

Retraction

Retracted: Types of Complex Fuzzy Relations with Applications in Future Commission Market

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

- (1) Discrepancies in scope
- (2) Discrepancies in the description of the research reported
- (3) Discrepancies between the availability of data and the research described
- (4) Inappropriate citations
- (5) Incoherent, meaningless and/or irrelevant content included in the article
- (6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

- [1] M. Khan, M. Zeeshan, S.-Z. Song, and S. Iqbal, "Types of Complex Fuzzy Relations with Applications in Future Commission Market," *Journal of Mathematics*, vol. 2021, Article ID 6685977, 14 pages, 2021.

Research Article

Types of Complex Fuzzy Relations with Applications in Future Commission Market

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In this paper, we introduce types of relations on complex fuzzy sets such as the complex fuzzy (CF) inverse relation, complex fuzzy reflexive relation, complex fuzzy symmetric relation, complex fuzzy antisymmetric relation, complex fuzzy transitive relation, complex fuzzy irreflexive relation, complex fuzzy asymmetric relation, complex fuzzy equivalence relation, and complex fuzzy-order relation. We study some basic results and particular examples of these relations. Moreover, we discuss the applications of complex fuzzy relations in Future Commission Market (FCM). We show that the introduction of CF relations to applications of FCMs can give a significant method for describing the temporal dependence between parameters of a Future Commission Market.

1. Introduction

Models reflecting the phenomena of real life with just choices of truth and falsehood are not enough to reflect the true reality of the problems. The explanation for this is that the models have many complications, which is why a framework needs to be built to deal with the models' ill-defined situations. There are now two ways to deal with these kinds of situations, one is to find the problems' numerical solutions and the other is to create a numerical model. We get numerical solutions to the problems in both cases. The second is about the fuzzy set theory, which includes the theory of probability, the theory of fuzzy soft sets, the theory of intuitionist fuzzy sets, and most specifically, the theory of neutrosophic sets. The later theory for dealing with problems involving complexities is more generalized. One of the acceptable examples of these theories is the fuzzy differential equations theory, which is more generalized than the differential equations to solve problems of everyday life with greater precision.

Zadeh [1] gave the description of a fuzzy set (FS) in 1965, which is similar to a probability function. For models of real-world problems in different branches of science, a fuzzy set

plays a vital role. Fuzzy set theory has many applications in operational research, decision making, medicine, engineering design, psychology, quantum physics, image processing, mathematical chemistry, biological classification, thermodynamics, economics, and nonequilibrium. Dubois et al. in [2] discussed the applications of fuzzy sets in approximate reasoning and information systems, pattern recognition and image processing, decision analysis, operation research and statistics, and modeling and control of systems. Ngan et al. in [3] provided two numerical examples of applying the complex t-norm and t-conorm to multicriteria decision making in the context of medicine-related problems using medical datasets. Nisren et al. introduced the concept of complex multifuzzy soft expert set (CMFSES) and discussed the application of a complex multifuzzy expert soft set in decision-making problems [4]. Poodeh studied and evaluated a randomized-learning approach to train this neurofuzzy system and proposed a machine-learning algorithm, which is designed for fast training of a compact, accurate forecasting model [5]. Singh in [6] introduced a method to provide an effective way to analyze the uncertainty and vagueness in a complex (or dynamic) dataset using a complex vague concept lattice. Xindong et al. in [7]

discussed the relationship between the distance measure, the similarity measure, the entropy, and the inclusion measure for Pythagorean fuzzy sets. They showed the efficiency of the proposed similarity measure in pattern recognition, clustering analysis, and medical diagnosis. Moreover, Xindong et al. in [8] studied deeper insights into the decision-making problem based on the interval-valued fuzzy soft set. Xindong et al. provided two novel algorithms in decision-making problems under a Pythagorean fuzzy environment [9]. Naz et al. developed a new decision-making approach based on graph theory to deal with the multiattribute decision-making problems. They utilized the numerical examples concerning the energy project selection and software evaluation to show the detailed implementation procedure and reliability of our method in solving multiattribute decision-making problems under hesitant fuzzy, interval-valued hesitant fuzzy, and a hesitant triangular fuzzy environment [10].

Ramot et al. in [11] first gave the concept of a complex fuzzy set (CFS). The generalization of a real number set introduced by Gauss in 1795 is the complex number set. Accordingly, a CFS is the extension of a fuzzy set, the range of which extends from a closed interval $[0, 1]$ to a disc of radius one in a complex plane. The membership function of CFS C is denoted as $\lambda_C(u)$ and defined on the universal U as for any $u \in U$ a complex value in the disc of radius one in a complex plane. Thus, all values of $\lambda_C(u)$ exist inside a circle of radius one in a complex plane and $\lambda_C(u) = a_C(u)e^{ip_C(u)}$, where $i = \sqrt{-1}$. The term $p_C(u)$ is said to be phase term, $a_C(u)$ is said to be an amplitude term, and both of these are real valued with $a_C(u) \in [0, 1]$. The CFS C is represented as $\{(u, \lambda_C(u)) \mid u \in U\}$.

Imprecise, inconsistent, and incomplete knowledge of the periodic nature cannot be treated by fuzzy sets and intuitionistic fuzzy sets. These theories refer to various fields of research, but in both sets, there is one significant weakness, that is, a lack of capacity to discuss two-dimensional phenomena. Ramot presented a complex fuzzy set to address this challenge. The phase term of the CFS plays a crucial role in defining the functionality of the complex fuzzy set model. This term differentiates a model of the CFS from all other models available in the literature. The ability of a complex fuzzy set to depict two-dimensional phenomena makes it superior to the handling of vague and intuitive details prevalent in time-periodic phenomena. Complex fuzzy sets, their classes, and logic play an important role in applications including periodic event prediction and advanced control systems. A complex fuzzy set is somewhat similar to a Fourier transform; in reality, it is the particular form of the Fourier transformation by limiting the range of the Fourier transformation to a complex disc unit. Fourier transform has a lot of applications in various fields such as in signals and systems, communication, astronomy, geology, and optics. A complex fuzzy set can also be used in models such as the Fourier transform. Several other real-life phenomena are vague and cannot be modeled using one-dimensional variables. For example, objects can be represented as a collection of measurements in pattern recognition and are seen as vectors in a multidimensional space. These

multidimensional variables cannot be expressed through a simple combination of variables, particularly the consideration of fuzzy sets. These types of sets can be expressed via complex classes. For periodic phenomena, a complex fuzzy set is very useful. Ramot et al. proposed that the intermittent problems or repeated-problem phenomenon be more precisely modeled using the phase term of the complex fuzzy set membership, such as describing the effect of two countries' financial measures on each other over time. He suggested that signal processing is yet another attractive area of operation for a complex fuzzy set. Xueling et al. in [12] proposed the model for identifying the reference signal out of largely interested signals by using complex fuzzy sets. In addition, it is used to convey solar activity (solar maximum and solar minimum) by means of the average sunspot number [11]. Dick suggested that one of the beneficial applications of complex fuzzy sets is to use it to represent relatively periodic behavioral phenomena [13]. Traffic congestions in a big city are aperiodic phenomena that never repeat themselves. Complex fuzzy logic can also be used to solve those forms of issues more easily and reliably than fuzzy logic. Akram et al. [14] introduced the concept of competition graphs under a complex fuzzy environment. They described an application in the ecosystem. Moreover, Akram et al. [15] discussed the complex Pythagorean Dombi fuzzy graph (CPDFG). They utilized CPDFAA and CPDFGA operators in solving a decision-making numerical example. Akram et al. [16] gave the notion of the complex Pythagorean fuzzy planar graph (CPFPG), and an extension of a Pythagorean fuzzy planar graph is presented to study the planarity. Moreover, Akram et al. proposed a new graph, a complex Pythagorean fuzzy competition graph by combining the complex Pythagorean fuzzy information with a competition graph. They also investigate the two extensions of complex Pythagorean fuzzy competition graphs, namely, complex Pythagorean fuzzy k -competition and complex Pythagorean fuzzy p -competition graphs [17].

The idea of relations is one of the most important notions in pure and applied science. Science has been defined as the discovery of the relation between events, objects, and states. Relations are associations that remain at the very core of the majority of science and engineering methodological approaches. Fuzzy relations in fuzzy theory are important concepts and have been commonly used in many fields, such as fuzzy control, fuzzy clustering, and uncertainty reasoning. In fuzzy diagnosis and fuzzy modeling, they also play a significant role. How to estimate and compare them is a significant issue when fuzzy relations are used in practice. Some researchers have carried out ambiguous measurements of fuzzy relations.

