 + 1984n^2 - 2048n - 1024 = -32(n - 4)(n - 2)(3n + 4) < 0. \quad (16)$$

This follows directly that $(\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4)^2 - a_1^2 > 32n - 64$. This completes the proof of Claim 1.

By Claim 1 with $a_1 = 2n - 8$, we obtain

$$\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 > \sqrt{4(n - 4)^2 + 32n - 64} = 2n. \quad (17)$$

Hence, we have

$$E_D(K_{p,q} - e) = 2(n - 4) + \lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 > 4(n - 2) = E_D(K_{p,q}). \quad (18)$$

This completes the proof of this theorem. \square

Remark 1. It is worth mentioning that the method we used in the above proof would be a tool to compare the sum of the absolute values of the roots of a quadratic polynomial with a certain value.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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