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Research Article

On the Shrinking Projection Method for the Split Feasibility Problem in Banach Spaces

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In this paper, we consider the split feasibility problem in Banach spaces. By applying the shrinking projection method, we propose an iterative method for solving this problem. It is shown that the algorithm under two different choices of the stepsizes is strongly convergent to a solution of the problem.

1. Introduction

In this paper, we consider the split feasibility problem [1]. It is very useful in dealing with problems arising from various applied disciplines (see, e.g., [2–6]). More precisely, the split feasibility problem requires to find a point $\hat{x} \in \mathbb{R}^n$ satisfying the following property:

$$\begin{aligned} &\hat{x} \in C, \\ &A\hat{x} \in Q, \end{aligned} \tag{1}$$

where C and Q are nonempty closed convex subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and A is a linear operator.

The split feasibility problem was first treated in Euclidean spaces and recently was extended to more general framework including Hilbert spaces and Banach spaces. In Hilbert spaces, Byrne [7] introduced the CQ algorithm:

$$x_{n+1} = P_C\left(x_n - r_n A^*\left(I - P_Q\right)Ax_n\right), \tag{2}$$

where $r_n > 0$ is a properly chosen stepsize, A^* is the conjugate of A, I is the identity operator, and P_C, P_Q denote the metric projections onto the respective sets. By using Polyak's gradient method, Wang [8] recently proposed another iterative algorithm:

$$x_{n+1} = x_n - r_n\left[\left(I - P_C\right)x_n + A^*\left(I - P_Q\right)Ax_n\right], \tag{3}$$

where $r_n > 0$ is a properly chosen stepsize (see also [4, 7–13] for some related works). In the framework of Banach spaces, Schöpfer et al. [14] extended the CQ method as

$$x_{n+1} = \Pi_C J_X^{-1}\left[J_X x_n - r_n A^* J_Y\left(I - P_Q\right)Ax_n\right], \tag{4}$$

where r_n is a positive parameter, J_X, J_Y are, respectively, the duality mappings on X and Y, Π_C denotes the Bregman projection, and P_C denotes the metric projection. The weak convergence of (4) is guaranteed if X is p-uniformly convex and uniformly smooth and J_X is sequentially weak-to-weak continuous. Recently, Takahashi [15] suggested a novel way for the split feasibility problem:

$$\begin{cases} z_n = x_n - r_n J_{X^*} A^* J_Y\left(I - P_Q\right)Ax_n, \\ C_n = \{z \in C\colon \langle z_n - z, J_X(x_n - z_n)\rangle \geq 0\}, \\ Q_n = \{z \in C\colon \langle x_n - z, J_X(x_0 - x_n)\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \quad \forall n \in \mathbb{N}. \end{cases} \tag{5}$$

By applying the shrinking projection method, he [16] also proposed another method:

$$\begin{cases} z_n = x_n - r_n J_{X^*} A^* J_Y \left(I - P_Q\right) A x_n, \\ Q_{n+1} = \{z \in Q_n: \langle z_n - z, J_X (x_n - z_n)\rangle \geq 0\}, \\ x_{n+1} = P_{Q_{n+1}} (x_0), \quad \forall n \in \mathbb{N}. \end{cases} \tag{6}$$

Instead of weak convergence, Takahashi proved the strong convergence of both methods, under the assumption that X is uniformly convex and smooth, which is clearly weaker than that used in [14]. Following the above works, Wang [10] recently proposed a new method, which generates a sequence as

$$\begin{cases} z_n = x_n - r_n J_{X^*} \left[J_X (I - P_C) x_n + A^* J_Y \left(I - P_Q\right) A x_n\right], \\ C_n = \{z \in X: \langle z_n - z, J_X (x_n - z_n)\rangle \geq 0\}, \\ Q_n = \{z \in X: \langle x_n - z, J_X (x_0 - x_n)\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} (x_0), \quad \forall n \in \mathbb{N}. \end{cases} \tag{7}$$

Our aim of this paper is to continue the above works by constructing new iterative methods in Banach spaces. By applying ideas (6) and (7), we introduce a new iterative algorithm and propose two different choices of the stepsize. We show that if the spaces involved are smooth and uniformly convex, then the algorithm converges strongly under both choices of the stepsize. It is also worth noting that one choice of the stepsize does not need any a priori knowledge of the operator norm.

