

Research Article

Hadamard Inequalities for Strongly (α, m) -Convex Functions via Caputo Fractional Derivatives

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In this paper, we present two versions of the Hadamard inequality for (α, m) convex functions via Caputo fractional derivatives. Several related results are analyzed for convex and m -convex functions along with their refinements and generalizations. The error bounds of the Hadamard inequalities are established by applying some known identities.

1. Introduction

Fractional calculus is a natural extension of classical calculus and the notions related to integer order derivatives and integrals have been extended to fractional order derivatives and integrals. Many classical inequalities related to integrals of real valued functions have been presented for fractional integrals. The Hadamard inequality is one of them which is studied extensively for different types of convex functions via fractional derivatives and integrals. For the detailed study of Hadamard fractional integral inequalities, we refer the readers to [1–13]. In 1967, Caputo made the most significant contributions to fractional calculus by reformulating the definition of the Riemann–Liouville fractional derivatives [14].

In the following, we give the definition of Caputo fractional derivatives.

Definition 1 [15]. Let $\psi \in AC^n[a, b]$ and $n = \Re(\beta) + 1$. Then, Caputo fractional derivatives of order $\beta \in \mathbb{C}$, $\Re(\beta) > 0$, of ψ are defined as follows:

$${}^C D_{a^+}^\beta \psi(x) = \frac{1}{\Gamma(n-\beta)} \int_a^x \frac{\psi^{(n)}(z)}{(x-z)^{\beta-n+1}} dz, \quad x > a, \quad (1)$$

$${}^C D_{b^-}^\beta \psi(x) = \frac{(-1)^n}{\Gamma(n-\beta)} \int_x^b \frac{\psi^{(n)}(z)}{(z-x)^{\beta-n+1}} dz, \quad x < b.$$

If $\beta = n \in \{1, 2, 3, \dots\}$ and usual derivative of order n exists, then Caputo fractional derivative $({}^C D_{a^+}^\beta \psi)(x)$ coincides with $\psi^{(n)}(x)$, where as $({}^C D_{b^-}^\beta \psi)(x)$ coincides with $\psi^{(n)}(x)$ with exactness to a constant multiplier $(-1)^n$. In particular, we have

$$({}^C D_{a^+}^0 \psi)(x) = ({}^C D_{b^-}^0 \psi)(x) = \psi(x), \quad (2)$$

where $n = 1$ and $\beta = 0$.

Convex functions are represented in terms of different inequalities. Many of the well-known inequalities are consequences of convex functions. Strongly convexity is a strengthening of the notion of convexity; some properties of strongly convex functions are just “stronger versions” of known properties of convex functions. Strongly convex function was introduced by Polyak [16].

Definition 2. Let D be a convex subset of L and $(L, \|\cdot\|)$ be a normed space. A function $\psi: D \subset L \rightarrow \mathbb{R}$ is called strongly convex function with modulus C if it satisfies

$$\begin{aligned} \psi(az + (1-z)b) &\leq z\psi(a) + (1-z)\psi(b) \\ &\quad - Cz(1-z)\|a-b\|^2, \end{aligned} \quad (3)$$

$$\forall a, b \in D, z \in [0, 1].$$

For $C = 0$, (3) gives the inequality satisfied by convex functions. Many authors have been inventing the properties and applications of strongly convex functions, see [17–21].

The concept of m -convex functions and strongly m -convex functions are introduced in [22, 23], respectively. In [23], the definition of m -convex functions is given.

Definition 3. The function $f: [0, b] \rightarrow \mathbb{R}$ is said to be m -convex, where $m \in [0, 1]$, if for every $x, y \in [0, b]$ and $t \in [0, 1]$ we have

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y). \quad (4)$$

In [22], strongly m -convex function is introduced as follows.

Definition 4. A function $\psi: [0, b] \rightarrow \mathbb{R}$ is called strongly m -convex function with modulus C if

$$\begin{aligned} \psi(zx + m(1-z)y) &\leq z\psi(x) + m(1-z)\psi(y) \\ &\quad - Cmz(1-z)|x-y|^2, \end{aligned} \quad (5)$$

where $x, y \in [0, b]$ and $z \in [0, 1]$.

In [24], (α, m) -convex function is introduced as follows.

Definition 5. A function $f: [0, b] \rightarrow \mathbb{R}$ is said to be (α, m) -convex function, where $(\alpha, m) \in [0, 1]^2$ and $b > 0$, if for every $x, y \in [0, b]$ and $z \in [0, 1]$, we have

$$f(zx + m(1-z)y) \leq z^\alpha f(x) + m(1-z)^\alpha f(y). \quad (6)$$

From (α, m) -convex functions, one can obtain star-shaped, m -convex, convex, and α -convex functions. In literature, (α, m) -convex functions are considered by many researchers and mathematicians and many properties especially inequalities have been obtained for them, for example (see [25–30]).

The strongly (α, m) -convex function is introduced as follows.

