Retraction

Retracted: Some Novel Cosine Similarity Measures Based on Complex Hesitant Fuzzy Sets and Their Applications

Journal of Mathematics

This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope

(2) Discrepancies in the description of the research reported

(3) Discrepancies between the availability of data and the research described

(4) Inappropriate citations

(5) Incoherent, meaningless and/or irrelevant content included in the article

(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

[1] R. Chinram, T. Mahmood, U. Ur Rehman, Z. Ali, and A. Iampan, "Some Novel Cosine Similarity Measures Based on Complex Hesitant Fuzzy Sets and Their Applications," *Journal of Mathematics*, vol. 2021, Article ID 6690728, 20 pages, 2021.

Research Article

Some Novel Cosine Similarity Measures Based on Complex Hesitant Fuzzy Sets and Their Applications

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The theory of complex hesitant fuzzy set (CHFS) is a modification technique of the complex fuzzy set (CFS) to cope with awkward and unreliable information's in daily life issues. CHFS contains the grade of truth in the form of complex number, whose real and imaginary parts are in the form of the finite subset of the unit interval. CHFS is the mixture of hesitant fuzzy set (HFS) and CFS, which handles the complex and uncertain information in real-world issues which is compared with fuzzy sets and complex fuzzy sets. The positive membership in CHFS is in the form a polar coordinate belonging to unit disc in the complex plane. The aims of this manuscript are to explore some similarity measures (SMs), weighted SMs (WSMs) such as cosine SMs, weighted cosine SMs, SMs based on cosine function, WSMs based on cosine function, SMs based on tangent function, and WSMs based on tangent function of CHFS. Some special cases of the presented measures are discussed in detail. Moreover, we use our described SMs and weighted SMs of CHFS in the environment of medical diagnosis and pattern recognition to assess the practicality and competence of the described SMs. Finally, to find the validity and proficiency of the investigated measures based on CHFSs, the comparison between explored measures with some already defined measures and their graphical representations are also discussed in detail.

1. Introduction

The fuzzy set (FS) is the modification of crisp set which was given by Zadeh [1] to manage the vagueness and uncertainty in the information in real-life decisions. In the theory of FS, the positive grade belongs to closed interval [0, 1], where greatest value designated greatest positive grade. FS has numerous applications in various fields [2–4]. Bustince et al. [5] operated on FSs and their models, extensions portrayal, and aggregation. SMs between FSs play an essential role in the theory of FS, which attracted a lot of attention from the authors. SMs have a lot of applications in real-world problems and are extremely useful in numerous fields [6, 7]. Chen [8] interpreted the similarity function to find the similarity degree among FSs. Pedrycz [9] presented fuzzy control and fuzzy systems. FSs in pattern recognition,

methodology, and methods are also presented by Pedrycz [10]. Rangel-Valdez et al. [11] described parallel designs for metaheuristics that solve portfolio selection problems using fuzzy outranking relations. Mahmood [12] described a novel approach towards bipolar soft set and their applications.

Numerous authors mentioned the issue, what will be the impact when we alter the range of FS into a unit circle of a complex plane. To deal with such sorts of circumstance, Ramot et al. [13] described the notion of CFS as a modification of FS to handle the complex and tricky data in real-world. The idea of CFS is represented by complex-valued positive grade, which carries two-dimensional data in a particular set. Moreover, Tamir et al. [14] presented the Cartesian form of CFS and the Cartesian complex fuzzy positive grade where both real and imaginary parts carry the fuzzy data. In polar portrayal, the fuzzy data carries the

phase value and absolute value of complex positive grade. The complex fuzzy number is not the same as the CFS. The δ-equalities and operation properties of CFS were introduced by Zhang et al. [15].

HFS are the significant expansions of the theory of FS. Torra [16, 17] described the notion of HFS. An HFS is represented by positive grade which is in the shape of a finite subset of closed interval [0, 1]. Torra and Narukawa [17] characterized some fundamental operations on HFS. Rodriguez et al. [18] built up the idea of hesitant fuzzy linguistic term sets. Farhadinia [19] interpreted the idea of similarity and distance measures for higher order HFS. The notion of hesitant fuzzy data aggregation in decision-making (DM) was described by Xia and Xu [20]. Wei et al. [21] interpreted the idea of hesitant fuzzy Choquet aggregation operators and their applications to multiple attribute DM (MADM). Zhang [22] characterized hesitant fuzzy aggregation operators and their application to MADM. Xu and Xia [23] explored the idea of separation and correlation measures of HFS. Zhu et al. [24] gave the idea of hesitant fuzzy geometric Bonferroni means. Herrera et al. [25] described HFSs, an emerging tool in decision-making. A review of HFSs, quantitative and qualitative extensions, was explored by Rodriguez et al. [26]. Li et al. [27] described the consistency of hesitant fuzzy linguistic preference relations. Muhiuddin et al. [28] interpreted the generalized hesitant fuzzy ideals in semigroups.

The idea of similarity is a fundamental idea in human cognizance. Similarity has a key role in recognition, taxonomy, and several different fields. There are numerous aspects of the notion of the similarity that have escaped formalization. As per (HFS) detailing of a substantial, broadly useful definition of similarity is a difficult issue. There does not exist a legitimate, universally useful definition of similarity. There exist numerous specific definitions that have been utilized with accomplishment in diagnostics, classification, cluster analysis, and recognition. There are a few comparability measures that are interpreted and utilized for different purposes [29]. The SMs are categorized into 3 classifications: (1) measures based on implicators. (2) Measure based on metric. (3) Measure based on set-theoretic. While managing SMs based on distance, examples have been developed for perceptual similarity where each distance adage is obviously damaged by dissimilarity measures, especially the triangle inequality [17], and thusly the relating SM ignores transitivity. This model hypothesizes that the perceptual distance fulfils the metric adages, the observational legitimacy of which has been tentatively tested by a few authors, especially the triangle inequality (for subtleties see [16] and [17, 30, 31]). Thus, in the event of set-theoretic SMs, it is seen that crisp transitivity is a lot more grounded condition to be put upon SM. Set-theoretic SMs are additionally partitioned in three gatherings: (i) measures dependent on crisp logic; (ii) measures dependent on fuzzy logic; (iii) measures dependent on HFSs.

In this paper, we present complex HFSs. The inspiration is that when characterizing the positive grade of the element, the struggle of establishing the positive grade is not as we have margin of error (as in complex intuitionistic FS [32]), or some chance circulations (as in type 2 CFSs) on the probable values. In the existing theories, numerous scholars have faced several troubles. When a decision-maker provides such types of information for the grade of truth in the form of $0.22e^{(0.3)}$ and $0.5e^{(0.31)}$, this circumstance can emerge in a multicriteria DM. Basically, the theory of complex hesitant fuzzy set contains the grade of truth in the form of complex number, whose real and imaginary parts are in the form of the finite subset of the unit interval. In this unique situation, rather than considering only an aggregation operator [12], it is helpful to manage all the possible values. This circumstance, as we will talk about later, can be demonstrated utilizing multisets. Therefore, the existing theories are not able to cope with such types of troubles. The investigated ideas are more able to cope with it effectivelyg.

Due to this and preserving the advantages of the SMs, in this manuscript, the notion of CHFS is explored, which is the fusion of HFS and CFS to manage the uncertainty and complicated data in real world. The positive membership in CHFS is in the form of a finite subset of unit disc in the complex plane. Moreover, in this manuscript, we interpreted some similarity measures (SMs) and weighted SMs (WSMs). Additionally, we use our explored SMs and weighted SMs of CHFS in the environment of medical diagnosis and pattern recognition to assess the practicality and competence of the described SMs. The comparison between explored measures with some already defined measures and their graphical representations are also discussed in detail.

The structure of this manuscript is given as follows: in Section 2 of this manuscript, we present preliminaries. In Section 3, the notion of the CHFS and its fundamental properties are explored. In Section 4 of this manuscript, we explore some similarity measures (SMs) and weighted SMs (WSMs) of CHFS. In Section 5, we use proposed SMs and weighted SMs in the environment of medical diagnosis and pattern recognition. The comparison between explored measure with some already defined measures and their graphical representations are also discussed in detail in Section 6. In Section 7, we discuss the conclusion of the article.

2. Preliminaries

In this section, we revise fundamental definitions such as FS, CFS, and HFS. Throughout this paper, x denotes the fix set.

Definition 1 (see [1]). An FS E is of the shape,

$$E = \{(x, \mu_E(x))|x \in \chi\}, \tag{1}$$

with a condition $0 \leq \mu_E(x) \leq 1$, where $\mu_E(x)$ stands for the grade of membership. Throughout this paper, the family of all FSs on X are designated by $FS(X)$. The pair $E = (x, \mu_E(x))$ is said to be fuzzy number (FN).

Definition 2 (see [13]). A CFS E is of the shape,

$$E = \{(x, \mu_E(x))|x \in \chi\}, \tag{2}$$

where $\mu_E(x) = \gamma_E(x).e^{i2\pi(\omega\gamma_E^{(x)})}$ stands for the complex-valued membership grade in the shape of polar coordinate, where

$\gamma_E(x), \omega_{\gamma_{E(x)}}(x) \in [0, 1]$. Moreover, the pair $E = (x, \gamma_E(x).e^{i2\pi(\omega\gamma_E^{(x)})})$ is said to be complex fuzzy number (CFN).

Definition 3 (see [16, 17]). An HFS E is of the shape,

$$E = \{(x, \mu_E(x)) | x \in \chi\}, \tag{3}$$

where $\mu_E(x)$ is a finite subset of $[0, 1]$ standing for the grade of membership for every element $x \in \chi$. Moreover, the pair $E = (x, \mu_E(x))$ is said to be hesitant fuzzy number (HFN).

Definition 4 (see [29]). For any two HFSs E and F, the SM $\mathbb{D}(E, F)$ fulfils the following axioms:

(1) $0 \leq \mathbb{S}(E, F) \leq 1$;

(2) $\mathbb{S}(E, F) = 1 \Longleftrightarrow E = F$;

(3) $\mathbb{S}(E, F) = \mathbb{S}(F, E)$.

$$\mu_E(x) = \left\{ \gamma_{E_j}(x).e^{i2\pi\left(\omega\gamma_{E_j}^{(x)}\right)}, \quad j = 1, 2, 3, \ldots, n \right\}$$

$$= \left\{ \gamma_{E_1}(x).e^{i2\pi\left(\omega_{\gamma_{E1}}(x)\right)}, \gamma E_2(x).e^{i2\pi\left(\omega_{\gamma_{E2}}(x)\right)}, \ldots \ldots, \gamma E_n(x).e^{i2\pi\left(\omega_{\gamma_{En}}(x)\right)} \right\}, \tag{5}$$

expressed the complex-valued grade of membership which is the subset of unit disc in complex plane with a condition $\gamma_{E_j}(x), \omega_{\gamma_{E_j}}(x) \in [0, 1]$. Further, $E = (x, \gamma_{E_j}(x).e^{i2\pi(\omega_{\gamma_{E_j}}^x)})$ is known as the complex hesitant fuzzy number (CHFN).

Definition 7. Let $E = (x, \gamma_{E_j}(x).e^{i2\pi(\omega_{\gamma_{E_j}}^x)})$ and $F = (x, \gamma_{E_j}(x).e^{i2\pi(\omega_{\gamma_{E_j}}^x)})$ be two CHFNs. Then,

(1) $c(\gamma_E(x)) = \left\{ \left(x, \left\{1 - \gamma_{E_j}(x)\right\}.e^{i2\pi\left(\left\{1-\omega_{\gamma_{E_j}}^{(x)}\right\}\right)}\right) \right\}$;

(2) $E \cup F = \left\{ \left(x, \max(\gamma_{E_j}(x), \gamma_{F_j}(x)).e^{i2\pi(\max(\omega_{\gamma_{E_j}}(x), \omega_{\gamma_{E_j}}(x)))}\right) \right\}$;

(3) $E \cap F = \left\{ \left(x, \min(\gamma_{E_j}(x), \gamma_{F_j}(x)).e^{i2\pi(\min(\omega_{\gamma_{E_j}}(x), \omega_{\gamma_{E_j}}(x)))}\right) \right\}$.

The theory of CHFS is a powerful tool to deal with unsure and complicated data in real-world issues. The CHFS holds the grade of membership in the shape of a finite subset of the unit disc in the complex plane, whose entities are in the shape of polar coordinates. Essentially, the CHFS holds two-dimensional data in a particular set. The explored CHFS is more general than the existing notions such as FS, CFS, and HFS.

Example 1. Let

$E = \left\{ \begin{array}{l} (x_1, \{0.8e^{i2\pi(0.9)}, 0.6e^{i2\pi(0.5)}\}), (x_2, \{0.5e^{i2\pi(0.7)}, 0.2e^{i2\pi(1)}, 1e^{i2\pi(0.5)}\}), (x_3, \{0.1e^{i2\pi(0.2)}\}) \\ (x_4, \{0.4e^{i2\pi(0.5)}, 0.5e^{i2\pi(0.6)}, 0.3e^{i2\pi(0.4)}\}), (x_5, \{0.1e^{i2\pi(0.3)}, 0.3e^{i2\pi(0.5)}\}) \end{array} \right\}$ and

$F = \left\{ \begin{array}{l} (x_1, \{0.5e^{i2\pi(0.8)}, 0.7e^{i2\pi(0.4)}, 1e^{i2\pi(0.5)}\}), (x_2, \{0.4e^{i2\pi(0.6)}\}), (x_3, \{0.8e^{i2\pi(0.6)}, 0.5e^{i2\pi(0.8)}\}) \\ (x_4, \{0.9e^{i2\pi(0.6)}, 0.4e^{i2\pi(0.3)}\}), (x_5, \{0.5e^{i2\pi(0.7)}, 0.3e^{i2\pi(0.6)}, 0.2e^{i2\pi(0.5)}\}), \end{array} \right\}$ be

two CHFSs. Then,

Definition 5 (see [29]). For any two HFSs E and F, the distance measure $\mathbb{d}(E, F)$ fulfils the following properties:

(1) $0 \leq \mathbb{D}(E, F) \leq 1$;

(2) $\mathbb{D}(E, F) = 1 \Longleftrightarrow E = F$;

(3) $\mathbb{D}(E, F) = \mathbb{D}(F, E)$.

