Research Article

Reinforcement Number of a Graph with respect to Half-Domination

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Received 17 December 2020; Revised 1 February 2021; Accepted 30 March 2021; Published 14 April 2021

Academic Editor: Ismail Naci Cangul

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In this paper, we introduce the concept of reinforcement number with respect to half-domination and initiate a study on this parameter. Furthermore, we obtain various upper bounds for this parameter. AMS subject classification: 05C38, 05C40, 05C69.

1. Introduction

Throughout our discussion, we consider only simple finite graphs. For graph theoretic terminologies, we refer the readers to [1]. If $G = (V, E)$ is a graph to each vertex v of G , $N_1(v)$ denotes the set of all vertices of G which are adjacent to v . For any subset $S \subseteq V$, $N_1(S) = \cup \{N_1(v) : v \in S\}$. For a vertex v of G , $N_1[v] = N_1(v) \cup \{v\}$ and $N_2[v] = \{u \in V : d(u, v) = 2\}$. A vertex $v \in V$ is said to dominate itself and its adjacent vertices. In other words, a vertex v dominates a vertex u if $u \in N_1[v]$. A subset D of $V(G)$ is said to be a dominating set of G if $V = \cup_{v \in D} N_1[v]$. The minimum cardinality of a dominating set D of G is denoted by $\gamma(G)$ and is called the domination number of G . The total domination number of a graph G , denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G and, for their properties, we refer the reader to [2–6]. In [7], Kulli introduced the concept of the cobondage number $cb(G)$ of a graph, which is the minimum number of edges to be added to reduce the domination number. The same concept has been independently introduced and studied earlier by others under the name “reinforcement number” (refer to chapter 17 of Haynes et al.’s work [8]). Total reinforcement number of a graph has been studied in [9–12]. Recently, Muhiuddin

et al. have studied various related concepts on graphs (see, e.g., [13–17]).

In [18, 19], a new domination parameter $\gamma_\lambda(G)$, where $0 < \lambda < 1$, was introduced and a study on $\gamma_{(1/2)}(G)$ had been initiated. In this paper, we introduce the concept of reinforcement number with respect to half-domination and initiate a study on this parameter.

2. Definition and Examples

If G is a graph for every vertex u of G , we define a map $f_u: V(G) \rightarrow \{0, (1/2), 1\}$ as follows:

$$f_u(v) = \begin{cases} 1, & \text{if } d(u, v) \leq 1, \\ \frac{1}{2}, & \text{if } d(v, u) = 2, \\ 0, & \text{otherwise,} \end{cases} \quad (1)$$

for all $v \in V(G)$. The map f_u is called the half-domination factor of u .

A subset D of $V(G)$ is said to be a $(1/2)$ -dominating set for G if, for each $v \in V(G)$, $\sum_{u \in D} f_u(v) \geq 1$. The minimum

cardinality of a $(1/2)$ -dominating set is called the $(1/2)$ -domination number of G and is denoted by $\gamma_{(1/2)}(G)$. A $(1/2)$ -dominating set of G with the minimum cardinality $\gamma_{(1/2)}(G)$ is said to be a $\gamma_{(1/2)}$ -set of G . The parameter $\gamma_{(1/2)}(G)$ was introduced in [18] and bounds for $\gamma_{(1/2)}(G)$ are obtained in [18].

Theorem 1 (see [18]). *If G is a graph on n vertices and $\Delta(G) = \Delta$, then*

$$\left\lceil \frac{n}{1 + (1/2)\Delta + (1/2)\Delta^2} \right\rceil \leq \gamma_{(1/2)}(G) \leq n - \Delta. \quad (2)$$

Theorem 2 (see [18]). *If G is a connected graph with $\Delta(G) \leq n - 3$, then $\gamma_{(1/2)}(G) \leq n - \Delta - 1$. Moreover, trees T with $\Delta(T) \leq n - 3$, and $\gamma_{(1/2)}(G) = n - \Delta - 1$ are characterized in [18].*

Definition 1 (reinforcement number with respect to half-domination). Let G be a graph with $\gamma_{(1/2)}(G) \geq 2$. The reinforcement number with respect to half-domination is denoted by $\mathcal{R}_{(1/2)}(G)$ and is given by $\mathcal{R}_{(1/2)}(G) = \min \{A: A \subseteq E(G) \text{ and } \gamma_{(1/2)}(G + A) < \gamma_{(1/2)}(G)\}$. For graphs G , with $\gamma_{(1/2)}(G) = 1$, $\mathcal{R}_{(1/2)}(G)$ is not defined.

Remark 1. If $\gamma_{(1/2)}(G) = 2$, then $\mathcal{R}_{(1/2)}(G) = n - \Delta - 1$ (let u be a vertex of G with $\deg(u) = \Delta$. If $A = \{uv: v \notin N_1[u]\}$, then $\{u\}$ is a $\gamma_{(1/2)}$ -dominating set for $G + A$).

$\mathcal{R}_{(1/2)}(G)$ for some standard graphs: here, we discuss path P_n on n vertices. As $\gamma_{(1/2)}(P_n) = 2$, if $n = 4, 5, 6, 7$, it follows that $\mathcal{R}_{(1/2)}(P_4) = 1$, $\mathcal{R}_{(1/2)}(P_5) = 2$, $\mathcal{R}_{(1/2)}(P_6) = 3$, and $\mathcal{R}_{(1/2)}(P_7) = 4$.

Theorem 3. *If $n \geq 8$, then $\mathcal{R}_{(1/2)}(P_n) = \begin{cases} 1 & \text{if } n \equiv 0, 1 \pmod{4} \\ 2 & \text{if } n \equiv 2, 3 \pmod{4} \end{cases}$.*

Proof. $\gamma_{(1/2)}(P_n) = \lfloor n/4 \rfloor + 1$ for all n . Let P_n be $v_1, v_2, \dots, v_n$. If $n = 4k$ or $4k + 1$ ($k \geq 2$), then $D = \{v_{2+4i}: 0 \leq i \leq k-1\} \cup \{v_{4k}\}$ is a $\gamma_{(1/2)}$ -set of P_n . If $e = v_2v_n$, then $e \in E(P_n)$ and $D - \{v_{4k}\}$ is a half-dominating set for $P_n + e$. Thus, $\mathcal{R}_{(1/2)}(P_n) = 1$ if $n = 4k$ or $4k + 1$, $k \geq 2$.