Definition 6. A function $f: [0, +\infty) \rightarrow \mathbb{R}$ is said to be strongly (α, m) -convex function with modulus $C \geq 0$, for $(\alpha, m) \in [0, 1]^2$, if

$$\begin{aligned} f(zx + m(1-z)y) &\leq z^\alpha f(x) + m(1-z)^\alpha f(y) \\ &\quad - Cmz^\alpha(1-z)^\alpha|y-x|^2, \end{aligned} \quad (7)$$

holds for all $x, y \in [0, +\infty)$ and $z \in [0, 1]$.

A well-known inequality named Hadamard inequality is another interpretation of convex function. It is stated as follows [31].

Theorem 1. Let $\psi: I \rightarrow \mathbb{R}$ be a convex function on interval $I \subset \mathbb{R}$ and $a, b \in I$, where $a < b$. Then, the following inequality holds:

$$\psi\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b \psi(x) dx \leq \frac{\psi(a) + \psi(b)}{2}. \quad (8)$$

If order in (8) is reversed, then it holds for concave function.

Fractional integral inequalities are useful in establishing the uniqueness of solutions for certain fractional partial differential equations. These inequalities also provide upper as well as lower bounds for solutions of the fractional boundary value problems. Fractional integral inequalities are in the study of several mathematicians. For fractional versions of Hadamard inequality, we refer the researchers and references [1–5, 11, 12, 32].

Farid et al. [33] established the following identity for Caputo fractional derivatives.

Lemma 1. Let $\psi: [a, b] \rightarrow \mathbb{R}$, $0 \leq a < b$, be the function such that $\psi \in C^n[a, b]$. Also, let $\psi^{(n+1)}$ be positive and convex function on $[a, b]$. Then, the following equality for Caputo fractional derivatives holds:

$$\begin{aligned} &\frac{\psi^{(n)}(a) + \psi^{(n)}(b)}{2} - \frac{\Gamma(n-\beta+1)}{2(b-a)^{n-\beta}} \\ &\quad \cdot \left[({}^C D_{a^+}^\beta \psi)(b) + (-1)^n ({}^C D_{b^-}^\beta \psi)(a) \right] \\ &= \frac{b-a}{2} \int_0^1 \left[(1-z)^{n-\beta} - z^{n-\beta} \right] \psi^{(n+1)}(za + (1-z)b) dz. \end{aligned} \quad (9)$$

The following identity is established in [34].

Lemma 2. Let $\psi: [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$. If $\psi \in C^{n+1}[a, b]$, then the following equality for Caputo fractional derivatives holds:

$$\begin{aligned} & \frac{2^{n-(\beta/k)-1} k \Gamma_k(n - (\beta/k) + k)}{(mb - a)^{n-(\beta/k)}} \left[\left({}^C D_{(a+bm/2)^+}^{\beta, k} \psi \right)(mb) + m^{n-(\beta/k)+1} (-1)^n \left({}^C D_{(a+mb/2m)^-}^{\beta, k} \psi \right) \left(\frac{a}{m} \right) \right] \\ & - \frac{1}{2} \left[\psi^{(n)} \left(\frac{a + mb}{2} \right) + m \psi^{(n)} \left(\frac{a + mb}{2m} \right) \right] \\ & = \frac{mb - a}{4} \left[\int_0^1 z^{n-(\beta/k)} \psi^{(n+1)} \left(\frac{z}{2} a + m \left(\frac{2-z}{2} \right) b \right) - \int_0^1 z^{n-(\beta/k)} \psi^{(n+1)} \left(\frac{2-z}{2m} a + \frac{z}{2} b \right) dz \right], \end{aligned} \tag{10}$$

with $\beta > 0$.

The Hadamard inequality for Caputo fractional derivatives of convex functions is studied in [7, 33, 34]; also, the error estimations are established by using aforementioned identities. The aim of this paper is to prove the Hadamard inequality for Caputo fractional derivatives of strongly (α, m) -convex functions. We have obtained refinements of various inequalities proved for convex and m -convex functions.

In Section 2, we will give two versions of the Hadamard inequality for Caputo fractional derivatives using strongly (α, m) -convex functions. Also, we connect the particular cases with some classical results. In Section 3, by applying

known identities, we will derive refinements of some well-known inequalities.

2. Main Results

The following results give the Hadamard inequality for Caputo fractional derivatives of strongly (α, m) -convex functions.

