Research Article

Common Fixed-Point Theorems of Generalized (ψ, φ) – Weakly Contractive Mappings in b – Metric-Like Spaces and Application

Hongyan Guan and Jianju Li

School of Mathematics and Systems Science, Shenyang Normal University, Shenyang 110034, China

Correspondence should be addressed to Jianju Li; 56278372@qq.com

Received 6 December 2020; Revised 26 December 2020; Accepted 30 December 2020; Published 20 January 2021

Academic Editor: Naeem Saleem

Copyright © 2021 Hongyan Guan and Jianju Li. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In this paper, we prove some common fixed-point theorems of generalized (ψ, φ) –weakly contractive mappings in b –metric-like spaces. We also give two examples to support our results. Meanwhile, we present an application to the existence of solutions for a system of integral equations by means of our results.

1. Introduction

The Banach contraction mapping theorem [1] popularly known as Banach contraction mapping principle is a rewarding result in fixed-point theory. It has widespread applications in both pure and applied mathematics. This celebrated principle has been generalized by several authors. Recently, Saleem et al. [2] obtained fixed-point results of Suzuki-type generalized multivalued (f, θ, L) -almost contractions and coincidence and common fixed-point results of Suzuki-type generalized multivalued (f, θ, L) -almost contraction mapping in the setting of metric spaces. In 2019, Li et al. [3] defined a new contractive-type mapping called Z_θ -contraction and proved some fixed-point and Suzuki-type fixed-point results in the context of complete metric spaces. Another extension idea is the promotion of spaces. In 1993, Czerwik [4] introduced the concept of the b –metric space, and he also obtained some fixed-point theorems of contractive mappings in b –metric space. Since then, some papers have generalized the fixed-point of different contractive conditions in b –metric space. For example, Aydi et al. [5] proved common fixed-point results for weak ϕ –contractions on b –metric space, and Abbas et al. [6] also proved common fixed-point theorems of four mappings in b –metric space. In [7], Hussain et al. established some best

proximity point and coupled best proximity point results in the context of complete b –metric spaces based on the concepts of α –proximal admissible mappings and simulation function. Lately, Lael et al. [8] replied to an open problem related to a b –metric version of Banach’s fixed-point theorem and obtained some new fixed-point theorems for multi-valued mappings in b –metric spaces.

Inspired by Czerwik’s work, some researchers had studied the fixed-point theorems of various new types of contractive conditions in the generalized metric space and b –metric space. In 2011, Hussain and Shah in [9] introduced the notion of cone b –metric spaces, which means that it is a generalization of b –metric spaces and cone metric spaces. They also considered topological properties of cone b –metric spaces and results on KKM mappings in the setting of cone b –metric spaces. Fernandez et al. [10] introduced the concept of N_b –cone metric spaces over a Banach algebra as a generalization of N –cone metric spaces over a Banach algebra and b –metric spaces and studied some coupled common fixed-point theorems for generalized Lipschitz mappings in this framework. In 2019, Kanwal et al. [11] generalized Nadler’s theorem in weak partial b –metric space by using weak partial Hausdorff b –metric spaces. In 2020, Abbas et al. [12] introduced the concepts of ψ –contraction and monotone ψ –contraction correspondence in fuzzy

b -metric spaces and obtained fixed-point results for these contractive mappings. Ansari et al. [13] introduced the concept of inverse C -class function in G -metric setting and established some fixed-point theorems. Recently, Saleem et al. [14] proved some new fixed-point theorems, coincidence point theorems, and common fixed-point theorems for multivalued F -contractions involving a binary relation that is not necessarily a partial order, in the context of generalized metric spaces (in the sense of Jleli and Samet).

The concept of φ -contractive mappings was introduced by Rhoades [15]. Afterwards, some researchers introduced a few φ - and ψ, φ -weakly contractive conditions and discussed the existence of fixed and common fixed-point for these mappings [16–18]. In particular, Aghajani et al. [19]

presented several common fixed-point results of generalized weak contractive mappings in partially ordered b -metric spaces.

Our main concern is to study common fixed-point results involving generalized (ψ, φ) -weakly contractive conditions in b -metric-like space. Furthermore, we apply the given results to obtain existence of solutions of integral equations.

2. Preliminaries

Throughout this paper, let $\mathbb{N}$ denote the set of all positive integers, $\mathbb{R}^+ = [0, +\infty)$ and $\mathbb{R} = (-\infty, +\infty)$. Put

$$\begin{aligned}\Psi &= \{\psi: [0, +\infty) \longrightarrow [0, +\infty) \text{ is an increasing and continuous function}\}, \\ \Phi &= \{\varphi: [0, +\infty) \longrightarrow [0, +\infty) \text{ is an increasing and continuous function and, } \varphi(t) = 0 \text{ if and only if } t = 0\}.\end{aligned}\quad (1)$$

In order to get our main results, we introduce some definitions and lemmas as follows.

Definition 1 (see [4]). Let X be a nonempty set and $s \geq 1$ be a given real number. A mapping $d: X \times X \longrightarrow [0, +\infty)$ is said to be a b -metric if and only if, for all $x, y, z \in X$, the following conditions are satisfied:

- (1) $d(x, y) = 0$ if and only if $x = y$
- (2) $d(x, y) = d(y, x)$
- (3) $d(x, y) \leq s(d(x, z) + d(y, z))$

The pair (X, d) is called a b -metric space with parameter $s \geq 1$.

In general, the class of b -metric space is effectively larger than that of metric space, since a b -metric is a metric with $s = 1$. We can find several examples of b -metric spaces which are not metric spaces (see [20]).

Definition 2 (see [21]). Let X be a nonempty set and $s \geq 1$ be a given real number. A mapping $d: X \times X \longrightarrow [0, +\infty)$ is said to be a b -metric-like if and only if, for all $x, y, z \in X$, the following conditions are satisfied:

- (1) $d(x, y) = 0$ implies $x = y$
- (2) $d(x, y) = d(y, x)$
- (3) $d(x, y) \leq s(d(x, z) + d(y, z))$

The pair (X, d) is called a b -metric-like space with parameter $s \geq 1$.

Remark 1. We should note that in a b -metric-like space (X, d) , if $x, y \in X$ and $d(x, y) = 0$, then $x = y$. But the converse need not be true, and $d(x, x)$ may be positive for $x \in X$.

Example 1. Let $X = \mathbb{R}^+$, and let the mapping $d: X \times X \longrightarrow \mathbb{R}^+$ be defined by $d(x, y) = (x + y)^2$ for all

$x, y \in X$. Then, (X, d) is a b -metric-like space with parameter $s \geq 2$.

Proof. We can infer from the convexity of the function $f(x) = x^2$ ($x > 0$) that $(a + b)^2 \leq 2(a^2 + b^2)$ holds. Then, for $x, y, z \in X$, we have

$$\begin{aligned}d(x, y) &= (x + y)^2 \leq (x + z + y + z)^2 \\ &\leq 2[(x + z)^2 + (y + z)^2] = 2d(x, z) + 2d(y, z).\end{aligned}\quad (2)$$

In this case, (X, d) is a b -metric-like space with parameter $s \geq 2$. $\square$

Definition 3 (see [21]). Let (X, d) be a b -metric-like space with parameter $s \geq 1$ and $\{x_n\}$ be a sequence in X .