Klir studied the crisp relations in [18]. A crisp relation shows the existence or absence of association, interconnectedness, or interaction between the parameters. The relation between two sets is denoted by $R(M, N)$, and its membership function is represented by $\lambda_R(m, n)$, where $m \in M$ and $n \in N$. The membership function $\lambda_R(m, n)$ has two values either 1 or 0. The generalization of a crisp relation is the fuzzy relation. The fuzzy relation was discussed by Mendel in [19]. Fuzzy relations show a degree of the

presence or absence of association, interaction, or interconnectedness between the elements of two or more fuzzy sets. A fuzzy relation between two fuzzy sets M and N is denoted by $R(M, N)$, and its membership function is represented by $\lambda_R(m, n)$, where $m \in M$ and $n \in N$. All the values of $\lambda_R(m, n) \in [0, 1]$. Fuzzy relations play a vital role in a fuzzy logic system. Tripathi et al. in [20] used the complex fuzzy relations to obtain diagnostic conclusions about diabetes by restricting grade values to symptoms of a disease from 0 to 1. Majid in [21] discussed some important compositions of fuzzy relations for predicting scores in cricket. Moreover, it studied the restoration and the identification of the causes (diagnosis) through the observed effects (symptoms) on the basis of fuzzy relations.

In crisp relations and fuzzy relations, there is one significant weakness, which is a lack of capacity to examine two-dimensional phenomena. Ramot discussed the complex fuzzy relations in [11], which is the generalization of crisp relations and fuzzy relations. Complex fuzzy relations represent both the degree of the presence or absence of association, interaction, or interconnectedness and the phase of association, interaction, or interconnectedness between the elements of two or more crisp sets. For any two crisp sets M and N , the complex fuzzy relation is denoted by $R(M, N)$. The relation $R(M, N)$ may be represented as the set of ordered pairs $R(M, N) = \{(m, n), \lambda_R(m, n) / (m, n) \in tMn \times qN\}$. The membership function of a complex fuzzy relation is denoted by $\lambda_R(m, n)$, and all its values lie within the unit circle in the complex plane. Ramot et al. in [11] discussed the applications of complex fuzzy relations in Future Commission Merchant.

The purpose of this article is two-fold. The first half aims to present the theoretical foundations of the types of complex fuzzy relations. In any field of mathematics, we have needed such relations to solve problems. We can solve lots of problems with the help of these relations easily. The second half aims to present these theoretical foundations and key techniques in Future Commission Market, decision making, and the principle of the types of complex fuzzy relations in a coherent manner. The purpose of these innovative concepts is to provide a new approach with useful mathematical tools to address the fundamental problem of decision making (FCM problem). The generality of these new concepts is given special importance, illustrating how many interesting uncertainty problems can be formulated easily. These applied contexts provide solid evidence of the wide applications of the complex fuzzy relation approach to the model. This article will stimulate the interest in types of complex fuzzy relations and their application in various fields of science.

Now in this paper, we define some types of complex fuzzy relations such as the complex fuzzy inverse relation, complex fuzzy reflexive relation, complex fuzzy symmetric relation, complex fuzzy antisymmetric relation, complex fuzzy transitive relation, complex fuzzy irreflexive relation, complex fuzzy asymmetric relation, complex fuzzy equivalence relation, complex fuzzy-order relation, and complex fuzzy equivalence class and discuss the particular examples

of these relations. We also study some basic results. Moreover, we discuss the applications of complex fuzzy relations in Future Commission Merchant.

2. Preliminaries

We will discuss here the types of complex fuzzy relations and also discuss particular examples of these relations.

Definition 1 (See [11]). A CFS S , defined on a universal set U , is represented by a grade value $\lambda_C(u)$ whose codomain is the disc of the radius on in the complex plane. Mathematically, the grade value of CFS C can be represented by $\lambda_C(u) = a_C(u)e^{ip_C(u)}$ where $a_C(u)$ and $p_C(u)$ are known as an amplitude term and phase term, respectively. These two terms are real valued and $a_C(u) \in [0, 1]$.

Mathematically, CFS can be expressed as a set of ordered pairs given by

$$C = \{(u; \lambda_C(u)) : u \in U\}. \quad (1)$$

Definition 2 (See [22]). Let $X_m, m = 1, 2, 3, \dots, M$ be M CFS defined on U and $\lambda_{C_m}(u) = a_{C_m}(u)e^{ip_{C_m}(u)}$ their membership functions. The complex fuzzy Cartesian product of C_m denoted by $C_1 \times C_2 \times C_3 \times \dots \times C_m$ is specified by a function

$$\begin{aligned} \lambda_{C_1 \times C_2 \times C_3 \times \dots \times C_m}(u) &= a_{C_1 \times C_2 \times C_3 \times \dots \times C_m}(u)e^{ip_{C_1 \times C_2 \times C_3 \times \dots \times C_m}(u)}, \\ &= \min(a_{C_1}(u_1), a_{C_2}(u_2), \dots, a_{C_m}(u_m)) \\ &\quad e^{i \min(p_{C_1}(u_1), p_{C_2}(u_2), \dots, p_{C_m}(u_m))}. \end{aligned} \quad (2)$$

Definition 3 (See [11]). For any two crisps sets X and Y , the fuzzy relation $R(X, Y)$ is a fuzzy subset of the product space $X \times Y$. The grade value of the fuzzy relation is represented by $\lambda_{X \times Y}(x, y)$, where $x \in X$ and $y \in Y$. All the values of grade value lie in the closed interval $[0, 1]$.

The fuzzy relation may be represented as the set of ordered pairs

$$R(X, Y) = \left\{ \frac{(x, y), \lambda_{X \times Y}(x, y)}{(x, y)} \in X \times Y \right\}. \quad (3)$$

Definition 4 (See [11]). For any two crisp sets X and Y , the complex fuzzy relation $R(X, Y)$ is a complex fuzzy subset of the product space $X \times Y$. The grade value of the complex fuzzy relation is represented by $\lambda_{X \times Y}(x, y)$, where $x \in X$ and $y \in Y$. All the values of the grade value of the complex fuzzy relation lie in unit disc in a complex plane.

The complex fuzzy relation may be represented as the set of ordered pairs:

$$R(X, Y) = G = \left\{ \frac{(x, y), \lambda_{X \times Y}(x, y)}{(x, y)} \in X \times Y \right\}, \quad (4)$$

where

$$\lambda_{X \times Y}(x, y) = \min(a_X(x), a_X(y)) \frac{e^{i \min(p_X(x), p_Y(y))}}{(x, y)}. \quad (5)$$

Example 1. For any complex fuzzy set $C = (0.8e^{i\pi}/1 + 1e^{i2\pi})/2$, the product space $X \times X$ is

$$C \times C = \frac{0.8e^{i\pi}}{(1, 1)} + \frac{0.8e^{i\pi}}{(1, 2)} + \frac{0.8e^{i\pi}}{(2, 1)} + \frac{1e^{i2\pi}}{(2, 2)}. \quad (6)$$

So, the complex fuzzy relation on C is subset of $C \times C$ is given by

$$R(C, C) = G = \frac{0.8e^{i\pi}}{(1, 1)} + \frac{0.8e^{i\pi}}{(1, 2)} + \frac{1e^{i2\pi}}{(2, 2)}. \quad (7)$$

3. Types of Complex Fuzzy Relations

Definition 5. If G is a complex fuzzy relation on a complex fuzzy set C , then the inverse relation G^{-1} in C is defined by

$$G^{-1} = \left\{ \frac{\lambda_{C \times C}(v, u)}{\lambda_{C \times C}(u, v)} \in G \right\}. \quad (8)$$

Definition 6. If G is a complex fuzzy relation on a complex fuzzy set C , G is said to be complex fuzzy reflexive relation if for all $u \in U$ and $\lambda_C(u) \in C \implies \lambda_{C \times C}(u, u) \in G$.

Example 2. For any complex fuzzy set $C = (0.4e^{i2\pi}/1) + (1e^{i(\pi/2)}/2) + (0.5e^{i\pi}/3)$ defined on any universal set $U = \{1, 2, 3\}$, the product space $C \times C$ is

$$\begin{aligned} C \times C = & \frac{0.4e^{i2\pi}}{(1, 1)} + \frac{0.4e^{i(\pi/2)}}{(1, 2)} + \frac{0.4e^{i\pi}}{(1, 3)} + \frac{0.4e^{i(\pi/2)}}{(2, 1)} + \frac{1e^{i(\pi/2)}}{(2, 2)} \\ & + \frac{0.5e^{i(\pi/2)}}{(2, 3)} + \frac{0.4e^{i\pi}}{(3, 1)} + \frac{0.5e^{i(\pi/2)}}{(3, 2)} + \frac{0.5e^{i\pi}}{(3, 3)}. \end{aligned} \quad (9)$$

So, the complex fuzzy reflexive relation on C is subset of $C \times C$ is given by

$$G = \frac{0.4e^{i2\pi}}{(1, 1)} + \frac{1e^{i(\pi/2)}}{(2, 2)} + \frac{0.5e^{i\pi}}{(3, 3)}. \quad (10)$$

Definition 7. If G is a complex fuzzy relation on a complex fuzzy set C , G is said to be a complex fuzzy symmetric relation if $\lambda_{C \times C}(u, v) \in G \implies \lambda_{C \times C}(v, u) \in G$.