If $n = 4k + 2$ or $4k + 3$ ($k \geq 2$), then $D = \{v_{2+4i}: 0 \leq i \leq k\}$ is a $\gamma_{(1/2)}$ -set of P_n . Let $e_1 = v_2v_7$ and $e_2 = v_5v_{4k+2}$. Then, $D - \{v_6\}$ is a half-dominating set for $P_n + A$, where $A = \{e_1, e_2\}$. Thus, $\mathcal{R}_{(1/2)}(P_n) \leq 2$. We claim that $\mathcal{R}_{(1/2)}(P_n) \geq 2$. If $\mathcal{R}_{(1/2)}(P_n) = 1$, then there is a subset S of $P(n)$ and an edge $e \in E(P_n)$ such that $|S| = k$ and S is a half-dominating set for $P_n + e$. With respect the path P_n , either $N[u_1] \cap S \neq \emptyset$ or $N[u_n] \cap S \neq \emptyset$ (otherwise, S cannot be a half-dominating set of $P_n + e$). In P_n (by ignoring the contribution by the edge e), $f_S(V) \leq 4k - (1/2)$. So, $n - f_S(V) \geq (5/2)$. But the contribution by the edge e will be at most 2. Thus, S cannot be a half-dominating set for $P_n + e$, which is a contradiction. Thus, $\mathcal{R}_{(1/2)}(P_n) \geq 2$, if $n = 4k + 2$ or $4k + 3$ ($k \geq 2$).

One can easily prove the following theorem. $\square$

Theorem 4. *If C_n is a cycle on n -vertices ($n \geq 3$), then*

$$\mathcal{R}_{(1/2)}(C_n) = \begin{cases} n - 3, & \text{if } 3 \leq n \leq 8, \\ 1, & \text{if } n = 9 \text{ or } 4k + 1 \text{ or } 4k + 2, k \geq 3, \\ 2, & \text{if } n = 10 \text{ or } 4k \text{ or } 4k + 3, k \geq 3. \end{cases} \quad (3)$$

Remark 2. We compare $\mathcal{R}_{(1/2)}(G)$, the reinforcement number of G with respect to $(1/2)$ -domination, and $\mathcal{R}_t(G)$, the reinforcement number with respect to the total-domination for some graphs, to show that there is no correlation between $\mathcal{R}_{(1/2)}(G)$ and $\mathcal{R}_t(G)$.

- (a) If $G = P_8$, the path on eight vertices, $\gamma_{(1/2)}(P_8) = 3$ and $\mathcal{R}_{(1/2)}(P_8) = 1$, while $\gamma_t(P_8) = 4$ and $\mathcal{R}_t(P_8) = 3$
- (b) If $G = P_9$, the path on nine vertices, $\gamma_{(1/2)}(P_9) = 3$ and $\mathcal{R}_{(1/2)}(P_9) = 1$, while $\gamma_t(P_9) = 5$ and $\mathcal{R}_t(P_9) = 1$
- (c) If $G = C_8$, the cycle on eight vertices, $\gamma_{(1/2)}(C_8) = 2$ and $\mathcal{R}_{(1/2)}(C_8) = 5$, while $\gamma_t(C_8) = 4$ and $\mathcal{R}_t(C_8) = 2$

Theorem 5. *If T is a tree with $\gamma_{(1/2)}(T) \geq 3$, then $\mathcal{R}_{(1/2)}(T) \leq 3$.*

Proof. As $\gamma_{(1/2)}(T) \geq 3$, $\text{diam}(T) \geq 3$. Let $u_1u_2, \dots, u_l$ be a diagonal path in T (so $d(u_1, u_l) = \text{diam}(T)$). Let D be a $\gamma_{(1/2)}$ -set of T . We can assume that $u_2 \in D$. If $u_3 \in D$, let $e = u_2x$ for some $x \neq u_2, u_3 \in D$. Then $D - u_2$ is a half-dominating set for $T + e$. Assume that $u_3 \notin D$. If $|N(u_3) \cap D| \geq 3$, select $x, y \in D' = D - u_2$. Let $A = \{u_2x, u_2y\}$. Then $D' = D - u_2$ is a half-dominating set of $T + A$. If $|N(u_3) \cap D| \leq 2$, find $x, y \in D' = D - u_2$, such that $y \notin N(u_3)$. Let $e_1 = u_2x$, $e_2 = u_2y$, and $e_3 = u_3y$. Then D' is a half-dominating set for $G + A$, where $A = \{e_1, e_2, e_3\}$. Thus, $\mathcal{R}_{(1/2)}(T) \leq 3$, for all trees T with $\gamma_{(1/2)}(T) \geq 3$. $\square$

Remark 3. An example for a tree T , with $\mathcal{R}_{(1/2)}(T) = 3$, is given in Figure 1.

3. Bounds for $\mathcal{R}_{(1/2)}(G)$

In this section, we obtain certain bounds for $\mathcal{R}_{(1/2)}(G)$. If $\gamma_{(1/2)}(G) = 2$, then $\mathcal{R}_{(1/2)}(G) = n - \Delta(G) - 1$. First, we have the following theorem.

Theorem 6. *If $\gamma_{(1/2)}(G) \geq 2$, then $\mathcal{R}_{(1/2)}(G) \leq n - \Delta(G) - \gamma_{(1/2)}(G) + 1 \leq n - \Delta - 1$.*

Proof. Let G be a graph with $\gamma_{(1/2)}(G) = k \geq 2$. Let u be a vertex of degree Δ . As $k \geq 2$, $V - N[u]$ contains at least $k - 1$ vertices. Select a subset S of $V(G)$ such that $|S| = k - 1$ and $S \cap N[u] = \{u\}$. Let $A = \{uv: v \notin S \cup N[u] \text{ and } uv \in E(G)\}$. Then S is a dominating set of $G + A$. Thus, $\gamma_{(1/2)}(G + A) \leq k - 1$ and $\mathcal{R}_{(1/2)}(G) \leq |A| \leq n - \Delta - (k - 1) = n - \Delta - k + 1 \leq n - \Delta - 1$, as $k \geq 2$. $\square$

FIGURE 1: Example for a tree T , with $\gamma_{(1/2)}(t) = 4$ and $\mathcal{R}_{(1/2)}(T) = 3$.