- (1) The sequence $\{x_n\}$ is said to be convergent to x if $\lim_{n \rightarrow +\infty} d(x_n, x) = d(x, x)$.
- (2) The sequence $\{x_n\}$ is said to be a Cauchy sequence if and only if $\lim_{n \rightarrow +\infty} d(x_n, x_m)$ exists and is finite.
- (3) (X, d) is said to be complete if for each Cauchy sequence $\{x_n\}$ in X , there exists an $x \in X$ such that $\lim_{n, m \rightarrow +\infty} d(x_n, x_m) = \lim_{n \rightarrow +\infty} d(x_n, x) = d(x, x)$.

Definition 4 (see [22]). Let f and g be two self-mappings on a nonempty set X . If $w = fx = gx$, for some $x \in X$, then x is said to be the coincidence point of f and g , where w is called the point of coincidence of f and g . Let $C(f, g)$ denote the set of all coincidence points of f and g .

Definition 5 (see [22]). Let f and g be two self-mappings defined on a nonempty set X . Then, f and g are said to be weakly compatible if they commute at every coincidence point, that is, $fx = gx \Rightarrow fgx = gfx$ for every $x \in C(f, g)$.

Lemma 1 (see [21]). Let (X, d) be a b -metric-like space with $s \geq 1$. We assume that $\{x_n\}$ and $\{y_n\}$ are convergent to x and y , respectively. Then, we have

$$\begin{aligned} & \frac{1}{s^2}d(x, y) - \frac{1}{s}d(x, x) - d(y, y) \\ & \leq \liminf_{n \rightarrow +\infty} d(x_n, y_n) \leq \limsup_{n \rightarrow +\infty} d(x_n, y_n) \\ & \leq sd(x, x) + s^2d(y, y) + s^2d(x, y). \end{aligned} \quad (3)$$

In particular, if $d(x, y) = 0$, then we have $\lim_{n \rightarrow +\infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$, we have

$$\begin{aligned} & \frac{1}{s}d(x, z) - d(x, x) \leq \liminf_{n \rightarrow +\infty} d(x_n, z) \\ & \leq \limsup_{n \rightarrow +\infty} d(x_n, z), \quad (4) \\ & \leq sd(x, z) + sd(x, x). \end{aligned}$$

In particular, if $d(x, x) = 0$, then

$$\begin{aligned} & \frac{1}{s}d(x, z) \leq \liminf_{n \rightarrow +\infty} d(x_n, z) \leq \limsup_{n \rightarrow +\infty} d(x_n, z) \\ & \leq sd(x, z). \end{aligned} \quad (5)$$

Lemma 2 (see [23]). Let (X, d) be a b -metric-like space with $s \geq 1$. Then,

- (1) If $d(x, y) = 0$, then $d(x, x) = d(y, y) = 0$
- (2) If $\{x_n\}$ is a sequence such that $\lim_{n \rightarrow +\infty} d(x_n, x_{n+1}) = 0$, then we have $\lim_{n \rightarrow +\infty} d(x_n, x_n) = \lim_{n \rightarrow +\infty} d(x_{n+1}, x_{n+1}) = 0$
- (3) If $x \neq y$, then $d(x, y) > 0$

3. Main Results

In this section, we will show the existence and uniqueness of common fixed-point for generalized (ψ, φ) -weakly contractive mappings in complete b -metric-like space. Meanwhile, we give two examples to support our results.

Theorem 1. Let (X, d) be a complete b -metric-like space with parameter $s \geq 1$ and let $f, g: X \rightarrow X$ be given self-mappings satisfying $f(X) \subset g(X)$ where $g(X)$ is a closed subset of X . If there are functions $\psi \in \Psi$ and $\varphi \in \Phi$ such that

$$\psi(s^2[d(fx, fy)]^2) \leq \psi(N_1(x, y)) - \varphi(M_1(x, y)), \quad (6)$$

where

$$\begin{aligned} N_1(x, y) &= \max\{[d(fx, gx)]^2, [d(gx, gy)]^2, [d(fy, gy)]^2, d(fx, gx)d(fx, fy), d(fx, gx)d(gx, gy)\}, \\ M_1(x, y) &= \max\left\{[d(fy, gy)]^2, [d(fx, gy)]^2, [d(gx, gy)]^2, \frac{[d(fx, gx)]^2[1 + [d(gx, gy)]^2]}{1 + [d(fx, gy)]^2}\right\}, \end{aligned} \quad (7)$$

then f and g have a unique coincidence point in X . Moreover, f and g have a unique common fixed-point provided that f and g are weakly compatible.

Proof. Let $x_0 \in X$. As $f(X) \subset g(X)$, there exists $x_1 \in X$ such that $fx_0 = gx_1$. Now we define the sequences $\{x_n\}$ and $\{y_n\}$ in X by $y_n = fx_n = gx_{n+1}$ for all $n \in \mathbb{N}$. If $y_n = y_{n+1}$ for some $n \in \mathbb{N}$, then we have $y_n = y_{n+1} = fx_{n+1} = gx_{n+1}$ and f and g have a coincidence point. Without loss of generality, we assume that $y_n \neq y_{n+1}$ (by Lemma 2, we know that

$d(y_n, y_{n+1}) > 0$) for all $n \in \mathbb{N}$. Applying (6) with $x = x_n$ and $y = x_{n+1}$, we obtain

$$\begin{aligned} \psi(s^2[d(y_n, y_{n+1})]^2) &= \psi(s^2[d(fx_n, fx_{n+1})]^2) \\ &\leq \psi(N_1(x_n, x_{n+1})) - \varphi(M_1(x_n, x_{n+1})), \end{aligned} \quad (8)$$

where

$$N_1(x_n, x_{n+1}) = \max\{[d(y_n, y_{n-1})]^2, [d(y_{n-1}, y_n)]^2, [d(y_{n+1}, y_n)]^2, d(y_n, y_{n-1})d(y_n, y_{n+1}), [d(y_n, y_{n-1})]^2\}, \quad (9)$$

$$M_1(x_n, x_{n+1}) = \max\left\{[d(y_{n+1}, y_n)]^2, [d(y_n, y_n)]^2, [d(y_{n-1}, y_n)]^2, \frac{[d(y_n, y_{n-1})]^2[1 + [d(y_{n-1}, y_n)]^2]}{1 + [d(y_n, y_n)]^2}\right\}. \quad (10)$$

If $d(y_n, y_{n+1}) \geq d(y_n, y_{n-1}) > 0$, for some $n \in \mathbb{N}$, in view of (9) and (10), we have

$$\begin{aligned} N_1(x_n, x_{n+1}) &= [d(y_n, y_{n+1})]^2, \\ M_1(x_n, x_{n+1}) &\geq [d(y_n, y_{n+1})]^2. \end{aligned} \quad (11)$$

It follows from inequality (8) and the above inequalities that

$$\begin{aligned} \psi([d(y_n, y_{n+1})]^2) &\leq \psi(s^2[d(y_n, y_{n+1})]^2) \\ &\leq \psi(N_1(x_n, x_{n+1})) - \varphi(M_1(x_n, x_{n+1})) \\ &\leq \psi([d(y_n, y_{n+1})]^2) - \varphi([d(y_n, y_{n+1})]^2), \end{aligned} \quad (12)$$

which implies $\varphi([d(y_n, y_{n+1})]^2) = 0$, that is, $y_n = y_{n+1}$, a contradiction. Hence, $d(y_n, y_{n+1}) < d(y_n, y_{n-1})$ and $\{d(y_n, y_{n+1})\}$ is a nonincreasing sequence, and so there exists $r \geq 0$ such that $\lim_{n \rightarrow +\infty} d(y_n, y_{n+1}) = r$.