Theorem 2. *Let $\psi: [a, b] \rightarrow \mathbb{R}$ be a positive function with $\psi \in C^n[a, b]$, $(\alpha, m) \in [0, 1]^2$, and $0 \leq a < mb, m \neq 0$. If $\psi^{(n)}$ is a strongly (α, m) -convex function with modulus C , then the following inequality for Caputo fractional derivatives holds:*

$$\begin{aligned} & \psi^{(n)} \left(\frac{bm + a}{2} \right) + \frac{mC(n - \beta)}{2^\alpha(n - \beta + 2)} \left(1 - \frac{1}{2^\alpha} \right) \left\{ (b - a)^2 + \frac{2(b - a)((a/m) - mb)}{(n - \beta + 1)} + \frac{2((a/m) - mb)^2}{(n - \beta)(n - \beta + 1)} \right\} \\ & \leq \frac{\Gamma(n - \beta + 1)}{(bm - a)^{n-\beta}} \left[\left(1 - \frac{1}{2^\alpha} \right) m^{n-\beta+1} (-1)^n \left({}^C D_b^\beta \psi \right) \left(\frac{a}{m} \right) + \frac{1}{2^\alpha} \left({}^C D_a^\beta \psi \right) (mb) \right] \\ & \leq \frac{n - \beta}{n - \beta + \alpha} \left\{ \left(1 - \frac{1}{2^\alpha} \right) \frac{[m^2 \psi^{(n)}(a/m^2) + m \psi^{(n)}(b)]}{n - \beta} + \frac{[m \psi^{(n)}(b) + \psi^{(n)}(a)]}{2^\alpha} - \frac{Cm\alpha \left(m(2^\alpha - 1)(b - (a/m^2))^2 + (b - a)^2 \right)}{2^\alpha(n - \beta + 2\alpha)} \right\}, \end{aligned} \tag{11}$$

with $\beta > 0$ and $n = [\beta] + 1$.

Proof. Since $\psi^{(n)}$ is strongly (α, m) -convex function with modulus C , for $x, y \in [a, b]$, we have

$$\begin{aligned} \psi^{(n)} \left(\frac{mx + y}{2} \right) & \leq \left(1 - \frac{1}{2^\alpha} \right) m \psi^{(n)}(x) + \frac{\psi^{(n)}(y)}{2^\alpha} \\ & \quad - \frac{mC}{2^\alpha} \left(1 - \frac{1}{2^\alpha} \right) |x - y|^2. \end{aligned} \tag{12}$$

Let $x = (1 - z)(a/m) + zb \leq b$ and $y = m(1 - z)b + za \geq a, z \in [0, 1]$. Then, we have

$$\begin{aligned} \psi^{(n)} \left(\frac{bm + a}{2} \right) & \leq \left(1 - \frac{1}{2^\alpha} \right) m \psi^{(n)} \left((1 - z) \frac{a}{m} + zb \right) + \frac{\psi^{(n)}(m(1 - z)b + za)}{2^\alpha} \\ & \quad - \frac{mC}{2^\alpha} \left(1 - \frac{1}{2^\alpha} \right) \left((1 - z) \frac{a}{m} + zb - (m(1 - z)b + za) \right)^2. \end{aligned} \tag{13}$$

Multiplying (13) with $z^{n-\beta-1}$ on both sides and making integration over $[0, 1]$, we get

$$\begin{aligned} \psi^{(n)}\left(\frac{bm+a}{2}\right) \int_0^1 z^{n-\beta-1} dz &\leq \left(1 - \frac{1}{2^\alpha}\right) \left(\int_0^1 m\psi^{(n)}\left((1-z)\frac{a}{m} + zb\right) z^{n-\beta-1} dz + \frac{1}{2^\alpha} \int_0^1 \psi^{(n)}(m(1-z)b + za) z^{n-\beta-1} dz \right) \\ &\quad - \frac{mC}{2^\alpha} \left(1 - \frac{1}{2^\alpha}\right) \int_0^1 \left(z^{n-\beta-1} \times \left((1-z)\frac{a}{m} + zb - (m(1-z)b + za) \right)^2 \right) dz. \end{aligned} \quad (14)$$

By using change of the variables and computing the last integral, from (14), we get

$$\begin{aligned} \frac{1}{n-\beta} \psi^{(n)}\left(\frac{bm+a}{2}\right) &\leq \frac{1}{bm-a} \left(\left(1 - \frac{1}{2^\alpha}\right) \int_{(a/m)}^b m^2 \psi^{(n)}(x) \left(\frac{mx-a}{bm-a}\right)^{n-\beta-1} dx + \frac{1}{2^\alpha} \int_a^{mb} \psi^{(n)}(y) \left(\frac{bm-y}{bm-a}\right)^{n-\beta-1} dy \right) \\ &\quad - \frac{mC}{2^\alpha} \left(1 - \frac{1}{2^\alpha}\right) \times \left\{ \frac{(b-a)^2}{n-\beta+2} + \frac{2(b-a)((a/m)-mb)}{(n-\beta+1)(n-\beta+2)} + \frac{2((a/m)-mb)^2}{(n-\beta)(n-\beta+1)(n-\beta+2)} \right\}. \end{aligned} \quad (15)$$

Further, it takes the following form:

$$\begin{aligned} \psi^{(n)}\left(\frac{bm+a}{2}\right) &\leq \frac{\Gamma(n-\beta+1)}{(bm-a)^{n-\beta}} \left[\left(1 - \frac{1}{2^\alpha}\right) m^{n-\beta+1} (-1)^n ({}^C D_b^\beta \psi)\left(\frac{a}{m}\right) + \frac{1}{2^\alpha} ({}^C D_a^\beta \psi)(mb) \right] \\ &\quad - \frac{mC(n-\beta)}{2^\alpha} \left(1 - \frac{1}{2^\alpha}\right) + \left\{ \frac{(b-a)^2}{n-\beta+2} + \frac{2(b-a)((a/m)-mb)}{(n-\beta+1)(n-\beta+2)} + \frac{2((a/m)-mb)^2}{(n-\beta)(n-\beta+1)(n-\beta+2)} \right\}. \end{aligned} \quad (16)$$