From the discussion we did above, we get that the $\mathbb{S}(E, F) = 1 = \mathbb{D}(E, F)$.

3. Complex Hesitant Fuzzy Sets

In this section, we explored the notion of complex hesitant fuzzy sets (CHFSs) and some of its properties.

Definition 6. A CHFS E is of the shape,

$$E = \{(x, \mu_E(x)) | x \in X\}, \tag{4}$$

where

(1) $E^c = \left\{ \begin{array}{l} \{0.2e^{i2\pi(0.1)}, 0.4e^{i2\pi(0.5)}\}, \{0.5e^{i2\pi(0.3)}, 0.8e^{i2\pi(0.0)}, 0.0e^{i2\pi(0.5)}\}, \{0.9e^{i2\pi(0.8)}\} \\ \{0.6e^{i2\pi(0.5)}, 0.5e^{i2\pi(0.4)}, 0.7e^{i2\pi(0.6)}\}, \{0.9e^{i2\pi(0.7)}, 0.7e^{i2\pi(0.5)}\} \end{array} \right\}$;

(2) $E \cup F = \left\{ \begin{array}{l} \{0.8e^{i2\pi(0.9)}, 0.7e^{i2\pi(0.5)}\}, \{0.5e^{i2\pi(0.7)}, 0.2e^{i2\pi(1)}, 1e^{i2\pi(0.5)}\}, \{0.8e^{i2\pi(0.6)}, 0.5e^{i2\pi(0.8)}\} \\ \{0.9e^{i2\pi(0.6)}, 0.5e^{i2\pi(0.6)}, 0.3e^{i2\pi(0.4)}\}, \{0.5e^{i2\pi(0.7)}, 0.3e^{i2\pi(0.6)}, 0.2e^{i2\pi(0.5)}\} \end{array} \right\}$;

(3) $E \cap F = \left\{ \begin{array}{l} \{0.5e^{i2\pi(0.8)}, 0.6e^{i2\pi(0.4)}\}, \{0.4e^{i2\pi(0.6)}\}, \{0.1e^{i2\pi(0.2)}\} \\ \{0.4e^{i2\pi(0.5)}, 0.4e^{i2\pi(0.3)}\}, \{0.1e^{i2\pi(0.3)}, 0.3e^{i2\pi(0.5)}\} \end{array} \right\}$;

4. Similarity Measures Based on the Cosine Function for CHFSs

In this section, we interpreted some SMs such as cosine SMs for CHFSs, SMs of CHFSs based on cosine function, and SMs of CHFSs based on cotangent function.

Definition 8. Let E and F be two CHFSs on set X. Then, SM between E and F is represented by $\mathbb{S}_c(E, F)$, which fulfils the following postulate:

(1) $0 \leq \mathbb{S}_c(E, F) \leq 1$;

(2) $\mathbb{S}_c(E, F) = 1$ if and only if $E = F$;

(3) $\mathbb{S}_c(E, F) = \mathbb{S}_c(F, E)$.

4.1. Cosine Similarity Measures for CHFS. Let E be a CHFS on a set X. Then, the elements contained in CHFS can be presented as the function of membership degree $\mu_E(x)$, which is a subset of a unit disc in a complex plane. Consequently, a cosine SM and weighted cosine SM with CHF data are expressed similarly to the cosine SM based on Bhattacharya's distance [33].

Definition 9. Let E and F be two CHFSs on a set X. Then, the cosine SM between E and F can be presented as

$$\mathbb{S}_c^1(E,F) = \frac{1}{n}\sum_{\kappa=1}^{n}\left(\frac{(1/\mathbb{L})\sum_{j=1}^{\mathbb{L}}\gamma_{E_j}(x_\kappa)\cdot\gamma_{F_j}(x_\kappa) + (1/\mathbb{L})\sum_{j}^{\mathbb{L}}\omega_{\gamma_{E_j}}(x_\kappa)\cdot\omega_{\gamma_{F_j}}(x_\kappa)}{\sqrt{1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\gamma_{E_j}^2(x_\kappa) + (1/\mathbb{L})\sum_{j=1}^{\mathbb{L}}\omega_{\gamma_{E_j}}^2(x_\kappa)}\sqrt{(1/\mathbb{L})\sum_{j=1}^{\mathbb{L}}\gamma_{E_j}^2(x_\kappa) + (1/\mathbb{L})\sum_{j=1}^{\mathbb{L}}\omega_{\gamma_{E_j}}^2(x_\kappa)}}\right). \tag{6}$$

In Definition 9, if we assume the imaginary parts zero, then the interpreted SM transforms for HFS. Likewise, if we assume the CHFS as a singleton set, then the interpreted SM transforms for CFS. Moreover, if we assume the CHFS as a singleton set and the imaginary part zero, then the interpreted SM transforms for FS. Its structure makes it important and expert to deal with unknown and undependable data in real decision theory.

Theorem 1. *The SM $\mathbb{S}_c^1(E,F)$ fulfils the following postulates:*

(1) $0 \le \mathbb{S}_c^1(E,F) \le 1$;

(2) $\mathbb{S}_c^1(E,F) = 1$ *if* $E = F$;

(3) $\mathbb{S}_c^1(E,F) = \mathbb{S}_c^1(F,E)$.

Proof

(1) Since $1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\gamma_{E_j}(x_\kappa)\cdot\gamma_{F_j}(x_\kappa)\in[0,1]$, $1/\mathbb{L}\sum_{j}^{\mathbb{L}}\omega_{\gamma_{E_j}}(x_\kappa)\cdot\omega_{\gamma_{F_j}}(x_\kappa)\in[0,1]$, $1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\gamma_{E_j}^2(x_\kappa)\in[0,1]$, $1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\omega_{\gamma_{E_j}}^2(x_\kappa)\in[0,1]$, and $1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\gamma_{E_j}^2(x_\kappa)\in[0,1]$, $1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\omega_{\gamma_{E_j}}^2(x_\kappa)\in[0,1]$ and denominator will always remain greater than the nominator. So, for $\kappa=1$, we have

$$\frac{1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\gamma_{E_j}(x_1)\cdot\gamma_{F_j}(x_1) + 1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\omega_{\gamma_{E_j}}(x_1)\cdot\omega_{\gamma_{F_j}}(x_1)}{\sqrt{1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\gamma_{E_j}^2(x_1) + 1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\omega_{\gamma_{E_j}}^2(x_1)}\sqrt{1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\gamma_{E_j}^2(x_1) + 1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\omega_{\gamma_{E_j}}^2(x_1)}} \in [0,1]. \tag{7}$$

For $\kappa = 2$, we have

$$\frac{1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\gamma_{E_j}(x_2)\cdot\gamma_{F_j}(x_2) + 1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\omega_{\gamma_{E_j}}(x_2)\cdot\omega_{\gamma_{F_j}}(x_2)}{\sqrt{1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\gamma_{E_j}^2(x_2) + 1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\omega_{\gamma_{E_j}}^2(x_2)}\sqrt{1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\gamma_{E_j}^2(x_2) + 1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\omega_{\gamma_{E_j}}^2(x_2)}} \in [0,1]. \tag{8}$$

By continuing this procedure, we obtain

$$\sum_{\kappa=1}^{n}\left(\frac{1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\gamma_{E_j}(x_\kappa)\cdot\gamma_{F_j}(x_\kappa) + 1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\omega_{\gamma_{E_j}}(x_\kappa)\cdot\omega_{\gamma_{F_j}}(x_\kappa)}{\sqrt{1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\gamma_{E_j}^2(x_\kappa) + 1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\omega_{\gamma_{E_j}}^2(x_\kappa)}\sqrt{1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\gamma_{E_j}^2(x_\kappa) + 1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\omega_{\gamma_{E_j}}^2(x_\kappa)}}\right) \in n[0,1]. \tag{9}$$

This implies that

$$0 \le \sum_{\kappa=1}^{n} \left(\frac{1/\mathbb{L} \sum_{j=1}^{\mathbb{L}} \gamma_{E_j}(x_\kappa) \cdot \gamma_{F_j}(x_\kappa) + 1/\mathbb{L} \sum_{j=1}^{\mathbb{L}} \omega_{\gamma_{E_j}}(x_\kappa) \cdot \omega_{\gamma_{F_j}}(x_\kappa)}{\sqrt{1/\mathbb{L} \sum_{j=1}^{\mathbb{L}} \gamma_{E_j}^2(x_\kappa) + 1/\mathbb{L} \sum_{j=1}^{\mathbb{L}} \omega_{\gamma_{E_j}}^2(x_\kappa)} \sqrt{1/\mathbb{L} \sum_{j=1}^{\mathbb{L}} \gamma_{E_j}^2(x_\kappa) + 1/\mathbb{L} \sum_{j=1}^{\mathbb{L}} \omega_{\gamma_{E_j}}^2(x_\kappa)}} \right) \le n,$$

$$0 \le \frac{1}{2} \sum_{\kappa=1}^{n} \left(\frac{1/\mathbb{L} \sum_{j=1}^{\mathbb{L}} \gamma_{E_j}(x_\kappa) \cdot \gamma_{F_j}(x_\kappa) + 1/\mathbb{L} \sum_{j=1}^{\mathbb{L}} \omega_{\gamma_{E_j}}(x_\kappa) \cdot \omega_{\gamma_{F_j}}(x_\kappa)}{\sqrt{1/\mathbb{L} \sum_{j=1}^{\mathbb{L}} \gamma_{E_j}^2(x_\kappa) + 1/\mathbb{L} \sum_{j=1}^{\mathbb{L}} \omega_{\gamma_{E_j}}^2(x_\kappa)} \sqrt{1/\mathbb{L} \sum_{j=1}^{\mathbb{L}} \gamma_{E_j}^2(x_\kappa) + 1/\mathbb{L} \sum_{j=1}^{\mathbb{L}} \omega_{\gamma_{E_j}}^2(x_\kappa)}} \right) \le 1, \tag{10}$$

which implies that

(2) We have

$$0 \le \mathbb{S}_c^1(x_\kappa) \le 1. \tag{11}$$

$$\mathbb{S}_c^1(E,F) = \frac{1}{n} \sum_{\kappa=1}^{n} \left(\frac{1/\mathbb{L} \sum_{j=1}^{\mathbb{L}} \gamma_{E_j}(x_\kappa) \cdot \gamma_{F_j}(x_\kappa) + 1/\mathbb{L} \sum_{j=1}^{\mathbb{L}} \omega_{\gamma_{E_j}}(x_\kappa) \cdot \omega_{\gamma_{F_j}}(x_\kappa)}{\sqrt{1/\mathbb{L} \sum_{j=1}^{\mathbb{L}} \gamma_{E_j}^2(x_\kappa) + 1/\mathbb{L} \sum_{j=1}^{\mathbb{L}} \omega_{\gamma_{E_j}}^2(x_\kappa)} \sqrt{1/\mathbb{L} \sum_{j=1}^{\mathbb{L}} \gamma_{E_j}^2(x_\kappa) + 1/\mathbb{L} \sum_{j=1}^{\mathbb{L}} \omega_{\gamma_{E_j}}^2(x_\kappa)}} \right). \tag{12}$$

Now, as $E = F \Longrightarrow \mu_E(x_\kappa) = \mu_F(x_\kappa)$, for $\kappa = 1, 2, \ldots, n \Longrightarrow \gamma_{E_j}(x_\kappa) e^{i2\pi(\omega_{E_j}(x_\kappa))} = \gamma_{F_j}(x_\kappa) e^{i2\pi(\omega_{F_j}(x_\kappa))}$ for $\kappa = 1, 2, \ldots, n \Longrightarrow \gamma_{E_j}(x_\kappa) = \gamma_{F_j}(x_\kappa)$ and $e^{i2\pi(\omega_{E_j}(x_\kappa))} = e^{i2\pi(\omega_{F_j}(x_\kappa))}$ for $\kappa = 1, 2, \ldots, n$. Then,

$$\mathbb{S}_c^1(E,F) = \frac{1}{n} \sum_{\kappa=1}^{n} \left(\frac{1/\mathbb{L} \sum_{j=1}^{\mathbb{L}} \gamma_{E_j}^2(x_\kappa) + 1/\mathbb{L} \sum_{j=1}^{\mathbb{L}} \omega_{\gamma_{E_j}}^2(x_\kappa)}{\left(\sqrt{1/\mathbb{L} \sum_{j=1}^{\mathbb{L}} \gamma_{E_j}^2(x_\kappa) + 1/\mathbb{L} \sum_{j=1}^{\mathbb{L}} \omega_{\gamma_{E_j}}^2(x_\kappa)} \right)^2} \right). \tag{13}$$

$$\mathbb{S}_c^1(E,F) = \frac{1}{n} \sum_{\kappa=1}^{n} \left(\frac{1/\mathbb{L} \left(\gamma_{E_1}^2(x_\kappa) + \gamma_{E_2}^2(x_\kappa) + \cdots + \gamma_{E_{\mathbb{L}}}^2(x_\kappa) \right) + 1/\mathbb{L} \left(\omega_{\gamma_{E_1}}^2(x_\kappa) + \omega_{\gamma_{E_2}}^2(x_\kappa) + \cdots + \omega_{\gamma_{E_{\mathbb{L}}}}^2(x_\kappa) \right)}{1/\mathbb{L} \left(\gamma_{E_1}^2(x_\kappa) + \gamma_{E_2}^2(x_\kappa) + \cdots + \gamma_{E_{\mathbb{L}}}^2(x_\kappa) \right) + 1/\mathbb{L} \left(\omega_{\gamma_{E_1}}^2(x_\kappa) + \omega_{\gamma_{E_2}}^2(x_\kappa) + \cdots + \omega_{\gamma_{E_{\mathbb{L}}}}^2(x_\kappa) \right)} \right).$$