By virtue of (9) and (10), we have

$$\begin{aligned} N_1(x_n, x_{n+1}) &= [d(y_n, y_{n-1})]^2, \\ M_1(x_n, x_{n+1}) &= [d(y_n, y_{n-1})]^2. \end{aligned} \quad (13)$$

It follows that

$$\begin{aligned} \psi([d(y_n, y_{n+1})]^2) &\leq \psi(N_1(x_n, x_{n+1})) - \varphi(M_1(x_n, x_{n+1})) \\ &\leq \psi([d(y_n, y_{n-1})]^2) - \varphi([d(y_n, y_{n-1})]^2). \end{aligned} \quad (14)$$

Now suppose that $r > 0$. By taking the limit as $n \rightarrow +\infty$ in (12), we have $\psi(r^2) \leq \psi(r^2) - \varphi(r^2)$, a contradiction. This yields that

$$\lim_{n \rightarrow +\infty} d(y_n, y_{n+1}) = r = 0. \quad (15)$$

Now we shall prove that $\lim_{n, m \rightarrow +\infty} d(y_n, y_m) = 0$. Suppose on the contrary that $\lim_{n, m \rightarrow +\infty} d(y_n, y_m) \neq 0$. It follows that there exists $\varepsilon > 0$ for which one can find sequences $\{y_{m_k}\}$ and $\{y_{n_k}\}$ of $\{y_n\}$ where n_k is the smallest index for which $n_k > m_k > k$, $\varepsilon \leq d(y_{m_k}, y_{n_k})$, and $d(y_{m_k}, y_{n_{k-1}}) < \varepsilon$.

In view of the triangle inequality in b -metric-like space, we get

$$\begin{aligned} \varepsilon^2 &\leq [d(y_{m_k}, y_{n_k})]^2 \leq [sd(y_{m_k}, y_{n_{k-1}}) + sd(y_{n_{k-1}}, y_{n_k})]^2 \\ &= s^2[d(y_{m_k}, y_{n_{k-1}})]^2 + s^2[d(y_{n_{k-1}}, y_{n_k})]^2 + 2s^2d(y_{m_k}, y_{n_{k-1}})d(y_{n_{k-1}}, y_{n_k}) \\ &\leq s^2\varepsilon^2 + s^2[d(y_{n_{k-1}}, y_{n_k})]^2 + 2s^2d(y_{m_k}, y_{n_{k-1}})d(y_{n_{k-1}}, y_{n_k}). \end{aligned} \quad (16)$$

Using equality (15) and taking the upper limit as $k \rightarrow +\infty$ in the above inequality, we obtain

$$\varepsilon^2 \leq \limsup_{k \rightarrow +\infty} [d(y_{m_k}, y_{n_k})]^2 \leq s^2\varepsilon^2. \quad (17)$$

As the same arguments, we deduce the following results:

$$\begin{aligned} \varepsilon^2 &\leq [d(y_{m_k}, y_{n_k})]^2 \leq [sd(y_{m_k}, y_{n_{k-1}}) + sd(y_{n_{k-1}}, y_{n_k})]^2 \\ &= s^2[d(y_{m_k}, y_{n_{k-1}})]^2 + s^2[d(y_{n_{k-1}}, y_{n_k})]^2 + 2s^2d(y_{m_k}, y_{n_{k-1}})d(y_{n_{k-1}}, y_{n_k}), \end{aligned} \quad (18)$$

$$\begin{aligned} [d(y_{m_k}, y_{n_k})]^2 &\leq [sd(y_{m_k}, y_{m_{k-1}}) + sd(y_{m_{k-1}}, y_{n_k})]^2 \\ &= s^2[d(y_{m_k}, y_{m_{k-1}})]^2 + s^2[d(y_{m_{k-1}}, y_{n_k})]^2 + 2s^2d(y_{m_k}, y_{m_{k-1}})d(y_{m_{k-1}}, y_{n_k}), \end{aligned} \quad (19)$$

$$\begin{aligned} [d(y_{m_{k-1}}, y_{n_k})]^2 &\leq [sd(y_{m_{k-1}}, y_{m_k}) + sd(y_{m_k}, y_{n_k})]^2 \\ &= s^2[d(y_{m_{k-1}}, y_{m_k})]^2 + s^2[d(y_{m_k}, y_{n_k})]^2 + 2s^2d(y_{m_{k-1}}, y_{m_k})d(y_{m_k}, y_{n_k}). \end{aligned} \quad (20)$$

In view of (18), we have

$$\frac{\varepsilon^2}{s^2} \leq \limsup_{k \rightarrow +\infty} [d(y_{m_k}, y_{n_{k-1}})]^2 \leq \varepsilon^2. \quad (21)$$

Using (19) and (20), we obtain

$$\frac{\varepsilon^2}{s^2} \leq \limsup_{k \rightarrow +\infty} [d(y_{m_{k-1}}, y_{n_k})]^2 \leq s^4\varepsilon^2. \quad (22)$$

Similarly, we deduce that

$$\begin{aligned}
 [d(y_{m_k-1}, y_{n_k-1})]^2 &\leq [sd(y_{m_k-1}, y_{m_k}) + sd(y_{m_k}, y_{n_k-1})]^2 \\
 &= s^2 [d(y_{m_k-1}, y_{m_k})]^2 + s^2 [d(y_{m_k}, y_{n_k-1})]^2 + 2s^2 d(y_{m_k-1}, y_{m_k}) d(y_{m_k}, y_{n_k-1}), \\
 [d(y_{m_k}, y_{n_k})]^2 &\leq [sd(y_{m_k}, y_{m_k-1}) + sd(y_{m_k-1}, y_{n_k})]^2 = s^2 [d(y_{m_k}, y_{m_k-1})]^2 + s^2 [d(y_{m_k-1}, y_{n_k})]^2 \\
 &\quad + 2s^2 d(y_{m_k}, y_{m_k-1}) d(y_{m_k-1}, y_{n_k}) \\
 &\leq s^2 [d(y_{m_k}, y_{m_k-1})]^2 + s^2 [sd(y_{m_k-1}, y_{n_k-1}) + sd(y_{n_k-1}, y_{n_k})]^2 + 2s^2 d(y_{m_k}, y_{m_k-1}) [sd(y_{m_k-1}, y_{n_k-1}) \\
 &\quad + sd(y_{n_k-1}, y_{n_k})].
 \end{aligned} \tag{23}$$

It follows that

$$\frac{\varepsilon^2}{4} \leq \limsup_{k \rightarrow +\infty} [d(y_{m_k-1}, y_{n_k-1})]^2 \leq s^2 \varepsilon^2. \tag{24}$$

Through the definition of $N_1(x, y)$, we have

$$\begin{aligned}
 N_1(x_{m_k}, x_{n_k}) &= \max \left\{ [d(y_{m_k}, y_{m_k-1})]^2, [d(y_{m_k-1}, y_{n_k-1})]^2, [d(y_{n_k}, y_{n_k-1})]^2, d(y_{m_k}, y_{m_k-1}) d(y_{m_k}, y_{n_k}), \right. \\
 &\quad \left. d(y_{m_k}, y_{m_k-1}) d(y_{m_k-1}, y_{n_k-1}) \right\},
 \end{aligned} \tag{25}$$

which yields that

$$\limsup_{k \rightarrow +\infty} N_1(x_{m_k}, x_{n_k}) \leq \max \{0, s^2 \varepsilon^2, 0, 0, 0\} = \varepsilon^2 s^2. \tag{26}$$