Example 3. For any complex fuzzy set $C = (0.4e^{i2\pi}/a) + (1e^{i(\pi/2)}/b) + (0.5e^{i\pi}/c)$ defined on any universal set $U = \{a, b, c\}$, the product space $C \times C$ is

$$\begin{aligned} C \times C = & \frac{0.4e^{i2\pi}}{(a, a)} + \frac{0.4e^{i(\pi/2)}}{(a, b)} + \frac{0.4e^{i\pi}}{(a, c)} + \frac{0.4e^{i(\pi/2)}}{(b, a)} + \frac{1e^{i(\pi/2)}}{(b, b)} \\ & + \frac{0.5e^{i(\pi/2)}}{(b, c)} + \frac{0.4e^{i\pi}}{(c, a)} + \frac{0.5e^{i(\pi/2)}}{(c, b)} + \frac{0.5e^{i\pi}}{(c, c)}. \end{aligned} \quad (11)$$

So, the complex fuzzy symmetric relation on C is given by

$$G = \frac{0.4e^{i2\pi}}{(a, a)} + \frac{1e^{i(\pi/2)}}{(b, b)} + \frac{0.5e^{i\pi}}{(c, c)} + \frac{0.4e^{i(\pi/2)}}{(a, b)} + \frac{0.4e^{i(\pi/2)}}{(b, a)}. \quad (12)$$

Definition 8. If G is a complex fuzzy relation on a complex fuzzy set C , G is said to be a complex fuzzy antisymmetric relation if $\lambda_{C \times C}(u, v) \in G$ and $\lambda_{C \times C}(v, u) \in G \implies \lambda_{C \times C}(u, v) = \lambda_{C \times C}(v, u)$.

Example 4. For any complex fuzzy set $C = (0.5e^{i2\pi}/x) + (1e^{i(\pi/2)}/y) + (0e^{i\pi}/z)$ defined on any universal set $U = \{a, b, c\}$, the product space $C \times C$ is

$$\begin{aligned} C \times C = & \frac{0.5e^{i2\pi}}{(x, x)} + \frac{0.5e^{i(\pi/2)}}{(x, y)} + \frac{0e^{i\pi}}{(x, z)} + \frac{0.5e^{i(\pi/2)}}{(y, x)} + \frac{1e^{i(\pi/2)}}{(y, y)} \\ & + \frac{0e^{i(\pi/2)}}{(y, z)} + \frac{0e^{i\pi}}{(z, x)} + \frac{0e^{i(\pi/2)}}{(z, y)} + \frac{0e^{i\pi}}{(z, z)}. \end{aligned} \quad (13)$$

So, the complex fuzzy antisymmetric relation on C is given by

$$G = \frac{0.5e^{i2\pi}}{(x, x)} + \frac{1e^{i(\pi/2)}}{(y, y)} + \frac{0e^{i\pi}}{(z, z)}. \quad (14)$$

Definition 9. If G is a complex fuzzy relation on a complex fuzzy set C , G is said to be complex fuzzy transitive relation if $\lambda_{C \times C}(u, v) \in G$ and $\lambda_{C \times C}(v, w) \in G \implies \lambda_{C \times C}(u, w) \in G$.

Example 5. For any complex fuzzy set $C = (0.5e^{i2\pi}/x) + (1e^{i(\pi/2)}/y) + (0e^{i\pi}/z)$ defined on any universal set $U = \{a, b, c\}$, the product space $C \times C$ is

$$\begin{aligned} C \times C = & \frac{0.5e^{i2\pi}}{(x, x)} + \frac{0.5e^{i(\pi/2)}}{(x, y)} + \frac{0e^{i\pi}}{(x, z)} + \frac{0.5e^{i(\pi/2)}}{(y, x)} + \frac{1e^{i(\pi/2)}}{(y, y)} \\ & + \frac{0e^{i(\pi/2)}}{(y, z)} + \frac{0e^{i\pi}}{(z, x)} + \frac{0e^{i(\pi/2)}}{(z, y)} + \frac{0e^{i\pi}}{(z, z)}. \end{aligned} \quad (15)$$

So, the complex fuzzy transitive relation on C is given by

$$G = \frac{0.5e^{i2\pi}}{(x, x)} + \frac{1e^{i(\pi/2)}}{(y, y)} + \frac{0e^{i\pi}}{(z, z)} + \frac{0.5e^{i(\pi/2)}}{(x, y)} + \frac{0e^{i\pi}}{(x, z)}. \quad (16)$$

Definition 10. If G is a complex fuzzy relation on a complex fuzzy set C , G is said to be a complex fuzzy irreflexive relation if for all $u \in U$ and $\lambda_C(u) \in C \implies \lambda_{C \times C}(u, u) \notin G$.

Example 6. For any complex fuzzy set $C = (0.4e^{i2\pi}/1) + (1e^{i(\pi/2)}/2) + (0.5e^{i\pi}/3)$ defined on any universal set $U = \{1, 2, 3\}$, the product space $C \times C$ is

$$C \times C = \frac{0.4e^{i2\pi}}{(1, 1)} + \frac{0.4e^{i(\pi/2)}}{(1, 2)} + \frac{0.4e^{i\pi}}{(1, 3)} + \frac{0.4e^{i(\pi/2)}}{(2, 1)} + \frac{1e^{i(\pi/2)}}{(2, 2)} + \frac{0.5e^{i(\pi/2)}}{(2, 3)} + \frac{0.4e^{i\pi}}{(3, 1)} + \frac{0.5e^{i(\pi/2)}}{(3, 2)} + \frac{0.5e^{i\pi}}{(3, 3)}. \tag{17}$$

So, the complex fuzzy irreflexive relation on C is subset of $C \times C$ is given by

$$G = \frac{0.4e^{i2\pi}}{(1, 1)} + \frac{1e^{i(\pi/2)}}{(2, 2)} + \frac{0.4e^{i\pi}}{(1, 3)} + \frac{0.4e^{i(\pi/2)}}{(2, 1)}. \tag{18}$$

Definition 11. If G is a complex fuzzy relation on a complex fuzzy set C , G is said to be a complex fuzzy asymmetric relation if $\lambda_{C \times C}(u, v) \in G \implies \lambda_{C \times C}(v, u) \notin G$.

Example 7. For any complex fuzzy set $C = (0.4e^{i2\pi}/a) + (1e^{i(\pi/2)}/b) + (0.5e^{i\pi}/c)$ defined on any universal set $U = \{a, b, c\}$, the product space $C \times C$ is

$$C \times C = \frac{0.4e^{i2\pi}}{(a, a)} + \frac{0.4e^{i(\pi/2)}}{(a, b)} + \frac{0.4e^{i\pi}}{(a, c)} + \frac{0.4e^{i(\pi/2)}}{(b, a)} + \frac{1e^{i(\pi/2)}}{(b, b)} + \frac{0.5e^{i(\pi/2)}}{(b, c)} + \frac{0.4e^{i\pi}}{(c, a)} + \frac{0.5e^{i(\pi/2)}}{(c, b)} + \frac{0.5e^{i\pi}}{(c, c)}. \tag{19}$$

So, the complex fuzzy asymmetric relation on C is given by

$$G = \frac{0.4e^{i2\pi}}{(a, a)} + \frac{1e^{i(\pi/2)}}{(b, b)} + \frac{0.5e^{i\pi}}{(c, c)} + \frac{0.4e^{i(\pi/2)}}{(a, b)}. \tag{20}$$

Definition 12. A relation G is said to be a complex fuzzy equivalence relation if it satisfies the following conditions:

- (i) G is reflexive
- (ii) G is symmetric
- (iii) G is transitive

Example 8. For any complex fuzzy set $C = (0.8e^{i2\pi}/u) + (0.5e^{i(\pi/2)}/v) + (0.2e^{i\pi}/v)$ defined on any universal set $U = \{1, 2, 3\}$, the product space $C \times C$ is given by

$$C \times C = \frac{0.8e^{i2\pi}}{(u, u)} + \frac{0.5e^{i(\pi/2)}}{(u, v)} + \frac{0.2e^{i\pi}}{(u, w)} + \frac{0.5e^{i(\pi/2)}}{(v, u)} + \frac{0.5e^{i(\pi/2)}}{(v, v)} + \frac{0.2e^{i(\pi/2)}}{(v, w)} + \frac{0.2e^{i\pi}}{(w, u)} + \frac{0.2e^{i(\pi/2)}}{(w, v)} + \frac{0.2e^{i\pi}}{(w, w)}. \tag{21}$$