Since $\psi^{(n)}$ is strongly (α, m) -convex function with modulus C , for $z \in [0, 1]$, then one has

$$\begin{aligned} m\psi^{(n)}\left((1-z)\frac{a}{m} + zb\right) &\leq mz^\alpha \psi^{(n)}(b) + m^2(1-z^\alpha) \psi^{(n)}\left(\frac{a}{m^2}\right) \\ &\quad - Cm^2 z^\alpha (1-z^\alpha) \left(b - \frac{a}{m^2}\right)^2. \end{aligned} \quad (17)$$

By multiplying (17) with $(1 - (1/2^\alpha))z^{n-\beta-1}$ on both sides and making integration over $[0, 1]$, we get

$$\begin{aligned} \left(1 - \frac{1}{2^\alpha}\right) \int_0^1 m\psi^{(n)}\left((1-z)\frac{a}{m} + zb\right) z^{n-\beta-1} dz \\ \leq \left(1 - \frac{1}{2^\alpha}\right) \left\{ \int_0^1 m\psi^{(n)}(b) z^{n-\beta+\alpha-1} dz + \int_0^1 m^2 \psi^{(n)}\left(\frac{a}{m^2}\right) (1-z^\alpha) z^{n-\beta-1} dz - Cm^2 \left(b - \frac{a}{m^2}\right)^2 \int_0^1 (1-z^\alpha) z^{n-\beta+\alpha-1} dz \right\}. \end{aligned} \quad (18)$$

By using change of the variables and computing the last integral, from (18), we get

$$\begin{aligned} & \left(1 - \frac{1}{2^\alpha}\right) \frac{1}{bm - a} \left(\int_{(a/m)}^b m^2 \psi^{(n)}(x) \left(\frac{mx - a}{bm - a}\right)^{n-\beta-1} dx \right) \\ & \leq \left(1 - \frac{1}{2^\alpha}\right) \left\{ \frac{[m^2 \psi^{(n)}(a/m^2) + m\psi^{(n)}(b)]}{(n - \beta)(n - \beta + \alpha)} - \frac{Cm^2 \alpha (b - (a/m^2))^2}{(n - \beta + \alpha)(n - \beta + 2\alpha)} \right\}. \end{aligned} \tag{19}$$

Further, it takes the following form:

$$\begin{aligned} & \frac{\Gamma(n - \beta + 1)}{(bm - a)^{n-\beta}} \left[\left(1 - \frac{1}{2^\alpha}\right) m^{n-\beta+1} (-1)^n ({}^C D_b^\beta \psi) \left(\frac{a}{m}\right) \right] \\ & \leq \left(1 - \frac{1}{2^\alpha}\right) (n - \beta) \left\{ \frac{[m^2 \psi^{(n)}(a/m^2) + m\psi^{(n)}(b)]}{(n - \beta)(n - \beta + \alpha)} - \frac{Cm^2 \alpha (b - (a/m^2))^2}{(n - \beta + \alpha)(n - \beta + 2\alpha)} \right\}. \end{aligned} \tag{20}$$

Since $\psi^{(n)}$ is strongly (α, m) -convex function with modulus C , for $z \in [0, 1]$, then one has

$$\begin{aligned} \psi^{(n)}(m(1 - z)b + za) & \leq z^\alpha \psi^{(n)}(a) + m(1 - z^\alpha) \psi^{(n)}(b) \\ & \quad - Cmz^\alpha (1 - z^\alpha)(b - a)^2. \end{aligned} \tag{21}$$

By multiplying (21) with $(1/2^\alpha)z^{n-\beta-1}$ on both sides and making integration over $[0, 1]$, we get

$$\begin{aligned} & \frac{1}{2^\alpha} \int_0^1 \psi^{(n)}(m(1 - z)b + za) z^{n-\beta-1} dz \\ & \leq \frac{1}{2^\alpha} \left(\int_0^1 \psi^{(n)}(a) z^{n-\beta+\alpha-1} dz \right. \\ & \quad \left. + \int_0^1 m\psi^{(n)}(b) (1 - z^\alpha) z^{n-\beta-1} dz \right. \\ & \quad \left. - Cm(b - a)^2 \int_0^1 (1 - z^\alpha) z^{n-\beta+\alpha-1} dz \right). \end{aligned} \tag{22}$$