Also,

$$M_1(x_{m_k}, x_{n_k}) = \max \left\{ [d(y_{n_k}, y_{n_k-1})]^2, [d(y_{m_k}, y_{n_k-1})]^2, [d(y_{m_k-1}, y_{n_k-1})]^2, \frac{[d(y_{m_k}, y_{m_k-1})]^2 [1 + [d(y_{m_k-1}, y_{n_k-1})]^2]}{1 + [d(y_{m_k}, y_{n_k-1})]^2} \right\}. \tag{27}$$

It is easy to show that

$$\liminf_{k \rightarrow +\infty} M_1(x_{m_k}, x_{n_k}) \geq \max \left\{ 0, \frac{\varepsilon^2}{s^2}, \frac{\varepsilon^2}{s^4}, 0 \right\} \geq \frac{\varepsilon^2}{s^4}. \tag{28}$$

Applying (6) with $x = x_{m_k}$ and $y = x_{n_k}$, we get

$$\begin{aligned}
 \psi([d(y_{m_k}, y_{n_k})]^2) &\leq \psi(s^2 [d(y_{m_k}, y_{n_k})]^2) \\
 &\leq \psi(N_1(x_{m_k}, x_{n_k})) - \varphi(M_1(x_{m_k}, x_{n_k})).
 \end{aligned} \tag{29}$$

In light of (26), one can obtain

$$\begin{aligned}
 \psi(s^2 \varepsilon^2) &\leq \psi(s^2 \limsup_{k \rightarrow +\infty} [d(fx_{m_k}, fx_{n_k})]^2) \\
 &\leq \psi(\limsup_{k \rightarrow +\infty} N_1(x_{m_k}, x_{n_k})) - \varphi(\liminf_{k \rightarrow +\infty} M_1(x_{m_k}, x_{n_k})) \\
 &\leq \psi(s^2 \varepsilon^2) - \varphi(\liminf_{k \rightarrow +\infty} M_1(x_{m_k}, x_{n_k})),
 \end{aligned} \tag{30}$$

which implies that

$$\liminf_{k \rightarrow +\infty} M_1(x_{n_k}, x_{n_k}) = 0, \quad (31)$$

a contradiction to (28). It follows that $\{y_n\}$ is a Cauchy sequence in X and $\lim_{n,m \rightarrow +\infty} d(y_n, y_m) = 0$. Since X is complete b -metric-like space, there exists $u \in X$ such that

$$\lim_{n \rightarrow +\infty} d(y_n, u) = \lim_{n \rightarrow +\infty} d(fx_n, u) = \lim_{n \rightarrow +\infty} d(gx_{n+1}, u) = \lim_{n,m \rightarrow +\infty} d(y_n, y_m) = d(u, u) = 0. \quad (32)$$

Furthermore, we have $u \in g(X)$ since $g(X)$ is closed. It follows that one can choose a $z \in X$ such that $u = gz$, and one can write (32) as

$$\begin{aligned} \lim_{n \rightarrow +\infty} d(y_n, gz) &= \lim_{n \rightarrow +\infty} d(fx_n, gz) \\ &= \lim_{n \rightarrow +\infty} d(gx_{n+1}, gz) = 0. \end{aligned} \quad (33)$$

where

$$\begin{aligned} \psi\left(s^2 [d(y_{n_k}, fz)]^2\right) &= \psi\left(s^2 [d(fx_{n_k}, fz)]^2\right) \\ &\leq \psi(N_1(x_{n_k}, z)) - \phi(M_1(x_{n_k}, z)), \end{aligned} \quad (34)$$

If $fz \neq gz$, taking $x = x_{n_k}$ and $y = z$ in contractive condition (6), we get

$$\begin{aligned} N_1(x_{n_k}, z) &= \max\left\{[d(y_{n_k}, y_{n_k-1})]^2, [d(y_{n_k-1}, gz)]^2, [d(fz, gz)]^2, d(y_{n_k}, y_{n_k-1})d(y_{n_k}, fz), d(y_{n_k}, y_{n_k-1})d(y_{n_k-1}, gz)\right\}, \\ M_1(x_{n_k}, z) &= \max\left\{[d(fz, gz)]^2, [d(y_{n_k}, gz)]^2, [d(y_{n_k-1}, gz)]^2, \frac{[d(y_{n_k}, y_{n_k-1})]^2 [1 + [d(y_{n_k-1}, gz)]^2]}{1 + [d(y_{n_k}, gz)]^2}\right\}, \end{aligned} \quad (35)$$

and then we obtain

$$\begin{aligned} \limsup_{k \rightarrow +\infty} N_1(x_{n_k}, z) &= \max\{0, 0, [d(gz, fz)]^2, 0, 0\} = [d(gz, fz)]^2, \\ \liminf_{k \rightarrow +\infty} M_1(x_{n_k}, z) &= \max\{[d(gz, fz)]^2, 0, 0, 0\} = [d(gz, fz)]^2. \end{aligned} \quad (36)$$

Taking the upper limit as $k \rightarrow +\infty$ in (34),

$$\begin{aligned} \psi([d(gz, fz)]^2) &= \psi\left(s^2 \cdot \frac{1}{s^2} [d(gz, fz)]^2\right) \leq \psi\left(s^2 \limsup_{k \rightarrow +\infty} [d(fx_{n_k}, fz)]^2\right) \\ &\leq \psi(\limsup_{k \rightarrow +\infty} N_1(x_{n_k}, z)) - \phi(\liminf_{k \rightarrow +\infty} M_1(x_{n_k}, z)) \\ &= \psi([d(gz, fz)]^2) - \phi([d(gz, fz)]^2), \end{aligned} \quad (37)$$

which implies that $\phi([d(fz, gz)]^2) = 0$.

It follows that $d(fz, gz) = 0$. That is, $fz = gz$. Therefore, $u = fz = gz$ is a point of coincidence for f and g . We also conclude that the point of coincidence is unique. Assume on the contrary that there exist $z, z' \in C(f, g)$ and $z \neq z'$; applying (6) with $x = z$ and $y = z'$, we obtain that

$$\begin{aligned} \psi([d(fz, fz')]^2) &\leq \psi(s^2 [d(fz, fz')]^2) \\ &\leq \psi(N_1(z, z')) - \phi(M_1(z, z')) \\ &\leq \psi([d(fz, fz')]^2) - \phi([d(fz, fz')]^2). \end{aligned} \quad (38)$$

Hence, $fz = fz'$. That is, the point of coincidence is unique. Considering the weak compatibility of f and g , it can be shown that z is a unique common fixed-point. This completes the proof. $\square$

Theorem 2. Let (X, d) be a complete b -metric-like space with parameter $s \geq 1$ and let $f, g: X \rightarrow X$ be given

self-mappings satisfying $f(X) \subset g(X)$ where $g(X)$ is a closed subset of X . If there are functions $\psi \in \Psi$ and $\varphi \in \Phi$ such that

$$\psi(s^2 [d(fx, fy)]^2) \leq \psi(N_2(x, y)) - \varphi(M_2(x, y)), \quad (39)$$

where

$$\begin{aligned} N_2(x, y) &= \max \left\{ d(fx, fy)d(gx, gy), [d(gx, gy)]^2, \frac{[d(fy, gy)]^2 + [d(fx, gy)]^2}{1 + 4s^2} \right\}, \\ M_2(x, y) &= \max \left\{ [d(fy, gy)]^2, [d(fx, gy)]^2, [d(gx, gy)]^2, \frac{[d(fx, gx)]^2 [1 + [d(gx, gy)]^2]}{1 + [d(fx, gy)]^2}, \right. \\ &\quad \left. \frac{[d(gx, gy)]^2 [1 + [d(gx, gy)]^2]}{1 + [d(fx, gx)]^2} \right\}, \end{aligned} \quad (40)$$

then f and g have a unique coincidence point in X . Moreover, f and g have a unique common fixed-point provided that f and g are weakly compatible.