$$\mathbb{S}_c^1(E,F) = \frac{1}{n} \left[\frac{1/\mathbb{L} \left(\gamma_{E_1}^2(x_1) + \gamma_{E_2}^2(x_1) + \cdots + \gamma_{E_{\mathbb{L}}}^2(x_1) \right) + 1/\mathbb{L} \left(\omega_{\gamma_{E_1}}^2(x_1) + \omega_{\gamma_{E_2}}^2(x_1) + \cdots + \omega_{\gamma_{E_{\mathbb{L}}}}^2(x_1) \right)}{1/\mathbb{L} \left(\gamma_{E_1}^2(x_1) + \gamma_{E_2}^2(x_1) + \cdots + \gamma_{E_{\mathbb{L}}}^2(x_1) \right) + 1/\mathbb{L} \left(\omega_{\gamma_{E_1}}^2(x_1) + \omega_{\gamma_{E_2}}^2(x_1) + \cdots + \omega_{\gamma_{E_{\mathbb{L}}}}^2(x_1) \right)} \right.$$

$$+ \frac{1/\mathbb{L} \left(\gamma_{E_1}^2(x_2) + \gamma_{E_2}^2(x_2) + \cdots + \gamma_{E_{\mathbb{L}}}^2(x_2) \right) + 1/\mathbb{L} \left(\omega_{\gamma_{E_1}}^2(x_2) + \omega_{\gamma_{E_2}}^2(x_2) + \cdots + \omega_{\gamma_{E_{\mathbb{L}}}}^2(x_2) \right)}{1/\mathbb{L} \left(\gamma_{E_1}^2(x_2) + \gamma_{E_2}^2(x_2) + \cdots + \gamma_{E_{\mathbb{L}}}^2(x_2) \right) + 1/\mathbb{L} \left(\omega_{\gamma_{E_1}}^2(x_2) + \omega_{\gamma_{E_2}}^2(x_2) + \cdots + \omega_{\gamma_{E_{\mathbb{L}}}}^2(x_2) \right)} + \cdots$$

$$\left. + \frac{1/\mathbb{L} \left(\gamma_{E_1}^2(x_n) + \gamma_{E_2}^2(x_n) + \cdots + \gamma_{E_{\mathbb{L}}}^2(x_n) \right) + 1/\mathbb{L} \left(\omega_{\gamma_{E_1}}^2(x_n) + \omega_{\gamma_{E_2}}^2(x_n) + \cdots + \omega_{\gamma_{E_{\mathbb{L}}}}^2(x_n) \right)}{1/\mathbb{L} \left(\gamma_{E_1}^2(x_n) + \gamma_{E_2}^2(x_n) + \cdots + \gamma_{E_{\mathbb{L}}}^2(x_n) \right) + 1/\mathbb{L} \left(\omega_{\gamma_{E_1}}^2(x_n) + \omega_{\gamma_{E_2}}^2(x_n) + \cdots + \omega_{\gamma_{E_{\mathbb{L}}}}^2(x_n) \right)} \right], \tag{14}$$

$$S_c^1(E,F) = 1.$$

(3) We have

$$
\mathbb{S}_c^1(E, F) = \frac{1}{n} \sum_{\mathcal{K}=1}^{n} \left(\frac{1/\mathbb{L} \sum_{j=1}^{\mathbb{L}} \gamma_{E_j}(x_{\mathcal{K}}).\gamma_{F_j}(x_{\mathcal{K}}) + 1/\mathbb{L} \sum_{j}^{\mathbb{L}} \omega_{\gamma_{E_j}}(x_{\mathcal{K}}).\omega_{\gamma_{F_j}}(x_{\mathcal{K}})}{\sqrt{1/\mathbb{L} \sum_{j=1}^{\mathbb{L}} \gamma_{E_j}^2(x_{\mathcal{K}}) + 1/\mathbb{L} \sum_{j=1}^{\mathbb{L}} \omega_{\gamma_{E_j}}^2(x_{\mathcal{K}})} \sqrt{1/\mathbb{L} \sum_{j=1}^{\mathbb{L}} \gamma_{F_j}^2(x_{\mathcal{K}}) + 1/\mathbb{L} \sum_{j=1}^{\mathbb{L}} \omega_{\gamma_{F_j}}^2(x_{\mathcal{K}})}} \right)
$$

$$
= \frac{1}{n} \sum_{\mathcal{K}=1}^{n} \left(\frac{1/\mathbb{L} \sum_{j=1}^{\mathbb{L}} \gamma_{F_j}(x_{\mathcal{K}}).\gamma_{E_j}(x_{\mathcal{K}}) + 1/\mathbb{L} \sum_{j}^{\mathbb{L}} \omega_{\gamma_{F_j}}(x_{\mathcal{K}}).\omega_{\gamma_{E_j}}(x_{\mathcal{K}})}{\sqrt{1/\mathbb{L} \sum_{j=1}^{\mathbb{L}} \gamma_{F_j}^2(x_{\mathcal{K}}) + 1/\mathbb{L} \sum_{j=1}^{\mathbb{L}} \omega_{\gamma_{F_j}}^2(x_{\mathcal{K}})} \sqrt{1/\mathbb{L} \sum_{j=1}^{\mathbb{L}} \gamma_{E_j}^2(x_{\mathcal{K}}) + 1/\mathbb{L} \sum_{j=1}^{\mathbb{L}} \omega_{\gamma_{E_j}}^2(x_{\mathcal{K}})}} \right), \tag{15}
$$

$$
= \mathbb{S}_c^1(F, E).
$$

We defined distance measure of the angle as $d(E, F) = \arccos(\mathbb{S}_c^1(E, F))$. It holds the following axioms:

(1) $d(E, F) \geq 0$ if $0 \leq \mathbb{S}_c^1(F, E) \leq 1$;

(2) $d(E, F) = \arccos(1) = 0$ if $\mathbb{S}_c^1(E, F) = 1$;

(3) $d(E, F) = d(F, E)$ if $\mathbb{S}_c(E, F) = \mathbb{S}_c(F, E)$. □

Definition 10. Let E and F be two CHFSs on a set X. Then, the weighted cosine SM between E and F can be presented as

$$
\mathbb{S}_{cw}^1(E, F) = \sum_{\mathcal{K}=1}^{n} w_{\mathcal{K}} \frac{1/\mathbb{L} \sum_{j=1}^{\mathbb{L}} \gamma_{E_j}(x_{\mathcal{K}}).\gamma_{F_j}(x_{\mathcal{K}}) + 1/\mathbb{L} \sum_{j}^{\mathbb{L}} \omega_{\gamma_{E_j}}(x_{\mathcal{K}}).\omega_{\gamma_{F_j}}(x_{\mathcal{K}})}{\sqrt{1/\mathbb{L} \sum_{j=1}^{\mathbb{L}} \gamma_{E_j}^2(x_{\mathcal{K}}) + 1/\mathbb{L} \sum_{j=1}^{\mathbb{L}} \omega_{\gamma_{E_j}}^2(x_{\mathcal{K}})} \sqrt{1/\mathbb{L} \sum_{j=1}^{\mathbb{L}} \gamma_{F_j}^2(x_{\mathcal{K}}) + 1/\mathbb{L} \sum_{j=1}^{\mathbb{L}} \omega_{\gamma_{F_j}}^2(x_{\mathcal{K}})}}, \tag{16}
$$

where $w = (w_1, w_2, \ldots, w_n)^{\mathcal{T}}$ represents the weight vector of every element $x_{\mathcal{K}}(\mathcal{K} = 1, 2, \ldots, .n)$ included in CHFS and the weight vector satisfies $w_{\mathcal{K}} \in [0, 1]$ for every $\mathcal{K} = 1, 2, \ldots, n, \sum_{\mathcal{K}=1}^{n} w_{\mathcal{K}} = 1$. When we suppose the weight vector to be $w = (1/n, 1/n, \ldots, 1/n)^{\mathcal{T}}$, the weighted cosine SM will transform into cosine SM. Otherwise speaking, when $w_{\mathcal{K}} = 1/n, \mathcal{K} = 1, 2, 3, \ldots, n,$ the $\mathbb{S}_{cw}^1(E, F) = \mathbb{S}_c^1(E, F)$.

4.2. Similarity Measures of CHFSs Based on Cosine Function. In this part of the paper, we interpreted SMs of CHFSs based on cosine function and studied their properties.

Definition 11. Let E and F be two CHFSs on a set X. Then, the SMs based on the cosine function between E and F can be presented as

$$
\mathbb{S}_c^2(E, F) = \frac{1}{n} \sum_{\mathcal{K}=1}^{n} \text{Cos}\left[\frac{\pi}{2} \left(\max\left(\frac{1}{\mathbb{L}} \sum_{j=1}^{\mathbb{L}} \left| \gamma_{E_j}(x_{\mathcal{K}}) - \gamma_{F_j}(x_{\mathcal{K}}) \right|, \frac{1}{\mathbb{L}} \sum_{j=1}^{\mathbb{L}} \left| \omega_{\gamma_{E_j}}(x_{\mathcal{K}}) - \omega_{\gamma_{F_j}}(x_{\mathcal{K}}) \right| \right) \right) \right], \tag{17}
$$

where $\mathbb{S}_c^2(E, F)$ means the SM based on the cosine function between E and F, which considers the maximum distance based on the amplitude and phase terms.

$$
\mathbb{S}_c^3(E, F) = \frac{1}{n} \sum_{\mathcal{K}=1}^{n} \text{Cos}\left[\frac{\pi}{4} \left(\frac{1}{\mathbb{L}} \sum_{j=1}^{\mathbb{L}} \left| \gamma_{E_j}(x_{\mathcal{K}}) - \gamma_{F_j}(x_{\mathcal{K}}) \right| + \frac{1}{\mathbb{L}} \sum_{j=1}^{\mathbb{L}} \left| \omega_{\gamma_{E_j}}(x_{\mathcal{K}}) - \omega_{\gamma_{F_j}}(x_{\mathcal{K}}) \right| \right) \right], \tag{18}
$$

where $\mathbb{S}_c^3(E, F)$ means the SM based on the cosine function between E and F, which considers the sum of the distance based on the amplitude and phase terms.

Theorem 2. *The SM $\mathbb{S}_c^2(E, F)$ fulfils the following postulates:*

(1) $0 \leq \mathbb{S}_c^2(E, F) \leq 1$;

(2) $\mathbb{S}_c^2(E, F) = 1$ *if* $E = F$;

(3) $\mathbb{S}_c^2(E, F) = \mathbb{S}_c^2(F, E)$.

Proof

1. Since $1/\mathbb{L} \sum_{j=1}^{\mathbb{L}} |\gamma_{E_j}(x_{\mathcal{K}}) - \gamma_{F_j}(x_{\mathcal{K}})| \in [0, 1]$, $1/\mathbb{L} \sum_{j}^{\mathbb{L}} |\omega_{\gamma_{E_j}}(x_{\mathcal{K}}) - \omega_{\gamma_{F_j}}(x_{\mathcal{K}})| \in [0, 1]$, this implies that $\max(1/\mathbb{L} \sum_{j=1}^{\mathbb{L}} |\gamma_{E_j}(x_{\mathcal{K}}) - \gamma_{F_j}(x_{\mathcal{K}})|, 1/\mathbb{L} \sum_{j=1}^{\mathbb{L}} |\omega_{\gamma_{E_j}}(x_{\mathcal{K}}) - \omega_{\gamma_{F_j}}(x_{\mathcal{K}})|) \in [0, 1]$. So, for $\mathcal{K} = 1$, we have

$$\mathrm{Cos}\left[\frac{\pi}{2}\left(\max\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_1) - \gamma_{F_j}(x_1)\right|, \frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_1) - \omega_{\gamma_{F_j}}(x_1)\right|\right)\right)\right] \in [0, 1]. \tag{19}$$

For $\mathcal{K} = 2$, we have

$$\mathrm{Cos}\left[\frac{\pi}{2}\left(\max\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_2) - \gamma_{F_j}(x_2)\right|, \frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_2) - \omega_{\gamma_{F_j}}(x_2)\right|\right)\right)\right] \in [0, 1]. \tag{20}$$

By continuing this procedure, we obtain

$$\sum_{\mathcal{K}=1}^{n}\mathrm{Cos}\left[\frac{\pi}{2}\left(\max\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_{\mathcal{K}}) - \gamma_{F_j}(x_{\mathcal{K}})\right|, \frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_{\mathcal{K}}) - \omega_{\gamma_{F_j}}(x_{\mathcal{K}})\right|\right)\right)\right] \in n[0, 1]. \tag{21}$$

This implies that

$$0 \le \sum_{\mathcal{K}=1}^{n}\mathrm{Cos}\left[\frac{\pi}{2}\left(\max\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_{\mathcal{K}}) - \gamma_{F_j}(x_{\mathcal{K}})\right|, \frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_{\mathcal{K}}) - \omega_{\gamma_{F_j}}(x_{\mathcal{K}})\right|\right)\right)\right] \le n,$$

$$0 \le \frac{1}{n}\sum_{\mathcal{K}=1}^{n}\mathrm{Cos}\left[\frac{\pi}{2}\left(\max\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_{\mathcal{K}}) - \gamma_{F_j}(x_{\mathcal{K}})\right|, \frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_{\mathcal{K}}) - \omega_{\gamma_{F_j}}(x_{\mathcal{K}})\right|\right)\right)\right] \le 1, \tag{22}$$

which implies that

$$0 \le \mathbb{S}_c^2(x_{\mathcal{K}}) \le 1. \tag{23}$$

(2) We have

$$\mathbb{S}_c^2(E, F) = \frac{1}{n}\sum_{\mathcal{K}=1}^{n}\mathrm{Cos}\left[\frac{\pi}{2}\left(\max\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_{\mathcal{K}}) - \gamma_{F_j}(x_{\mathcal{K}})\right|, \frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_{\mathcal{K}}) - \omega_{\gamma_{F_j}}(x_{\mathcal{K}})\right|\right)\right)\right]. \tag{24}$$