Corollary 1. If $\gamma_{(1/2)}(G) \neq 1$, $\mathcal{R}_{(1/2)}(G) = n - \Delta - 1$ if $\gamma_{(1/2)}(G) = 2$.

As a corollary, we get a better upper bound for $\gamma_{(1/2)}(G)$ which is given by inequality (1).

Corollary 2. For all graphs G with $1 \leq \delta \leq \Delta \leq n - 2$, $\gamma_{(1/2)}(G) \leq n - \Delta - \mathcal{R}_{(1/2)}(G) + 1$.

Proof. Let G be a connected graph with $\gamma_{(1/2)}(G) = k \geq 3$. Let $D = \{u_1, u_2, \dots, u_k\}$ be a $\gamma_{(1/2)}$ -set of G . Let $V_0 = D$, $V_1 = N_1(D) - D$, and $V_2 = V - N_1[D]$. Note that if $v \in V_2$, then $v \in N_2(u_i) \cap N_2(u_j)$ for some i, j , $1 \leq i < j \leq k$. Then, $V = V_0 \cup V_1 \cup V_2$ is a partition of $V(G)$. As D is a $\gamma_{(1/2)}$ -set of G , to each u_i , $D - u_i$ is not a half-dominating set of G and hence the set $P(u_i, D)$, called private neighbors of u_i , consisting of all $v \in V(G)$ which are not half-dominated by the set $D - u_i$, is not empty. Note that $P(u_i, D) = P_1(u_i, D) \cup P_2(u_i, D)$, where $P_1(u_i, D) = \{v \in V : N[v] \cap D = \{u_i\} \text{ and } |N_2(v) \cap D| \leq 1\}$ and $P_2(u_i, D) = \{v \in V : v \notin N[D], u_i \in N_2(v) \text{ and } |N_2(v) \cap D| = 2\}$. As $P(u_i, D) \neq \emptyset$, either $P_1(u_i, D) \neq \emptyset$ or $P_2(u_i, D) \neq \emptyset$. We write $P(u_i, D)$, $P_1(u_i, D)$, and $P_2(u_i, D)$ for $P(u_i, D)$, $P_1(u_i, D)$, and $P_2(u_i, D)$, respectively. Now, we observe the following. $\square$

Observation 1. $|D| = \gamma_{(1/2)}(G) \leq \gamma(G) \leq |V_1| = |N_1(D) - D|$.

Proof. As D is a $\gamma_{(1/2)}$ -set of G , for each u_i , either $P_1(u_i, D) \neq \emptyset$ or $P_2(u_i, D) \neq \emptyset$, so $N(u_i) - D \neq \emptyset$ for each i , $1 \leq i \leq k$. Hence, V_1 is a dominating set of G . Thus, $|D| = \gamma_{(1/2)}(G) \leq \gamma(G) \leq |V_1|$. $\square$

Observation 2. If $\gamma_{(1/2)}(G) \geq 3$ and $|D| = |N_1(D)|$ for at least one $\gamma_{(1/2)}$ -set D of G , then $\mathcal{R}_{(1/2)}(G) \leq 2$.

Proof. If $|D| = |N_1(D)|$, then $N_1(D) = V_1$ is also a $\gamma_{(1/2)}$ -set of G and the maximum degree of the induced subgraph $\langle V_1 \rangle \leq 1$ (if, for some $x \in V_1$, $\deg(x) \geq 2$ in $\langle V_1 \rangle \leq 1$, then $V_1 - x$ is also a half-dominating set of G , which is a contradiction as $|V_1| = |D| = \gamma_{(1/2)}(G)$). If xy is an edge in $\langle V_1 \rangle$, select a vertex $z \neq x, y \in V_1$ (note that $|V_1| = |D| \geq 3$). Then $V_1 - \{x\}$ is a half-dominating set for $G + e$, where $e = xy$. Thus, in this case, $\mathcal{R}_{(1/2)}(G) = 1$. If V_1 is an independent set in G , select $x, y, z \in V_1$. Let $A = \{xy, xz\} \subseteq E(\overline{G})$. Then, $V_1 - \{x\}$ is a half-dominating set for $G + A$. Thus, $\mathcal{R}_{(1/2)}(G) \leq |A| = 2$. $\square$

Observation 3. If $V_2 = \emptyset$ for some $\gamma_{(1/2)}$ -set D of G , then $\mathcal{R}_{(1/2)}(G) \leq 2$.

Proof. If $V_2 = \emptyset$, $P_2(u_i) = \emptyset$ and hence $D - \{u_i\}$ is a half-dominating set of $G + \{u_1u_2, u_1u_3\}$. $\square$

Observation 4. If G is a connected graph with $\gamma_{(1/2)}(G) = k \geq 3$, then $\mathcal{R}_{(1/2)}(G) \leq (2n/k) - 2 \leq (2n/3) - 2$.

Proof. If $P_2(u_i) = \emptyset$ for any i , $1 \leq i \leq k$ (with respect to some $\gamma_{(1/2)}$ -set D of G), then, by our Observation 1, $\mathcal{R}_{(1/2)}(G) \leq 2$ and so $\mathcal{R}_{(1/2)}(G) \leq (2n/k) - 2$ (as G is connected, $2k \leq n$). If $|D| = |V_0| = |V_1|$, then also $\mathcal{R}_{(1/2)}(G) \leq 2$ and hence the result is true. So assume that $|D| < |V_1|$ and $P_2(u_i) \neq \emptyset$ for all i , $1 \leq i \leq k$. If $x \in P_2(u_i, D)$, for some i , then $x \in P_2(u_j, D)$ for exactly one $j \neq i$. Thus, each $x \in P_2(u_j, D)$ is counted twice in $\sum_{i=1}^k P_2(u_i, D)$. As $|V_1| \geq k + 1$, $|V_2| = V - N_1[D] \leq n - (2k + 1)$. So, $\sum |P_2(u_i)| \leq 2|V_2| \leq 2n - 4k - 2$ and hence we can find one u_j such that $|P_2(u_j)| \leq ((2n - 4k - 2)/k)$. To each vertex $w \in P_2(u_j)$, select a vertex $x \in N_1(u_j)$ such that xw is an edge in G , and select a vertex u_i such that $w \notin N_2(u_i)$ (since $|N_2(w) \cap D| = 2$, there is a vertex $u_i \in D$ such that $w \notin N_2(u_i)$), and let $e_w = xv \in E(\overline{G})$. Let $A = \{e_w : w \in P_2(u_j)\} \cup \{u_ju_i, u_ju_{i_2}\}$, where $i_1 \neq i_2$ and $j \notin \{i_1, i_2\}$. Then, $\gamma_{(1/2)}(G + A) \leq k - 1$ and hence $\mathcal{R}_{(1/2)}(G) \leq |A| + 2 = |P_2(u_j)| + 2 \leq ((2n - 4k - 2)/k) + 2 = ((2(n - 1))/k) - 2$. $\square$