Proof. It is the same as the proof of Theorem 1, and we also define the sequences $\{x_n\}$ and $\{y_n\}$ in X by $y_n = fx_n = gx_{n+1}$ for all $n \in \mathbb{N}$. We also suppose that $y_n \neq y_{n+1}$ for each $n \in \mathbb{N}$, and it follows from (39) that

$$\begin{aligned} \psi(s^2 [d(y_n, y_{n+1})]^2) &= \psi(s^2 [d(fx_n, fx_{n+1})]^2) \\ &\leq \psi(N_2(x_n, x_{n+1})) - \varphi(M_2(x_n, x_{n+1})), \end{aligned} \quad (41)$$

where

$$\begin{aligned} N_2(x_n, x_{n+1}) &= \max \left\{ d(y_n, y_{n+1})d(y_{n-1}, y_n), [d(y_{n-1}, y_n)]^2, \frac{[d(y_{n+1}, y_n)]^2 + [d(y_n, y_n)]^2}{1 + 4s^2} \right\}, \\ M_2(x_n, x_{n+1}) &= \max \left\{ [d(y_{n+1}, y_n)]^2, [d(y_n, y_n)]^2, [d(y_{n-1}, y_n)]^2, \frac{[d(y_n, y_{n-1})]^2 [1 + [d(y_{n-1}, y_n)]^2]}{1 + [d(y_n, y_n)]^2}, \right. \\ &\quad \left. \frac{[d(y_{n-1}, y_n)]^2 [1 + [d(y_{n-1}, y_n)]^2]}{1 + [d(y_n, y_{n-1})]^2} \right\}. \end{aligned} \quad (42)$$

$$\quad (43)$$

If we assume that for some $n \in \mathbb{N}$,

$$d(y_n, y_{n+1}) \geq d(y_{n-1}, y_n) > 0, \quad (44)$$

then from inequalities (42) and (43), we get that

$$\begin{aligned} N_2(x_n, x_{n+1}) &\leq [d(y_{n+1}, y_n)]^2, \\ M_2(x_n, x_{n+1}) &\geq [d(y_{n+1}, y_n)]^2. \end{aligned} \quad (45)$$

In view of (41), we have the following inequality:

$$\begin{aligned} \psi([d(y_n, y_{n+1})]^2) &\leq \psi(s^2 [d(y_n, y_{n+1})]^2) \\ &\leq \psi(N_2(x_n, x_{n+1})) - \varphi(M_2(x_n, x_{n+1})) \\ &\leq \psi([d(y_n, y_{n+1})]^2) - \varphi([d(y_n, y_{n+1})]^2), \end{aligned} \quad (46)$$

which gives that $d(y_n, y_{n+1}) = 0$, a contradiction to $d(y_n, y_{n+1}) > 0$. It follows that $d(y_n, y_{n+1}) < d(y_n, y_{n-1})$. Hence, $\{d(y_n, y_{n+1})\}$ is a nonincreasing sequence. Consequently, the limit of the sequence is a nonnegative number, say $r \geq 0$. That is, $\lim_{n \rightarrow +\infty} d(y_n, y_{n+1}) = r$.

According to (42) and (43), we have

$$\begin{aligned} N_2(x_n, x_{n+1}) &\leq [d(y_n, y_{n-1})]^2, \\ M_2(x_n, x_{n+1}) &\geq [d(y_n, y_{n-1})]^2. \end{aligned} \quad (47)$$

So,

$$\begin{aligned} \psi([d(y_n, y_{n+1})]^2) &\leq \psi(N_2(x_n, x_{n+1})) - \varphi(M_2(x_n, x_{n+1})) \\ &\leq \psi([d(y_n, y_{n-1})]^2) - \varphi([d(y_n, y_{n-1})]^2). \end{aligned} \quad (48)$$

If $r > 0$, then letting $n \rightarrow +\infty$ in above inequality, we obtain that $\psi(r^2) = \psi(r^2) - \varphi(r^2)$ which implies that $r = 0$, i.e., $\lim_{n \rightarrow +\infty} d(y_n, y_{n+1}) = 0$.

Now we prove that $\lim_{n,m \rightarrow +\infty} d(y_n, y_m) = 0$. If not, as the proof of Theorem 1, there exists $\varepsilon > 0$ for which one can find sequences $\{y_{m_k}\}$ and $\{y_{n_k}\}$ of $\{y_n\}$ so that n_k is the smallest index for which $n_k > m_k > k$, and the following inequalities hold:

$$\begin{aligned} \varepsilon &\leq \limsup_{k \rightarrow +\infty} d(y_{m_k}, y_{n_k}) \leq s\varepsilon, \\ \frac{\varepsilon}{s} &\leq \limsup_{k \rightarrow +\infty} d(y_{m_k}, y_{n_{k-1}}) \leq \varepsilon, \\ \frac{\varepsilon}{s} &\leq \limsup_{k \rightarrow +\infty} d(y_{m_{k-1}}, y_{n_k}) \leq s^2\varepsilon, \\ \frac{\varepsilon}{s^2} &\leq \limsup_{k \rightarrow +\infty} d(y_{m_{k-1}}, y_{n_{k-1}}) \leq s\varepsilon. \end{aligned} \quad (49)$$

$$\begin{aligned} N_2(x_{m_k}, x_{n_k}) &= \max \left\{ d(y_{m_k}, y_{n_k})d(y_{m_{k-1}}, y_{n_{k-1}}), [d(y_{m_{k-1}}, y_{n_{k-1}})]^2, \frac{[d(y_{n_k}, y_{n_{k-1}})]^2 + [d(y_{m_k}, y_{n_{k-1}})]^2}{1 + 4s^2} \right\}, \\ M_2(x_{m_k}, x_{n_k}) &= \max \left\{ \begin{aligned} &[d(y_{n_k}, y_{n_{k-1}})]^2, [d(y_{m_k}, y_{n_{k-1}})]^2, [d(y_{m_{k-1}}, y_{n_{k-1}})]^2, \frac{[d(y_{m_k}, y_{m_{k-1}})]^2 [1 + [d(y_{m_{k-1}}, y_{n_{k-1}})]^2]}{1 + [d(y_{m_k}, y_{n_{k-1}})]^2}, \\ &\frac{[d(y_{m_{k-1}}, y_{n_{k-1}})]^2 [1 + [d(y_{m_{k-1}}, y_{n_{k-1}})]^2]}{1 + [d(y_{m_k}, y_{m_{k-1}})]^2} \end{aligned} \right\}, \end{aligned} \quad (50)$$

---

using (49), one can obtain that

$$\limsup_{k \rightarrow +\infty} N_2(x_{m_k}, x_{n_k}) \leq \max \left\{ s^2\varepsilon^2, s^2\varepsilon^2, \frac{\varepsilon^2}{1 + 4s^2} \right\} = s^2\varepsilon^2, \quad (51)$$

$$\liminf_{k \rightarrow +\infty} M_2(x_{m_k}, x_{n_k}) \geq \max \left\{ 0, \frac{\varepsilon^2}{s^2}, \frac{\varepsilon^2}{s^4}, 0, \frac{\varepsilon^2}{s^4} \left(1 + \frac{\varepsilon^2}{s^4} \right) \right\} \geq \frac{\varepsilon^2}{s^4}. \quad (52)$$