So, the complex fuzzy equivalence relation on C is given by

$$G = \frac{0.8e^{i2\pi}}{(u, u)} + \frac{0.5e^{i(\pi/2)}}{(v, v)} + \frac{0.2e^{i\pi}}{(w, w)} + \frac{0.2e^{i(\pi/2)}}{(v, w)} + \frac{0.2e^{i(\pi/2)}}{(w, v)}. \tag{22}$$

Definition 13. A relation G is said to be a complex fuzzy-order relation if it satisfies the following conditions:

- (i) G is reflexive
- (ii) G is antisymmetric
- (iii) G is transitive

Example 9. For any complex fuzzy set $C = (0.8e^{i2\pi}/u) + (0.5e^{i(\pi/2)}/v) + (0.2e^{i\pi}/v)$ defined on any universal set $U = \{1, 2, 3\}$, the product space $C \times C$ is given by

$$C \times C = \frac{0.8e^{i2\pi}}{(u, u)} + \frac{0.5e^{i(\pi/2)}}{(u, v)} + \frac{0.2e^{i\pi}}{(u, w)} + \frac{0.5e^{i(\pi/2)}}{(v, u)} + \frac{0.5e^{i(\pi/2)}}{(v, v)} + \frac{0.2e^{i(\pi/2)}}{(v, w)} + \frac{0.2e^{i\pi}}{(w, u)} + \frac{0.2e^{i(\pi/2)}}{(w, v)} + \frac{0.2e^{i\pi}}{(w, w)}. \tag{23}$$

So, the complex fuzzy-order relation on C is given by

$$G = \frac{0.8e^{i2\pi}}{(u, u)} + \frac{0.5e^{i(\pi/2)}}{(v, v)} + \frac{0.2e^{i\pi}}{(w, w)} + \frac{0.5e^{i(\pi/2)}}{(u, v)} + \frac{0.2e^{i(\pi/2)}}{(v, w)}. \tag{24}$$

Definition 14. Let C be a complex fuzzy set and G be a complex fuzzy equivalence relation in C . If $\lambda_C(u) \in C$, then the complex fuzzy equivalence class of $\lambda_C(u)$ modulo G is the set $G_{\lambda_C(u)}$ defined by

$$G_{\lambda_C(u)} = \left\{ \frac{\lambda_C(v)}{\lambda_C(v, u)} \in G \right\}. \tag{25}$$

Example 10. For any complex fuzzy set $C = (1e^{i1.2\pi}/1) + (0.5e^{i\pi}/2) + (0e^{i(\pi/2)}/3)$ defined on any universal set $U = \{1, 2, 3\}$, the product space $C \times C$ is given by

$$C \times C = \frac{1e^{i1.2\pi}}{(1, 1)} + \frac{0.5e^{i\pi}}{(1, 2)} + \frac{0e^{i(\pi/2)}}{(1, 3)} + \frac{0.5e^{i\pi}}{(2, 1)} + \frac{0.5e^{i\pi}}{(2, 2)} + \frac{0e^{i(\pi/2)}}{(2, 3)} + \frac{0e^{i(\pi/2)}}{(3, 1)} + \frac{0e^{i(\pi/2)}}{(3, 2)} + \frac{0e^{i(\pi/2)}}{(3, 3)}. \tag{26}$$

Also, let

$$G = \frac{1e^{i1.2\pi}}{(1,1)} + \frac{0.5e^{i\pi}}{(2,2)} + \frac{0.5e^{i\pi}}{(2,1)} + \frac{0e^{i(\pi/2)}}{(3,2)} + \frac{0e^{i(\pi/2)}}{(3,3)}, \quad (27)$$

be the complex fuzzy relation in C . Then, the complex fuzzy equivalence class of $\lambda_C(1)$ is given by

$$G_{\lambda_C(1)} = \{\lambda_C(1), \lambda_C(2)\}. \quad (28)$$

The complex fuzzy equivalence class of $\lambda_C(2)$ is given by

$$G_{\lambda_C(2)} = \{\lambda_C(2), \lambda_C(3)\}. \quad (29)$$

Also, the complex fuzzy equivalence class of $\lambda_C(3)$ is given by

$$G_{\lambda_C(3)} = \{\lambda_C(3)\}. \quad (30)$$

Definition 15. Let C be a complex fuzzy set and G be a complex fuzzy relation in C . For any $\lambda_{C \times C}(u, v) \in G$ and $\lambda_{C \times C}(v, w) \in G$, $\lambda_{C \times C}(u, w) \in G^\circ G$ for all $u, v, w \in U$ (universal set). $G^\circ G$ is called the complex fuzzy composition relations.

4. Main Results

Proposition 1. If G and H are symmetric relations in a complex fuzzy set C , then $G \cap H$ is also a complex fuzzy symmetric relation in C .

Proof. Suppose that G and H are complex fuzzy symmetric relations in a complex fuzzy set C . Since $G \subseteq C \times C$ and $H \subseteq C \times C$, $G \cap H \subseteq C \times C$. Therefore, $G \cap H$ is a complex fuzzy relation in C .

Let $\lambda_{C \times C}(u, v) \in G \cap H$; then, $\lambda_{C \times C}(u, v) \in G$ and $\lambda_{C \times C}(u, v) \in H$. But, G and H are symmetric. Therefore $\lambda_{C \times C}(v, u) \in G$ and $\lambda_{C \times C}(v, u) \in H$, so that $\lambda_{C \times C}(v, u) \in G \cap H$. \square

Proposition 2. Let G be a complex fuzzy relation in a complex fuzzy set C ; then, G is symmetric if $G = G^{-1}$.

Proof. Suppose G is a complex fuzzy symmetric relation; then,

$$\lambda_{C \times C}(u, v) \in G \iff \lambda_{C \times C}(v, u) \in G \iff \lambda_{C \times C}(u, v) \in G^{-1}. \quad (31)$$

Thus,

$$G = G^{-1}. \quad (32)$$

Conversely, let $G = G^{-1}$; then,

$$\lambda_{C \times C}(u, v) \in G \iff \lambda_{C \times C}(u, v) \in G^{-1} \iff \lambda_{C \times C}(v, u) \in G. \quad (33)$$

Thus, G is a complex fuzzy symmetric relation. \square

Proposition 3. Let G be a complex fuzzy relation in a complex fuzzy set C ; then, G is transitive if $G^\circ G \subseteq G$.

Proof. Suppose G is transitive. Assume that $\lambda_{C \times C}(u, w) \in G^\circ G$. Then, there exists $v \in U$ such that $\lambda_{C \times C}(u, v) \in G$ and $\lambda_{C \times C}(v, w) \in G$. Since G is complex fuzzy transitive relation and, hence, by the transitive property, $\lambda_{C \times C}(u, w) \in G$.

Thus,

$$G^\circ G \subseteq G. \quad (34)$$

Conversely, suppose that $G^\circ G \subseteq G$. Then, $\lambda_{C \times C}(u, v) \in G$ and $\lambda_{C \times C}(v, w) \in G \iff \lambda_{C \times C}(u, w) \in G^\circ G \subseteq G$. Thus, $\lambda_{C \times C}(u, w) \in G$, and hence, G is transitive. \square

Proposition 4. If G is a complex fuzzy equivalence relation in a complex fuzzy set C , then $G^\circ G = G$.

Proof. Let $\lambda_{C \times C}(u, v) \in G^\circ G$; then, there exists $w \in U$ such that $\lambda_{C \times C}(u, w) \in G$ and $\lambda_{C \times C}(w, v) \in G$, but G is a complex fuzzy equivalence relation in C , so G is a complex fuzzy transitive relation. Therefore, by the transitive property, $\lambda_{C \times C}(u, v) \in G$. Thus,

$$G^\circ G \subseteq G. \quad (35)$$

Conversely, assume that $\lambda_{C \times C}(u, v) \in G$. Since G is a complex fuzzy reflexive relation, for $v \in U \implies \lambda_{C \times C}(v, v) \in G$. Now, $\lambda_{C \times C}(u, v) \in G$ and $\lambda_{C \times C}(v, v) \in G$. Since G is a complex fuzzy transitive relation, so by the transitive property, $\lambda_{C \times C}(u, v) \in G^\circ G$. Thus,

$$G \subseteq G^\circ G. \quad (36)$$

From (35) and (36), we have

$$G = G^\circ G. \quad (37)$$

\square

Proposition 5. The inverse of a complex fuzzy-order relation in a complex fuzzy set C is also a complex fuzzy-order relation in C .

Proof. Let G be a complex fuzzy-order relation in a complex fuzzy set C . To show that G^{-1} is also a complex fuzzy-order relation in C , we have to satisfy the three conditions of a complex fuzzy-order relation.