2. Preliminaries

In what follows, we shall assume that the split feasibility problem is consistent, that is, its solution set denoted by $\mathcal{S}$ is nonempty. The notation " $\longrightarrow$ " stands for strong convergence, "$\rightharpoonup$" represents weak convergence, and $\omega_w\{x_n\}$ is the set of weak cluster points of a sequence $\{x_n\}$. Let $S_X = \{x \in X: \|x\| = 1\}$ and $B_X = \{x \in X: \|x\| \leq 1\}$, respectively, be the unit sphere and unit ball of X. $T^{-1}(0) = \{x \in X: Tx = 0\}$ denotes the null-point set of an operator T defined on X. For $x \in X$, we let $J_X (I - P_C) x = J_X (x - P_C x)$ and $A^* J_Y \left(I - t P_Q \leftharpoondown A x = \right. A^* \leftharpoondown J_Y A x - P_Q (A x (\leftharpoondown$.

Definition 1. Let X be a real Banach space.

(1) The modulus of convexity $\delta_X (\varepsilon)$: $[0, 2] \longrightarrow [0, 1]$ is defined as

$$\delta_X (\varepsilon) = \inf\left\{1 - \frac{\|x + y\|}{2}: x, y \in B_X, \|x - y\| \geq \varepsilon\right\}. \tag{8}$$

(2) The modulus of smoothness $\rho_X (\tau (: 0, \infty \longrightarrow 0, \infty$ is defined by

$$\rho_X (\tau) = \sup\left\{\frac{\|x + \tau y\| + \|x - \tau y\|}{2} - 1: x, y \in S_X\right\}. \tag{9}$$

(3) The duality mapping J_X: $X \longrightarrow 2^{X^*}$ is defined by

$$J_X x = \left\{x^* \in X^*: \langle x, x^*\rangle = \|x\|^2 = \left\|x^*\right\|^2\right\}. \tag{10}$$

Definition 2. Let X be a real Banach space.

(1) X is called strictly convex if $\delta_X (2) = 1$.
(2) X is called smooth if $\lim_{t \longrightarrow 0} (\|x + t y\| - \|x\|/t)$ exists for each $x, y \in S_X$.
(3) X is called uniformly convex if $\delta_X (\varepsilon) > 0$ for any $\varepsilon \in \leftharpoondown 0, 2 \leftharpoondown$.
(4) X is called uniformly smooth if $\lim_{\tau \longrightarrow 0} \rho_X (\tau)/\tau = 0$.
(5) X is called p-uniformly convex if there exist $p \geq 2$ and a constant $c > 0$ such that $\delta_X (\varepsilon (\geq c\varepsilon^p$, $\forall \varepsilon \in \leftharpoondown 0, 2 \leftharpoondown$.

Lemma 1 (see [17–19]). *If X is uniformly convex, then X^* is uniformly smooth; X is strictly convex and reflexive.*

Lemma 2 (see [17–19]). *Let $\{x_n\}$ be a sequence in X such that $x_n \rightharpoonup x$ and $\|x_n\| \longrightarrow \|x\|$ as $n \longrightarrow \infty$. If X is uniformly convex, then* $\lim_n x_n = x$.

Lemma 3 (see [17–19]). *Let $M > 0$ and $\{x_n\}, \{y_n\}$ be two sequences in X such that $\|x_n\| \longrightarrow M$, $\|y_n\| \longrightarrow M$ and $\|x_n + y_n\| \longrightarrow 2M$ as $n \longrightarrow \infty$. If X is uniformly convex, then* $\lim_n \|x_n - y_n\| = 0$.