Now, as $E = F \implies \mu_E(x_{\mathcal{K}}) = \mu_F(x_{\mathcal{K}})$, for $\mathcal{K} = 1, 2, \ldots, n \implies \gamma_{E_j}(x_{\mathcal{K}})e^{i2\pi(\omega_{E_j}(x_{\mathcal{K}}))} = \gamma_{F_j}(x_{\mathcal{K}})e^{i2\pi(\omega_{F_j}(x_{\mathcal{K}}))}$ for $\mathcal{K} = 1, 2, \ldots, n \implies \gamma_{E_j}(x_{\mathcal{K}}) = \gamma_{F_j}(x_{\mathcal{K}})$ and $e^{i2\pi(\omega_{E_j}(x_{\mathcal{K}}))} = e^{i2\pi(\omega_{F_j}(x_{\mathcal{K}}))}$ for $\mathcal{K} = 1, 2, \ldots, n$. Then, $|\gamma_{E_j}(x_{\mathcal{K}}) - \gamma_{F_j}(x_{\mathcal{K}})| = 0$ and $|\omega_{\gamma_{E_j}}(x_{\mathcal{K}}) - \omega_{\gamma_{F_j}}(x_{\mathcal{K}})| = 0$ for $\mathcal{K} = 1, 2, \ldots, n$. This implies that

$$\mathbb{S}_c^2(E, F) = 1. \tag{25}$$

(3) We have

$$
\begin{aligned}
\mathbb{S}_c^2(E, F) &= \frac{1}{n}\sum_{\mathcal{K}=1}^{n}\mathrm{Cos}\left[\frac{\pi}{2}\left(\max\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_{\mathcal{K}})-\gamma_{F_j}(x_{\mathcal{K}})\right|,\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_{\mathcal{K}})-\omega_{\gamma_{F_j}}(x_{\mathcal{K}})\right|\right)\right)\right], \\
&= \frac{1}{n}\sum_{\mathcal{K}=1}^{n}\mathrm{Cos}\left[\frac{\pi}{2}\left(\max\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{F_j}(x_{\mathcal{K}})-\gamma_{E_j}(x_{\mathcal{K}})\right|,\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{F_j}}(x_{\mathcal{K}})-\omega_{\gamma_E}(x_{\mathcal{K}})\right|\right)\right)\right], \\
&= \mathbb{S}_c^2(F, E).
\end{aligned}
\tag{26}
$$

□

Theorem 3. *The SM $\mathbb{S}_c^3(E, F)$ fulfils the following postulates:*

(4) $0 \leq \mathbb{S}_c^3(E, F) \leq 1$;

(5) $\mathbb{S}_c^3(E, F) = 1$ *if* $E = F$;

(6) $\mathbb{S}_c^3(E, F) = \mathbb{S}_c^3(F, E)$.

Proof

(1) Since $1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}|\gamma_{E_j}(x_{\mathcal{K}})-\gamma_{F_j}(x_{\mathcal{K}})| \in [0,1]$, $1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}|\omega_{E_j}(x_{\mathcal{K}})-\omega_{F_j}(x_{\mathcal{K}})| \in [0,1]$, this implies that, $1/2\max(1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}|\gamma_{E_j}(x_{\mathcal{K}})-\gamma_{F_j}(x_{\mathcal{K}})| \in [0,1]$, $1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}|\omega_{E_j}(x_{\mathcal{K}})-\omega_{F_j}(x_{\mathcal{K}})| \in [0,1]$). So, for $\mathcal{K} = 1$, we have

$$\mathrm{Cos}\left[\frac{\pi}{4}\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_1)-\gamma_{F_j}(x_1)\right|+\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_1)-\omega_{\gamma_{F_j}}(x_1)\right|\right)\right] \in [0,1]. \tag{27}$$

For $\mathcal{K} = 2$, we have

$$\mathrm{Cos}\left[\frac{\pi}{4}\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_2)-\gamma_{F_j}(x_2)\right|+\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_2)-\omega_{\gamma_{F_j}}(x_2)\right|\right)\right] \in [0,1]. \tag{28}$$

By continuing this procedure, we obtain

$$\sum_{\mathcal{K}=1}^{n}\mathrm{Cos}\left[\frac{\pi}{4}\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_{\mathcal{K}})-\gamma_{F_j}(x_{\mathcal{K}})\right|+\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_{\mathcal{K}})-\omega_{\gamma_{F_j}}(x_{\mathcal{K}})\right|\right)\right] \in n[0,1]. \tag{29}$$

This implies that

$$
\begin{aligned}
&0 \leq \sum_{\mathcal{K}=1}^{n}\mathrm{Cos}\left[\frac{\pi}{4}\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_{\mathcal{K}})-\gamma_{F_j}(x_{\mathcal{K}})\right|+\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_{\mathcal{K}})-\omega_{\gamma_{F_j}}(x_{\mathcal{K}})\right|\right)\right] \leq n, \\
&0 \leq \frac{1}{n}\sum_{\mathcal{K}=1}^{n}\mathrm{Cos}\left[\frac{\pi}{4}\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_{\mathcal{K}})-\gamma_{F_j}(x_{\mathcal{K}})\right|+\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_{\mathcal{K}})-\omega_{\gamma_{F_j}}(x_{\mathcal{K}})\right|\right)\right] \leq 1,
\end{aligned}
\tag{30}
$$

which implies that

$$0 \leq \mathbb{S}_c^3(x_{\mathscr{K}}) \leq 1. \tag{31}$$

$$\mathbb{S}_c^3(E,F) = \frac{1}{n}\sum_{\mathscr{K}=1}^{n}\mathrm{Cos}\left[\frac{\pi}{4}\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_{\mathscr{K}}) - \gamma_{F_j}(x_{\mathscr{K}})\right| + \frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_{\mathscr{K}}) - \omega_{\gamma_{F_j}}(x_{\mathscr{K}})\right|\right)\right]. \tag{32}$$

Now, as $E = F \implies \mu_E(x_{\mathscr{K}}) = \mu_F(x_{\mathscr{K}})$. for $\mathscr{K} = 1, 2, \ldots, n \implies \gamma_{E_j}(x_{\mathscr{K}})e^{i2\pi(\omega_{E_j}(x_{\mathscr{K}}))} = \gamma_{F_j}(x_{\mathscr{K}})e^{i2\pi}$
$(\omega_{F_j}(x_{\mathscr{K}}))$ for $\mathscr{K} = 1, 2, \ldots, n \implies \gamma_{E_j}(x_{\mathscr{K}}) = \gamma_{F_j}(x_{\mathscr{K}})$
and $e^{i2\pi(\omega_{E_j}(x_{\mathscr{K}}))} = e^{i2\pi(\omega_{F_j}(x_{\mathscr{K}}))}$ for $\mathscr{K} = 1, 2, \ldots, n$.

(2) We have

Then, $\left|\gamma_{E_j}(x_{\mathscr{K}}) - \gamma_{F_j}(x_{\mathscr{K}})\right| = 0$ and $\left|\omega_{\gamma_{E_j}}(x_{\mathscr{K}}) - \omega_{\gamma_{F_j}}(x_{\mathscr{K}})\right| = 0$ for $\mathscr{K} = 1, 2, \ldots, n$. This implies that

$$\mathbb{S}_c^3(E,F) = 1. \tag{33}$$

(3) We have

$$\begin{aligned}\mathbb{S}_c^3(E,F) &= \frac{1}{n}\sum_{\mathscr{K}=1}^{n}\mathrm{Cos}\left[\frac{\pi}{4}\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_{\mathscr{K}}) - \gamma_{F_j}(x_{\mathscr{K}})\right| + \frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_{\mathscr{K}}) - \omega_{\gamma_{F_j}}(x_{\mathscr{K}})\right|\right)\right],\\ &= \frac{1}{n}\sum_{\mathscr{K}=1}^{n}\mathrm{Cos}\left[\frac{\pi}{4}\left(\frac{1}{\mathbb{L}}\left|\gamma_{F_j}(x_{\mathscr{K}}) - \gamma_{E_j}(x_{\mathscr{K}})\right|, \frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{F_j}}(x_{\mathscr{K}}) - \omega_{\gamma_E}(x_{\mathscr{K}})\right|\right)\right],\\ &= \mathbb{S}_c^3(F,E). \end{aligned} \tag{34}$$

$\square$

Definition 12. Let E and F be two CHFSs on a set X. Then, the weighted SMs based on the cosine function between E and F can be presented as

$$\mathbb{S}_{cw}^2(E,F) = \sum_{\mathscr{K}=1}^{n}w_{\mathscr{K}}\mathrm{Cos}\left[\frac{\pi}{2}\left(\max\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_{\mathscr{K}}) - \gamma_{F_j}(x_{\mathscr{K}})\right|, \frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_{\mathscr{K}}) - \omega_{\gamma_{F_j}}(x_{\mathscr{K}})\right|\right)\right)\right],$$

$$\mathbb{S}_{cw}^3(E,F) = \sum_{\mathscr{K}=1}^{n}w_{\mathscr{K}}\mathrm{Cos}\left[\frac{\pi}{4}\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_{\mathscr{K}}) - \gamma_{F_j}(x_{\mathscr{K}})\right| + \frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_{\mathscr{K}}) - \omega_{\gamma_{F_j}}(x_{\mathscr{K}})\right|\right)\right], \tag{35}$$

where $w = (w_1, w_2, \ldots, w_n)^{\mathscr{T}}$ represents the weight vector of every element $x_{\mathscr{K}}$ $(\mathscr{K} = 1, 2, , , . n)$ carried in CHFS and the weight vector satisfies $w_{\mathscr{K}} \in [0, 1]$ for every $\mathscr{K} = 1, 2, 3, , , . n$, $\sum_{\mathscr{K}=1}^{n}w_{\mathscr{K}} = 1$. When we assume the weight vector to be $w = (1/n, 1/n, \ldots, 1/n)^{\mathscr{T}}$, the weighted SMs based on the cosine function will transform into SMs based on the cosine function. Otherwise speaking, when $w_{\mathscr{K}} = 1/n$, $\mathscr{K} = 1, 2, 3, \ldots, n$, the $\mathbb{S}_{cw}^m(E,F) = \mathbb{S}_c^m(E,F)$ $m = 2, 3$

4.3. Similarity Measures of CHFSs Based on Cotangent Function. In this section, according to the cotangent function, we interpreted some cotangent SMs between CHFSs and studied their properties.

Definition 13. Let E and F be two CHFSs on a set X. Then, the cotangent SMs between E and F can be presented as

$$\mathbb{S}_c^4(E,F) = \frac{1}{n}\sum_{\mathscr{K}=1}^{n}\mathrm{Cot}\left[\frac{\pi}{4} + \frac{\pi}{4}\left(\max\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_{\mathscr{K}}) - \gamma_{F_j}(x_{\mathscr{K}})\right|, \frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_{\mathscr{K}}) - \omega_{\gamma_{F_j}}(x_{\mathscr{K}})\right|\right)\right)\right], \tag{36}$$

where $\mathbb{S}_c^4(E, F)$ means the cotangent SM between E and F, which considers the maximum distance based on the amplitude and phase terms.

$$\mathbb{S}_c^5(E, F) = \frac{1}{n} \sum_{\mathcal{K}=1}^{n} \operatorname{Cot}\left[\frac{\pi}{4} + \frac{\pi}{8}\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_{\mathcal{K}}) - \gamma_{F_j}(x_{\mathcal{K}})\right|, \frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_{\mathcal{K}}) - \omega_{\gamma_{F_j}}(x_{\mathcal{K}})\right|\right)\right], \tag{37}$$

where $\mathbb{S}_c^5(E, F)$ means the cotangent SM between E and F, which considers the sum of distance based on the amplitude and phase terms.

Theorem 4. *The SM $\mathbb{S}_c^4(E, F)$ fulfils the following postulates:*

(7) $0 \le \mathbb{S}_c^4(E, F) \le 1$;

(8) $\mathbb{S}_c^4(E, F) = 1$ *if* $E = F$;

(9) $\mathbb{S}_c^4(E, F) = \mathbb{S}_c^2(F, E)$.