By using change of the variables and computing the last integral, from (22), we get

$$\begin{aligned} & \frac{1}{2^\alpha (bm - a)} \left(\int_a^{mb} \psi^{(n)}(y) \left(\frac{bm - y}{bm - a}\right)^{n-\beta-1} dy \right) \\ & \leq \frac{[m\psi^{(n)}(b) + \psi^{(n)}(a)]}{2^\alpha (n - \beta + \alpha)} - \frac{Cm\alpha (b - a)^2}{2^\alpha (n - \beta + \alpha)(n - \beta + 2\alpha)}. \end{aligned} \tag{23}$$

Further, it takes the following form:

$$\begin{aligned} & \frac{\Gamma(n - \beta + 1)}{(bm - a)^{n-\beta}} \left[\frac{1}{2^\alpha} ({}^C D_a^\beta \psi)(mb) \right] \\ & \leq \frac{(n - \beta)}{2^\alpha} \left\{ \frac{[m\psi^{(n)}(b) + \psi^{(n)}(a)]}{n - \beta + \alpha} - \frac{Cm\alpha (b - a)^2}{(n - \beta + \alpha)(n - \beta + 2\alpha)} \right\}. \end{aligned} \tag{24}$$

By adding (20) and (24), we have

$$\begin{aligned} & \frac{\Gamma(n - \beta + 1)}{(bm - a)^{n-\beta}} \left[\left(1 - \frac{1}{2^\alpha}\right) m^{n-\beta+1} (-1)^n ({}^C D_b^\beta \psi) \left(\frac{a}{m}\right) + \frac{1}{2^\alpha} ({}^C D_a^\beta \psi)(mb) \right] \\ & \leq (n - \beta) \left\{ \left(1 - \frac{1}{2^\alpha}\right) \frac{[m^2 \psi^{(n)}(a/m^2) + m\psi^{(n)}(b)]}{(n - \beta)(n - \beta + \alpha)} + \frac{[m\psi^{(n)}(b) + \psi^{(n)}(a)]}{2^\alpha (n - \beta + \alpha)} - \frac{Cm\alpha (m(2^\alpha - 1)(b - (a/m^2))^2 + (b - a)^2)}{2^\alpha (n - \beta + \alpha)(n - \beta + 2\alpha)} \right\}. \end{aligned} \tag{25}$$

Inequalities (16) and (25) constituted the required inequality.

The consequences of Theorem 2 are stated in the following remark. \square

Remark 1. If $C = 0, m = 1,$ and $\alpha = 1$ in (11), then we get the fractional Hadamard inequality for convex function given in [33], Theorem 3.

The upcoming result is the refinement of another version of the Hadamard inequality for Caputo fractional integrals stated in [7], Theorem 2.

Theorem 3. Let $\psi: [a, b] \rightarrow \mathbb{R}$ be a positive function with $\psi \in C^n[a, b], (\alpha, m) \in [0, 1]^2,$ and $0 \leq a < mb, m \neq 0.$ If $\psi^{(n)}$ is a strongly (α, m) -convex function with modulus $C,$ then the following inequality for Caputo fractional derivatives holds:

$$\begin{aligned} & \psi^{(n)}\left(\frac{bm+a}{2}\right) + \frac{mC(n-\beta)(1-(1/2^\alpha))}{2^{\alpha+1}(n-\beta+2)} \left\{ \frac{(b-a)^2}{2} + \frac{(b-a)((a/m)-mb)(n-\beta+3)}{(n-\beta+1)} \right. \\ & \left. + \frac{((a/m)-mb)^2[(n-\beta)^2+5n-5\beta+8]}{2(n-\beta)(n-\beta+1)} \right\} \leq \frac{2^{n-\beta}\Gamma(n-\beta+1)}{(bm-a)^{n-\beta}} \\ & \times \left[\left(1-\frac{1}{2^\alpha}\right)m^{n-\beta+1}(-1)^n({}^C D_{(a+bm/2m)^-}^\beta \psi)\left(\frac{a}{m}\right) + \left(\frac{1}{2^\alpha}\right)({}^C D_{(a+bm/2)^+}^\beta \psi)(mb) \right] \tag{26} \\ & \leq \frac{(n-\beta)}{2^\alpha(n+\alpha-\beta)} \left\{ \left(1-\frac{1}{2^\alpha}\right) \frac{[m\psi^{(n)}(b) + m^2\psi^{(n)}(a/m^2)][2^\alpha(n+\alpha-\beta)-(n-\beta)]}{(n-\beta)} + \frac{m\psi^{(n)}(a) + \psi^{(n)}(a)}{2^\alpha} \right. \\ & \left. \frac{mC[2^\alpha(2\alpha+n-\beta)-(\alpha+n-\beta)]}{2^\alpha(2\alpha+n-\beta)} \times \left[\left(1-\frac{1}{2^\alpha}\right) \left(m\left(b-\frac{a}{m^2}\right)^2 + \frac{(b-a)^2}{2^\alpha} \right) \right] \right\}. \end{aligned}$$

with $\beta > 0$ and $n = [\beta] + 1.$

Proof. Let $x = (a/m)(2-z/2) + b(z/2)$ and $y = a(z/2) + m(2-z/2)b, z \in [0, 1]$ in (12), then we have

$$\begin{aligned} \psi^{(n)}\left(\frac{bm+a}{2}\right) & \leq \left(1-\frac{1}{2^\alpha}\right)m\psi^{(n)}\left(\frac{a}{m}\left(\frac{2-z}{2}\right) + b\frac{z}{2}\right) + \left(\frac{1}{2^\alpha}\right)\psi^{(n)}\left(a\frac{z}{2} + m\left(\frac{2-z}{2}\right)b\right) \\ & - \frac{mC}{2^\alpha}\left(1-\frac{1}{2^\alpha}\right)\left(\frac{a}{m}\left(\frac{2-z}{2}\right) + b\frac{z}{2} - a\frac{z}{2} - m\left(\frac{2-z}{2}\right)b\right)^2. \end{aligned} \tag{27}$$