Lemma 4 (see [17–19]). *Let J_X be the duality mapping on X.*

(1) J_X is surjective if and only if X is reflexive.
(2) J_X is injective if and only if X is strictly convex.
(3) J_X is single-valued if and only if X is smooth.
(4) If X is smooth, then J_X is monotone, that is,

$$\langle x - y, J_X x - J_X y\rangle \geq 0, \quad \forall x, y \in X. \tag{11}$$

Moreover, if X is further strictly convex, then J_X is strictly monotone, that is,

$$\langle x - y, J_X x - J_X y\rangle = 0 \Rightarrow x = y. \tag{12}$$

(5) If X is reflexive, smooth, and strictly convex, then J_X is one-to-one single-valued and $J_X^{-1} = J_{X^*}$, where J_{X^*} is the duality mapping of X^*.

The Bregman distance with respect to $\|\cdot\|$ is given by

$$\Delta (x, y) = \frac{1}{2}\|x\|^2 - \langle J_X x, y\rangle + \frac{1}{2}\|y\|^2. \tag{13}$$

This notion goes back to Bregman [20] and now is successfully used in various optimization problems in Banach spaces (see, e.g., [21, 22]). In general, the Bregman distance is not a metric due to the absence of symmetry, but it has some distance-like properties.

Definition 3. Let C be a nonempty closed convex subset of X.

(1) The metric projection P_C: $X \longrightarrow C$ is defined as

$$P_C x := \arg\min_{y\in C} \|x - y\|, \quad x \in X. \tag{14}$$

(2) The Bregman projection Π_C: $X \longrightarrow C$ is defined as

$$\Pi_C x = \arg\min_{y\in C} \triangle(x, y), \quad x \in X. \tag{15}$$

In Hilbert spaces, the metric and Bregman projections are the same, but in general they are completely different. More importantly, the metric projection cannot share the descent property as the Bregman projection in Banach spaces. We now collect some properties of metric projections.

Lemma 5 (see [14]). *Let $\{x_n\}$ be a sequence in X and $C\subseteq X$ be a nonempty closed convex subset. Then, for $x \in X$, the following holds.*

(1) $\langle z - P_C x, J_X(x - P_C x)\rangle \le 0$, $\forall z \in C$.

(2) $\|x - P_C x\|^2 \le \langle x - z, J_X(x - P_C x)\rangle$, $\forall z \in C$.

(3) If $x_n \rightharpoonup x$ and $\|x_n - P_C x_n\| \longrightarrow 0$, then $x \in C$.

3. Convergence Analysis

To construct our algorithm, we need the following lemma. It converts the split feasibility problem to an equivalent null-point problem, which indeed amounts to a fixed point problem.

Lemma 6 (see [10]). *Let $T: = J_{X^*}[J_X(I - P_C) + A^* J_Y(I - P_Q)A]$. Then, $\mathcal{S} = T^{-1}(0)$.*

By applying idea (5) to Lemma 6, we thus can propose the following algorithm for solving the split feasibility problem in Banach spaces. Choose $x_0 \in X$ and $Q_0 = X$. Given x_n, update x_{n+1} by the iteration formula:

$$\begin{cases} z_n = x_n - r_n J_{X^*}\left[J_X(I - P_C)x_n + A^* J_Y\left(I - P_Q\right)Ax_n\right], \\ Q_{n+1} = \{z \in Q_n: \langle z_n - z, J_X(x_n - z_n)\rangle \ge 0\}, \\ x_{n+1} = P_{Q_{n+1}}(x_0), \quad \forall n \in \mathbb{N}. \end{cases} \tag{16}$$

Lemma 7. *Assume that both X and Y are reflexive, smooth, and strictly convex. If r_n is chosen so that $0 < a \le r_n \le (1/1 + \|A\|^2)$, then for each $n \in \mathbb{N}$, the set Q_n is nonempty, closed, and convex. Hence, the proposed algorithm is well defined.*