Proof

1. Since $1/\mathbb{L} \sum_{j=1}^{\mathbb{L}} |\gamma_{E_j}(x_{\mathcal{K}}) - \gamma_{F_j}(x_{\mathcal{K}})| \in [0, 1]$, $1/\mathbb{L} \sum_{j=1}^{\mathbb{L}} |\omega_{E_j}(x_{\mathcal{K}}) - \omega_{F_j}(x_{\mathcal{K}})| \in [0, 1]$, this implies that $\max(1/\mathbb{L} \sum_{j=1}^{\mathbb{L}} |\gamma_{E_j}(x_{\mathcal{K}}) - \gamma_{F_j}(x_{\mathcal{K}})|, 1/\mathbb{L} \sum_{j=1}^{\mathbb{L}} |\omega_{E_j}(x_{\mathcal{K}}) - \omega_{F_j}(x_{\mathcal{K}})|) \in [0, 1]$. So, for $\mathcal{K} = 1$, we have

$$\operatorname{Cot}\left[\frac{\pi}{4} + \frac{\pi}{4}\left(\max\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_1) - \gamma_{F_j}(x_1)\right|, \frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_1) - \omega_{\gamma_{F_j}}(x_1)\right|\right)\right)\right] \in [0, 1]. \tag{38}$$

For $\mathcal{K} = 2$, we have

$$\operatorname{Cot}\left[\frac{\pi}{4} + \frac{\pi}{4}\left(\max\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_2) - \gamma_{F_j}(x_2)\right|, \frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_2) - \omega_{\gamma_{F_j}}(x_2)\right|\right)\right)\right] \in [0, 1]. \tag{39}$$

By continuing this procedure, we obtain

$$\sum_{\mathcal{K}=1}^{n} \operatorname{Cot}\left[\frac{\pi}{4} + \frac{\pi}{4}\left(\max\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_{\mathcal{K}}) - \gamma_{F_j}(x_{\mathcal{K}})\right|, \frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_{\mathcal{K}}) - \omega_{\gamma_{F_j}}(x_{\mathcal{K}})\right|\right)\right)\right] \in n[0, 1]. \tag{40}$$

This implies that

$$0 \le \sum_{\mathcal{K}=1}^{n} \operatorname{Cot}\left[\frac{\pi}{4} + \frac{\pi}{4}\left(\max\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_{\mathcal{K}}) - \gamma_{F_j}(x_{\mathcal{K}})\right|, \frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_{\mathcal{K}}) - \omega_{\gamma_{F_j}}(x_{\mathcal{K}})\right|\right)\right)\right] \le n,$$

$$0 \le \frac{1}{n}\sum_{\mathcal{K}=1}^{n} \operatorname{Cot}\left[\frac{\pi}{4} + \frac{\pi}{4}\left(\max\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_{\mathcal{K}}) - \gamma_{F_j}(x_{\mathcal{K}})\right|, \frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_{\mathcal{K}}) - \omega_{\gamma_{F_j}}(x_{\mathcal{K}})\right|\right)\right)\right] \le 1, \tag{41}$$

which implies that

$$0 \le \mathbb{S}_c^4(x_{\mathcal{K}}) \le 1. \tag{42}$$

$$\mathbb{S}_c^4(E, F) = \frac{1}{n} \sum_{\mathcal{K}=1}^{n} \mathrm{Cot}\left[\frac{\pi}{4} + \frac{\pi}{4}\left(\max\left(\frac{1}{\mathbb{L}} \sum_{j=1}^{\mathbb{L}} \left|\gamma_{E_j}(x_{\mathcal{K}}) - \gamma_{F_j}(x_{\mathcal{K}})\right|, \frac{1}{\mathbb{L}} \sum_{j=1}^{\mathbb{L}} \left|\omega_{\gamma_{E_j}}(x_{\mathcal{K}}) - \omega_{\gamma_{F_j}}(x_{\mathcal{K}})\right|\right)\right)\right]. \tag{43}$$

Now, as $E = F \implies \mu_E(x_{\mathcal{K}}) = \mu_F(x_{\mathcal{K}})$, for $\mathcal{K} = 1, 2, \ldots, n \implies \gamma_{E_j}(x_{\mathcal{K}}) e^{i2\pi(\omega_{E_j}(x_{\mathcal{K}}))} = \gamma_{F_j}(x_{\mathcal{K}}) e^{i2\pi(\omega_{F_j}(x_{\mathcal{K}}))}$ for $\mathcal{K} = 1, 2, \ldots, n \implies \gamma_{E_j}(x_{\mathcal{K}}) = \gamma_{F_j}(x_{\mathcal{K}})$ and $e^{i2\pi(\omega_{E_j}(x_{\mathcal{K}}))} = e^{i2\pi(\omega_{F_j}(x_{\mathcal{K}}))}$ for $\mathcal{K} = 1, 2, \ldots, n$. Then,

(2) We have

$|\gamma_{E_j}(x_{\mathcal{K}}) - \gamma_{F_j}(x_{\mathcal{K}})| = 0$ and $|\omega_{\gamma_{E_j}}(x_{\mathcal{K}}) - \omega_{\gamma_{F_j}}(x_{\mathcal{K}})| = 0$ for $\mathcal{K} = 1, 2, \ldots, n$. This implies that

$$\mathbb{S}_c^4(E, F) = 1. \tag{44}$$

(3) We have

$$\mathbb{S}_c^4(E, F) = \frac{1}{n} \sum_{\mathcal{K}=1}^{n} \mathrm{Cot}\left[\frac{\pi}{4} + \frac{\pi}{4}\left(\max\left(\frac{1}{\mathbb{L}} \sum_{j=1}^{\mathbb{L}} \left|\gamma_{E_j}(x_{\mathcal{K}}) - \gamma_{F_j}(x_{\mathcal{K}})\right|, \frac{1}{\mathbb{L}} \sum_{j=1}^{\mathbb{L}} \left|\omega_{\gamma_{E_j}}(x_{\mathcal{K}}) - \omega_{\gamma_{F_j}}(x_{\mathcal{K}})\right|\right)\right)\right]$$

$$= \frac{1}{n} \sum_{\mathcal{K}=1}^{n} \mathrm{Cot}\left[\frac{\pi}{4} + \frac{\pi}{4}\left(\max\left(\frac{1}{\mathbb{L}} \sum_{j=1}^{\mathbb{L}} \left|\gamma_{E_j}(x_{\mathcal{K}}) - \gamma_{F_j}(x_{\mathcal{K}})\right|, \frac{1}{\mathbb{L}} \sum_{j=1}^{\mathbb{L}} \left|\omega_{\gamma_{E_j}}(x_{\mathcal{K}}) - \omega_{\gamma_{F_j}}(x_{\mathcal{K}})\right|\right)\right)\right] \tag{45}$$

$$= \mathbb{S}_c^4(F, E).$$

$\square$

Theorem 5. *The SM $\mathbb{S}_c^5(E, F)$ fulfils the following postulates:*

(10) $0 \le \mathbb{S}_c^5(E, F) \le 1$;
(11) $\mathbb{S}_c^5(E, F) = 1$ *if* $E = F$;
(12) $\mathbb{S}_c^5(E, F) = \mathbb{S}_c^5(F, E)$.

Proof

(1) Since $1/\mathbb{L} \sum_{j=1}^{\mathbb{L}} |\gamma_{E_j}(x_{\mathcal{K}}) - \gamma_{F_j}(x_{\mathcal{K}})| \in [0, 1]$, $1/\mathbb{L} \sum_{j=1}^{\mathbb{L}} |\omega_{E_j}(x_{\mathcal{K}}) - \omega_{F_j}(x_{\mathcal{K}})| \in [0, 1]$, this implies that, $1/2 \max(1/\mathbb{L} \sum_{j=1}^{\mathbb{L}} |\gamma_{E_j}(x_{\mathcal{K}}) - \gamma_{F_j}(x_{\mathcal{K}})|, \sum_{j=1}^{\mathbb{L}} |\omega_{E_j}(x_{\mathcal{K}}) - \omega_{F_j}(x_{\mathcal{K}})|) \in [0, 1]$. So, for $\mathcal{K} = 1$, we have

$$\mathrm{Cot}\left[\frac{\pi}{4} + \frac{\pi}{8}\left(\frac{1}{\mathbb{L}} \sum_{j=1}^{\mathbb{L}} \left|\gamma_{E_j}(x_1) - \gamma_{F_j}(x_1)\right| + \frac{1}{\mathbb{L}} \sum_{j=1}^{\mathbb{L}} \left|\omega_{\gamma_{E_j}}(x_1) - \omega_{\gamma_{F_j}}(x_1)\right|\right)\right] \in [0, 1]. \tag{46}$$

For $\mathcal{K} = 2$, we have

$$\mathrm{Cot}\left[\frac{\pi}{4} + \frac{\pi}{8}\left(\frac{1}{\mathbb{L}} \sum_{j=1}^{\mathbb{L}} \left|\gamma_{E_j}(x_2) - \gamma_{F_j}(x_2)\right| + \frac{1}{\mathbb{L}} \sum_{j=1}^{\mathbb{L}} \left|\omega_{\gamma_{E_j}}(x_2) - \omega_{\gamma_{F_j}}(x_2)\right|\right)\right] \in [0, 1]. \tag{47}$$

By continuing this procedure, we obtain

$$\sum_{\mathcal{K}=1}^{n} \operatorname{Cot}\left[\frac{\pi}{4}+\frac{\pi}{8}\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_{\mathcal{K}})-\gamma_{F_j}(x_{\mathcal{K}})\right|+\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_{\mathcal{K}})-\omega_{\gamma_{F_j}}(x_{\mathcal{K}})\right|\right)\right] \in n[0,1]. \tag{48}$$

This implies that

$$0 \leq \sum_{\mathcal{K}=1}^{n} \operatorname{Cot}\left[\frac{\pi}{4}+\frac{\pi}{8}\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_{\mathcal{K}})-\gamma_{F_j}(x_{\mathcal{K}})\right|+\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_{\mathcal{K}})-\omega_{\gamma_{F_j}}(x_{\mathcal{K}})\right|\right)\right] \leq n,$$

$$0 \leq \frac{1}{n}\sum_{\mathcal{K}=1}^{n} \operatorname{Cot}\left[\frac{\pi}{4}+\frac{\pi}{8}\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_{\mathcal{K}})-\gamma_{F_j}(x_{\mathcal{K}})\right|+\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_{\mathcal{K}})-\omega_{\gamma_{F_j}}(x_{\mathcal{K}})\right|\right)\right] \leq 1, \tag{49}$$

which implies that

$$0 \leq \mathbb{S}_c^5(x_{\mathcal{K}}) \leq 1. \tag{50}$$

(2) We have

$$\mathbb{S}_c^5(E,F)=\frac{1}{n}\sum_{\mathcal{K}=1}^{n} \operatorname{Cot}\left[\frac{\pi}{4}+\frac{\pi}{8}\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_{\mathcal{K}})-\gamma_{F_j}(x_{\mathcal{K}})\right|+\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_{\mathcal{K}})-\omega_{\gamma_{F_j}}(x_{\mathcal{K}})\right|\right)\right]. \tag{51}$$

Now, as $E=F \implies \mu_E(x_{\mathcal{K}})=\mu_F(x_{\mathcal{K}})$, for $\mathcal{K}=1,2,\ldots,n \implies \gamma_{E_j}(x_{\mathcal{K}})e^{i2\pi(\omega_{E_j}(x_{\mathcal{K}}))}=\gamma_{F_j}(x_{\mathcal{K}})e^{i2\pi(\omega_{F_j}(x_{\mathcal{K}}))}$ for $\mathcal{K}=1,2,\ldots,n \implies \gamma_{E_j}(x_{\mathcal{K}})=\gamma_{F_j}(x_{\mathcal{K}})$ and $e^{i2\pi(\omega_{E_j}(x_{\mathcal{K}}))}=e^{i2\pi(\omega_{F_j}(x_{\mathcal{K}}))}$ for $\mathcal{K}=1,2,\ldots,n$. Then,

$|\gamma_{E_j}(x_{\mathcal{K}})-\gamma_{F_j}(x_{\mathcal{K}})|=0$ and $|\omega_{\gamma_{E_j}}(x_{\mathcal{K}})-\omega_{\gamma_{F_j}}(x_{\mathcal{K}})|=0$ for $\mathcal{K}=1,2,\ldots,n$. This implies that

$$\mathbb{S}_c^5(E,F)=1. \tag{52}$$

(3) We have

$$\mathbb{S}_c^5(E,F)=\frac{1}{n}\sum_{\mathcal{K}=1}^{n} \operatorname{Cot}\left[\frac{\pi}{4}+\frac{\pi}{8}\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_{\mathcal{K}})-\gamma_{F_j}(x_{\mathcal{K}})\right|+\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_{\mathcal{K}})-\omega_{\gamma_{F_j}}(x_{\mathcal{K}})\right|\right)\right],$$

$$=\frac{1}{n}\sum_{\mathcal{K}=1}^{n} \operatorname{Cot}\left[\frac{\pi}{4}+\frac{\pi}{8}\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_{\mathcal{K}})-\gamma_{F_j}(x_{\mathcal{K}})\right|+\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_{\mathcal{K}})-\omega_{\gamma_{F_j}}(x_{\mathcal{K}})\right|\right)\right], \tag{53}$$

$$=\mathbb{S}_c^5(F,E).$$

□

Definition 14. Let E and F be two CHFSs on a set X. Then, the weighted cotangent SMs between E and F can be presented as

$$\mathbb{S}_{\mathbf{cw}}^4(E, F) = \sum_{\mathcal{K}=1}^n w_{\mathcal{K}} \mathrm{Cot}\left[\frac{\pi}{4} + \frac{\pi}{4}\left(\max\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_{\mathcal{K}}) - \gamma_{F_j}(x_{\mathcal{K}})\right|, \frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_{\mathcal{K}}) - \omega_{\gamma_{F_j}}(x_{\mathcal{K}})\right|\right)\right)\right],$$

(54)

$$\mathbb{S}_{\mathbf{cw}}^5(E, F) = \sum_{\mathcal{K}=1}^n w_{\mathcal{K}} \mathrm{Cot}\left[\frac{\pi}{4} + \frac{\pi}{8}\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_{\mathcal{K}}) - \gamma_{F_j}(x_{\mathcal{K}})\right|, \frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_{\mathcal{K}}) - \omega_{\gamma_{F_j}}(x_{\mathcal{K}})\right|\right)\right],$$

where $w = (w_1, w_2, \ldots, w_n)^{\mathcal{T}}$ represents the weight vector of every element $x_{\mathcal{K}}(\mathcal{K} = 1, 2, ,, .n)$ carried in CHFS and the weight vector satisfies $w_{\mathcal{K}} \in [0, 1]$ for every $\mathcal{K} = 1, 2, 3, ,, ,.n$, $\sum_{\mathcal{K}=1}^n w_{\mathcal{K}} = 1$. When we assume the weight vector to be $w = (1/n, 1/n, \ldots, 1/n)^{\mathcal{T}}$, the weighted cotangent SMs will transform into cotangent SMs. Otherwise speaking, when $w_{\mathcal{K}} = 1/n$, $\mathcal{K} = 1, 2, \ldots, n$, the $\mathbb{S}_{cw}^m(E, F) = \mathbb{S}_c^m(E, F) \, m = 4, 5$.

5. Applications

In this section, we gave two applications about cosine SM, SMs based on cosine function, and cotangent SM under CHF environment. The interpreted SMs are applied to pattern recognition and medical diagnosis to express the usefulness of these SMs.