Observation 5. If $|N_1(u_i)| = 1$, for some u_i in a $\gamma_{(1/2)}$ -set D , then $\mathcal{R}_{(1/2)} \leq 2$.

Observation 6. For a connected graph G , with radius of $G \geq 3$ and $\gamma_{(1/2)}(G) \geq 3$, $\mathcal{R}_{(1/2)}(G) \leq n - \Delta - \delta - \gamma_{(1/2)} + 1$ (radius of $G \geq 3 \implies$ to each vertex $x \in v$, there is a vertex y such that $d(x, y) \geq 3$).

Proof. Let $k = \gamma_{(1/2)}(G)$. Let v_1 be a vertex of G with $\deg(v_1) = \Delta$. Let v_2 be a vertex such that $d(v_1, v_2) \geq 3$. Then, $N_1[v_1] \cap N_1[v_2] = \emptyset$. Let $B = V - (N_1[v_1] \cup N_1[v_2])$. Then, B contains at least $k - 3$ vertices. Select $k - 2$ vertices $v_3, v_4, \dots, v_{k-1}$ in B . Let $A = \{v_1x : x \in B - \{v_3, \dots, v_{k-1}\}\}$. Note that $A \subseteq E(\overline{G})$. Now, $S = \{v_i : 1 \leq i \leq k - 1\}$ is a half-dominating set for $G + A$. So $\mathcal{R}_{(1/2)}(G) \leq n - \Delta - \delta - k + 1$, where $k = \gamma_{(1/2)}(G)$. $\square$

Remark 4. The upper bound given in Observation 6 is attained by the graph in Figure 2. For this graph, $n = 9$, $\gamma_{(1/2)}(G) = 3$, $\Delta = 4$, $\delta = 2$, and $\mathcal{R}_{(1/2)}(G) = 1$.

Remark 5. For a connected graph G with radius ≥ 3 and $\gamma_{(1/2)}(G) \geq 3$, Observation 6 yields a better bound for $\gamma_{(1/2)}(G)$ than the bound given by inequality (1). $\gamma_{(1/2)}(G) \leq n - \Delta - \delta - \mathcal{R}_{(1/2)}(G) + 1$. In the following theorem, we obtain an upper bound for $\mathcal{R}_{(1/2)}(G)$ in terms of $\Delta(G)$.

FIGURE 2: A group G for which $\mathcal{R}_{(1/2)}(G) = n - \Delta - \delta - \gamma_{(1/2)}(G) + 1$.

Theorem 7. For any graph G , with $\gamma_{(1/2)}(G) \geq 3$, $\mathcal{R}_{(1/2)}(G) \leq 2\Delta$.

Proof. Let D be a $\gamma_{(1/2)}$ -set of G . Select three vertices u, v, w in D . Let $A = \{xv, xw : x \in N(u) \cap \overline{E(G)}\}$. Then, $|A| \leq 2 \deg(u) \leq 2\Delta(G)$ and $D - \{u\}$ is a half-dominating set of $G + A$; thus, $\mathcal{R}_{(1/2)}(G) \leq 2\Delta$. $\square$

Remark 6. There are graphs G , for which $\gamma_{(1/2)}(G) \geq 3$ and $\mathcal{R}_{(1/2)}(G) = 2\Delta$.

Example 1. (Example for a graph G with $\gamma_{(1/2)}(G) \geq 3$ and $\mathcal{R}_{(1/2)}(G) = 2\Delta$). Consider the graph G given in Figure 3. First, we note that for this graph $n = 21$, $\Delta = 3$, $\delta = 2$, and $(n/(1 + (\Delta/2) + (\Delta^2/2))) = \gamma_{(1/2)}(G) = 3$. ($\gamma_{(1/2)}(G)$ attains its lower bound given in (1). The set $D = \{a, b, c\}$ is the unique $\gamma_{(1/2)}$ -set of G (for any vertex $s \in V(G)$, $f_s(V) \leq (n/(1 + (\Delta/2) + (\Delta^2/2))) = 7$ and $f_s(V) = 7$ if $s \in D$ and $f_s(V) < 7$ for all $s \notin D$. If D' is any three-element subset of V , other than D , then $f_{D'}(V) < 21$). $V(G) = V_1 \cup V_2 \cup V_3$, where $V_1 = \{a, b, c\}$, $V_2 = \{x_i, y_i, z_i : 1 \leq i \leq 3\}$, and $V_3 = \{u_{ij}, v_{ij}, w_{ij} : 1 \leq i < j \leq 3\}$. Assume that B is a two-element subset of V and $A \subset \overline{E(G)}$ such that B is a half-domination set of $G + A$. In order to show that $\mathcal{R}_{(1/2)}(G) = 6$, it is enough to prove that $|A| \geq 6$. Claim: $|A| \geq 6$.

In our further discussion, we consider the following notations.