Proof. It suffices to show that Q_n is nonempty since it is clearly closed and convex. To this end, we now show by induction that $\mathcal{S}\subseteq Q_n$ for all $n \in \mathbb{N}$ and that $\mathcal{S}\subseteq Q_1$ is obvious. Suppose that $\mathcal{S}\subseteq Q_k$ for some $k \in \mathbb{N}$. Take any $z \in \mathcal{S}$. Then, we have $z \in Q_k$. Furthermore, we have

$$\begin{aligned} &\langle x_k - z, J_X(x_k - z_k)\rangle \\ &= r_k\langle x_k - z, J_X(I - P_C)x_k + A^* J_Y\left(I - P_Q\right)Ax_k\rangle \\ &= r_k\langle x_k - z, J_X(I - P_C)x_k\rangle + r_k\langle Ax_k - Az, J_Y\left(I - P_Q\right)Ax_k\rangle, \end{aligned} \tag{17}$$

which from Lemma 5 yields

$$\langle x_k - z, J_X(x_k - z_k)\rangle \ge r_k\left(\left\|(I - P_C)x_k\right\|^2 + \left\|\left(I - P_Q\right)Ax_k\right\|^2\right). \tag{18}$$

By a simple calculation, we have

$$\begin{aligned} \left\|x_k - z_k\right\|^2 &= r_k^2\left\|J_X(I - P_C)x_k + A^* J_Y\left(\left(I - P_Q\right)Ax_k\right)\right\|^2 \\ &\le r_k^2\left(\left\|(I - P_C)x_k\right\| + \|A\|\left\|\left(I - P_Q\right)Ax_k\right\|\right)^2 \\ &= r_k^2\left(\left\|(I - P_C)x_k\right\|^2 + \|A\|^2\left\|\left(I - P_Q\right)Ax_k\right\|^2\right) \\ &\quad + 2r_k^2\|A\|\left\|(I - P_C)x_k\right\|\left\|\left(I - P_Q\right)Ax_k\right\| \\ &\le r_k^2\left(\left\|(I - P_C)x_k\right\|^2 + \|A\|^2\left\|\left(I - P_Q\right)Ax_k\right\|^2\right) \\ &\quad + r_k^2\left(\|A\|^2\left\|(I - P_C)x_k\right\|^2 + \left\|\left(I - P_Q\right)Ax_k\right\|^2\right). \end{aligned} \tag{19}$$

Hence, we have

$$\left\|x_k - z_k\right\|^2 \le r_k^2\left(1 + \|A\|^2\right)\left(\left\|(I - P_C)x_k\right\|^2 + \left\|\left(I - P_Q\right)Ax_k\right\|^2\right). \tag{20}$$

It then follows from (18) that

$$\begin{aligned} \langle z_k - z, J_X(x_k - z_k)\rangle &= \langle z_k - x_k, J_X(x_k - z_k)\rangle \\ &\quad + \langle x_k - z, J_X(x_k - z_k)\rangle \\ &= -\left\|z_k - x_k\right\|^2 + \langle x_k - z, J_X(x_k - z_k)\rangle \\ &\ge r_k\left(\left\|(I - P_C)x_k\right\|^2 + \left\|\left(I - P_Q\right)Ax_k\right\|^2\right) \\ &\quad - \left\|x_k - z_k\right\|^2. \end{aligned} \tag{21}$$

Substituting (20) into the above inequality, we have

$$\begin{aligned} &\langle z_k - z, J_X(x_k - z_k)\rangle \\ &\ge r_k\left(1 - r_k\left(1 + \|A\|^2\right)\right)\left(\left\|(I - P_C)x_k\right\|^2 + \left\|\left(I - P_Q\right)Ax_k\right\|^2\right). \end{aligned} \tag{22}$$

By our choice of r_k, we see that

$$\langle z_k - z, J_X(x_k - z_k)\rangle \ge 0. \tag{23}$$

This implies that $z \in Q_{k+1}$. Since z is chosen in $\mathcal{S}$ arbitrarily, we conclude that $\mathcal{S}\subseteq Q_{k+1}$. Consequently, $\mathcal{S}\subseteq Q_n$ for all $n \in \mathbb{N}$. Now it is clear that the set Q_n is nonempty, closed, and convex. Thus, the proposed algorithm is well defined.