Taking $x = x_{m_k}$ and $y = x_{n_k}$ in (39), we get

$$\begin{aligned}\psi\left([d(y_{m_k}, y_{n_k})]^2\right) &\leq \psi\left(s^2[d(y_{m_k}, y_{n_k})]^2\right) \\ &\leq \psi(N_2(x_{m_k}, x_{n_k})) - \varphi(M_2(x_{m_k}, x_{n_k})).\end{aligned}\quad (53)$$

Therefore, we have

$$\begin{aligned}\psi(s^2\varepsilon^2) &\leq \psi\left(s^2\limsup_{k\rightarrow+\infty}[d(fx_{m_k}, fx_{n_k})]^2\right) \\ &\leq \psi(\limsup_{k\rightarrow+\infty}N_2(x_{m_k}, x_{n_k})) - \varphi(\liminf_{k\rightarrow+\infty}M_2(x_{m_k}, x_{n_k})) \\ &\leq \psi(s^2\varepsilon^2) - \varphi(\liminf_{k\rightarrow+\infty}M_2(x_{m_k}, x_{n_k})),\end{aligned}\quad (54)$$

and we conclude that $\liminf_{k\rightarrow+\infty}M_2(x_{m_k}, x_{n_k}) = 0$ which gives a contradiction to (52). Hence, $\lim_{n,m\rightarrow+\infty}d(y_n, y_m) = 0$.

The completeness of X ensures that there exists $u \in X$ such that

$$\lim_{n\rightarrow+\infty}d(y_n, u) = \lim_{n\rightarrow+\infty}d(fx_n, u) = \lim_{n\rightarrow+\infty}d(gx_{n+1}, u) = \lim_{n,m\rightarrow+\infty}d(y_n, y_m) = d(u, u) = 0. \quad (55)$$

In view of the hypothesis $g(X)$ is closed, we obtain that $u \in g(X)$. It follows that one can choose $z \in X$ such that $u = gz$, and we write the above equality as

$$\begin{aligned}\lim_{n\rightarrow+\infty}d(y_n, gz) &= \lim_{n\rightarrow+\infty}d(fx_n, gz) \\ &= \lim_{n\rightarrow+\infty}d(gx_{n+1}, gz) = 0.\end{aligned}\quad (56)$$

If $fx \neq gz$, putting $x = x_{n_k}$ and $y = z$ into contractive condition (39), we have

$$\psi\left(s^2[d(fx_{n_k}, fz)]^2\right) \leq \psi(N_2(x_{n_k}, z)) - \varphi(M_2(x_{n_k}, z)), \quad (57)$$

where

$$\begin{aligned}N_2(x_{n_k}, z) &= \max\left\{d(y_{n_k}, fz)d(y_{n_k-1}, gz), [d(y_{n_k-1}, gz)]^2, \frac{[d(fz, gz)]^2 + [d(y_{n_k}, gz)]^2}{1 + 4s^2}\right\}, \\ M_2(x_{n_k}, z) &= \max\left\{[d(fz, gz)]^2, [d(y_{n_k}, gz)]^2, [d(y_{n_k-1}, gz)]^2, \frac{[d(y_{n_k}, y_{n_k-1})]^2[1 + [d(y_{n_k-1}, gz)]^2]}{1 + [d(y_{n_k}, gz)]^2}, \right. \\ &\quad \left. \frac{[d(y_{n_k-1}, gz)]^2[1 + [d(y_{n_k-1}, gz)]^2]}{1 + d(y_{n_k}, y_{n_k-1})}\right\}.\end{aligned}\quad (58)$$

Consequently, we get

$$\begin{aligned}\limsup_{k\rightarrow+\infty}N_2(x_{n_k}, z) &= \max\left\{0, 0, \frac{[d(fz, gz)]^2}{1 + 4s^2}\right\} \leq [d(fz, gz)]^2, \\ \liminf_{k\rightarrow+\infty}M_2(x_{n_k}, z) &= \max\{[d(fz, gz)]^2, 0, 0, 0, 0\} = [d(fz, gz)]^2.\end{aligned}\quad (59)$$

Taking the upper limit as $k \rightarrow +\infty$ in (57), we have

$$\begin{aligned}
 \psi([d(gz, fz)]^2) &= \psi\left(s^2 \cdot \frac{1}{s^2} [d(gz, fz)]^2\right) \\
 &\leq \psi\left(s^2 \limsup_{k \rightarrow +\infty} [d(fx_{n_k}, fz)]^2\right) \\
 &\leq \psi\left(\limsup_{k \rightarrow +\infty} N_2(x_{n_k}, z)\right) \\
 &\quad - \varphi\left(\liminf_{k \rightarrow +\infty} M_2(x_{n_k}, z)\right) \\
 &\leq \psi([d(gz, fz)]^2) - \varphi([d(gz, fz)]^2), \tag{60}
 \end{aligned}$$

which implies that $d(fz, gz) = 0$. That is, $u = fz = gz$ is a point of coincidence for f and g . Using the same technique in the proof of Theorem 1, it can be proved that z is a unique common fixed-point. This completes the proof. $\square$

The following examples support Theorems 1 and 2.

Example 2. Let $X = [0, 1]$ be endowed with the b -metric-like $d(x, y) = (x + y)^2$ for all $x, y \in X$ and $s = 2$. Define mappings $f, g: X \rightarrow X$ by $fx = (x/64)$ and $gx = (x/2)$.

The control functions $\psi, \varphi: [0, +\infty) \rightarrow [0, +\infty)$ are defined as $\psi(t) = (5t/4)$, $\varphi(t) = (48545t/87846)$ for all $t \in [0, +\infty)$. It is clear that $f(X) \subset g(X)$ and $g(X)$ is closed. For all $x, y \in X$, we have

$$\begin{aligned}
 \psi(s^2 [d(fx, fy)]^2) &= \psi\left(4 \cdot \left(\frac{x}{64} + \frac{y}{64}\right)^4\right) = \frac{5}{4} \cdot 4 \left(\frac{x}{64} + \frac{y}{64}\right)^4 = \frac{5}{64^4} (x + y)^4, \\
 \psi(N_1(x, y)) &\geq \psi([d(gx, gy)]^2) = \frac{5}{4} \cdot \left(\frac{x}{2} + \frac{y}{2}\right)^4 = \frac{5}{64} (x + y)^4, \tag{61} \\
 \varphi(M_1(x, y)) &= \max \left\{ \left(\frac{y}{64} + \frac{y}{2}\right)^4, \left(\frac{x}{64} + \frac{y}{2}\right)^4, \left(\frac{x}{2} + \frac{y}{2}\right)^4, \frac{((x/64) + (x/2))^4 [1 + ((x/2) + (y/2))^4]}{1 + ((x/64) + (y/2))^4} \right\}.
 \end{aligned}$$

It follows that

$$\begin{aligned}
 \varphi(M_1(x, y)) &\leq \varphi\left(2 \cdot \left(\frac{33x}{64} + \frac{33y}{64}\right)^4\right) \\
 &= \frac{5 \cdot 64^3 - 5}{2 \cdot 33^4} \cdot 2 \cdot \left(\frac{33}{64}\right)^4 (x + y)^4 \tag{62} \\
 &= \frac{1310715}{64^4} (x + y)^4.
 \end{aligned}$$

It is easy to see that

$$\begin{aligned}
 \psi(s^2 [d(fx, fy)]^2) &= \psi([d(gx, gy)]^2) \\
 &\quad - \varphi\left(2 \cdot \left(\frac{33}{64}\right)^4 (x + y)^4\right) \tag{63} \\
 &\leq \psi(N_1(x, y)) - \varphi(M_1(x, y)).
 \end{aligned}$$

Therefore, the conditions of Theorem 1 are satisfied. It is obvious that 0 is the unique common fixed-point of f and g .

