Research Article

Nonlocal Fractional Hybrid Boundary Value Problems Involving Mixed Fractional Derivatives and Integrals via a Generalization of Darbo's Theorem

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In this work, a new existence result is established for a nonlocal hybrid boundary value problem which contains one left Caputo and one right Riemann–Liouville fractional derivatives and integrals. The main result is proved by applying a new generalization of Darbo's theorem associated with measures of noncompactness. Finally, an example to justify the theoretical result is also presented.

1. Introduction

In the past years, fractional differential equations have attracted a lot of attention from many research studies as they have played a key role in many basic sciences such as chemistry, control theory, biology, and other arenas [1–3]. In addition, boundary conditions of differential models are the strongest tools to extend applications of those equations [4–6]. In fact, fractional differential equations can be extended by creating different types of boundary conditions. Newly, many authors have studied various types of boundary conditions to obtain new results of differential models.

The following hybrid differential equation was studied by Dhage and Lakshmikantham [7]:

$$\begin{cases} \frac{d}{dt} \left[\frac{x(t)}{h(t, x(t))} \right] = \omega(t, x(t)), & a.e. t \in J, \\ x(t_0) = x_0 \in \mathbb{R}, \end{cases} \quad (1)$$

where h and ω are continuous functions from $J \times \mathbb{R}$ into $\mathbb{R} \setminus \{0\}$ and $\mathbb{R}$, respectively. Based on the above work, the Caputo hybrid boundary value problem of the form

$$\begin{cases} {}^C D_{0^+}^p \left[\frac{x(t)}{h(t, x(t))} \right] = \omega(t, x(t)), & a.e. t \in I := [0, L], \\ a_1 \frac{x(0)}{h(0, x(0))} + a_2 \frac{x(L)}{h(L, x(L))} = d, \end{cases} \quad (2)$$

was studied by Hilal and Kajouni [8] in which $0 < p < 1$, h and ω are continuous functions from $J \times \mathbb{R}$ into $\mathbb{R} \setminus \{0\}$ and $\mathbb{R}$, respectively, and a_1, a_2 , and d are real constants with $a_1 + a_2 \neq 0$. For some recent results on hybrid fractional differential equations, see [9–12].

In [13], the authors proved the following integro-differential equation:

$$\begin{aligned}
{}^c D_{1-}^{\alpha_1} {}^{\text{RL}} D_{0+}^{\alpha_2} u(t) + \theta I_{-1}^p I_{0+}^q f_1(t, u(t)) &= f(t, u(t)), \quad t \in [0, 1], \\
u(0) &= u(\xi) = 0, \\
u(1) &= \delta u(\mu), \quad 0 < \xi < \mu < 1,
\end{aligned} \tag{3}$$

where ${}^c D_{1-}^{\alpha_1}$ and ${}^{\text{RL}} D_{0+}^{\alpha_2}$ indicate right Caputo and left Riemann–Liouville fractional derivatives of orders $\alpha_1 \in (1, 2]$ and $\alpha_2 \in (0, 1]$, respectively, $f_1, f_2: [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions, and the symbols I_{-1}^p and I_{0+}^q denote both right and left Riemann–Liouville fractional integrals of orders $p, q > 0$, respectively. Ahmad et al. [13] applied Banach and Krasnosel'skii fixed point theorems as well

Leray–Schauder nonlinear alternative to obtain main results. We point out that fractional differential equations containing mixed fractional derivatives appear in the study of variational principles [14].

For some recent results for boundary value problems involving left or/and right fractional derivatives, we refer to the papers [15–31] and references therein.

In the present paper, we combine mixed fractional derivatives and hybrid fractional differential equations. More precisely, we investigate the existence of solutions for the following hybrid boundary value problem which contains both left Caputo and right Riemann–Liouville fractional derivatives and integrals and nonlocal hybrid conditions of the form:

$${}^c D_{1-}^{\alpha_1} {}^{\text{RL}} D_{1+}^{\alpha_2} \frac{u(t)}{g(t, u(t))} + \theta I_{-1}^p I_{0+}^q f_1(t, u(t)) = f_2(t, y(t)), \quad t \in J := [0, 1],$$