Now let us state the convergence of the proposed algorithm. □

Theorem 1. *Assume that X is uniformly convex and smooth and Y is reflexive, smooth, and strictly convex. If r_n is chosen so that $0 < a \le r_n \le (1/1 + \|A\|^2)$, then the sequence $\{x_n\}$ generated by (16) converges strongly to $\widehat{z} \in \mathcal{S}$, where $\widehat{z} = P_{\mathcal{S}}(x_0)$.*

Proof. We first show the following equality:

$$\lim_{n\longrightarrow\infty} \|x_n - z_n\| = 0. \tag{24}$$

To this end, let $z \in S$. From the previous lemma, it is clear that $z \in Q_n$, $x_{n+1} \in Q_{n+1} \subseteq Q_n$. Thus, we have for each $n \in \mathbb{N}$,

$$\|x_0 - x_n\| = \|x_0 - P_{Q_n}(x_0)\| \le \min\left(\|x_0 - z\|, \|x_0 - x_{n+1}\|\right). \tag{25}$$

This indicates that $\{\|x_0 - x_n\|\}$ is nondecreasing and bounded above; thus, the limit of $\{\|x_0 - x_n\|\}$ exists. Now set $M := \lim_n \|x_0 - x_n\|$. We have

$$\begin{aligned}\limsup_{n\longrightarrow\infty} \|(x_n - x_0) + (x_{n+1} - x_0)\| &\le \lim_{n\longrightarrow\infty} \left(\|x_n - x_0\| + \|x_{n+1} - x_0\|\right) \\ &= 2M.\end{aligned} \tag{26}$$

On the other hand, we have

$$\begin{aligned}\liminf_{n\longrightarrow\infty} \|(x_n - x_0) + (x_{n+1} - x_0)\| &= \liminf_{n\longrightarrow\infty} 2\left\|\frac{x_n + x_{n+1}}{2} - x_0\right\| \\ &\ge \lim_{n\longrightarrow\infty} 2\|x_n - x_0\| \\ &= 2M,\end{aligned} \tag{27}$$

where the inequality follows from the fact $(x_n + x_{n+1})/2 \in Q_n$. Altogether, we have $\lim_n \|(x_n - x_0) + (x_{n+1} - x_0)\| = 2M$. Since X is uniformly convex, this yields that

$$\lim_{n\longrightarrow\infty} \|x_n - x_{n+1}\| = \lim_{n\longrightarrow\infty} \|(x_n - x_0) - (x_{n+1} - x_0)\| = 0. \tag{28}$$

Furthermore, since $x_{n+1} \in Q_{n+1}$, we have

$$\langle z_n - x_{n+1}, J_X(x_n - z_n)\rangle \ge 0, \tag{29}$$

which clearly implies that

$$\begin{aligned}\|x_n - z_n\|^2 &= \langle x_n - z_n, J_X(x_n - z_n)\rangle \\ &\le \langle x_n - x_{n+1}, J_X(x_n - z_n)\rangle \\ &\le \|x_n - x_{n+1}\| \|x_n - z_n\|.\end{aligned} \tag{30}$$

Hence, we have $\|x_n - z_n\| \le \|x_n - x_{n+1}\| \longrightarrow 0$, which yields (24).