By multiplying (27) with $z^{n-\beta-1}$ on both sides and making integration over $[0, 1],$ we get

$$\begin{aligned} \psi^{(n)}\left(\frac{bm+a}{2}\right) \int_0^1 z^{n-\beta-1} dz & \leq \int_0^1 \left(\left(1-\frac{1}{2^\alpha}\right)m\psi^{(n)}\left(\frac{a}{m}\left(\frac{2-z}{2}\right) + b\frac{z}{2}\right) \right. \\ & \left. + \left(\frac{1}{2^\alpha}\right)\psi^{(n)}\left(a\frac{z}{2} + m\left(\frac{2-z}{2}\right)b\right) \right) z^{n-\beta-1} dz \\ & - \frac{mC}{2^\alpha}\left(1-\frac{1}{2^\alpha}\right) \int_0^1 \left(\frac{a}{m}\left(\frac{2-z}{2}\right) + b\frac{z}{2} - a\frac{z}{2} - m\left(\frac{2-z}{2}\right)b\right)^2 z^{n-\beta-1} dz. \end{aligned} \tag{28}$$

By using change of variables and computing the last integral, from (14), we get

$$\begin{aligned} \frac{1}{n-\beta} \psi^{(n)}\left(\frac{bm+a}{2}\right) &\leq \left(\left(1 - \frac{1}{2^\alpha}\right) m \int_{(a/m)}^{(a+bm/2m)} \left(\frac{2m(y-(a/m))}{bm-a}\right)^{n-\beta-1} \psi^{(n)}(y) \frac{2m dy}{bm-a} \right. \\ &\quad \left. + \left(\frac{1}{2^\alpha}\right) \int_{(a+mb/2)}^{mb} \left(\frac{2(mb-x)}{bm-a}\right)^{n-\beta-1} \psi^{(n)}(x) \frac{2 dx}{bm-a} \right) - \frac{mC}{2^\alpha(n-\beta+2)} \left(1 - \frac{1}{2^\alpha}\right) \\ &\quad \left\{ \frac{(b-a)^2}{4} + \frac{(b-a)((a/m)-mb)(n-\beta+3)}{2(n-\beta+1)} + \frac{((a/m)-mb)^2[(n-\beta)^2+5n-5\beta+8]}{4(n-\beta)(n-\beta+1)} \right\}. \end{aligned} \tag{29}$$

Further, it takes the following form:

$$\begin{aligned} \psi^{(n)}\left(\frac{bm+a}{2}\right) &\leq \frac{2^{n-\beta} \Gamma(n-\beta+1)}{(bm-a)^{n-\beta}} \left[\left(1 - \frac{1}{2^\alpha}\right) m^{n-\beta+1} (-1)^n ({}^C D_{(a+bm/2m)^-}^\beta \psi)\left(\frac{a}{m}\right) \right. \\ &\quad \left. + \left(\frac{1}{2^\alpha}\right) ({}^C D_{(a+bm/2)^+}^\beta \psi)(mb) \right] - \frac{mC(n-\beta)}{2^{\alpha+1}} \left(1 - \frac{1}{2^\alpha}\right) \left\{ \frac{(b-a)^2}{2(n-\beta+2)} + \frac{(b-a)((a/m)-mb)(n-\beta+3)}{(n-\beta+1)(n-\beta+2)} + \right. \\ &\quad \left. \frac{((a/m)-mb)^2[(n-\beta)^2+5n-5\beta+8]}{2(n-\beta)(n-\beta+1)(n-\beta+2)} \right\}. \end{aligned} \tag{30}$$

Since $\psi^{(n)}$ is strongly (α, m) -convex function and $z \in [0, 1]$, we have the following inequality:

$$\begin{aligned} m \psi^{(n)}\left(\frac{a}{m} \left(\frac{2-z}{2}\right) + b \frac{z}{2}\right) &\leq m^2 \left(1 - \left(\frac{z}{2}\right)^\alpha\right) \psi^{(n)}\left(\frac{a}{m^2}\right) \\ &\quad + m \left(\frac{z}{2}\right)^\alpha \psi^{(n)}(b) - m^2 C \left(\frac{z}{2}\right)^\alpha \left(1 - \left(\frac{z}{2}\right)^\alpha\right) \left(b - \frac{a}{m^2}\right)^2. \end{aligned} \tag{31}$$