$$\frac{u(0)}{g(0, u(0))} = \frac{u(\xi)}{g(\xi, u(\xi))} = 0, \tag{4}$$

$$\frac{u(1)}{g(1, u(1))} = \delta \frac{u(\mu)}{g(\mu, u(\mu))}, \quad 0 < \xi < \mu < 1,$$

where ${}^c D_{1-}^{\alpha_1}$ and ${}^{\text{RL}} D_{0+}^{\alpha_2}$ are right Caputo and left Riemann–Liouville fractional derivatives of orders $\alpha_1 \in (1, 2]$ and $\alpha_2 \in (0, 1]$, respectively, and the symbols I_{-1}^p and I_{0+}^q denote both right and left Riemann–Liouville fractional integrals of orders $p, q > 0$, respectively, $f_1, f_2 \in C(J \times \mathbb{R}, \mathbb{R})$, $g \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, and $\delta, \theta \in \mathbb{R}$. An existence result is obtained via a new extension of Darbo's theorem associated with measures of noncompactness.

The structure of the paper has been organized as follows. Section 2 presents some basic definitions and lemmas which will be applied in the future. In Section 3, we prove an existence result for problem (4). Finally, we present an example to illustrate the obtained result.

2. Preliminaries

Now, some basic notations are recalled from [2].

Definition 1. For an integrable function $\phi: (0, \infty) \rightarrow \mathbb{R}$, we define the left and right Riemann–Liouville fractional integrals of order $\beta > 0$, respectively, by

$$\begin{aligned}
I_{0+}^{\beta} \phi(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \phi(s) ds, \\
I_{1-}^{\beta} \phi(t) &= \int_t^1 \frac{(s-t)^{\beta-1}}{\Gamma(\beta)} \phi(s) ds.
\end{aligned} \tag{5}$$

Definition 2. For the function $\phi: (0, \infty) \rightarrow \mathbb{R}$ in which $\phi \in C^n(0, \infty)$, we define the left Riemann–Liouville

fractional derivative and the right Caputo fractional derivative of order $\beta \in (n-1, n]$, respectively, by

$$\begin{aligned}
{}^{\text{RL}} D_{0+}^{\beta} \phi(t) &= \frac{d^n}{dt^n} \int_0^t \frac{(t-s)^{n-\beta-1}}{\Gamma(n-\beta)} \phi(s) ds, \\
{}^c D_{1-}^{\beta} \phi(t) &= (-1)^n \int_t^1 \frac{(s-t)^{n-\beta-1}}{\Gamma(n-\beta)} \phi^{(n)}(s) ds.
\end{aligned} \tag{6}$$

Lemma 1. If $p > 0$ and $q > 0$, then the following relations hold almost everywhere on $[a, b]$:

$$\begin{aligned}
I_{1-}^p I_{-1}^q f(x) &= I_{1-}^{p+q} f(x), \\
I_{0+}^p I_{0+}^q f(x) &= I_{0+}^{p+q} f(x).
\end{aligned} \tag{7}$$

As the technique of measure of noncompactness will be applied to obtain our main result, we recall some basic facts about the notion of measure of noncompactness.

Assume that Z is the real Banach space with the norm $\|\cdot\|$ and zero element θ . For a nonempty subset X of Z , the closure and the closed convex hull of X will be denoted by $\overline{X}$ and $\text{Conv}(X)$, respectively. Also, M_Z and N_Z denote the family of all nonempty and bounded subsets of Z and its subfamily consisting of all relatively compact sets, respectively.

Definition 3 (see [32]). We say that a mapping $h: M_Z \rightarrow [0, \infty)$ is a measure of noncompactness, if the following conditions hold true:

- (1) The family $\text{Ker}h = \{X \in M_Z : h(X) = 0\}$ is nonempty and $\text{Ker} h \subseteq N_Z$
- (2) $X_1 \subseteq Y_1 \Rightarrow h(X_1) \leq h(Y_1)$
- (3) $h(\overline{X}) = h(X)$
- (4) $h(\text{Conv}(X)) = h(X)$
- (5) $h(\alpha X + (1 - \alpha)Y) \leq \alpha h(X) + (1 - \alpha)h(Y)$ for $\alpha \in [0, 1]$
- (6) For the sequence $\{X_n\}$ of closed sets from M_Z in which $X_{n+1} \subseteq X_n$ for $n = 1, 2, \dots$ and $\lim_{n \rightarrow \infty} h(X_n) = 0$, we have $\bigcap_{n=1}^{\infty} X_n \neq \emptyset$

In [33], some generalizations of Darbo's theorem have been proved by Samadi and Ghaemi. Also, in [34], Darbo's theorem was extended, and the following result was presented which is basis for our main result.