5.1. Pattern Recognition

Example 2. Without any hesitancy, the quantity of construction usually relies on the standard of building materials. Accordingly, building material scrutiny is the assumption of good engineering standards. The selection of material must be strictly controlled. Scrutiny authorizes the builders to correctly recognize qualified materials and upgrade the standard of the project. To resolve the abovementioned issues, we choose the building meterials E_j(j = 1, 2, 3, 4, 5), which are discussed as follows:

$$E_1 = \left\{ \begin{array}{l} \left(x_1, \left\{0.6e^{i2\pi(1)}, 0.5e^{i2\pi(0.5)}\right\}\right), \left(x_2, \left\{0.7e^{i2\pi(0.4)}\right\}\right), \left(x_3, \left\{0.6e^{i2\pi(0.8)}, 0.4e^{i2\pi(0.7)}\right\}\right) \\ \left(x_4, \left\{0.8e^{i2\pi(0.9)}, 0.2e^{i2\pi(0.7)}\right\}\right), \left(x_5, \left\{0.2e^{i2\pi(0.3)}, 0.6e^{i2\pi(0.5)}, 0.4e^{i2\pi(0.6)}\right\}\right) \end{array} \right\},$$

$$E_2 = \left\{ \begin{array}{l} \left(x_1, \left\{0.1e^{i2\pi(0.4)}\right\}\right), \left(x_2, \left\{0.5e^{i2\pi(0.1)}, 0.1e^{i2\pi(0.6)}\right\}\right), \left(x_3, \left\{0.2e^{i2\pi(0.6)}, 0.7e^{i2\pi(0.4)}\right\}\right) \\ \left(x_4, \left\{0.1e^{i2\pi(0.4)}, 0.3e^{i2\pi(0.1)}\right\}\right), \left(x_5, \left\{0.5e^{i2\pi(0.6)}\right\}\right) \end{array} \right\},$$

$$E_3 = \left\{ \begin{array}{l} \left(x_1, \left\{1e^{i2\pi(0.8)}, 0.6e^{i2\pi(0.8)}, 0.5e^{i2\pi(0.9)}\right\}\right), \left(x_2, \left\{0.5e^{i2\pi(0.7)}\right\}\right), \left(x_3, \left\{0.8e^{i2\pi(1)}, 0.7e^{i2\pi(0.9)}\right\}\right) \\ \left(x_4, \left\{0.9e^{i2\pi(0.8)}, 0.7e^{i2\pi(0.6)},\right\}\right), \left(x_5, \left\{0.7e^{i2\pi(0.5)}, 0.2e^{i2\pi(0.4)}, 0.3e^{i2\pi(0.7)}\right\}\right) \end{array} \right\},$$

(55)

$$E_4 = \left\{ \begin{array}{l} \left(x_1, \left\{0.3e^{i2\pi(0.9)}, 1\,e^{i2\pi(1)},\right\}\right), \left(x_2, \left\{0.4.e^{i2\pi(0.2)}, 0.2e^{i2\pi(0.5)}\right\}\right), \left(x_3, \left\{0.2e^{i2\pi(1)}\right\}\right) \\ \left(x_4, \left\{0.8e^{i2\pi(0.6)}\right\}\right), \left(x_5, \left\{0.5e^{i2\pi(0.1)}, 0.6e^{i2\pi(0.3)}, 0.8e^{i2\pi(0.5)}\right\}\right) \end{array} \right\},$$

$$E_5 = \left\{ \begin{array}{l} \left(x_1, \left\{0.4e^{i2\pi(0.2)}, 0.2e^{i2\pi(0.5)}\right\}\right), \left(x_2, \left\{0.4.e^{i2\pi(0.2)}, 0.4e^{i2\pi(0.1)}\right\}\right), \left(x_3, \left\{0.2e^{i2\pi(0.4)}, 0.3e^{i2\pi(0.1)}\right\}\right) \\ \left(x_4, \left\{0.6e^{i2\pi(0.7)}, 0.3e^{i2\pi(0.5)}0.6e^{i2\pi(0.1)}\right\}\right), \left(x_5, \left\{0.1e^{i2\pi(0.3)}\right\}\right) \end{array} \right\}.$$

To resolve the abovementioned issue, we choose the complex hesitant fuzzy set in the form of unknown materials.

$$E = \left\{ \begin{array}{l} \left(x_1, \left\{0.9e^{i2\pi(0.8)}, 0.7e^{i2\pi(0.4)}, 1e^{i2\pi(0.8)}\right\}\right), \left(x_2, \left\{0.4e^{i2\pi(0.6)}\right\}\right), \left(x_3, \left\{0.8e^{i2\pi(0.6)}, 0.5e^{i2\pi(0.8)}\right\}\right) \\ \left(x_4, \left\{0.9e^{i2\pi(0.6)}, 0.4e^{i2\pi(0.3)}\right\}\right)\left(x_5, \left\{0.5e^{i2\pi(0.7)}, 0.3e^{i2\pi(0.9)}, 0.2e^{i2\pi(0.5)}\right\}\right), \end{array} \right\}.$$

(56)

The aim of this issue is to categorize the unspecified building material E in one of the categories E_j ($j = 1, 2, 3, 4, 5$). For it, the cosine SM, SMs based on cosine function, and cotangent SMs which are explored in this paper have been used to determine the similarity from E to E_j ($j = 1, \ldots, 5$) and calculations are introduced in Tables 1 and 2.

As stated by the above-computed calculations given in Table 1, we simply note that the degree of similarity between E and E_3 is the greatest one as an extract by all five SMs. This specifies that all five SMs assign the unspecified building material E to the specified building material E_3 based on the principle of the maximum degree of similarity. Ranking of the explored cosine and cotangent SMs between E and E_j ($j = 1, \ldots, 5$) is also introduced in Table 1. The graphical representation of the interpreted SMs between E and E_j ($j = 1, \ldots, 5$) is indicated in Figure 1.

The weight of elements has great significance to suppose in real decision-making problems. If we suppose the weight of elements $x_{\mathscr{K}}$ ($\mathscr{K} = 1, 2, 3, 4, 5$) to be $w_{\mathscr{K}} = (0.15, 0.1, 0.25, 0.2, 0.3)$, respectively, then the interpreted WSMs (weighted cosine SMs and weighted cotangent SMs) have been used to determine the similarity from E to E_j ($j = 1, \ldots, 5$) and calculations are introduced in Table 2.

As stated by the above-computed calculations given in Table 2, we simply note that the degree of similarity between E and E_3 is the greatest one as an extract by all five WSMs. This specifies that all five WSMs assign the unspecified building material E to the specified building material E_3 based on the principle of the maximum degree of similarity. Ranking of the explored weighted cosine SMs, weighted SMs based on cosine function, and weighted cotangent SMs between E and E_j ($j = 1, \ldots, 5$) is also introduced in Table 2. The graphical representation of the interpreted WSMs between E and E_j ($j = 1, \ldots, 5$) is indicated in Figure 2.

5.2. Medical Diagnosis. Symptoms of every diseases are almost different. To examine that the victim is suffering from what type of diseases, the medical diagnosis relies on the victim's symptoms. The victim's symptoms are a set of symptoms and unspecified diseases will be a set of diagnostic diseases. The interpreted SMs are illustrated by a following numerical example of medical diagnosis.

Example 3. Let a set of diagnosis $D = \{D_1$ (Typhoid), D_2 (Flu), D_3 (Heart problem), D_4 (Pneumonia), D_5 (Coronavirus)$\}$ and set of symptoms $\mathscr{X} = \{x_1$ (fever), x_2 (cough), x_3 (heart pain), x_4 (loss of ppetite), x_5 (short of breath)$\}$. The victim's symptoms are represented in the form of CHFSs as follows:

$$P = \left\{ \begin{array}{c} \left(x_1, \left\{0.8e^{i2\pi(0.9)}, 0.6e^{i2\pi(0.5)}\right\}\right), \left(x_2, \left\{0.5e^{i2\pi(0.7)}0.9e^{i2\pi(1)}, 1e^{i2\pi(0.5)}\right\}\right), \left(x_3, \left\{0.1e^{i2\pi(0.2)}\right\}\right) \\ \left(x_4, \left\{0.4e^{i2\pi(0.5)}, 0.5e^{i2\pi(0.6)}, 0.7e^{i2\pi(0.4)}\right\}\right), \left(x_5, \left\{0.1e^{i2\pi(0.3)}, 0.3e^{i2\pi(0.5)}\right\}\right) \end{array} \right\}. \tag{57}$$

The indications of each disease D_j ($j = 1, 2, 3, 4, 5$) are represented in the form CHFSs as follows:

$$\begin{aligned}
D_1 &= \left\{ \begin{array}{c} \left(x_1, \left\{0.7e^{i2\pi(1)}, 0.9e^{i2\pi(0.8)}\right\}\right), \left(x_2, \left\{1e^{i2\pi(0.8)}, 0.5e^{i2\pi(0.6)}, 0.6e^{i2\pi(0.9)}\right\}\right), \left(x_3, \left\{0.4e^{i2\pi(0.6)},\right\}\right) \\ \left(x_4, \left\{0.9e^{i2\pi(0.8)}, 0.7e^{i2\pi(0.6)}, 0.5e^{i2\pi(0.7)}\right\}\right), \left(x_5, \left\{0.2e^{i2\pi(0.4)}, 0.3e^{i2\pi(0.4)}\right\}\right) \end{array} \right\}, \\
D_2 &= \left\{ \begin{array}{c} \left(x_1, \left\{0.5e^{i2\pi(0.6)}, 0.9e^{i2\pi(0.8)}\right\}\right), \left(x_2, \left\{0.8e^{i2\pi(1)}, 0.7e^{i2\pi(0.8)}\right\}\right), \left(x_3, \left\{0.1e^{i2\pi(0.05)}\right\}\right) \\ \left(x_4, \left\{0.2e^{i2\pi(0.1)}, 0.5e^{i2\pi(0.2)}\right\}\right), \left(x_5, \left\{0.5e^{i2\pi(0.3)}\right\}\right) \end{array} \right\}, \\
D_3 &= \left\{ \begin{array}{c} \left(x_1, \left\{0.6e^{i2\pi(0.5)}\right\}\right), \left(x_2, \left\{0.2e^{i2\pi(0.2)}, 0.4e^{i2\pi(0.1)}\right\}\right), \left(x_3, \left\{0.8e^{i2\pi(1)}, 1e^{i2\pi(1)}, 0.7e^{i2\pi(0.9)}\right\}\right) \\ \left(x_4, \left\{0.5e^{i2\pi(0.7)}, 0.3e^{i2\pi(0.4)},\right\}\right), \left(x_5, \left\{0.7e^{i2\pi(0.6)}, 0.2e^{i2\pi(0.7)}\right\}\right) \end{array} \right\}, \\
D_4 &= \left\{ \begin{array}{c} \left(x_1, \left\{0.6e^{i2\pi(0.9)}, 0.7e^{i2\pi(0.8)}, 0.4e^{i2\pi(0.7)}\right\}\right), \left(x_2, \left\{0.5e^{i2\pi(0.7)}, 0.7e^{i2\pi(0.3)}, 0.1e^{i2\pi(0.6)}\right\}\right), \\ \left(x_3, \left\{0.1e^{i2\pi(0.4)}\right\}\right), \left(x_4, \left\{0.6e^{i2\pi(0.8)}\right\}\right), \left(x_5, \left\{0.4e^{i2\pi(0.1)}, 0.2e^{i2\pi(0.4)}\right\}\right) \end{array} \right\}, \\
D_5 &= \left\{ \begin{array}{c} \left(x_1, \left\{0.8e^{i2\pi(0.4)}, 0.5e^{i2\pi(0.7)}\right\}\right), \left(x_2, \left\{0.6e^{i2\pi(0.7)}, 0.7e^{i2\pi(0.9)}\right\}\right), \left(x_3, \left\{0.1e^{i2\pi(0.4)}, 0.3e^{i2\pi(0.2)}\right\}\right) \\ \left(x_4, \left\{0.3e^{i2\pi(0.4)}\right\}\right), \left(x_5, \left\{0.8e^{i2\pi(0.7)}, 0.9e^{i2\pi(1)}, 1e^{i2\pi(0.7)}\right\}\right) \end{array} \right\}.
\end{aligned} \tag{58}$$

The aim of this issue is to find the disease of the victim P in one of the diseases D_j ($j = 1, 2, 3, 4, 5$). For it, the cosine SM, SMs based on cosine function, and cotangent SMs which are explored in this paper have been utilized to determine the similarity from P to D_j ($j = 1, \ldots, 5$) and calculations are introduced in Tables 3 and 4.

Table 1: The explored SMs between E and E_j ($j = 1, 2, 3, 4, 5$).

SMs	$(\mathbf{E}, \mathbf{E_1})$	$(\mathbf{E}, \mathbf{E_2})$	$(\mathbf{E}, \mathbf{E_3})$	$(\mathbf{E}, \mathbf{E_4})$	$(\mathbf{E}, \mathbf{E_5})$	Ranking
$\mathbb{S}^1_{\varsigma}(\mathbf{E}, \mathbf{E_j})$	0.4733	0.2829	0.517	0.3429	0.2844	$E_3 \geq E_1 \geq E_4 \geq E_5 \geq E_2$
$\mathbb{S}^2_{\varsigma}(\mathbf{E}, \mathbf{E_j})$	0.8674	0.6372	0.9321	0.6047	0.7278	$E_3 \geq E_1 \geq E_5 \geq E_2 \geq E_4$
$\mathbb{S}^3_{\varsigma}(\mathbf{E}, \mathbf{E_j})$	0.9022	0.796	0.959	0.7405	0.8054	$E_3 \geq E_1 \geq E_5 \geq E_2 \geq E_4$
$\mathbb{S}^4_{\varsigma}(\mathbf{E}, \mathbf{E_j})$	0.6056	0.3777	0.6988	0.3762	0.4605	$E_3 \geq E_1 \geq E_5 \geq E_2 \geq E_4$
$\mathbb{S}^5_{\varsigma}(\mathbf{E}, \mathbf{E_j})$	0.6528	0.5264	0.7563	0.4746	0.7479	$E_3 \geq E_5 \geq E_1 \geq E_2 \geq E_4$

Table 2: The explored WSMs between E and E_j ($j = 1, 2, 3, 4, 5$).