If $e \in A$, we write $T_e(s)$, where $s \in V(G)$, and the number $T_e(s) = (\sum_{t \in B} f_t(s) \text{ in } G + e) - (\sum_{t \in B} f_t(s) \text{ in } G)$. We say that the edge $e \in A$ contributes the value λ to a vertex s of G if $\lambda = T_e(s)$. Similarly, we say that e contributes the value λ to $V(G)$ if $\lambda = T_e(V) = \sum_{s \in V} T_e(s)$. If $A' \subset A$, we define $T_{A'}(s) = (\sum_{t \in B} f_t(s) \text{ in } G + A) - (\sum_{t \in B} f_t(s) \text{ in } G)$ and $T_{A'}(V) = \sum_{s \in V} T_{A'}(s)$. We note that $T_e(s) \leq 1$ for all $e \in A$ and $s \in V(G)$. We can also assume that every edge e in A has one end in B . Now, we establish our claim $|A| \geq 6$ by considering various possibilities for B (for each possibility, we consider one subcase; other subcases can be proved similarly):

Case (i): $B \subset V_1$. Assume that $B = \{a, b\}$. Then $f_B(s) = (1/2)$ if $s \in \{u_{13}, v_{13}, w_{13}, u_{23}, v_{23}, w_{23}\}$. If, for an edge $e \in A$, $T_e(s_0) \geq (1/2)$ for one of the elements s_0 in $\{u_{13}, v_{13}, w_{13}\}$, then $T_s(s) = 0$ for $s \neq s_0 \in \{u_{13}, v_{13}, w_{13}\}$ and b is one of the ends of e . Similarly, if $T_e(s_0) \geq (1/2)$

FIGURE 3: A graph G with $\gamma_{(1/2)}(G) = 3$ and $\mathcal{R}_{(1/2)}(G) = 2\Delta = 6$.

for one $s_0 \in \{u_{23}, v_{23}, w_{23}\}$, then $T_e(s) = 0$ for $s \neq s_0 \in \{u_{23}, v_{23}, w_{23}\}$ and a is one of the ends of e . Thus, $|A| \geq 6$.

Case (ii): $B \subset V_2$. Assume that $B = \{x_1, x_2\}$. Then $f_B(s) = (1/2)$ if $s \in \{y_1, y_2, x_3, z_3\}$ and $f_B(s) = 0$ if $s \in \{c, y_3, u_{23}, w_{13}, v_{12}, v_{13}, v_{23}\}$. Let $A' \subset A$ be such that $T_{A'}(y_1), T_{A'}(y_2) \geq (1/2)$; $T_{A'}(s) \geq 1$ for $s \in \{v_{12}, v_{13}, v_{23}, z_3\}$; and $T_e(Y) \geq (1/2)$ for every $e \in A'$, where $Y = \{y_1, y_2, y_3, v_{12}, v_{13}, v_{23}\}$. Then $|A'| \geq 3$; $T_{A'}(c) \geq (1/2)$ only if $|A'| \geq 4$. If $|A'| = 4$, $A - A'$ contains at least two edges e_1 and e_2 so that $T_{e_1}(u_{23}) \geq (1/2)$ and $T_{e_2}(w_{13}) \geq (1/2)$. So, in this case, $|A| \geq 6$. If $|A'| = 3$, $A - A'$ contains a set A'' of at least three edges such that $T_{A''}(s) \geq 1$, where $s \in \{c, u_{23}, v_{23}\}$. Thus, in this case, also $|A| \geq 6$.

Case (iii): $B \subset V_3$. Assume that $B = \{u_{12}, w_{13}\}$. Then $f_B(s) = (1/2)$ if $s \in \{u_{13}, u_{23}, w_{12}, w_{23}, b, c\}$ and $f_B(s) = 0$ if $s \in \{y_1, y_2, y_3, v_{12}, v_{13}, v_{23}, x_3, z_2\}$. If $A' \subset A$ such that $T_{A'}(s) \geq 1$ for $s \in \{y_1, y_2, y_3, v_{12}, v_{13}, v_{23}\}$, then $|A'| \geq 4$ (A' contains at least two edges with one end at u_{12} and at least two edges with one end at w_{13}). $A - A'$ contains an edge e_1 with one end at w_{13} and $T_{e_1}(u_{13}) \geq (1/2)$ and contains another edge e_2 with one end at u_{12} and $T_{e_2}(w_{23}) \geq (1/2)$. Thus, $|A| \geq 6$.

Case (iv): $B \cap V_1 \neq \emptyset$ and $B \cap V_2 \neq \emptyset$. Assume that $B = \{a, x_2\}$. Then, $f_B(s) = (1/2)$ if $s \in \{u_{13}, v_{13}, w_{13}, v_{12}, w_{12}, x_2, y_2, z_2\}$ and $f_B(s) = 0$ if $s \in \{c, y_3, z_3, w_{23}, v_{23}\}$. The set A contains a subset A' , with $|A'| = 3$, $T_{A'}(s) \geq (1/2)$ for $s \in \{u_{12}, v_{13}, w_{13}\}$, and each edge in A' has one end at x_2 . Note that $T_{A'}(c) \leq (1/2)$, either $T_{A'}(w_{23}) = 0$ or $T_{A'}(w_{12}) = 0$, and either $T_{A'}(v_{12}) = 0$ or $T_{A'}(v_{23}) = 0$. So, there is a set A'' contained in $A - A'$ with $|A''| \geq 3$ such that $T_{B \cup A' \cup A''}(s) \geq 1$ for $s \in \{c, w_{23}, v_{12}, w_{12}, w_{23}\}$. Thus, $6 \leq |A'| + |A''| \leq |A|$.

Case (v): $B \cap V_1 \neq \emptyset$ and $B \cap V_3 \neq \emptyset$. Assume that $B = \{a, u_{23}\}$. Then, $f_B(s) = (1/2)$ if $s \in \{u_{12}, u_{13}, v_{12}, v_{13}, b, c\}$ and $f_B(s) = 0$ if $s \in \{x_2, x_3, y_2, y_3, u_{23}, v_{23}\}$. There are edges e_1 and e_2 in A with one end at a and

$T_{e_1}(c) \geq (1/2)$. Note that $T_{e_1}(u_{23}) + T_{e_2}(u_{23}) \leq (1/2)$ and $T_{e_1}(v_{23}) + T_{e_2}(v_{23}) \leq (1/2)$. There are edges e_3 and e_4 in A , with one end at w_{23} and $T_{e_3}(u_{13}) \geq (1/2)$; $T_{e_4}(v_{13}) \geq (1/2)$. Now either $T_{e_3}(u_{23}) = 0$ or $T_{e_3}(u_{12}) = 0$, and either $T_{e_4}(v_{23}) = 0$ or $T_{e_4}(v_{12}) = 0$. As $A_1 = \{e_1, e_2, e_3, e_4\} \subset A$ and $T_A(s) \geq (1/2)$ for $s \in \{c, b, u_{13}, u_{12}, v_{12}, v_{13}\}$ and $T_A(s) \geq 1$ for $s \in \{u_{23}, v_{23}\}$, we have $|A| \geq 6$.