Example 3. Let $X = [-1, 1]$ and $d(x, y) = (x + y)^2$ for all $x, y \in X$, so (X, d) is a b -metric-like space with parameter $s = 2$. Let $f, g: X \rightarrow X$ be defined by the formulas

$$f(x) = \begin{cases} 0, & x \in [-1, 1), \\ \frac{1}{8}, & x = 1, \end{cases} \tag{64}$$

$$g(x) = \frac{x^2}{2}.$$

The control functions $\psi, \varphi: [0, +\infty) \rightarrow [0, +\infty)$ are defined as $\psi(t) = bt$ and $\varphi(t) = (b - 1)t$, for all $t \in [0, +\infty)$, where $1 < b \leq (2048/2045)$.

Now we consider four cases:

Case 1. $x \neq 1, y \neq 1$. It is clear that

$$\psi(s^2[d(fx, fy)]^2) = \psi(0) = 0,$$

$$\psi(N_2(x, y)) = bN_2(x, y) \geq b[d(gx, gy)]^2 = b\left(\frac{x^2}{2} + \frac{y^2}{2}\right)^4, \quad (65)$$

$$\varphi(M_2(x, y)) = (b-1)M_2(x, y) \leq (b-1) \cdot 2\left(\frac{x^2}{2} + \frac{y^2}{2}\right)^4,$$

so we have

$$\begin{aligned} \psi(s^2[d(fx, fy)]^2) &= 0 \leq b\left(\frac{x^2}{2} + \frac{y^2}{2}\right)^4 \\ &\quad - 2(b-1)\left(\frac{x^2}{2} + \frac{y^2}{2}\right)^4 \\ &\leq \psi(N_2(x, y)) - \varphi(M_2(x, y)). \end{aligned} \quad (66)$$

Case 2. $x = 1, y \neq 1$. It follows that

$$\psi(s^2[d(fx, fy)]^2) = 4b\left(\frac{1}{8}\right)^4 = \frac{b}{1024},$$

$$\begin{aligned} \psi(N_2(x, y)) &= bN_2(x, y) \geq bd(fx, fy)d(gx, gy) \\ &= \frac{b}{64}\left(\frac{1}{2} + \frac{y^2}{2}\right)^2 \\ &\geq \frac{1}{64} \cdot \frac{b}{4} = \frac{b}{256}, \end{aligned}$$

$$\begin{aligned} \varphi(M_2(x, y)) &= (b-1)M_2(x, y) \leq 2(b-1), \end{aligned} \quad (67)$$

and we obtain that

$$\begin{aligned} \psi(s^2[d(fx, fy)]^2) &= \frac{b}{1024} \leq \frac{b}{256} - 2(b-1) \\ &\leq \psi(N_2(x, y)) - \varphi(M_2(x, y)). \end{aligned} \quad (68)$$

Case 3. $x \neq 1, y = 1$. By concise calculation, we get

$$\psi(s^2[d(fx, fy)]^2) = 4b\left(\frac{1}{8}\right)^4 = \frac{b}{1024},$$

$$\begin{aligned} \psi(N_2(x, y)) &= bN_2(x, y) \\ &\geq bd(fx, fy)d(gx, gy) \\ &= \frac{b}{64}\left(\frac{1}{2} + \frac{x^2}{2}\right)^2 \\ &\geq \frac{1}{64} \cdot \frac{b}{4} = \frac{b}{256}, \end{aligned}$$

$$\varphi(M_2(x, y)) = (b-1)M_2(x, y) \leq 2(b-1). \quad (69)$$

It follows that

$$\begin{aligned} \psi(s^2[d(fx, fy)]^2) &= \frac{b}{1024} \leq \frac{b}{256} - 2(b-1) \\ &\leq \psi(N_2(x, y)) - \varphi(M_2(x, y)). \end{aligned} \quad (70)$$

Case 4. $x = 1, y = 1$. It is easy to see that

$$\psi(s^2[d(fx, fy)]^2) = 4b\left(\frac{1}{8} + \frac{1}{8}\right)^4 = \frac{b}{64},$$

$$\psi(N_2(x, y)) = bN_2(x, y)$$

$$\geq bd(fx, fy)d(gx, gy) = b\left(\frac{1}{8} + \frac{1}{8}\right)^2 \cdot \left(\frac{1}{2} + \frac{1}{2}\right)^2$$

$$= \frac{b}{16} \varphi(M_2(x, y)) = (b-1)M_2(x, y) \leq 2(b-1). \quad (71)$$

Obviously, for all $x, y \in [-1, 1]$, we obtain

$$\begin{aligned} \psi(s^2[d(fx, fy)]^2) &= \frac{b}{64} \leq \frac{b}{16} - 2(b-1) \\ &\leq \psi(N_2(x, y)) - \varphi(M_2(x, y)). \end{aligned} \quad (72)$$

It follows from Theorem 2 that f and g have a unique common fixed-point in X . It is easy to show that 0 is the unique common fixed-point of f and g .

According to Theorems 1 and 2, one can get the following results.

Corollary 1. Let (X, d) be a complete b -metric-like space with constant $s \geq 1$ and let $f, g: X \rightarrow X$ be given self-mappings satisfying $f(X) \subset g(X)$ where $g(X)$ is a closed subset of X . If the following condition is satisfied:

$$s^2[d(fx, fy)]^2 \leq N_1(x, y) - L[d(gx, gy)]^2, \quad (73)$$

where $L \in (0, 1)$ is a constant and then f and g have a unique coincidence point in X . Moreover, f and g have a unique common fixed-point provided that f and g are weakly compatible.

Corollary 2. Let (X, d) be a complete b -metric-like space with constant $s \geq 1$ and let $f, g: X \rightarrow X$ be given self-mappings satisfying $f(X) \subset g(X)$ where $g(X)$ is a closed subset of X . If the following condition is satisfied:

$$s^2[d(fx, fy)]^2 \leq N_2(x, y) - M_2(x, y), \quad (74)$$

then f and g have a unique coincidence point in X . Moreover, if f and g are weakly compatible, then f and g have a unique common fixed-point.

4. Application

In this section, we will use Corollary 1 to show that there is a solution to the following system of integral equations:

$$\begin{cases} x(t) = \int_0^t K(r, x(r))dr, \\ x(t) = \int_0^t x(r)dr. \end{cases} \quad (75)$$

Let $X = C([0, T])$ be the set of real continuous functions defined on $[0, T]$ for $T > 0$. Define the b -metric-like mapping $d: X \times X \rightarrow [0, +\infty)$ by

$$d(x, y) = \max_{t \in [0, T]} (|x(t)| + |y(t)|)^{(m/2)}, \quad (76)$$

for all $x, y \in X$, where $m > 2$. It is evident that (X, d) with $s = 2^{(m/2)} - 1$ is a complete b -metric-like space. Consider the mappings $f, g: X \rightarrow X$ by

$$\begin{aligned} fx(t) &= \int_0^t K(r, x(r))dr, \\ gx(t) &= \int_0^t x(r)dr. \end{aligned} \quad (77)$$

Theorem 3. Consider the system of integral equations (75) and suppose that the following conditions hold:

- (i) $K: [0, T] \times \mathbb{R} \rightarrow \mathbb{R}^+$ is continuous
- (ii) If $K(r, x(r)) = x(r)$ for all $r \in [0, T]$, then we have $K(r, \int_0^t x(\omega)d\omega) = \int_0^t K(\omega, x(\omega))d\omega$
- (iii) There exists a continuous function $\gamma: [0, T] \rightarrow [0, 1]$ such that $|K(r, x(r))| \leq \gamma(r)|x(r)|$, for all $r \in [0, T]$, $x, y \in X$
- (iv) There exists constant L such that for all $r \in [0, T]$,

$$\sup_{t \in [0, T]} \gamma(t) \leq \left(\frac{1-L}{s^2} \right)^{1/m}. \quad (78)$$

Then, the system of integral equations (75) has a unique solution in $x \in X$.