We next show that every weak cluster point of $\{x_n\}$ is a solution of the split feasibility problem. To this end, let x be any weak cluster point of $\{x_n\}$ and take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging weakly to x. In view of (18) and (24), we have

$$\lim_{n\longrightarrow\infty} \|(I - P_C)x_n\| = \lim_{n\longrightarrow\infty} \|(I - P_Q)Ax_n\| = 0. \tag{31}$$

By Lemma 5, $x \in C$. On the other hand, for any $x^* \in X^*$, it follows that

$$\lim_{k\longrightarrow\infty} \langle Ax_{n_k}, x^*\rangle = \lim_{k\longrightarrow\infty} \langle x_{n_k}, A^* x^*\rangle = \langle x, A^* x^*\rangle = \langle Ax, x^*\rangle, \tag{32}$$

implying $Ax_{n_k} \rightharpoonup Ax$. By Lemma 5, $Ax \in Q$. Altogether, $x \in \mathcal{S}$. Since x is arbitrary, we obtain the desired conclusion.

Finally, we prove that $\{x_n\}$ converges strongly to $\widehat{z}$. Now take any $x \in \omega_w(x_n)$. Then, $x \in \mathcal{S}$, and there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging weakly to x. It then follows that

$$\begin{aligned}\|x_0 - \widehat{z}\| &= \|x_0 - P_{\mathcal{S}}(x_0)\| \le \|x_0 - x\| \le \lim_{k\longrightarrow\infty} \|x_0 - x_{n_k}\| \\ &= \lim_{k\longrightarrow\infty} \|x_0 - P_{Q_{n_k}} x_0\| \le \|x_0 - \widehat{z}\|,\end{aligned} \tag{33}$$

where the first and the last inequalities follow from the property of metric projections and the second one follows from the weak lower semicontinuity of the norm. Hence,

$$\widehat{z} = x, \lim_{k\longrightarrow\infty} \|x_0 - x_{n_k}\| = \|x_0 - \widehat{z}\|. \tag{34}$$

Since x is chosen arbitrarily, this implies that $\omega_w(x_n)$ is exactly a single-point set, that is, $\{x_n\}$ converges weakly to $\widehat{z}$. Note that $x_0 - x_{n_k} \rightharpoonup x_0 - \widehat{z}$. By Lemma 2, the uniform convexity implies $\lim_k x_{n_k} = \widehat{z}$. Since $\{x_n\}$ converges weakly, this yields $\lim_n x_n = \widehat{z}$ as desired.

As we see from the previous theorem, the choice of r_n is related to $\|A\|$. Thus, to implement this algorithm, one has to compute the norm $\|A\|$, which is generally not easy in practice. In what follows, we introduce another choice of r_n, which ultimately has no relation with $\|A\|$. By applying an idea in [4], we can propose another choice of the parameter r_n as follows:

$$r_n = \frac{\|(I - P_C)x_n\|^2 + \|(I - P_Q)Ax_n\|^2}{\|J_X(I - P_C)x_n + A^* J_Y(I - P_Q)Ax_n\|^2}. \tag{35}$$

Now let us state the convergence of $\{x_n\}$ under this choice of r_n. □

Theorem 2. *Assume that X is uniformly convex and smooth and Y is reflexive, smooth, and strictly convex. Then, the proposed algorithm with (35) is well defined. Moreover, the sequence $\{x_n\}$ generated by (16) converges strongly to $\widehat{z} \in \mathcal{S}$, where $\widehat{z} = P_{\mathcal{S}}(x_0)$.*

Proof. We first show that the algorithm under (35) is well defined. To this end, it suffices to show that for each $n \in \mathbb{N}$, Q_n is nonempty. We now prove this by induction. Suppose that $\mathcal{S}\subseteq Q_k$ for some $k \in \mathbb{N}$. Take any $z \in \mathcal{S}$. Then, we have $z \in Q_k$. Furthermore, by Lemma 5, we have