Case (vi): $B \cap V_2 \neq \emptyset$ and $B \cap V_3 \neq \emptyset$. Let $B = \{x_1, v_{23}\}$. Then $f_B(s) = (1/2)$ for $s \in \{c, b, u_{13}, u_{12}, v_{12}, v_{13}\}$ and $f_B(s) = 0$ if $s \in \{z_2, z_3, w_{12}, w_{23}, w_{13}, u_{23}\}$. The set A should contain at least three edges to attribute positive values to $z_1, z_2, z_3, w_{12}, w_{23}$; it should contain at least two edges with one end at v_{23} to contribute positive values to y_1 and x_3 and one edge with one end at x_1 to contribute a positive value to v_{13} . Thus, $|A| \geq 6$. The upper bound 2Δ for $\mathcal{R}_{(1/2)}(G)$ can be reduced further if $\Delta < \gamma_{(1/2)}(G)$.

Theorem 8. If $\gamma_{(1/2)}(G) \geq 3$ and $\Delta(G) < \gamma_{(1/2)}(G)$, then

$$\mathcal{R}_{(1/2)}(G) \leq \begin{cases} \Delta, & \text{if } \Delta(G) < \gamma_{(1/2)}(G) - 1, \\ \Delta + 1, & \text{if } \Delta(G) = \gamma_{(1/2)}(G) - 1. \end{cases} \quad (4)$$

Proof. Let $\gamma_{(1/2)}(G) = k \geq 3$. Let $D = \{u_1, u_2, \dots, u_k\}$ be a $\gamma_{(1/2)}$ -set of G . Let $N_1(u_1) - D = \{v_1, v_2, \dots, v_l\}$. Let $1 \leq j \leq l$. If $P_2(u_1) \cap N_1(v_j) = \emptyset$, select one i_j ($2 \leq i_j \leq k$) such that $u_{i_j} v_j \in E(\overline{G})$. If $P_2(u_1) \cap N_1(v_j) \neq \emptyset$, then either $N_1(v_j) \cap D = \{u_1\}$ or $|N_1(v_j) \cap D| = 2$. If $N_1(v_j) \cap D = \{u_1\}$, then, for each vertex $w \in P_2(u_1) \cap N_1(v_j)$, $|\{i: i \neq 1 \text{ and } w \in N_2(u_i)\}| = 1$. So, we can select one i_j ($2 \leq i_j \leq k$) such that $P_2(u_1) \cap N_1(v_j) \cap N_2(u_{i_j}) = \emptyset$ (this is possible since $|P_2(u_1) \cap N_1(v_j)| \leq \delta - 1 < \gamma_{(1/2)}(G) - 1$). Thus, to each v_j , ($1 \leq j \leq l$), we have selected one $u_{i_j} \in D - \{u_1\}$. If $\Delta(G) < \gamma_{(1/2)}(G) - 1$, we can select u_{i_j} such that $u_{i_j} \neq u_{i_1}$. Let $A = \{v_j u_{i_j}: 1 \leq j \leq l\}$. If $\Delta(G) < \gamma_{(1/2)}(G) - 1$, $D - u_1$ is a half-dominating set of $G + A$. If $\Delta(G) < \gamma_{(1/2)}(G) - 1$, then $D - u_1$ is a half-dominating set of $G + (A + e_1)$, where $e_1 = u_1 u_j$ for some $u_j \notin N(u_1)$ (as $D - u_1$ is not a half-dominating set of G , $N(u_1) \neq D - u_1$). Thus, $\mathcal{R}_{(1/2)}(G) \leq l \leq \Delta$ if $\Delta(G) < k - 1$ and $\mathcal{R}_{(1/2)}(G) \leq l + 1 \leq \Delta + 1$ if $\Delta(G) = k - 1$. In the following theorem, we obtain a lower bound for $\mathcal{R}_{(1/2)}(G)$ of some classes of graphs. $\square$

Theorem 9. If G is a connected graph with $\gamma_{(1/2)}(G) = k \geq 3$ and the order $n = k(1 + (\Delta/2) + (\Delta^2/2))$, then $\mathcal{R}_{(1/2)}(G) \geq \Delta$.

Proof. If $S = \{u_1, u_2, \dots, u_{k-1}\}$ is a set of vertices of G , then $f_s(V) \leq (k-1)(1 + (\Delta/2) + (\Delta^2/2))$. If S is a half-dominating set of $G + A$, where $A \subseteq E(\overline{G})$, then A should contribute at least $1 + (\Delta/2) + (\Delta^2/2)$ to the total $\sum f_s(V)$ in $G + A$. But T_e is an edge e in A which can contribute at most

$1 + (\Delta/2)$. So, $|A| \geq ((1 + (\Delta/2) + (\Delta^2/2))/(1 + (\Delta/2))) = \Delta - 1 + (2/(1 + \Delta))$. Hence, $|A| \geq \Delta$ whenever $\gamma_{(1/2)}(G + A) \leq k - 1$. Thus, $\mathcal{R}_{(1/2)}(G) \geq \Delta$. $\square$

Corollary 3. If, for a graph G with $\gamma_{(1/2)}(G) = k \geq 3$, $\Delta < k - 1$ and $n = k(1 + (\Delta/2) + (\Delta^2/2))$, then $\mathcal{R}_{(1/2)}(G) = \Delta$.

Example 2. If $\Delta = 2$, $\gamma_{(1/2)}(G) = k \geq 4$, and $n = k(1 + (\Delta/2) + (\Delta^2/2))$ for a connected graph G , then $G = C_{4k}$ for $k \geq 4$. We know that $\mathcal{R}_{(1/2)}(C_{4k}) = 2$ for all $k \geq 3$.