Theorem 1. Let T be a continuous self-mapping operator on the set D , where D denotes a nonempty, bounded, closed, and convex subset of a Banach space Z . Assume that, for all nonempty subset X of D , we have

$$\theta_1((h(X)) + \theta_2(h(T(X))) \leq \theta_2(h(X)), \quad (8)$$

where h is an arbitrary measure of noncompactness defined in Z and $(\theta_1, \theta_2) \in U$. Then, T has a fixed point in D .

In Theorem 1, let U indicate the set of all pairs (θ_1, θ_2) where the following conditions hold true:

- (U₁) $\theta_1(t_n) \rightarrow 0$ for each strictly increasing sequence $\{t_n\}$
- (U₂) θ_2 is strictly increasing function

(U₃) If $\{\alpha_n\}$ be a sequence of positive numbers, then $\lim_{n \rightarrow \infty} \alpha_n = 0 \Leftrightarrow \lim_{n \rightarrow \infty} \theta_2(\alpha_n) = -\infty$

(U₄) Let $\{l_n\}$ be a decreasing sequence in which $l_n \rightarrow 0$ and $\theta_1(l_n) < \theta_2(l_n) - \theta_2(l_{n+1})$, then we have $\sum_{n=1}^{\infty} l_n < \infty$

Next, the definition of a measure of noncompactness in the space $C([0, 1])$ is recalled which will be applied later. Fix $Y \in M_{C[0,1]}$, and for $\varepsilon > 0$ and $y \in Y$, we define

$$\begin{aligned} \varphi(y, \varepsilon) &= \sup \{|y(t) - y(s)| : t, s \in [0, 1], |t - s| \leq \varepsilon\}, \\ \varphi(Y, \varepsilon) &= \sup \{\varphi(y, \varepsilon) : y \in Y\}, \\ \varphi_0(Y) &= \lim_{\varepsilon \rightarrow 0} \varphi(Y, \varepsilon). \end{aligned} \quad (9)$$

Banas and Goebel [32] proved that $\varphi_0(Y)$ is a measure of noncompactness in the space $C([0, 1])$.

Lemma 2 (see [32]). The measure of noncompactness φ_0 on $C(I)$ satisfies the following condition:

$$\varphi_0(XY) \leq \|X\| \varphi_0(Y) + \|Y\| \varphi_0(X), \quad (10)$$

for all $X, Y \subseteq C(I)$.

3. Main Existence Result

In this section, an existence result of problem (4) is investigated. In view of [13], Lemma 2, we present the following lemma which is an essential tool in our consideration.

Lemma 3. Let $H_1, F_1 \in C[0, 1] \cap L(0, 1)$, $g \in C([0, 1], \mathbb{R} \setminus \{0\})$, and $\Lambda \neq 0$. Then, the solution of the problem

$$\begin{cases} {}^c D_{1-}^{\alpha_1, \text{RL}} D_{0+}^{\beta} \frac{u(t)}{g(t, u(t))} + \lambda I_{-1}^p I_{0+}^q H_1(t) = F_1(t), & t \in [0, 1], \\ \frac{u(0)}{g(0, u(0))} = \frac{u(\xi)}{g(\xi, u(\xi))} = 0, \\ \frac{u(1)}{g(1, u(1))} = \delta \frac{u(\mu)}{g(\mu, u(\mu))}, & 0 < \xi < \mu < 1, \end{cases} \quad (11)$$

has the form:

$$u(t) = g(t, u(t)) \left[\int_0^t \frac{(t-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} (I_{-1}^{\alpha_1} F_1(s) - \lambda I_{-1}^{\alpha_1+p} I_{0+}^q H_1(s)) ds + a_1(t) \left\{ \begin{aligned} & \delta \int_0^{\mu} \frac{(\mu-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} (I_{-1}^{\alpha_1} F_1(s) - \lambda I_{-1}^{\alpha_1+p} I_{0+}^q H_1(s)) ds \\ & - \int_0^1 \frac{(1-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} (I_{-1}^{\alpha_1} F_1(s) - \lambda I_{-1}^{\alpha_1+p} I_{0+}^q H_1(s)) ds \end{aligned} \right\} + a_2(t) \int_0^{\xi} \frac{(\xi-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} (I_{-1}^{\alpha_1} F_1(s) - \lambda I_{-1}^{\alpha_1+p} I_{0+}^q H_1(s)) ds \right], \quad (12)$$