By multiplying (31) with $(1 - (1/2^\alpha))z^{n-\beta-1}$ on both sides and making integration over $[0, 1]$, we get

$$\begin{aligned} \left(1 - \frac{1}{2^\alpha}\right) \int_0^1 m \psi^{(n)}\left(\frac{a}{m} \left(\frac{2-z}{2}\right) + b \frac{z}{2}\right) z^{n-\beta-1} dz &\leq \left(1 - \frac{1}{2^\alpha}\right) \\ &\quad \times \left(\int_0^1 m^2 \left(1 - \left(\frac{z}{2}\right)^\alpha\right) \psi^{(n)}\left(\frac{a}{m^2}\right) z^{n-\beta-1} dz + \int_0^1 m \left(\frac{z}{2}\right)^\alpha \psi^{(n)}(b) z^{n-\beta-1} dz \right) \\ &\quad - \left(1 - \frac{1}{2^\alpha}\right) \int_0^1 m C \left(\frac{z}{2}\right)^\alpha \left(1 - \left(\frac{z}{2}\right)^\alpha\right) \left[m \left(b - \frac{a}{m^2}\right)^2 \right] z^{n-\beta-1} dz. \end{aligned} \tag{32}$$

By using change of variables and computing the last integral, from (32), we get

$$\begin{aligned} & \frac{2(1 - (1/2^\alpha))}{bm - a} \left\{ \int_{(a/m)}^{(a+bm/2m)} \left(\frac{2m(y - (a/m))}{bm - a} \right)^{n-\beta-1} m^2 \psi^{(n)}(y) dy \right. \\ & \leq \frac{(1 - (1/2^\alpha)) [m\psi^{(n)}(b) + m^2\psi^{(n)}(a/m^2)] [2^\alpha(n + \alpha - \beta) - (n - \beta)]}{2^\alpha(n - \beta)(n + \alpha - \beta)} \\ & \quad - \frac{mC(1 - (1/2^\alpha)) [2^\alpha(2\alpha + n - \beta) - (\alpha + n - \beta)] (m(b - (a/m^2)))^2}{2^{2\alpha}(\alpha + n - \beta)(2\alpha + n - \beta)}. \end{aligned} \tag{33}$$

Further, it takes the following form:

$$\begin{aligned} & \frac{2^{n-\beta}\Gamma(n - \beta + 1)}{(bm - a)^{n-\beta}} \left[\left(1 - \frac{1}{2^\alpha}\right) m^{n-\beta+1} (-1)^n ({}^C D_{(a+bm/2m)}^\beta \psi) \left(\frac{a}{m}\right) \right] \\ & \leq \frac{(n - \beta)}{2^\alpha(n + \alpha - \beta)} \left(1 - \frac{1}{2^\alpha}\right) \left\{ \begin{aligned} & \frac{[m\psi^{(n)}(b) + m^2\psi^{(n)}(a/m^2)] [2^\alpha(n + \alpha - \beta) - (n - \beta)]}{(n - \beta)} \\ & - \frac{mC[2^\alpha(2\alpha + n - \beta) - (\alpha + n - \beta)] (m(b - (a/m^2)))^2}{2^\alpha(2\alpha + n - \beta)} \end{aligned} \right\}. \end{aligned} \tag{34}$$

Again by using strongly (α, m) -convexity on the function $\psi^{(n)}$ for $z \in [0, 1]$, we have the following inequality:

$$\begin{aligned} \psi^{(n)}\left(a\frac{z}{2} + m\left(\frac{2-z}{2}\right)b\right) & \leq \left(\frac{z}{2}\right)^\alpha \psi^{(n)}(a) + m\left(1 - \left(\frac{z}{2}\right)^\alpha\right) \psi^{(n)}(b) \\ & \quad - mC\left(\frac{z}{2}\right)^\alpha \left(1 - \left(\frac{z}{2}\right)^\alpha\right) (b - a)^2. \end{aligned} \tag{35}$$

By multiplying (35) with $(1/2^\alpha)z^{n-\beta-1}$ on both sides and making integration over $[0, 1]$, we get

$$\begin{aligned} & \left(\frac{1}{2^\alpha}\right) \int_0^1 \psi^{(n)}\left(a\frac{z}{2} + m\left(\frac{2-z}{2}\right)b\right) z^{n-\beta-1} dz \leq \left(\frac{1}{2^\alpha}\right) \\ & \quad \times \left(\int_0^1 \left(\frac{z}{2}\right)^\alpha \psi^{(n)}(a) z^{n-\beta-1} dz + \int_0^1 m\left(1 - \left(\frac{z}{2}\right)^\alpha\right) \psi^{(n)}(b) z^{n-\beta-1} dz \right) \\ & \quad - \left(\frac{1}{2^\alpha}\right) \int_0^1 mC\left(\frac{z}{2}\right)^\alpha \left(1 - \left(\frac{z}{2}\right)^\alpha\right) (b - a)^2 z^{n-\beta-1} dz. \end{aligned} \tag{36}$$