Remark 7. For all $k \geq 3$ and $\Delta < k - 1$, we can construct graphs G with $\gamma_{(1/2)}(G) = k$, $\Delta(G) = \Delta$, and $n = k(1 + (\Delta/2) + (\Delta^2/2))$. Now, we list some properties of graphs G with $\gamma_{(1/2)}(G) \geq 3$ and $\mathcal{R}_{(1/2)}(G) = 2\Delta$. Let $D = \{u_1, u_2, \dots, u_k\}$ be any $\gamma_{(1/2)}$ -set of G (where $k = \gamma_{(1/2)}(G)$). Then,

- (1) $k = \gamma_{(1/2)} \leq \Delta$
- (2) $\deg(u_i) = \Delta$ for all i , $1 \leq i \leq k$
- (3) $N[u_i] \cap N[u_j] = \emptyset$ for all $1 \leq i < j \leq k$
- (4) $|P_2(u_i) \cap N_1(v) \cap N_2(u_j)| \neq 0$ for all $i \neq j \in \{1, 2, \dots, k\}$ and for all $v \in N_1(u_i)$. So $|P_2(u_i) \cap N_1(v)| \geq k - 1 \forall 1 \leq i \leq k$ and $v \in N_1(u_i)$

Problem 1. Characterize the graphs G with $\gamma_{(1/2)}(G) \geq 3$ and $\mathcal{R}_{(1/2)}(G) = 2\Delta(G)$.

4. Nordhaus-Gaddum-Type Results

If G is a graph with $1 \leq \delta \leq \Delta \leq n - 2$, both $\mathcal{R}_{(1/2)}(G)$ and $\mathcal{R}_{(1/2)}(\overline{G})$ are defined. In this section, we obtain certain bounds for the sum $\mathcal{R}_{(1/2)}(G) + \mathcal{R}_{(1/2)}(\overline{G})$. We use the following notations:

If $v \in V(G)$, then $\overline{N}_1(v) = \{u \in (V/v) \neq u \text{ and } vu \in E(\overline{G})\}$ and $\overline{N}_2(v) = \{u \in (V/d(u, v)) = 2 \text{ in } \overline{G}\}$. Also, $\overline{\delta}$ and $\overline{\Delta}$ denote the minimum degree and maximum degree of $\overline{G}$, respectively. Note that $\overline{\delta} = n - \Delta(G) - 1$ and $\overline{\Delta} = n - \delta(G) - 1$.

Theorem 10. If $1 \leq \delta(G) \leq \Delta(G) \leq n - 2$, then $\mathcal{R}_{(1/2)}(G) + \mathcal{R}_{(1/2)}(\overline{G}) \leq n - (\Delta - \delta) - 1 \leq n - 1$.

Proof. By Theorem 6, $\mathcal{R}_{(1/2)}(G) \leq n - \Delta(G) - 1$ and $\mathcal{R}_{(1/2)}(\overline{G}) \leq n - \overline{\Delta} - 1$.

$$\begin{aligned} \mathcal{R}_{(1/2)}(G) + \mathcal{R}_{(1/2)}(\overline{G}) &\leq n - \Delta(G) - 1 + \delta(G) \\ &= n - (\Delta - \delta) - 1 \leq n - 1. \end{aligned} \quad (5)$$

$\square$

Corollary 4. If $1 \leq \delta(G) \leq \Delta(G) \leq n - 2$, $\mathcal{R}_{(1/2)}(G) + \mathcal{R}_{(1/2)}(\overline{G}) = n - 1$ if G is regular and $\gamma_{(1/2)}(G) = \gamma_{(1/2)}(\overline{G}) = 2$. Also, $\mathcal{R}_{(1/2)}(G) + \mathcal{R}_{(1/2)}(\overline{G}) = n - 2$ if $\Delta(G) = \delta(G) + 1$ and $\gamma_{(1/2)}(G) = \gamma_{(1/2)}(\overline{G}) = 2$.

Theorem 11. If $1 \leq \delta(G) \leq \Delta(G) \leq n - 2$, then $\mathcal{R}_{(1/2)}(G) + \mathcal{R}_{(1/2)}(\overline{G}) \leq n - (\Delta - \delta) - \gamma_{(1/2)}(G) + 1$.

Proof. By Theorem 6, $\mathcal{R}_{(1/2)}(G) \leq n - \Delta(G) - \gamma_{(1/2)}(G) + 1$ and $\mathcal{R}_{(1/2)}(\overline{G}) \leq n - \overline{\Delta} - 1 = \delta(G)$. So, $\mathcal{R}_{(1/2)}(G) + \mathcal{R}_{(1/2)}(\overline{G}) \leq n - (\Delta + \delta) - \gamma_{(1/2)}(G) + 1$. Now, we show that if $\gamma_{(1/2)}(G) \geq 3$, then either $\gamma_{(1/2)}(G) = 2$ or $\gamma_{(1/2)}(\overline{G}) = 2$. $\square$

Theorem 12. *If $1 \leq \delta \leq \Delta \leq n - 2$, then either $\gamma_{(1/2)}(G) = 2$ or $\gamma_{(1/2)}(\overline{G}) = 2$.*

Proof. Assume that $\gamma_{(1/2)}(G) \geq 3$. Let $D = \{u_1, \dots, u_k\}$ be a $\gamma_{(1/2)}$ -set of G . Let $V = V_0 \cup V_1 \cup V_2$, where $V_0 = D$, $V_2 = V - N[D]$, and $V_1 = V - (V_0 \cup V_2)$. We claim that $\gamma_{(1/2)}(\overline{G}) = 2$ (as $1 \leq \delta(G)$, $\gamma_{(1/2)}(\overline{G}) \neq 1$).

Case (i): $|\cup_{i=1}^k P_2(u_i, D)| \geq 2$. Select $x \neq y \in \cup P_2(u_i, D)$. Assume that $x \in P_2(u_1, D) \cap P_2(u_2, D)$ and $y \in P_2(u_1, D) \cap P_2(u_2, D)$, where $i_1 \neq i_2$ ($\{i_1, i_2\}$ may be the set $\{1, 2\}$). Clearly, $D = V_0 \subset \overline{N}_1(x) \cap \overline{N}_1(y)$, and $V_2 \subset \overline{N}_1(D)$. So, $V_2 - \{x, y\} \subset \overline{N}_2(x) \cap \overline{N}_2(y)$. Let $v \in V_1$. Then $v \in N_1(u_j)$ for some j . If $j \notin \{1, 2\}$, then either $v \in \overline{N}_1(x)$ or $v \in \overline{N}_1(u_1)$ and hence either $v \in \overline{N}_1(x)$ or $v \in \overline{N}_2(x) \cap \overline{N}_2(y)$. If $j \in \{1, 2\}$, then either $v \in \overline{N}_1(x)$ or $v \in \overline{N}_1(u_3)$. Thus, if $|\cup_{i=1}^k P_2(u_i, D)| \geq 2$, $\{x, y\}$ is a half-dominating set for $\overline{G}$.