where

$$\begin{aligned} a_1(t) &= \frac{1}{\Lambda} \left[\xi^{\alpha_2+1} t^{\alpha_2} - \xi^{\alpha_2} t^{\alpha_2+1} \right], \\ a_2(t) &= \frac{1}{\Lambda} \left[t^{\alpha_2} (1 - \delta \mu^{\alpha_2+1}) - t^{\alpha_2+1} (1 - \delta \mu^{\alpha_2}) \right], \\ \Lambda &= \xi^{\alpha_2+1} (1 - \delta \mu^{\alpha_2}) - \xi^{\alpha_2} (1 - \delta \mu^{\alpha_2+1}). \end{aligned} \quad (13)$$

Now, the hypotheses which will be applied to prove the main result of this section are presented.

(H₁) $g: [0, 1] \times \mathbb{R} \rightarrow \mathbb{R} \setminus \{0\}$ is a continuous function, and there exists a positive real number $d > 0$ provided that

$$|g(t, x_1) - g(t, x_2)| \leq e^{-d} |x_1 - x_2|, \quad (14)$$

where $t \in I$ and $x_1, x_2 \in \mathbb{R}$. Moreover, assume that $\bar{g} = \sup \{|g(t, 0)|; t \in [0, 1]\}$.

(H₂) $f_1, f_2: [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions provided that

$$\begin{aligned} |f_1(t, u)| &\leq M_1, \\ |f_1(t, u) - f_1(t, v)| &\leq k_1 |u - v|, \\ |f_2(t, u)| &\leq M_2, \\ |f_2(t, u) - f_2(t, v)| &\leq k_2 |u - v|, \end{aligned} \quad (15)$$

where $M_1, M_2, k_1, k_2 \geq 0$ and $u, v \in \mathbb{R}$.

(H₃) The inequality

$$\left[e^{-d} r_0 + \bar{g} \right] \left\{ \frac{M_1}{\Gamma(\alpha_1 + 1)} + \frac{|\theta| M_2}{\Gamma(\alpha_1 + p + 1) \Gamma(q + 1)} \right\} \Delta \leq r_0, \quad (16)$$

has a positive solution r_0 . Also, assume that

$$\left\{ \frac{M_1}{\Gamma(\alpha_1 + 1)} + \frac{|\theta| M_2}{\Gamma(\alpha_1 + p + 1) \Gamma(q + 1)} \right\} \Delta < 1, \quad (17)$$

where

$$\begin{aligned} \Delta &= \frac{1}{\Gamma(\alpha_2 + 1)} \left[1 + \bar{a}_1 (|\delta| \mu^{\alpha_2} + 1) + \bar{a}_2 \xi^{\alpha_2} \right], \\ \bar{a}_1 &= \max_{t \in [0, 1]} |a_1(t)|, \\ \bar{a}_2 &= \max_{t \in [0, 1]} |a_2(t)|. \end{aligned} \quad (18)$$

Theorem 2. Suppose that the hypotheses (H₁) – (H₃) are true. Then, the hybrid boundary value problem (4) has at least one solution on $[0, 1]$.

Proof. Due to Lemma 3, assume that the operator T has been defined on $C(I)$, $I := [0, 1]$ as follows:

$$T_1(u)(t) = (F_1 u(t) + \bar{F}_1 u(t))(G_1 u(t)), \quad (19)$$

where

$$\begin{aligned} G_1 u(t) &= g(t, u(t)), \\ F_1 u(t) &= \int_0^t \frac{(t-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} \left(I_{1-}^{\alpha_1} f_2(s, u(s)) - \theta I_{1-}^{\alpha_1+p} I_{0+}^q f_1(s, u(s)) \right) ds, \\ \bar{F}_1 u(t) &= a_1(t) \left\{ \begin{aligned} &\delta \int_0^\mu \frac{(\mu-s)^{\alpha_1-1}}{\Gamma(\alpha_2)} \left(I_{1-}^{\alpha_1} f_2(s, u(s)) - \theta I_{1-}^{\alpha_1+p} I_{0+}^q f_1(s, u(s)) \right) ds \\ &- \int_0^1 \frac{(1-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} \left(I_{1-}^{\alpha_1} f_2(s, u(s)) - \theta I_{1-}^{\alpha_1+p} I_{0+}^q f_1(s, u(s)) \right) ds \end{aligned} \right\} \\ &\quad + a_2(t) \int_0^\xi \frac{(\xi-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} \left(I_{1-}^{\alpha_1} f_2(s, u(s)) - \theta I_{1-}^{\alpha_1+p} I_{0+}^q f_1(s, u(s)) \right) ds. \end{aligned} \quad (20)$$

First, we show that $T_1 u \in C(I)$ in which $u \in C(I)$. In view of assumption (H₁), we conclude that $G_1 u \in C(I)$, $u \in C(I)$. Consequently, by proving $F_1 u, \bar{F}_2 u \in C(I)$, the

claim is obtained. Let l_n be a sequence in $[0, 1]$ such that $l_n \rightarrow l$. Then, due to our assumptions, we get

$$\begin{aligned}
|F_1 u(l_n) - F_1 u(l)| &\leq \left| \int_0^l \frac{[(l_n - s)^{\alpha_2 - 1} - (l - s)^{\alpha_2 - 1}]}{\Gamma(\alpha_2)} (I_{1-}^{\alpha_1} f_2(s, u(s)) - \theta I_{1-}^{\alpha_1 + p} I_{0+}^q f_1(s, u(s))) ds \right. \\
&\quad \left. + \int_l^{l_n} \frac{(l_n - s)^{\alpha_2 - 1}}{\Gamma(\alpha_2)} (I_{1-}^{\alpha_1} f_2(s, u(s)) - \theta I_{1-}^{\alpha_1 + p} I_{0+}^q f_1(s, u(s))) ds \right| \\
&\leq \frac{M_1}{\Gamma(\alpha_2)} \left| \int_0^{l_n} [(l_n - s)^{\alpha_2 - 1} - (l - s)^{\alpha_2 - 1}] I_{1-}^{\alpha_1}(1) ds \right| \\
&\quad + M_1 \left| \frac{1}{\Gamma(\alpha_2)} \int_l^{l_n} (l_n - s)^{\alpha_2 - 1} I_{1-}^{\alpha_1}(1) ds \right| + \frac{|\theta| M_2}{\Gamma(\alpha_2)} \left| \int_0^l [(l_n - s)^{\alpha_2 - 1} - (l - s)^{\alpha_2 - 1}] I_{1-}^{\alpha_1 + p} I_{0+}^q(1) ds \right| \\
&\quad + \frac{|\theta| M_2}{\Gamma(\alpha_2)} \left| \int_l^{l_n} (l_n - s)^{\alpha_2 - 1} I_{1-}^{\alpha_1 + p} I_{0+}^q(1) ds \right| \leq \left\{ \frac{M_1}{\Gamma(\alpha_1 + 1) \Gamma(\alpha_2 + 1)} + \frac{|\theta| M_2}{\Gamma(q + 1) \Gamma(\alpha_2 + 1) \Gamma(\alpha_1 + p + 1)} \right\} \\
&\quad \times [2(l_n - l)^{\alpha_2} + |l_n^{\alpha_2} - l^{\alpha_2}|] \longrightarrow 0.
\end{aligned} \tag{21}$$

Hence, $F_1 u \in C(I)$. To obtain that $\overline{F_1} u \in C(I)$, by the definitions of a_1 and a_2 , we have $|a_1(l_n) - a_1(l)| \longrightarrow 0$ and $|a_2(l_n) - a_2(l)| \longrightarrow 0$. Hence, we have $|\overline{F_1} u(l_n) - \overline{F_1} u(l)| \longrightarrow 0$. Consequently, $T_1 u \in C(I)$ for all $x \in C(I)$.