Since G is a complex fuzzy-order relation, for any $u \in U \implies \lambda_{C \times C}(u, u) \in G \implies \lambda_{C \times C}(u, u) \in G^{-1}$. Thus, G^{-1} is a complex fuzzy reflexive relation.

Let $\lambda_{C \times C}(u, v) \in G^{-1}$ and $\lambda_{C \times C}(v, u) \in G^{-1}$. Then, $\lambda_{C \times C}(v, u) \in G$ and $\lambda_{C \times C}(u, v) \in G$, but G is a complex fuzzy-order relation in C . Therefore, $\lambda_{C \times C}(u, v) = \lambda_{C \times C}(v, u)$, and hence, G^{-1} is antisymmetric.

Assume that $\lambda_{C \times C}(u, v) \in G^{-1}$ and $\lambda_{C \times C}(v, w) \in G^{-1}$. Then, $\lambda_{C \times C}(v, u) \in G$ and $\lambda_{C \times C}(w, v) \in G$. Since G is a complex fuzzy transitive relation. By the transitive property, $\lambda_{C \times C}(w, u) \in G$. Thus, $\lambda_{C \times C}(u, w) \in G^{-1}$, and hence, G^{-1} is a complex fuzzy transitive relation in C . Since G^{-1} satisfies all

the properties of a complex fuzzy-order relation, G^{-1} is a complex fuzzy-order relation. \square

Theorem 1. Let G be any complex fuzzy equivalence relation in a complex fuzzy set C . Then, $\lambda_{C \times C}(v, u) \in G$ if and only if $G_{\lambda_C(u)} = G_{\lambda_C(v)}$.

Proof. Suppose that $\lambda_{C \times C}(u, v) \in G$. Let $\lambda_C(w) \in G_{\lambda_C(u)}$. Then, $\lambda_{C \times C}(w, u) \in G$. Now, $\lambda_{C \times C}(w, u) \in G$ and $\lambda_{C \times C}(u, v) \in G$. But, since G is a complex fuzzy equivalence relation in a complex fuzzy set C , by the transitive property, $\lambda_{C \times C}(w, v) \in G$, so that $\lambda_C(w) \in G_{\lambda_C(v)}$. Thus,

$$G_{\lambda_C(u)} \subseteq G_{\lambda_C(v)}. \quad (38)$$

Let $\lambda_C(w) \in G_{\lambda_C(v)}$; then, $\lambda_{C \times C}(w, v) \in G$ and also $\lambda_{C \times C}(u, v) \in G$. Since G is a complex fuzzy symmetric relation, so by the symmetric property, $\lambda_{C \times C}(v, u) \in G$. Also, G is a complex fuzzy equivalence relation. Therefore, by the transitive property, $\lambda_{C \times C}(w, u) \in G$. Thus, $\lambda_C(w) \in G_{\lambda_C(u)}$, and hence,

$$G_{\lambda_C(v)} \subseteq G_{\lambda_C(u)}. \quad (39)$$

From (38) and (39), we have

$$G_{\lambda_C(v)} = G_{\lambda_C(u)}. \quad (40)$$

Conversely, suppose that $G_{\lambda_C(v)} = G_{\lambda_C(u)}$. Since G is a complex fuzzy equivalence relation, so by the reflexive property, $\lambda_{C \times C}(u, u) \in G$. By definition of a complex fuzzy equivalence class of modulo G ,

$$\lambda_C(u) \in G_{\lambda_C(u)} = G_{\lambda_C(v)} \lambda_C(u) \in G_{\lambda_C(v)}. \quad (41)$$

Hence, $\lambda_{C \times C}(u, v) \in G$. \square

5. Applications

In this section, we will discuss the application of complex fuzzy relations in Future Commission Market.

We are going to discuss a real-life application of newly defined types/properties of complex fuzzy relations. Ramot et al. in [11] discussed the application of the complex fuzzy relation in Future Commission Merchant. The physical meaning of the Ramot and proposed model is the same, but here, we will show that how the types of complex fuzzy relations play a significant role in real-life applications.

5.1. Decision-Making Method. Let U be a collection of financial indicators or indexes of any country. Possible components of this collection are import, export, agriculture, unemployment, and development rate, that is, $U = \{\text{import, export, agriculture, unemployment, development rate}\}$. Let

$$C = \{\lambda_C(\text{import}), \lambda_C(\text{export}), \lambda_C(\text{agriculture}), \lambda_C(\text{unemployment}), \lambda_C(\text{development rate})\} \quad (42)$$

represent the complex fuzzy sets and $\lambda_{C \times C}(x, y)$ represent the membership function of the complex fuzzy relations.

The membership function $\lambda_{C \times C}(x, y)$ is complex-valued, with a phase term and amplitude term. The amplitude term shows the grade of influence of one parameter on another parameter. The amplitude term with a value close to zero shows more influence, while a value close to zero shows a small influence. The phase term refers to the "phase" of control or time lag that characterizes one parameter's effect on another parameter.

Consider, for example, we find the relation set G for the complex fuzzy set C , that is, the cross product of C .

$$\begin{aligned} G = \{ & \lambda_{C \times C}(\text{import, import}), \lambda_{C \times C}(\text{import, export}), \lambda_{C \times C}(\text{import, agriculture}), \lambda_{C \times C}(\text{import, unemployment}), \\ & \lambda_{C \times C}(\text{import, development rate}), \\ & \lambda_{C \times C}(\text{export, import}), \lambda_{C \times C}(\text{export, export}), \lambda_{C \times C}(\text{export, agriculture}), \\ & \lambda_{C \times C}(\text{export, unemployment}), \lambda_{C \times C}(\text{export, development rate}), \\ & \lambda_{C \times C}(\text{agriculture, import}), \lambda_{C \times C}(\text{agriculture, export}), \\ & \lambda_{C \times C}(\text{agriculture, agriculture}), \lambda_{C \times C}(\text{agriculture, unemployment}), \\ & \lambda_{C \times C}(\text{agriculture, development rate}), \lambda_{C \times C}(\text{unemployment, import}), \\ & \lambda_{C \times C}(\text{unemployment, export}), \lambda_{C \times C}(\text{unemployment, agriculture}), \\ & \lambda_{C \times C}(\text{unemployment, unemployment}), \lambda_{C \times C}(\text{unemployment, development rate}), \\ & \lambda_{C \times C}(\text{development rate, import}), \lambda_{C \times C}(\text{development rate, export}), \\ & \lambda_{C \times C}(\text{development rate, agriculture}), \lambda_{C \times C}(\text{development rate, unemployment}), \\ & \lambda_{C \times C}(\text{development rate, development rate})\}. \end{aligned} \quad (43)$$

The relation G is an equivalence relation, that is, G is a complex fuzzy reflexive, complex fuzzy symmetric, and complex fuzzy transitive relation. From this relation, we can easily determine the influence of one parameter on another parameter. For example, we take $\lambda_{C \times C}(\text{import, agriculture}) = (\min$

$$\{a_C(\text{import}), a_C(\text{agriculture})\} e^{i \min\{p_C(\text{import}), p_C(\text{agriculture})\}} / (\text{import, agriculture}).$$

The $\min\{a_C(\text{import}), a_C(\text{agriculture})\}$ shows the degree of influence of import on agriculture or agriculture on import of a country with respect to time.

$$\min\{p_C(\text{import}), p_C(\text{agriculture})\}. \quad (44)$$

This means that the degree of influence depends on the min value of the parameter. Moreover, if we know the influence of import on agriculture and agriculture on development rate, then by the complex fuzzy transitive relation, we can determine the influence of import on the development rate. Similarly, we have to find the influence of each parameter on the other. The more significant relation is the complex fuzzy transitive relation because if we know the degree of influence of the first parameter on the second parameter and second parameter on the third parameter, then by the complex fuzzy transitive relation, we can find the

degree of influence of the first parameter on the third parameter. Moreover, the complex fuzzy symmetric relation shows that the degree of influence of the first parameter on the second parameter and the second parameter on the first parameter is the same.