Proof. Obviously, by condition (ii), f, g are weakly compatible. Let $x, y \in X$; from conditions (i), (iii), and (iv), for all $t \in [0, T]$, we have

$$\begin{aligned}
s^2(|fx(t)| + |fy(t)|)^m &= s^2\left(\left|\int_0^t K(r, x(r))dr\right| + \left|\int_0^t K(r, y(r))dr\right|\right)^m \\
&\leq s^2\left(\int_0^t |K(r, x(r))| + |K(r, y(r))|dr\right)^m \\
&\leq s^2\left(\int_0^t \gamma(r)(|x(r)| + |y(r)|)dr\right)^m \\
&\leq s^2\left(\sup_{t \in [0, T]} \gamma(t)\right)^m \cdot [d(gx(t), gy(t))]^2 \\
&\leq (1 - L)[d(gx(t), gy(t))]^2 \\
&\leq N_1(x(t), y(t)) - L[d(gx(t), gy(t))]^2,
\end{aligned} \tag{79}$$

which implies that

$$s^2[d(fx(t), fy(t))]^2 \leq N_1(x(t), y(t)) - L[d(gx(t), gy(t))]^2. \tag{80}$$

Consequently, letting $\psi(x) = x$ and $\varphi(x) = Lx$, all of the conditions of Corollary 1 are satisfied. As a result, the mappings f, g have a unique common fixed-point in X , which is the solution of the system of integral equations (75). $\square$

5. Conclusions

In this paper, we introduced new generalized (ψ, φ) -weakly contractive mappings and obtained common fixed-point theorems in the framework of b -metric-like space. Further we provided examples that elaborated the useability of our results. As an application of our result, we obtained a solution of the system of integral equations.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Acknowledgments

This study was financially supported by the Science and Research Project Foundation of Liaoning Province Education Department (nos. LQN201902 and LJC202003).

References

- [1] S. Banach, "Sur les opérations dans ensembles abstraits et leur application aux équations intégrales," *Fundamenta Mathematicae*, vol. 3, pp. 51–57, 1922.
- [2] N. Saleem, M. Abbas, B. Ali, and Z. Raza, "Fixed points of Suzuki-type generalized multivalued (f, θ, L) -almost contractions with applications," *Filomat*, vol. 33, no. 2, pp. 499–518, 2019.
- [3] X. Li, A. Hussain, M. Adeel, and E. Savas, "Fixed point theorems for Z_θ -contraction and applications to nonlinear integral equations," *IEEE Access*, vol. 7, pp. 120023–120029, 2019.
- [4] S. Czerwik, "Contraction mappings in b -metric spaces," *Acta Mathematica et Informatica Universitatis Ostraviensis*, vol. 1, pp. 5–11, 1993.
- [5] H. Aydi, M. Bota, and S. Moradi, "A common fixed points for weak ϕ -contractions on b -metric spaces," *Fixed Point Theory*, vol. 13, pp. 337–346, 2012.
- [6] M. Abbas, J. R. Roshan, and S. Sedghi, "Common fixed point of four maps in b -metric spaces," *Journal of Mathematics and Statistics*, vol. 43, no. 4, pp. 613–624, 2014.
- [7] A. Hussain, T. Kanwal, Z. Mitrovic, and S. Radenovic, "Optimal solutions and applications to nonlinear matrix and integral equations via simulation function," *Filomat*, vol. 32, no. 7, pp. 6087–6106, 2018.
- [8] F. Lael, N. Saleem, and M. Abbas, "On the fixed points of multivalued mappings in b -metric spaces and their application to linear systems," *UPB Scientific Bulletin, Series A*, vol. 82, no. 4, pp. 121–130, 2020.
- [9] N. Hussain and M. Shah, "KKM mappings in cone b -metric spaces," *Computers & Mathematics with Applications*, vol. 61, pp. 1677–1684, 2011.
- [10] J. Fernandez, N. Malviya, Z. D. Mitrovic, A. Hussain, and V. Parvaneh, "Some fixed point results on N_b -cone metric spaces over Banach algebra," *Advances in Difference Equations*, vol. 2020, p. 529, 2020.
- [11] T. Kanwal, A. Hussain, P. Kumam, and E. Savas, "Weak partial b -metric spaces and nadler's theorem," *Mathematics*, vol. 7, no. 4, p. 332, 2019.
- [12] M. Abbas, F. Lael, and N. Saleem, "Fuzzy b -metric spaces: fixed point results for ψ -contraction correspondences and their application," *Axioms*, vol. 9, no. 2, p. 36, 2020.
- [13] A. Ansari, J. Kumar, and N. Saleem, "Inverse-C-class function on weak semi compatibility and fixed point theorems for expansive mappings in G -metric spaces," *Mathematica Moravica*, vol. 24, no. 1, pp. 93–108, 2020.
- [14] N. Saleem, I. Iqbal, B. Iqbal, and S. Radenovic, "Coincidence and fixed points of multivalued F -contractions in generalized metric space with application," *Journal of Fixed Point Theory and Applications*, vol. 22, no. 2020, pp. 1–24, 2020.
- [15] B. E. Rhoades, "Some theorems on weakly contractive maps," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 47, no. 4, pp. 2683–2693, 2001.

- [16] M. Abbas and D. Djoric, "Common fixed point theorem for four mappings satisfying generalized weak contractive condition," *Filomat*, vol. 24, no. 2, pp. 1–10, 2010.
- [17] S. Moradi, Z. Fathi, and E. Analouee, "The common fixed point of single-valued generalized φ_f -weakly contractive mappings," *Applied Mathematics Letters*, vol. 24, pp. 771–776, 2011.
- [18] O. Popescu, "Fixed point for ψ, φ -weak contractions," *Applied Mathematics Letters*, vol. 24, pp. 1–4, 2011.
- [19] A. Aghaiani, M. Abbas, and J. R. Roshan, "Common fixed point of generalized weak contractive mappings in partially ordered b -metric spaces," *Math. Slovaca*, vol. 4, pp. 941–960, 2014.
- [20] S. L. Singh and B. Prasad, "Some coincidence theorems and stability of iterative procedures," *Computers & Mathematics with Applications*, vol. 55, no. 11, pp. 2512–2520, 2008.
- [21] M. A. Alghamdi, N. Hussain, and P. Salimi, "Fixed point and coupled fixed point theorems on b -metric-like spaces," *Journal of Inequalities and Applications*, vol. 2013, pp. 1–25, 2013.
- [22] G. Jungck, "Compatible mappings and common fixed points," *International Journal of Mathematics and Mathematical Sciences*, vol. 9, no. 4, pp. 771–779, 1986.
- [23] K. Zoto, B. E. Rhoades, and S. Radenović, "Some generalizations for $(\alpha - \psi, \phi)$ contractions in b -metric-like spaces and an application," *Fixed Points Theory and Applications*, vol. 26, 2017.