Now, we prove that the ball $D_{r_0} = \{u \in C(I) : \|u\| \leq r_0\}$ is mapped into itself by the operator T . Let us fix $u \in C(I)$. Hence, due to existence assumptions, for $t \in I$, we have

$$\begin{aligned}
|(T_1 u)(t)| &\leq [e^{-d} |u(t)| + \overline{g}] \\
&\cdot \left[\left| \int_0^t \frac{(t - s)^{\alpha_2 - 1}}{\Gamma(\alpha_2)} (I_{1-}^{\alpha_1} f_2(s, u(s)) - \theta I_{1-}^{\alpha_1 + p} I_{0+}^q f_1(s, u(s))) ds \right| \right. \\
&\quad + \left. \left| a_1(t) \left\{ \begin{aligned} &\delta \int_0^\mu \frac{(\mu - s)^{\alpha_2 - 1}}{\Gamma(\alpha_2)} (I_{1-}^{\alpha_1} f_2(s, u(s)) - \theta I_{1-}^{\alpha_1 + p} I_{0+}^q f_1(s, u(s))) ds \\ &- \int_0^1 \frac{(1 - s)^{\alpha_2 - 1}}{\Gamma(\alpha_2)} (I_{1-}^{\alpha_1} f_2(s, u(s)) - \theta I_{1-}^{\alpha_1 + p} I_{0+}^q f_1(s, u(s))) ds \end{aligned} \right\} \right| \right. \\
&\quad \left. + \left| a_2(t) \int_0^\xi \frac{(\xi - s)^{\alpha_2 - 1}}{\Gamma(\alpha_2)} (I_{1-}^{\alpha_1} f_2(s, u(s)) - \theta I_{1-}^{\alpha_1 + p} I_{0+}^q f_1(s, u(s))) ds \right| \right] \\
&\leq [e^{-d} r_0 + \overline{g}] \left\{ \frac{M_1}{\Gamma(\alpha_1 + 1)} + \frac{|\theta| M_2}{\Gamma(\alpha_1 + p + 1) \Gamma(q + 1)} \right\} \Delta.
\end{aligned} \tag{22}$$

Consequently, according to assumption (H_3) we conclude that T maps the ball D_{r_0} into itself.

Now, the continuity property of the operator T is considered on the ball D_{r_0} . To do this, fix $\varepsilon > 0$ and take $u, v \in D_{r_0}$ such that $\|u - v\| \leq \varepsilon$. Then, for $t \in I$, we have

$$\begin{aligned}
|(T_1 u)(t) - (T_1 v)(t)| &= |F_1 u(t)G_1 u(t) - F_1 v(t)G_1 v(t) + \overline{F_1} u(t)G_1 u(t) - \overline{F_1} v(t)G_1 v(t)| \\
&\leq |F_1 u(t)G_1 u(t) - F_1 v(t)G_1 u(t)| + |F_1 v(t)G_1 u(t) - F_1 v(t)G_1 v(t)| \\
&\quad + |\overline{F_1} u(t)G_1 u(t) - \overline{F_1} v(t)G_1 u(t)| + |\overline{F_1} v(t)G_1 u(t) - \overline{F_1} v(t)G_1 v(t)| \\
&\leq |g(t, u(t)) - g(t, v(t))| \|F_1 v(t)\| + |g(t, u(t))| |F_1 u(t) - F_1 v(t)| \\
&\quad + |g(t, u(t)) - g(t, v(t))| \|\overline{F_1} v(t)\| + |g(t, u(t))| |\overline{F_1} u(t) - \overline{F_1} v(t)|.
\end{aligned} \tag{23}$$

Then, we have

$$\begin{aligned}
|(T_1 u)(t) - (T_1 v)(t)| &\leq e^{-d} \varepsilon (F_1 v(t) + \overline{F_1} v(t)) + |g(t, u(t))| \varepsilon \left[\int_0^t \frac{(t-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} (k_1 I_{1-}^{\alpha_1}(1) + k_2 |\theta| I_{1-}^{\alpha_1+p} I_{0+}^q(1)) ds \right] \\
&\quad + |g(t, u(t))| \varepsilon \left[\begin{aligned} &\left\{ \delta \int_0^\mu \frac{(\mu-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} (k_1 I_{1-}^{\alpha_1}(1) + k_2 |\theta| I_{1-}^{\alpha_1+p} I_{0+}^q(1)) ds \right. \\ &\quad \left. + \int_0^1 \frac{(1-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} (k_1 I_{1-}^{\alpha_1}(1) + k_2 |\theta| I_{1-}^{\alpha_1+p} I_{0+}^q(1)) ds \right\} \\ &\quad + |a_2(t)| \int_0^\xi \frac{(\xi-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} (k_1 I_{1-}^{\alpha_1}(1) + k_2 |\theta| I_{1-}^{\alpha_1+p} I_{0+}^q(1)) ds \end{aligned} \right] \\
&\leq \varepsilon \left[\frac{k_1}{\Gamma(\alpha_1+1)} + \frac{|\theta| k_2}{\Gamma(\alpha_1+p+1)\Gamma(q+1)} \right] \{e^{-d} \Delta + (e^{-d} r_0 + \overline{g})(\Delta+1)\}.
\end{aligned} \tag{24}$$