$$\begin{aligned}\langle x_k - z, J_X(x_k - z_k)\rangle &= r_k\langle x_k - z, J_X(I - P_C)x_k \\ &\quad + A^* J_Y\left(I - P_Q\right)Ax_k\rangle \\ &= r_k\langle x_k - z, J_X(I - P_C)x_k\rangle \\ &\quad + r_k\langle Ax_k - Az, J_Y\left(I - P_Q\right)Ax_k\rangle \\ &\geq r_k\left(\left\|(I - P_C)x_k\right\|^2 + \left\|\left(I - P_Q\right)Ax_k\right\|^2\right),\end{aligned} \tag{36}$$

and

$$\begin{aligned}\left\|x_k - z_k\right\|^2 &= r_k^2\left\|J_X(I - P_C)x_k + A^* J_Y\left(I - P_Q\right)Ax_k\right\|^2 \\ &= r_k\left(\left\|(I - P_C)x_k\right\|^2 + \left\|\left(I - P_Q\right)Ax_k\right\|^2\right).\end{aligned} \tag{37}$$

Consequently, we have

$$\begin{aligned}\langle z_k - z, J_X(x_k - z_k)\rangle &= \langle z_k - x_k, J_X(x_k - z_k)\rangle \\ &\quad + \langle x_k - z, J_X(x_k - z_k)\rangle \\ &\geq r_k\left(\left\|(I - P_C)x_k\right\|^2 + \left\|\left(I - P_Q\right)Ax_k\right\|^2\right) \\ &\quad - \left\|z_k - x_k\right\|^2 \\ &= 0.\end{aligned} \tag{38}$$

This implies that $z \in Q_{k+1}$. Since z is chosen in $\mathcal{S}$ arbitrarily, we conclude that $\mathcal{S}\subseteq Q_{k+1}$. Consequently, $\mathcal{S}\subseteq Q_n$ for all $n \in \mathbb{N}$. Now it is clear that the set Q_n is nonempty, closed, and convex. Thus, the proposed algorithm is well defined.

We now prove that the sequence $\{x_n\}$ generated by (16) converges strongly to $\widehat{z} \in \mathcal{S}$. From the proof of the previous theorem, it suffices to verify that (31) still holds. Similarly, we obtain $\lim_n\|x_n - z_n\| = 0$. From (37), we have

$$\lim_{n\longrightarrow\infty} r_n\left(\left\|(I - P_C)x_n\right\|^2 + \left\|\left(I - P_Q\right)Ax_n\right\|^2\right) = 0. \tag{39}$$

On the other hand, we see that

$$\begin{aligned}r_n &= \frac{\left\|(I - tP_C(x_n\right\|^2 + \left\|\left(I - tP_Q\left(Ax_n\right\|^2\right.\right.}{\left\|J_X(I - tP_C(x_n + A^* J_Y I - P_Q Ax_n\right\|^2} \\ &\geq \frac{\left\|I - P_C x_n\right\|^2 + \left\|I - P_Q Ax_n\right\|^2}{\left(\left\|J_X I - P_C x_n\right\| + \|A\|\left\|J_Y I - P_Q Ax_n\right\|\right)^2} \\ &\geq \frac{\left\|I - P_C x_n\right\|^2 + \left\|I - P_Q Ax_n\right\|^2}{\left(1 + \|A\|^2\left\|I - P_C x_n\right\|^2 + \left\|I - P_Q Ax_n\right\|^2\right)} \\ &= \frac{1}{1 + \|A\|^2} > 0.\end{aligned} \tag{40}$$

This together with (39) yields (31) as desired. Hence, the proof is complete. □

Remark 1. It is worth noting that our algorithm is new even in Hilbert spaces. Indeed, in Hilbert spaces, our algorithm is reduced to

$$\begin{cases} z_n = x_n - r_n\left[(I - P_C)x_n + A^*\left(I - P_Q\right)Ax_n\right], \\ Q_{n+1} = \left\{z \in Q_n\colon \langle z_n - z, x_n - z_n\rangle \geq 0\right\}, \\ x_{n+1} = P_{Q_{n+1}}(x_0), \quad \forall n \in \mathbb{N}. \end{cases} \tag{41}$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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