Example 11. Consider a complex fuzzy set

$$C = \left\{ \frac{0.4e^{i\pi}}{\text{development rate}} + \frac{0.8e^{i(\pi/2)}}{\text{agriculture}} + \frac{0.2e^{i2\pi}}{\text{export}} \right\}. \quad (45)$$

Then, the relation G on $C \times C$ is

$$G = C \times C = \left\{ \frac{\min\{0.4, 0.4\}e^{i \min\{\pi, \pi\}}}{(\text{development rate}, \text{development rate})} + \frac{\min\{0.4, 0.8\}e^{i \min\{\pi, (\pi/2)\}}}{(\text{development rate}, \text{agriculture})} + \frac{\min\{0.4, 0.2\}e^{i \min\{\pi, 2\pi\}}}{(\text{development rate}, \text{export})} + \frac{\min\{0.8, 0.4\}e^{i \min\{(\pi/2), \pi\}}}{(\text{agriculture}, \text{development rate})} + \frac{\min\{0.8, 0.8\}e^{i \min\{(\pi/2), (\pi/2)\}}}{(\text{agriculture}, \text{agriculture})} + \frac{\min\{0.8, 0.2\}e^{i \min\{\pi/2, 2\pi\}}}{(\text{agriculture}, \text{export})} + \frac{\min\{0.2, 0.4\}e^{i \min\{2\pi, \pi\}}}{(\text{export}, \text{development rate})} + \frac{\min\{0.2, 0.8\}e^{i \min\{2\pi, (\pi/2)\}}}{(\text{export}, \text{agriculture})} + \frac{\min\{0.2, 0.2\}e^{i \min\{(\pi/2), \pi\}}}{(\text{export}, \text{export})} \right\}, \quad (46)$$

$$G = C \times C = \left\{ \frac{0.4e^{i\pi}}{(\text{development rate}, \text{development rate})} + \frac{0.4, e^{i(\pi/2)}}{(\text{development rate}, \text{agriculture})} + \frac{0.2e^{i\pi}}{(\text{development rate}, \text{export})} + \frac{0.4e^{i(\pi/2)}}{(\text{agriculture}, \text{development rate})} + \frac{0.8e^{i(\pi/2)}}{(\text{agriculture}, \text{agriculture})} + \frac{0.2e^{i(\pi/2)}}{(\text{agriculture}, \text{export})} + \frac{0.2e^{i\pi}}{(\text{export}, \text{development rate})} + \frac{0.2e^{i(\pi/2)}}{(\text{export}, \text{agriculture},)} + \frac{0.2e^{i(\pi/2)}}{(\text{export}, \text{export})} \right\}.$$

From the abovementioned relation G (complex fuzzy equivalence relation), we can determine the influence of the development rate, agriculture, and export on each other. For example,

$$\lambda_{C \times C}(\text{agriculture, development rate}) = 0.4e^{i(\pi/2)} \in G. \quad (47)$$

Here, 0.4 shows the degree of influence of agriculture on the development rate with respect to half a month. Moreover, the membership function $\lambda_{C \times C}(\text{agriculture, export})$, that is,

$$\lambda_{C \times C}(\text{agriculture, export}) = 0.2e^{i(\pi/2)} \in G, \quad (48)$$

shows that the degree of influence of agriculture on export is 0.2 with respect to the half a month.

If we compare the influence of agriculture on the development rate and export, we find that the degree of influence of agriculture on export is less than the degree of influence on the development rate.

From the abovementioned relation G , we have the same degree of influence of agriculture on the development rate and the development rate on agriculture.

Moreover, if we have the degree of influence of agriculture on the development rate and the degree of influence of the development rate on export, then by complex fuzzy transitive relations, we can obtain the degree of influence of agriculture on export.

In the second example, we will discuss the degree of influence of American financial indexes on China's financial indexes and China's financial indexes on Saudi Arabia's financial indexes.

Example 12. Let C_1 , C_2 , and C_3 represent the set of America, China, and Saudi Arabia's financial indexes. A possible collection of these sets are import, export, growth rate, interest rate, and unemployment rate. They are all complex-valued functions. Let the complex fuzzy relation $\lambda_{C_1 \times C_2}(x, y)$ represent the relation of influence of American financial indexes on China financial indexes and the complex fuzzy relation $\lambda_{C_2 \times C_3}(x, y)$ represent the relation of influence of China financial indexes on Saudi Arabia financial indexes, where x, y represent any two parameters.

Consider

$$\begin{aligned} C_1 &= \{\lambda_{C_1}(\text{import}), \lambda_{C_1}(\text{export}), \lambda_{C_1}(\text{growth rate}), \lambda_{C_1}(\text{interest rate}), \lambda_{C_1}(\text{unemployment rate})\}, \\ C_2 &= \{\lambda_{C_2}(\text{import}), \lambda_{C_2}(\text{export}), \lambda_{C_2}(\text{growth rate}), \lambda_{C_2}(\text{interest rate}), \lambda_{C_2}(\text{unemployment rate})\}, \\ C_3 &= \{\lambda_{C_3}(\text{import}), \lambda_{C_3}(\text{export}), \lambda_{C_3}(\text{growth rate}), \lambda_{C_3}(\text{interest rate}), \lambda_{C_3}(\text{unemployment rate})\}. \end{aligned} \quad (49)$$

Now, the complex fuzzy relation G_1 between C_1 and C_2 is given by

$$\begin{aligned} G_1 = C_1 \times C_2 = & \{\lambda_{C_1 \times C_2}(\text{import, import}), \lambda_{C_1 \times C_2}(\text{import, export}), \\ & \lambda_{C_1 \times C_2}(\text{import, growth rate}), \lambda_{C_1 \times C_2}(\text{import, interest rate}), \\ & \lambda_{C_1 \times C_2}(\text{import, unemployment rate}), \lambda_{C_1 \times C_2}(\text{export, import}), \\ & \lambda_{C_1 \times C_2}(\text{export, export}), \lambda_{C_1 \times C_2}(\text{export, growth rate}), \\ & \lambda_{C_1 \times C_2}(\text{export, interest rate}), \lambda_{C_1 \times C_2}(\text{export, unemployment rate}), \\ & \lambda_{C_1 \times C_2}(\text{growth rate, import}), \lambda_{C_1 \times C_2}(\text{growth rate, export}), \\ & \lambda_{C_1 \times C_2}(\text{growth rate, growth rate}), \lambda_{C_1 \times C_2}(\text{growth rate, interest rate}), \\ & \lambda_{C_1 \times C_2}(\text{growth rate, unemployment rate}), \lambda_{C_1 \times C_2}(\text{interest rate, import}), \\ & \lambda_{C_1 \times C_2}(\text{interest rate, export}), \lambda_{C_1 \times C_2}(\text{interest rate, growth rate}), \\ & \lambda_{C_1 \times C_2}(\text{interest rate, interest rate}), \lambda_{C_1 \times C_2}(\text{interest rate, unemployment rate}), \\ & \lambda_{C_1 \times C_2}(\text{unemployment rate, import}), \lambda_{C_1 \times C_2}(\text{unemployment rate, export}), \\ & \lambda_{C_1 \times C_2}(\text{unemployment rate, growth rate}), \lambda_{C_1 \times C_2}(\text{unemployment rate, interest rate}), \\ & \lambda_{C_1 \times C_2}(\text{unemployment rate, unemployment rate})\}. \end{aligned} \quad (50)$$

The relation G_1 shows the relation of influence of American financial indexes on China financial indexes.

The complex fuzzy relation between C_2 and C_3 is given by

$$\begin{aligned}
 G_2 = C_2 \times C_3 = \{ & \lambda_{C_2 \times C_3}(\text{import, import}), \lambda_{C_2 \times C_3}(\text{import, export}), \\
 & \lambda_{C_2 \times C_3}(\text{import, growth rate}), \lambda_{C_2 \times C_3}(\text{import, interest rate}), \\
 & \lambda_{C_2 \times C_3}(\text{import, unemployment rate}), \lambda_{C_2 \times C_3}(\text{export, import}), \\
 & \lambda_{C_2 \times C_3}(\text{export, export}), \lambda_{C_2 \times C_3}(\text{export, growth rate}), \\
 & \lambda_{C_2 \times C_3}(\text{export, interest rate}), \lambda_{C_2 \times C_3}(\text{export, unemployment rate}), \\
 & \lambda_{C_2 \times C_3}(\text{growth rate, import}), \lambda_{C_2 \times C_3}(\text{growth rate, export}), \\
 & \lambda_{C_2 \times C_3}(\text{growth rate, growth rate}), \lambda_{C_2 \times C_3}(\text{growth rate, interest rate}), \\
 & \lambda_{C_2 \times C_3}(\text{growth rate, unemployment rate}), \lambda_{C_2 \times C_3}(\text{interest rate, import}), \\
 & \lambda_{C_2 \times C_3}(\text{interest rate, export}), \lambda_{C_2 \times C_3}(\text{interest rate, growth rate}), \\
 & \lambda_{C_2 \times C_3}(\text{interest rate, interest rate}), \lambda_{C_2 \times C_3}(\text{interest rate, unemployment rate}), \\
 & \lambda_{C_2 \times C_3}(\text{unemployment rate, import}), \lambda_{C_2 \times C_3}(\text{unemployment rate, export}), \\
 & \lambda_{C_2 \times C_3}(\text{unemployment rate, growth rate}), \lambda_{C_2 \times C_3}(\text{unemployment rate, interest rate}), \\
 & \lambda_{C_2 \times C_3}(\text{unemployment rate, unemployment rate}) \}.
 \end{aligned} \tag{51}$$

The relation G_2 represents the relation of influence of China's financial indexes on Saudi Arabia's financial indexes.