Case (ii): $|\cup_{i=1}^k P_2(u_i, D)| = 1$. Let $\cup P_2(u_i, D) = \{x\}$. Assume that $x \in N_2(u_1) \cap N_2(u_2)$. Then, $x \notin N_2(u_i)$ for all $i \geq 3$. As $P_2(u_3, D) = \emptyset$, $P_1(u_3, D) \neq \emptyset$. Select $y \in P_1(u_3, D)$. Then, $V_0 \subset \overline{N}_1(x)$, and $V_0 - u_3 \subset \overline{N}_1(y)$. $V_2 \subset \overline{N}_2(x) \cap \overline{N}_2(y)$ as $V_2 \subset \overline{N}_2(u_3)$. Let $v \neq y \in V_1$ and $v \in N_1(u_j)$ for some j . If $j \notin \{1, 2\}$, $v \in \overline{N}_1(x)$. If $j \in \{1, 2\}$, say $j = 1$; then either $v \in \overline{N}_1(y)$ or $v \in \overline{N}_1(u_2)$. Thus, in this case, $\{x, y\}$ is a half-dominating set of $\overline{G}$.

Case (iii): $P_2(u_i, D) = \emptyset$ for all i . Then, $P_1(u_i, D) \neq \emptyset$ for all i . Let $x \in P_1(u_1, D)$, and $y \in P_1(u_2, D)$. Then, $u_1 \in \overline{N}_1(y)$, $u_2 \in \overline{N}_1(x)$, and $u_i \in \overline{N}_1(x) \cap \overline{N}_1(y) \forall i \geq 3$. $V_2 \subset \overline{N}_1(u_3) \subset \overline{N}_2(x) \cap \overline{N}_2(y)$. Let $v \neq x, y \in V_1$. Then, either $v \in \overline{N}_1(x)$ or $v \in \overline{N}_1(u_3)$. Then, in this case, also $\{x, y\}$ is a half-dominating set for $\overline{G}$. Now it follows that $\gamma_{(1/2)}(\overline{G}) = 2$. $\square$

Remark 8. We obtain the following bounds for $\mathcal{R}_{(1/2)}(G) + \mathcal{R}_{(1/2)}(\overline{G})$, using Theorems 7, 8, and 12. If $1 \leq \delta(G)$ and $\gamma_{(1/2)}(G) \geq 3$, then

$$\mathcal{R}_{(1/2)}(G) + \mathcal{R}_{(1/2)}(\overline{G}) \leq \begin{cases} 2\Delta + \delta, & \text{if } \gamma_{(1/2)}(G) \leq \Delta(G), \\ \Delta + \delta + 1, & \text{if } \gamma_{(1/2)}(G) - 1 = \Delta(G), \\ \Delta + \delta, & \text{if } \Delta(G) < \gamma_{(1/2)}(G) - 1. \end{cases} \quad (6)$$

Finally, we conclude this paper with the following remark on the bounds for the product $\mathcal{R}_{(1/2)}(G) \cdot \mathcal{R}_{(1/2)}(\overline{G})$.

Remark 9. Let $k \geq 2$ be an integer. If $x + y = k$, where x and y are nonnegative integers, then the product xy attains the maximum value if $x = y = (k/2)$ when k is even and $\{x, y\} = \{(k-1)/2, (k+1)/2\}$ when k is odd. Thus,

$$\max_{x+y=k} xy = \begin{cases} \left(\frac{k}{2}\right)^2, & \text{if } k \text{ is even,} \\ \frac{(k-1)(k+1)}{4}, & \text{if } k \text{ is odd.} \end{cases} \quad (7)$$

Also, if $2 \leq k_1 \leq k_2$, then $(\max_{x+y=k_1} xy) \leq (\max_{x+y=k_2} xy)$. As for any graph $\mathcal{R}_{(1/2)}(G) + \mathcal{R}_{(1/2)}(\overline{G}) \leq n - 1$, where $|V(G)| = n$, we have an upper bound for the product $\mathcal{R}_{(1/2)}(G) \cdot \mathcal{R}_{(1/2)}(\overline{G})$. In fact, $\mathcal{R}_{(1/2)}(G) \cdot \mathcal{R}_{(1/2)}(\overline{G}) \leq \begin{cases} ((n-1)/2)^2, & \text{if } n \text{ is odd,} \\ ((n(n-2))/4), & \text{if } n \text{ is even.} \end{cases}$

There are graphs G for which the product $\mathcal{R}_{(1/2)}(G) \cdot \mathcal{R}_{(1/2)}(\overline{G})$ attains this upper bound.

If $n \geq 5$ is odd and G is a $((n-1)/2)$ -regular graph, then $\mathcal{R}_{(1/2)}(G) \cdot \mathcal{R}_{(1/2)}(\overline{G}) = ((n-1)/2)^2$ and $\mathcal{R}_{(1/2)}(G) = \mathcal{R}_{(1/2)}(\overline{G}) = 2$. Note that Harary graphs $H_{2n+1,n}$ satisfy these properties. If $n \geq 3$, then, in this case, Harary graph $H_{2n,n-1}$ can be considered as an example.

5. Conclusion

In this paper, we introduced the concept $\mathcal{R}_{(1/2)}(G)$, the reinforcement number of a graph G with respect to the $(1/2)$ -dominating number of G . Also, we studied various upper bounds for $\mathcal{R}_{(1/2)}(G)$ and exhibited that these bounds are sharp. Furthermore, we obtained Nordhaus-Gaddum-type bounds for the sum $\mathcal{R}_{(1/2)}(G) + \mathcal{R}_{(1/2)}(\overline{G})$ and the product $\mathcal{R}_{(1/2)}(G) \cdot \mathcal{R}_{(1/2)}(\overline{G})$.

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported by the Taif University Researchers Supporting Project (TURSP-2020/246), Taif University, Taif, Saudi Arabia.

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