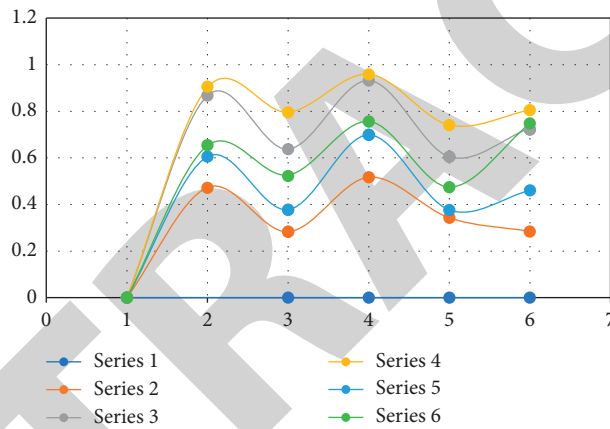| SMs	$(\mathbf{E}, \mathbf{E_1})$	$(\mathbf{E}, \mathbf{E_2})$	$(\mathbf{E}, \mathbf{E_3})$	$(\mathbf{E}, \mathbf{E_4})$	$(\mathbf{E}, \mathbf{E_5})$	Ranking
$\mathbb{S}^1_{cw}(\mathbf{E}, \mathbf{E_j})$	0.4247	0.2933	0.4557	0.3128	0.2826	$E_3 \geq E_1 \geq E_4 \geq E_2 \geq E_5$
$\mathbb{S}^2_{cw}(\mathbf{E}, \mathbf{E_j})$	0.8829	0.6705	0.9219	0.672	0.7072	$E_3 \geq E_1 \geq E_5 \geq E_4 \geq E_2$
$\mathbb{S}^3_{cw}(\mathbf{E}, \mathbf{E_j})$	0.911	0.816	0.9551	0.7603	0.7964	$E_3 \geq E_1 \geq E_2 \geq E_5 \geq E_4$
$\mathbb{S}^4_{cw}(\mathbf{E}, \mathbf{E_j})$	0.628	0.4011	0.6737	0.4207	0.4359	$E_3 \geq E_1 \geq E_5 \geq E_4 \geq E_2$
$\mathbb{S}^5_{cw}(\mathbf{E}, \mathbf{E_j})$	0.6664	0.5434	0.7431	0.4933	0.7386	$E_3 \geq E_5 \geq E_1 \geq E_2 \geq E_4$

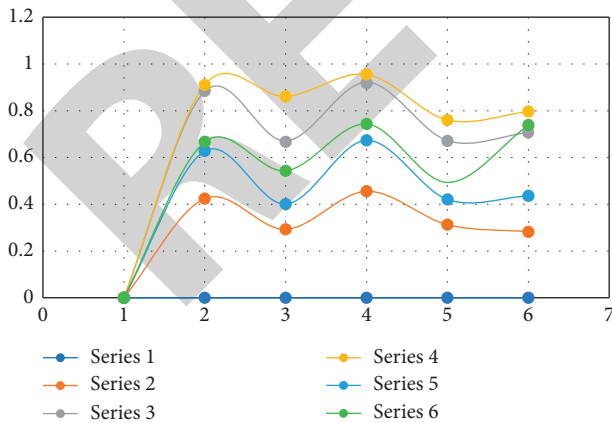Figure 1: The graphical representation of interpreted SMs.

Figure 2: The graphical representation of interpreted WSMs.

As stated by the above-computed calculations described in Table 3, we simply note that the degree of similarity between P and D_1 is the greatest one as an extract by five SMs. This specifies that all five SMs express that the victim P has typhoid based on the principle of the maximum similarity degree. Ranking of the explored cosine and cotangent SMs between P and D_j ($j = 1, \ldots, 5$) is also introduced in Table 3. Next, the graphical representation of the interpreted SMs between P and D_j ($j = 1, \ldots, 5$) is indicated in Figure 3.

The weight of elements has great significance to suppose in real decision-making problems. If we suppose the weight of elements $x_{\mathcal{K}}$ ($\mathcal{K} = 1, 2, 3, 4, 5$) to be $w_{\mathcal{K}} = (0.15, 0.1, 0.25, 0.2, 0.3)$, respectively, then the interpreted WSMs (weighted cosine SMs and weighted cotangent SMs) have been used to determine the similarity from P to D_j ($j = 1, \ldots, 5$) and calculations are introduced in Table 4.

As stated by the above-computed calculations described in Table 3, we simply note that the degree of similarity between P and D_1 is the greatest one as an extract by WSMs, except $\mathbb{S}^5_{cw}$. This specifies that WSMs $\mathbb{S}^1_{cw}$, $\mathbb{S}^2_{cw}$, $\mathbb{S}^3_{cw}$, and $\mathbb{S}^4_{cw}$ show that the victim P has typhoid based on the principle of the maximum similarity degree. We also note that degree of similarity between P and D_5 is the highest one as an extract by WSM $\mathbb{S}^5_{cw}$. This specifies that the WSM $\mathbb{S}^5_{cw}$ shows that the victim P has coronavirus. Ranking of the explored weighted cosine SMs, weighted SMs based on cosine function, and

TABLE 3: The explored SMs between P and D_j ($j = 1, 2, 3, 4, 5$).

Similarity measures	$(\mathbf{P}, \mathbf{D}_1)$	$(\mathbf{P}, \mathbf{D}_2)$	$(\mathbf{P}, \mathbf{D}_3)$	$(\mathbf{P}, \mathbf{D}_4)$	$(\mathbf{P}, \mathbf{D}_5)$	Ranking
$\mathbb{S}_C^1(\mathbf{P}, \mathbf{D}_j)$	0.5132	0.4279	0.3019	0.4216	0.3039	$D_1 \geq D_2 \geq D_4 \geq D_5 \geq D_3$
$\mathbb{S}_C^2(\mathbf{P}, \mathbf{D}_j)$	0.8833	0.8465	0.6363	0.8701	0.7434	$D_1 \geq D_4 \geq D_2 \geq D_5 \geq D_3$
$\mathbb{S}_C^3(\mathbf{P}, \mathbf{D}_j)$	0.9119	0.8843	0.6736	0.8985	0.8249	$D_1 \geq D_4 \geq D_2 \geq D_5 \geq D_3$
$\mathbb{S}_C^4(\mathbf{P}, \mathbf{D}_j)$	0.6367	0.5777	0.3953	0.6044	0.4858	$D_1 \geq D_4 \geq D_2 \geq D_5 \geq D_3$
$\mathbb{S}_C^5(\mathbf{P}, \mathbf{D}_j)$	0.7288	0.6797	0.6361	04314	0.6621	$D_1 \geq D_2 \geq D_5 \geq D_3 \geq D_4$

TABLE 4: The explored WSMs between P and D_j ($j = 1, 2, 3, 4, 5$)

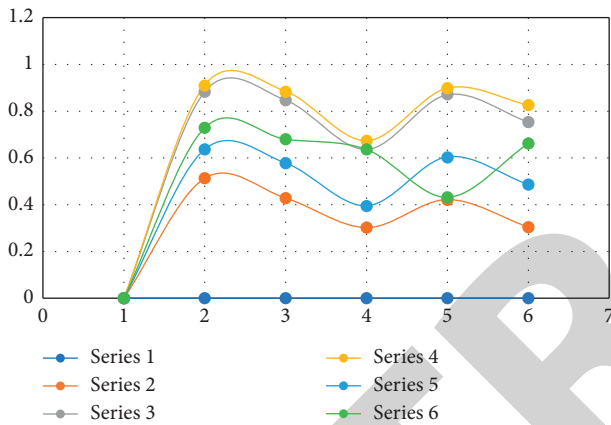| Similarity measures	$(\mathbf{P}, \mathbf{D}_1)$	$(\mathbf{P}, \mathbf{D}_2)$	$(\mathbf{P}, \mathbf{D}_3)$	$(\mathbf{P}, \mathbf{D}_4)$	$(\mathbf{P}, \mathbf{D}_5)$	Ranking
$\mathbb{S}_{cw}^1(\mathbf{P}, \mathbf{D}_j)$	0.5575	0.393	0.3089	0.4598	0.2974	$D_1 \geq D_4 \geq D_2 \geq D_3 \geq D_5$
$\mathbb{S}_{cw}^2(\mathbf{P}, \mathbf{D}_j)$	0.8972	0.8652	0.6369	0.8856	0.7065	$D_1 \geq D_4 \geq D_2 \geq D_5 \geq D_3$
$\mathbb{S}_{cw}^3(\mathbf{P}, \mathbf{D}_j)$	0.9224	0.8992	0.6785	0.9111	0.7898	$D_1 \geq D_4 \geq D_2 \geq D_5 \geq D_3$
$\mathbb{S}_{cw}^4(\mathbf{P}, \mathbf{D}_j)$	0.6635	0.601	0.4026	0.6302	0.465	$D_1 \geq D_4 \geq D_2 \geq D_5 \geq D_3$
$\mathbb{S}_{cw}^5(\mathbf{P}, \mathbf{D}_j)$	0.7058	0.6618	0.4433	0.6893	0.7526	$D_5 \geq D_1 \geq D_4 \geq D_2 \geq D_3$

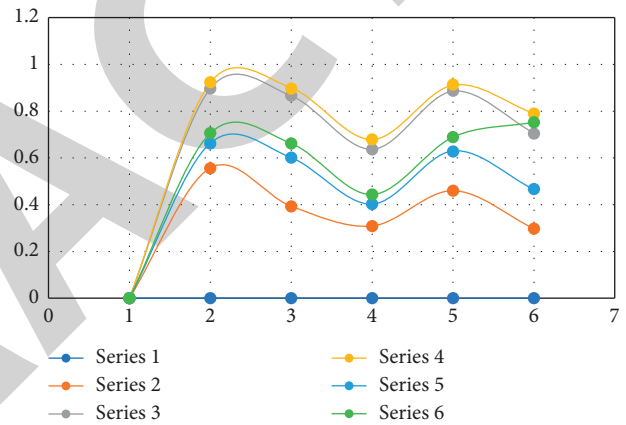FIGURE 3: The graphical representation of interpreted SMs.

FIGURE 4: The graphical representation of interpreted WSMs.

weighted cotangent SMs between P and D_j ($j = 1, \ldots, 5$) is also introduced in Table 4. Next, we have the graphical representation of the interpreted WSMs between P and D_j ($j = 1, \ldots, 5$) in Figure 4.

6. Comparison

In this section of the paper, we expressed the effectiveness and advantages of the interpreted SMs by comparing with some already defined SMs.

Example 4. Without any hesitancy, the quantity of construction usually relies on the standard of building materials. Accordingly, building material scrutiny is the assumption of good engineering standards. The selection of material must be strictly controlled. Scrutiny authorizes the builders to correctly recognize qualified materials and upgrade the standard of the project. Suppose pattern recognition problem about the categorization of building materials, Let five specified building materials E_j ($j = 1, 2, 3, 4, 5$) which are represented in the form of HFSs as follows:

$$E_1 = \left\{ \begin{array}{c} (x_1, \{0.6, 0.5\}), (x_2, \{0.7\}), (x_3, \{0.6, 0.4\}) \\ (x_4, \{0.8, 0.2\}), (x_5, \{0.2, 0.6, 0.4\}) \end{array} \right\},$$

$$E_2 = \left\{ \begin{array}{c} (x_1, \{0.1\}), (x_2, \{0.5, 0.1\}), (x_3, \{0.2, 0.7\}) \\ (x_4, \{0.1, 0.3\}), (x_5, \{0.5\}) \end{array} \right\},$$

$$E_3 = \left\{ \begin{array}{c} (x_1, \{1, 0.6, 0.5\}), (x_2, \{0.5\}), (x_3, \{0.8, 0.7\}) \\ (x_4, \{0.9, 0.7,\}), (x_5, \{0.7, 0.2, 0.3\}) \end{array} \right\},$$

$$E_4 = \left\{ \begin{array}{c} (x_1, \{0.3, 1\}), (x_2, \{0.4, 0.2\}), (x_3, \{0.2\}) \\ (x_4, \{0.8\}), (x_5, \{0.5, 0.6, 0.8\}) \end{array} \right\},$$

$$E_5 = \left\{ \begin{array}{c} (x_1, \{0.4, 0.2\}), (x_2, \{0.4, 0.4\}), (x_3, \{0.2, 0.3\}) \\ (x_4, \{0.6, 0.3, 0.6\}), (x_5, \{0.1\}) \end{array} \right\}.$$

$$(59)$$

Next, let an unspecified building material E in the form of CHFS which needs to be recognized be

$$E = \left\{ \begin{array}{c} (x_1, \{0.9, 0.7, 1\}), (x_2, \{0.4\}), (x_3, \{0.8, 0.5\}) \\ (x_4, \{0.9, 0.4\}) (x_5, \{0.5, 0.3, 0.2\}), \end{array} \right\}.$$

$$(60)$$

Table 5: The comparison between interpreted and some already defined SMs of Example 4.