From relation G_1 and G_2 , we have the relation of influence of American financial indexes on China's financial indexes and China financial indexes on Saudi Arabia

financial indexes. By the complex fuzzy composition relation, we can find the relation of the influence of American financial indexes on Saudi Arabia's financial indexes.

The relation G represents the relation of influence of American financial indexes on Saudi Arabia financial indices, that is,

$$\begin{aligned}
 G_1 = C_1 \times C_3 = \{ & \lambda_{C_1 \times C_3}(\text{import, import}), \lambda_{C_1 \times C_3}(\text{import, export}), \\
 & \lambda_{C_1 \times C_3}(\text{import, growth rate}), \lambda_{C_1 \times C_3}(\text{import, interest rate}), \\
 & \lambda_{C_1 \times C_3}(\text{import, unemployment rate}), \lambda_{C_1 \times C_3}(\text{export, import}), \\
 & \lambda_{C_1 \times C_3}(\text{export, export}), \lambda_{C_1 \times C_3}(\text{export, growth rate}), \\
 & \lambda_{C_1 \times C_3}(\text{export, interest rate}), \lambda_{C_1 \times C_3}(\text{export, unemployment rate}), \\
 & \lambda_{C_1 \times C_3}(\text{growth rate, import}), \lambda_{C_1 \times C_3}(\text{growth rate, export}), \\
 & \lambda_{C_1 \times C_3}(\text{growth rate, growth rate}), \lambda_{C_1 \times C_3}(\text{growth rate, interest rate}), \\
 & \lambda_{C_1 \times C_3}(\text{growth rate, unemployment rate}), \lambda_{C_1 \times C_3}(\text{interest rate, import}), \\
 & \lambda_{C_1 \times C_3}(\text{interest rate, export}), \lambda_{C_1 \times C_3}(\text{interest rate, growth rate}), \\
 & \lambda_{C_1 \times C_3}(\text{interest rate, interest rate}), \lambda_{C_1 \times C_3}(\text{interest rate, unemployment rate}), \\
 & \lambda_{C_1 \times C_3}(\text{unemployment rate, import}), \lambda_{C_1 \times C_3}(\text{unemployment rate, export}), \\
 & \lambda_{C_1 \times C_3}(\text{unemployment rate, growth rate}), \lambda_{C_1 \times C_3}(\text{unemployment rate, interest rate}), \\
 & \lambda_{C_1 \times C_3}(\text{unemployment rate, unemployment rate}) \}.
 \end{aligned} \tag{52}$$

For example, the membership function $\lambda_{C_1 \times C_2}(\text{import, export}) \in G_1$ shows the degree of influence of American import on a China export and the membership function

$\lambda_{C_2 \times C_3}(\text{export, growth rate}) \in G_2$ shows the degree of influence of China export on a Saudi Arabia growth rate. By the complex fuzzy composition relation of these two relations,

we have the membership function $\lambda_{C_1 \times C_3}(\text{import, growth rate}) \in G$, which is more significant and gives the degree of influence of American import on the Saudi Arabia growth rate.

Moreover, the types of complex fuzzy relations play a vital role in applications. If we have known the degree of influence of American financial indexes on China financial indexes, then by the inverse complex fuzzy relations, we can find the degree of China financial indexes on American financial indexes.

Similarly, the complex fuzzy transitive relation is the most important type of complex fuzzy relations and plays a major role in applications. For example, if we have the degree of the influence of import on the interest rate and the interest rate on the unemployment rate, that is,

$\lambda_{C_1 \times C_3}(\text{import, interest rate})$ and $\lambda_{C_1 \times C_3}(\text{interest rate, unemployment rate})$, then by complex fuzzy transitive relations, we can easily find the degree of influence of import on interest rate $\lambda_{C_1 \times C_3}(\text{import, interest rate})$.

The abovementioned process may be applied to discuss the degree of financial indexes of more than three countries.

Example 13. Let $C_1 = \{0.8e^{i(\pi/2)}/\text{import}, (0.5e^{i\pi}/\text{export}), (1e^{i2\pi}/\text{interest rate}), (0.7e^{i(3\pi/2)}/\text{unemployment rate})\}$, $C_2 = \{0.5e^{i(3\pi/2)}/\text{import}, 0.9e^{i2\pi}/\text{export}, (0.3e^{i\pi}/\text{interest rate}), 0.4e^{i(\pi/2)}/\text{unemployment rate}\}$, and $C_3 = \{0.6e^{i(5\pi/2)}/\text{import}, 1e^{i(\pi/2)}/\text{export}, 0.7e^{i(\pi/3)}/\text{interest rate}, 0.5e^{i\pi}/\text{unemployment rate}\}$, represent the sets of American, China, and Saudi Arabia's financial indexes. Then, the relation G_1 on $C_1 \times C_2$ is

$$\begin{aligned}
 G_1 = C_1 \times C_2 = & \left\{ \frac{0.5e^{i(\pi/2)}}{(\text{import, import})}, \frac{0.8e^{i(\pi/2)}}{(\text{import, export})}, \right. \\
 & \frac{0.3e^{i(\pi/2)}}{(\text{import, interest rate})}, \frac{0.4e^{i(\pi/2)}}{(\text{import, unemployment rate})}, \\
 & \frac{0.5e^{i(3\pi/2)}}{(\text{export, import})}, \frac{0.5e^{i\pi}}{(\text{export, export})}, \\
 & \frac{0.3e^{i\pi}}{(\text{export, interest rate})}, \frac{0.4e^{i(\pi/2)}}{(\text{export, unemployment rate})}, \\
 & \frac{0.5e^{i(3\pi/2)}}{(\text{interest rate, import})}, \frac{0.9e^{i2\pi}}{(\text{interest rate, export})}, \\
 & \frac{0.3e^{i\pi}}{(\text{interest rate, interest rate})}, \frac{0.4e^{i(\pi/2)}}{(\text{interest rate, unemployment rate})}, \\
 & \frac{0.5e^{i(3\pi/2)}}{(\text{unemployment rate, import})}, \frac{0.7e^{i(3\pi/2)}}{(\text{unemployment rate, export})}, \\
 & \left. \frac{0.3e^{i\pi}}{(\text{unemployment rate, interest rate})}, \frac{0.4e^{i(\pi/2)}}{(\text{unemployment rate, unemployment rate})} \right\}.
 \end{aligned} \tag{53}$$

The relation G_1 shows the relation of influence of American financial indexes on China financial indexes.

The complex fuzzy relation between C_2 and C_3 is given by

$$\begin{aligned}
G_2 = C_2 \times C_3 = & \left\{ \frac{0.5e^{i(3\pi/2)}}{(\text{import, import})}, \frac{0.5e^{i(\pi/2)}}{(\text{import, export})}, \right. \\
& \frac{0.5e^{i(\pi/3)}}{(\text{import, interest rate})}, \frac{0.5e^{i(3\pi/2)}}{(\text{import, unemployment rate})}, \\
& \frac{0.6e^{i(5\pi/2)}}{(\text{export, import})}, \frac{0.9e^{i(\pi/2)}}{(\text{export, export})}, \\
& \frac{0.7e^{i(\pi/3)}}{(\text{export, interest rate})}, \frac{0.5e^{i\pi}}{(\text{export, unemployment rate})}, \\
& \frac{0.3e^{i(5\pi/2)}}{(\text{interest rate, import})}, \frac{0.3e^{i(\pi/2)}}{(\text{interest rate, export})}, \\
& \frac{0.3e^{i(\pi/3)}}{(\text{interest rate, interest rate})}, \frac{0.3e^{i\pi}}{(\text{interest rate, unemployment rate})}, \\
& \frac{0.4e^{i(\pi/2)}}{(\text{unemployment rate, import})}, \frac{0.4e^{i(\pi/2)}}{(\text{unemployment rate, export})}, \\
& \left. \frac{0.4e^{i(\pi/3)}}{(\text{unemployment rate, interest rate})}, \frac{0.4e^{i(\pi/2)}}{(\text{unemployment rate, unemployment rate})} \right\}. \tag{54}
\end{aligned}$$