Consequently, the continuity property of T is obtained on the ball D_{r_0} .

To finish the proof, condition (8) of Theorem 1 is proved. Consider X as a nonempty subset of the ball D_{r_0} and assume that $u \in X$, $\varepsilon > 0$ be arbitrarily constant. Choose $l_1, l_2 \in [0, 1]$ such that $l_1 < l_2$ and $|l_2 - l_1| < \varepsilon$. Taking into account our assumptions, we get

$$\begin{aligned}
|(G_1 u)(l_1) - (G_1 u)(l_2)| &= |g(l_1, u(l_1)) - g(l_2, u(l_2))| \\
&\leq |g(l_1, u(l_1)) - g(l_1, u(l_2))| \\
&\quad + |g(l_1, u(l_2)) - g(l_2, u(l_2))| \\
&\leq e^{-d} \varphi(X, \varepsilon) + \varphi(g, \varepsilon),
\end{aligned} \tag{25}$$

where

$$\begin{aligned}
\varphi(g, \varepsilon) &= \sup\{|g(l_1, u) - g(l_2, u)|; l_1, l_2 \in I, |l_1 - l_2| \\
&\quad < \varepsilon, u \in [-r_0, r_0]\}.
\end{aligned} \tag{26}$$

Consequently,

$$\varphi(G_1 X, \varepsilon) \leq e^{-d} \varphi(X, \varepsilon) + \varphi(g, \varepsilon). \tag{27}$$

As g is uniformly continuous on $I \times [-r_0, r_0]$, we have $\varphi(g, \varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Thus, from (27), we conclude that

$$\varphi_0(G_1 X) \leq e^{-d} \varphi_0(X). \tag{28}$$

Next, we estimate $\varphi_0(F_1 X)$ and $\varphi_0(\overline{F_1}(X))$. In view of (21), since $F_1 x$ is uniformly continuous on $[0, 1]$, then for fixed $\varepsilon > 0$, there exists $\delta > 0$ such that, for $l_1, l_2 \in I$ with $|l_2 - l_1| < \delta \leq \varepsilon$, we have

$$\varphi_0(F_1 X) \leq \varepsilon. \tag{29}$$

Besides, since a_1 and a_2 are uniformly continuous on $[0, 1]$, for $l_1, l_2 \in [0, 1]$ with $|l_2 - l_1| < \delta \leq \varepsilon$, we have $|a_2(l_2) - a_2(l_1)| < \varepsilon$ and also $|a_1(l_2) - a_1(l_1)| < \varepsilon$. Consequently, we conclude that $\varphi_0(\overline{F_1}(X)) = 0$. Now, we estimate $\varphi_0(T_1 X)$ for $X \subseteq D_{r_0}$. By applying (28) and (29) and Lemma 2 and using the fact that $\varphi_0(\overline{F_1}(X)) = 0$, we get

$$\begin{aligned}
\varphi_0(T_1 X) &= \varphi_0(G_1)X(F_1)(X) + \overline{F_1}((X)) \\
&\leq (\|F_1 X\| + \|\overline{F_1}(X)\|)\varphi_0(G_1 X) \\
&\quad + \|G_1 X\|(\varphi_0)(F_1 X) + \varphi_0(\overline{F_1})(X) \\
&\leq e^{-d}\varphi_0(X) \left\{ \frac{M_1}{\Gamma(\alpha_1 + 1)} + \frac{|\theta|M_2}{\Gamma(\alpha_1 + p + 1)\Gamma(q + 1)} \right\} \Delta.
\end{aligned} \tag{30}$$