Method	Score value	Ranking
Xu and Xia [29]	$\mathbb{S}(E, E_1) = 0.7667,$ $\mathbb{S}(E, E_2) = 0.5967$ $\mathbb{S}(E, E_3) = 0.7733,$ $\mathbb{S}(E, E_4) = 0.3467$ $\mathbb{S}(E, E_5) = 0.7067$	$E_3 \geq E_1 \geq E_5 \geq E_2 \geq E_4$
Zeng et al. [30]	$\mathbb{S}_h(E, E_1) = 0.8367,$ $\mathbb{S}_h(E, E_2) = 0.5283,$ $\mathbb{S}_h(E, E_3) = 0.845,$ $\mathbb{S}_h(E, E_4) = 0.415,$ $\mathbb{S}_h(E, E_4) = 0.6872$	$E_3 \geq E_1 \geq E_5 \geq E_2 \geq E_4$
Jun [31]	$\mathrm{Cos}_{\mathrm{HFS}}(E, E_1) = 0.963,$ $\mathrm{Cos}_{\mathrm{HFS}}(E, E_2) = 0.841,$ $\mathrm{Cos}_{\mathrm{HFS}}(E, E_3) = 0.911,$ $\mathrm{Cos}_{\mathrm{HFS}}(E, E_4) = 0.883,$ $\mathrm{Cos}_{\mathrm{HFS}}(E, E_5) = 0.925$	$E_1 \geq E_5 \geq E_3 \geq E_4 \geq E_2$
Proposed SM	$\mathbb{S}_c^1(E, E_1) = 0.5006,$ $\mathbb{S}_c^1(E, E_2) = 0.3334,$ $\mathbb{S}_c^1(E, E_3) = 0.5234$ $\mathbb{S}_c^1(E, E_4) = 0.453,$ $\mathbb{S}_c^1(E, E_5) = 0.3147$	$E_3 \geq E_1 \geq E_4 \geq E_2 \geq E_5$
Proposed SM	$\mathbb{S}_c^2(E, E_1) = 0.8914,$ $\mathbb{S}_c^2(E, E_2) = 0.7565,$ $\mathbb{S}_c^2(E, E_3) = 0.9719,$ $\mathbb{S}_c^2(E, E_4) = 0.7922,$ $\mathbb{S}_c^2(E, E_5) = 0.8036$	$E_3 \geq E_1 \geq E_5 \geq E_4 \geq E_2$
Proposed SM	$\mathbb{S}_c^3(E, E_1) = 0.9721,$ $\mathbb{S}_c^3(E, E_2) = 0.9342,$ $\mathbb{S}_c^3(E, E_3) = 0.9929,$ $\mathbb{S}_c^3(E, E_4) = 0.9456,$ $\mathbb{S}_c^3(E, E_5) = 0.9486$	$E_3 \geq E_1 \geq E_5 \geq E_4 \geq E_2$
Proposed SM	$\mathbb{S}_c^4(E, E_1) = 0.6513,$ $\mathbb{S}_c^4(E, E_2) = 0.5454,$ $\mathbb{S}_c^4(E, E_3) = 0.7994,$ $\mathbb{S}_c^4(E, E_4) = 0.5352,$ $\mathbb{S}_c^4(E, E_5) = 0.5395$	$E_3 \geq E_1 \geq E_2 \geq E_5 \geq E_4$
Proposed SM	$\mathbb{S}_c^5(E, E_1) = 0.8105,$ $\mathbb{S}_c^5(E, E_2) = 0.7457,$ $\mathbb{S}_c^5(E, E_3) = 0.8953,$ $\mathbb{S}_c^5(E, E_4) = 0.7418,$ $\mathbb{S}_c^5(E, E_5) = 0.8671$	$E_3 \geq E_5 \geq E_1 \geq E_2 \geq E_4$

We convert the HFSs in the CHFSs by taking $1 = e^0$ as follows:

$$
\begin{aligned}
E_1 &= \left\{ \begin{array}{l} \left(x_1, \left\{0.6e^{i2\pi(0.0)}, 0.5e^{i2\pi(0.0)}\right\}\right), \left(x_2, \left\{0.7e^{i2\pi(0.0)}\right\}\right), \left(x_3, \left\{0.6e^{i2\pi(0.0)}, 0.4e^{i2\pi(0.0)}\right\}\right) \\ \left(x_4, \left\{0.8e^{i2\pi(0.0)}, 0.2e^{i2\pi(0.0)}\right\}\right), \left(x_5, \left\{0.2e^{i2\pi(0.0)}, 0.6e^{i2\pi(0.0)}, 0.4e^{i2\pi(0.0)}\right\}\right) \end{array} \right\}, \\
E_2 &= \left\{ \begin{array}{l} \left(x_1, \left\{0.1e^{i2\pi(0.0)}\right\}\right), \left(x_2, \left\{0.5e^{i2\pi(0.0)}, 0.1e^{i2\pi(0.0)}\right\}\right), \left(x_3, \left\{0.2e^{i2\pi(0.0)}, 0.7e^{i2\pi(0.0)}\right\}\right) \\ \left(x_4, \left\{0.1e^{i2\pi(0.0)}, 0.2e^{i2\pi(0.0)}\right\}\right), \left(x_5, \left\{0.5e^{i2\pi(0.0)}\right\}\right) \end{array} \right\}, \\
E_3 &= \left\{ \begin{array}{l} \left(x_1, \left\{1e^{i2\pi(0.0)}, 0.6e^{i2\pi(0.0)}, 0.5e^{i2\pi(0.0)}\right\}\right), \left(x_2, \left\{0.5e^{i2\pi(0.0)}\right\}\right), \left(x_3, \left\{0.8e^{i2\pi(0.0)}, 0.7e^{i2\pi(0.0)}\right\}\right) \\ \left(x_4, \left\{0.9e^{i2\pi(0.0)}, 0.7e^{i2\pi(0.0)}\right\}\left(x_5, \left\{0.7e^{i2\pi(0.0)}, 0.2e^{i2\pi(0.0)}, 0.3e^{i2\pi(0.0)}\right\}\right)\right) \end{array} \right\}, \\
E_4 &= \left\{ \begin{array}{l} \left(x_1, \left\{0.3e^{i2\pi(0.0)}, 1e^{i2\pi(0.0)}\right\}\right), \left(x_2, \left\{0.4e^{i2\pi(0.0)}, 0.2e^{i2\pi(0.0)}\right\}\right), \left(x_3, \left\{0.2e^{i\pi(0.0)}\right\}\right) \\ \left(x_4, \left\{0.8e^{i\pi(0.0)}\right\}\right)\left(x_5, \left\{0.5e^{i\pi(0.0)}, 0.6e^{i2\pi(0.0)}, 0.8e^{i2\pi(0.0)}\right\}\right) \end{array} \right\},
\end{aligned}
\tag{61}
$$

Comparison of interpreted SMs with some existing SMs

- Reference
- Xu and Xia [27]
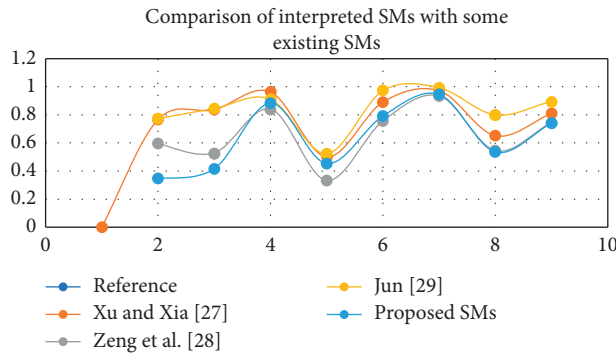- Zeng et al. [28]
- Jun [29]
- Proposed SMs

Figure 5: The graphical representation of interpreted SMs with some existing SMs for Example 4.

Table 6: The comparison between interpreted and some already defined SMs of Example 2.

Method	Score value	Ranking
Xu and Xia [29]	Unsuccessful	Unsuccessful
Zeng et al. [30]	Unsuccessful	Unsuccessful
Jun [31]	Unsuccessful	Unsuccessful
Proposed SM	$\mathbb{S}_c^1(E, E_1) = 0.4733, \ \mathbb{S}_c^1(E, E_2) = 0.2829$ $\mathbb{S}_c^1(E, E_3) = 0.517$ $\mathbb{S}_c^1(E, E_4) = 0.3429,$ $\mathbb{S}_c^1(E, E_5) = 0.2844$	$E_3 \geq E_1 \geq E_4 \geq E_5 \geq E_2$
Proposed SM	$\mathbb{S}_c^2(E, E_1) = 0.8674,$ $\mathbb{S}_c^2(E, E_2) = 0.6372,$ $\mathbb{S}_c^2(E, E_3) = 0.9321,$ $\mathbb{S}_c^2(E, E_4) = 0.6047,$ $\mathbb{S}_c^2(E, E_5) = 0.7278$	$E_3 \geq E_1 \geq E_5 \geq E_2 \geq E_4$
Proposed SM	$\mathbb{S}_c^3(E, E_1) = 0.9022,$ $\mathbb{S}_c^3(E, E_2) = 0.796,$ $\mathbb{S}_c^3(E, E_3) = 0.959,$ $\mathbb{S}_c^3(E, E_4) = 0.7405,$ $\mathbb{S}_c^3(E, E_5) = 0.8054$	$E_3 \geq E_1 \geq E_5 \geq E_2 \geq E_4$
Proposed SM	$\mathbb{S}_c^4(E, E_1) = 0.6056,$ $\mathbb{S}_c^4(E, E_2) = 0.3777,$ $\mathbb{S}_c^4(E, E_3) = 0.6988,$ $\mathbb{S}_c^4(E, E_4) = 0.3762,$ $\mathbb{S}_c^4(E, E_5) = 0.4605$	$E_3 \geq E_1 \geq E_5 \geq E_2 \geq E_4$
Proposed SM	$\mathbb{S}_c^5(E, E_1) = 0.6528,$ $\mathbb{S}_c^5(E, E_2) = 0.5264,$ $\mathbb{S}_c^5(E, E_3) = 0.7563,$ $\mathbb{S}_c^5(E, E_4) = 0.4746,$ $\mathbb{S}_c^5(E, E_5) = 0.7479$	$E_3 \geq E_5 \geq E_1 \geq E_2 \geq E_4$

and

$$E = \left\{ \begin{array}{c} \left(x_1, \left\{0.9e^{i2\pi(0.0)}, 0.7e^{i2\pi(0.0)}, 1e^{i2\pi(0.0)}\right\}\right), \left(x_2, \left\{0.4e^{i2\pi(0.0)}\right\}\right), \left(x_3, \left\{0.8e^{i2\pi(0.0)}, 0.5e^{i2\pi(0.0)}\right\}\right) \\ \left(x_4, \left\{0.9e^{i2\pi(0.0)}, 0.4e^{i2\pi(0.0)}\right\}\right)\left(x_5, \left\{0.5e^{i2\pi(0.0)}, 0.3e^{i2\pi(0.0)}, 0.2e^{i2\pi(0.0)}\right\}\right) \end{array} \right\}. \tag{62}$$

For Example 4, we need to find that the unknown building material E belongs to which of the specified building material E_j ($j = 1, 2, 3, 4, 5$). In Example 4, the data are in the shape of HFSs. We found similarity between E and E_j ($j = 1, \ldots, 5$) through some already defined SMs for HFSs, as shown in Table 5. As $1 = e^0$, then the data given in Example 4 are transformed into CHFSs. Then, through

interpreted SMs, we found the similarity between E and E_j ($j = 1, 2, 3, 4, 5$) which is also given in Table 5. Our interpreted SMs showed that unspecified building material E belongs to the specified building material E_3 because the similarity between E and E_3 is the greatest one. Ranking of the interpreted and already defined SMs is also described in Table 5. Next, we have the graphical representation of the
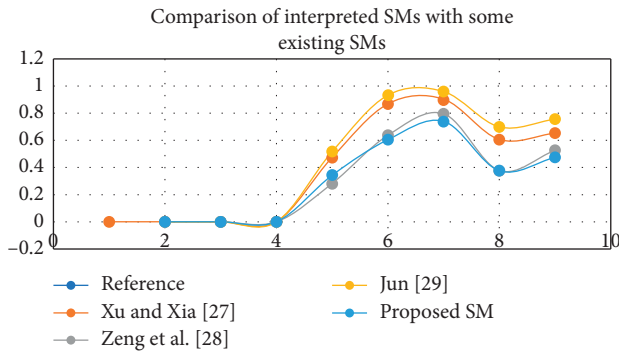
Figure 6: The graphical representation of interpreted SMs with some existing SMs for Example 2.

comparison of the proposed and already defined SMs which is represented in Figure 5.

Now, we discuss the comparison between interpreted and already defined SMs for Example 2. In Example 2, the data are in the shape of CHFSs. We know that no SM exists in the literature to solve this kind of data. The existing SMs are ineffective to find the similarity between E and E_j ($j = 1, \ldots, 5$) as demonstrated in Table 6. From Table 6, we observe that the data given in Example 2 are solvable by the interpreted SMs. The interpreted SMs get the similarity between E and E_j ($j = 1, \ldots, 5$), as demonstrated in Table 6. Our interpreted SMs showed that unspecified building material E belongs to the specified building material E_3 because the similarity between E and E_3 is the greatest one. Ranking of the explored SMs is also introduced in Table 6. Next, we have the graphical representation of the comparison of proposed and already defined SMs which is represented in Figure 6.

From the above discussion, our explored SMs can represent extra fuzzy information and put it broadly in circumstances in real-life problems. Based on CHFS, we explored the SMs; our SMs are more satisfactory for real-life problems, and the existing SMs and our SMs are more general than the existing SMs.

7. Conclusion

The CHFS is one of the enlargements of the CFS in which the possibility of the enrollment work is stretched out from the subset of the genuine number to the unit disc which is interpreted. In this article, we explored another type of similarity measure (SM) which relies on the cosine and cotangent functions. At that stage, we use our introduced SMs and weighted SMs (based on the cosine and cotangent functions) between CHFSs to manage pattern recognition and medical diagnosis problems including design acknowledgment and plan choice. Finally, two numerical models are given to represent the logic and effectiveness of the likeness measures for design acknowledgment and conspire choice. The comparison between explored measure with some existing measures and their graphical representations are also discussed in detail.

Consequently, the measures defined in this manuscript can be utilized in a larger range of applications. In future research,

we will extend this work to suppose the two facts: (1) similarity measures and aggregation operators [34–42]; (2) methods [43].

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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