The relation G_2 represents the relation of influence of China's financial indexes on Saudi Arabia's financial indexes. By the complex fuzzy composition relation, we can find the relation of the influence of American financial

indexes on Saudi Arabia's financial indexes. The relation G_3 represents the relation of influence of American financial indexes on Saudi Arabia financial indices, that is,

$$\begin{aligned}
G_3 = C_1 \times C_3 = & \left\{ \frac{0.6e^{i(\pi/2)}}{(\text{import, import})}, \frac{0.8e^{i(\pi/2)}}{(\text{import, export})}, \right. \\
& \frac{0.7e^{i(\pi/2)}}{(\text{import, interest rate})}, \frac{0.5e^{i(\pi/2)}}{(\text{import, unemployment rate})}, \\
& \frac{0.5e^{i\pi}}{(\text{export, import})}, \frac{0.5e^{i(\pi/2)}}{(\text{export, export})}, \\
& \frac{0.5e^{i(\pi/3)}}{(\text{export, interest rate})}, \frac{0.5e^{i\pi}}{(\text{export, unemployment rate})}, \\
& \frac{0.6e^{i2\pi}}{(\text{interest rate, import})}, \frac{1e^{i(\pi/2)}}{(\text{interest rate, export})}, \\
& \frac{0.7e^{i(\pi/3)}}{(\text{interest rate, interest rate})}, \frac{0.5e^{i\pi}}{(\text{interest rate, unemployment rate})}, \\
& \frac{0.6e^{i(3\pi/2)}}{(\text{unemployment rate, import})}, \frac{0.7e^{i(\pi/2)}}{(\text{unemployment rate, export})}, \\
& \left. \frac{0.7e^{i(\pi/3)}}{(\text{unemployment rate, interest rate})}, \frac{0.5e^{i\pi}}{(\text{unemployment rate, unemployment rate})} \right\}. \tag{55}
\end{aligned}$$

The abovementioned three relations G_1 , G_2 , and G_3 show the degree of influence of American financial indexes on China financial indexes, China financial indexes on Saudi Arabia financial indexes, and American financial indexes on Saudi Arabia financial indexes, respectively. Moreover, by the inverse complex fuzzy relations, we can obtain the degree of influence of China financial indexes on American financial indexes, Saudi Arabia financial indexes on China financial indexes, and Saudi Arabia financial indexes on American financial indexes.

Moreover, if we have the degree of the influence of import on the interest rate and the interest rate on the unemployment rate, then by complex fuzzy transitive relations, we can easily find the degree of influence of import on the interest rate. For example,

$0.5e^{i(\pi/3)}$ /import, interest rate, $\in \in G$ and $0.3e^{i\pi}$ /(interest rate, unemployment rate) $\in \in G$; then, by complex fuzzy transitive relations, we have $0.5e^{i(3\pi/2)}$ /(import, unemployment rate).

6. Comparison

There are many applications of crisp relations and fuzzy relations, particularly in fuzzy logic systems, diagnostic of symptoms, and decision making. But, there is one significant weakness, which is a lack of capacity to examine two-dimensional phenomena. They cannot deal with two-dimensional parameters. Ramot et al. in [11] introduced complex fuzzy relations which is the generalization of a fuzzy relation. Complex fuzzy relations show a degree of the presence or absence of association, interaction, or interconnectedness between two-dimensional parameters. Ramot et al. in [11] discussed the application of the complex fuzzy relation in Future Commission Merchant. He studied the degree of influence of parameters with respect to complex fuzzy relations. We explore this concept in detail and used the types of complex fuzzy relations such as the complex fuzzy transitive relation and complex fuzzy equivalence relation. The method we proposed here gives the degree of influence of the financial indexes of the three countries. This method can be used to find the degree of influence of financial indexes of more than three countries by using complex fuzzy transitive relations or composition of complex fuzzy relations. Moreover, this approach provides the degree of influence of the two countries financial indexes that do not have a direct relation by using the types of complex fuzzy relations. Similarly, by the inverse complex fuzzy relations, we can obtain the degree of influence of financial indexes on each other. However, our designed model is not complete, but it is stuck with a shortage of theoretical support. For applications, the types of complex fuzzy relations can be useful. Therefore, it will be significant for future work.

7. Conclusions

In this paper, we have discussed some new types of complex fuzzy relations such as the complex fuzzy inverse relation, complex fuzzy reflexive relation, complex fuzzy symmetric

relation, complex fuzzy antisymmetric relation, complex fuzzy transitive relation, complex fuzzy irreflexive relation, complex fuzzy asymmetric relation, complex fuzzy equivalence relation, complex fuzzy-order relation, and complex fuzzy equivalence class. We have presented some basic results and examples of these relations. Moreover, we have discussed the application of complex fuzzy relations in Future Commission Market. The complex fuzzy relation may be used in geology, signals and systems, and engineering fields for the identification of reference signals [23].

Data Availability

No data were used in this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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References

- [1] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.
- [2] D. Dubois and H. Prade, *Fundamentals of Fuzzy Sets*, Kluwer Academic Publisher, Boston, MA, USA, 2000.
- [3] T. T. Ngan, L. T. H. Lan, M. Ali et al., "Logic connectives of complex fuzzy sets," *Romanian Journal of Information Science and Technology*, vol. 21, no. 4, pp. 344–357, 2018.
- [4] G. S. Nisren, A. Hafeed, and A. R. Salleh, "Complex fuzzy soft expert sets," *AIP Conference Proceedings*, vol. 1830, pp. 1–8, Article ID 070020, 2017.
- [5] O. Y. Poodeh, *Applications of complex fuzzy sets in time-series prediction*, Ph.D Thesis, University of Alberta, Edmonton, Canada, 2017.
- [6] P. K. Singh, "Complex vague set based concept lattice," *Chaos, Solitons & Fractals*, vol. 96, pp. 145–153, 2017.
- [7] P. Xindong, H. Yuan, and Y. Yang, "Pythagorean fuzzy information measures and their applications," *International Journal of Intelligent Systems*, vol. 32, no. 10, pp. 991–1029, 2017.
- [8] P. Xindong and H. Garg, "Algorithms for interval-valued fuzzy soft sets in emergency decision making based on WDBA and CODAS with new information measure," *Computers & Industrial Engineering*, vol. 119, pp. 439–452, 2018.
- [9] P. Xindong and G. Selvachandran, "Pythagorean fuzzy set: state of the art and future directions," *Artificial Intelligence Review*, vol. 52, no. 3, pp. 1–55, 2017.
- [10] S. Naz and M. Akram, "Novel decision-making approach based on hesitant fuzzy sets and graph theory," *Computational and Applied Mathematics*, vol. 38, no. 1, p. 7, 2019.
- [11] D. Ramot, R. Milo, M. Friedman, and A. Kandel, "Complex fuzzy sets," *IEEE Transactions on Fuzzy Systems*, vol. 10, no. 2, pp. 171–186, 2002.
- [12] M. Xueling, J. Zhan, M. Khan, M. Zeeshan, S. Anis, and A. S. Awan, "Complex fuzzy sets with applications in signals,"

- Computational and Applied Mathematics*, vol. 38, no. 4, p. 150, 2019.
- [13] S. Dick, "Toward complex fuzzy logic," *IEEE Transactions on Fuzzy Systems*, vol. 13, no. 3, pp. 405–414, 2005.
- [14] M. Akram, A. Sattar, F. Karaaslan, and S. Samanta, "Extension of competition graphs under complex fuzzy environment," *Complex & Intelligent Systems*, vol. 7, no. 1, pp. 1–20, 2020.
- [15] M. Akram and A. Khan, "Complex pythagorean dombi fuzzy graphs for decision making," *Granular Computing*, pp. 1–25, 2020.
- [16] M. Akram, A. Bashir, and S. Samanta, "Complex pythagorean fuzzy planar graphs," *International Journal of Applied and Computational Mathematics*, vol. 6, no. 3, pp. 1–27, 2020.
- [17] M. Akram and A. Sattar, "Competition graphs under complex Pythagorean fuzzy information," *Journal of Applied Mathematics and Computing*, vol. 63, no. 1-2, pp. 543–583, 2020.
- [18] G. J. Klir and T. A. Folger, *Fuzzy Sets, Uncertainty, and Information*, Prentice-Hall, Englewood Cliffs, NJ, USA, 1988.
- [19] J. M. Mendel, "Fuzzy logic systems for engineering: a tutorial," *Proceedings of the IEEE*, vol. 83, no. 3, pp. 345–377, 1995.
- [20] R. K. Tripathi, K. Dookhitram, V. Raich, and H. Tomaskova, "Modified fuzzy relation and its applications," *International Journal of Fuzzy Mathematics and Systems*, vol. 3, no. 2, pp. 101–107, 2013.
- [21] M. Hussain, *Fuzzy relations*, Bleking Institute of Technology School of Engineering Department of Mathematics and Science, Karlskrona, Sweden, 2010.
- [22] G. Zhang, K.-Y. Cai, J. Ma, and J. Lu, "Operation properties and δ -equalities of complex fuzzy sets," *International Journal of Approximate Reasoning*, vol. 50, no. 8, pp. 1227–1249, 2009.
- [23] D. E. Tamir, L. Jin, and A. Kandel, "A new interpretation of complex membership grade," *International Journal of Intelligent Systems*, vol. 26, no. 4, pp. 285–312, 2011.