Consequently, we derive that

$$\begin{aligned}
D_{1-}^{(3/2)} D_{0+}^{(1/2)} \frac{u(t)}{(e^{-d}/1 + t + |u(t)|)} + 2I_{1-}^{4/3} I_{0+}^{5/4} \frac{e^{-t}}{100} \cos u(t) &= \frac{e^{-t}}{100} \sin u(t), \\
\frac{u(0)}{(e^{-d}/1 + |u(0)|)} &= \frac{u(2/3)}{(e^{-d}/1 + |u(2/3)|)} = 0, \quad \frac{u(1)}{(e^{-d}/2 + |u(1)|)} = \frac{1}{2} \frac{u(3/4)}{(e^{-d}/1 + (3/4) + |u(3/4)|)} = 0.
\end{aligned} \tag{32}$$

By putting

$$\begin{aligned}
g(t, u(t)) &= \frac{e^{-d}}{1 + t + |u(t)|}, \\
f_1(t, u(t)) &= \frac{e^{-t}}{100} \cos u(t), \\
f_2(t, u(t)) &= \frac{e^{-t}}{100} \sin u(t),
\end{aligned} \tag{33}$$

in problem (4), we conclude the above hybrid boundary value problem as a special case of problem (4). Now, the conditions of Theorem 2 are checked. For all $l \in [0, 1]$ and $u_1, u_2 \in \mathbb{R}$, we have

$$|g(l, u_1) - g(l, u_2)| \leq e^{-d} |u_1 - u_2|. \tag{34}$$

Moreover, we have $\overline{g} = \sup\{|g(l, 0); l \in [0, 1]\} = e^{-d}$. Besides, the functions f_1 and f_2 are continuous, and for all $l \in [0, 1]$ and $u, v \in \mathbb{R}$, we have

$$\begin{aligned}
|f_2(t, u)| &\leq \frac{1}{100}, \\
|f_2(t, u) - f_2(t, v)| &\leq \frac{1}{100} |u - v|, \\
|f_1(t, u)| &\leq \frac{1}{100}, \\
|f_1(t, u) - f_1(t, v)| &\leq \frac{1}{100} |u - v|.
\end{aligned} \tag{35}$$

In this example $\alpha_1 = 3/2$, $\alpha_2 = 1/2$, $\theta = 2$, $p = 4/3$, $q = 5/4$, $\mu = 3/43/4$, $\delta = 1/2$, and $\xi = 2/3$. Hence, $|\Lambda| \approx 0.242$, $\overline{a_1} = 1.121$ and $\overline{a_2} = 1.168$. Consequently, the existent inequality in condition (H_3) has the form:

$$d + \ln(\varphi_0(T_1 X)) \leq \ln(\varphi_0(X)). \tag{31}$$

Thus, we conclude the contractive condition in Theorem 1 with $\theta_1(t) = d$ and $\theta_2(t) = \ln(t)$. Thus, by Theorem 1, at least one solution is obtained for the operator T in D_{r_0} which is a solution of problem (4) and the proof is completed. $\square$

Now, the following example is investigated to show the applicability of the obtained result.

Example 1. Consider the following hybrid boundary value problem:

$$\left[e^{-d} r_0 + \frac{1}{100} \right] \left\{ \frac{(1/100)}{\Gamma(5/2)} + \frac{(2/100)}{\Gamma(23/6)\Gamma(9/4)} \right\} \Delta \leq r_0. \tag{36}$$

Obviously, the above inequality has the positive solution r_0 , for example $r_0 = 1$. Moreover, according to the obtained values we have

$$\left\{ \frac{M_1}{\Gamma(\alpha_1 + 1)} + \frac{|\theta|M_2}{\Gamma(\alpha_1 + p + 1)\Gamma(q + 1)} \right\} \Delta < 1. \tag{37}$$

Thus, we conclude all conditions of Theorem 2, and hence at least one solution of the mapping T_1 is obtained on $[0, 1]$ which is a solution of problem (32).

4. Conclusion

We have studied a nonlocal hybrid boundary value problem which contains both left Caputo and right Riemann–Liouville fractional derivatives and integrals and nonlocal hybrid conditions. An existence result is proved by applying a new generalization of Darbo's fixed point theorem associated with measures of noncompactness. The result obtained in this paper is new and significantly contributes to the existing literature on the topic.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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