By using change of variables and computing the last integral, from (36), we get

$$\frac{2}{2^\alpha (bm - a)} \left\{ \int_{(a+mb/2)}^{mb} \left(\frac{2(mb-x)}{bm-a} \right)^{n-\beta-1} \psi^{(n)}(x) dx \right\} \leq \frac{m\psi^{(n)}(b) + \psi^{(n)}(a)}{2^{2\alpha}(\alpha+n-\beta)} - \frac{mC[2^\alpha(2\alpha+n-\beta) - (\alpha+n-\beta)](b-a)^2}{2^{3\alpha}(\alpha+n-\beta)(2\alpha+n-\beta)} \tag{37}$$

Further, it takes the following form:

By adding (34) and (38), we get

$$\frac{2^{n-\beta}\Gamma(n-\beta+1)}{(bm-a)^{n-\beta}} \left[\left(\frac{1}{2^\alpha} \right) \left({}^C D_{(a+bm/2)^+}^\beta \psi \right) (mb) \right] \leq \frac{(n-\beta)}{2^{2\alpha}(n+\alpha-\beta)} \left\{ m\psi^{(n)}(b) + \psi^{(n)}(a) - \frac{mC[2^\alpha(2\alpha+n-\beta) - (\alpha+n-\beta)](b-a)^2}{2^\alpha(2\alpha+n-\beta)} \right\} \tag{38}$$

$$\frac{2^{n-\beta}\Gamma(n-\beta+1)}{(bm-a)^{n-\beta}} \left[\left(1 - \frac{1}{2^\alpha} \right) m^{n-\beta+1} (-1)^n \left({}^C D_{(a+bm/2m)^-}^\beta \psi \right) \left(\frac{a}{m} \right) + \left(\frac{1}{2^\alpha} \right) \left({}^C D_{(a+bm/2)^+}^\beta \psi \right) (mb) \right] \leq \frac{(n-\beta)}{2^\alpha(n+\alpha-\beta)} \left\{ \left(1 - \frac{1}{2^\alpha} \right) \frac{[m\psi^{(n)}(b) + m^2\psi^{(n)}(a/m^2)][2^\alpha(n+\alpha-\beta) - (n-\beta)]}{(n-\beta)} + \frac{m\psi^{(n)}(a) + \psi^{(n)}(a)}{2^\alpha} - \frac{mC[2^\alpha(2\alpha+n-\beta) - (\alpha+n-\beta)]}{2^\alpha(2\alpha+n-\beta)} \times \left[\left(1 - \frac{1}{2^\alpha} \right) \left(m \left(b - \frac{a}{m^2} \right)^2 \right) + \frac{(b-a)^2}{2^\alpha} \right] \right\} \tag{39}$$

From (30) and (39), (26) can be obtained. □

3. Error Bounds of Fractional Hadamard Inequalities

Remark 2. If $C = 0, m = 1,$ and $\alpha = 1$ in (26), then we get the fractional Hadamard inequality for convex function given in [7], Theorem 2.2.

In this section, we give refinements of the error bounds of fractional Hadamard inequalities for Caputo fractional derivatives.

Theorem 4. Let $\psi: [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) with $\psi \in C^{n+1}[a, b]$, $(\alpha, m) \in [0, 1]^2$, and $0 \leq a < mb, m \neq 0$. If $|\psi^{(n+1)}|$ is a strongly (α, m) -convex

function on $[a, b]$, then the following inequality for Caputo fractional derivatives holds:

$$\begin{aligned} & \left| \frac{\psi^{(n)}(a) + \psi^{(n)}(b)}{2} - \frac{\Gamma(n - \beta + 1)}{2(b - a)^{n - \beta}} \left[({}^C D_{a^+}^\beta \psi)(b) + (-1)^n ({}^C D_{b^-}^\beta \psi)(a) \right] \right| \\ & \leq |\psi^{(n+1)}(a)| \left(\left(\frac{1 - (1/2)^{np - \beta p + 1}}{np - \beta p + 1} \right)^{1/p} \left(\frac{1}{2^{q\alpha + 1}(q\alpha + 1)} \right)^{1/q} \right. \\ & \quad + \frac{2^{n - \beta + \alpha + 1} - 2}{2^{n - \beta + \alpha + 1}(n - \beta + \alpha + 1)} - \left. \left(\frac{(1/2)^{np - \beta p + 1}}{np - \beta p + 1} \right)^{1/p} \left(\frac{2^{q\alpha + 1} - 1}{2^{q\alpha + 1}(q\alpha + 1)} \right)^{1/q} \right) \\ & \quad + m \left| \psi^{(n+1)}\left(\frac{b}{m}\right) \right| \left(\frac{2 \left[2^{\alpha + 1} (2^{n - \beta} - 1)(n - \beta + \alpha + 1) + (n - \beta + 1)(1 - 2^{n - \beta + \alpha}) \right]}{2^{n - \beta + \alpha + 1}(n - \beta + 1)(n - \beta + \alpha + 1)} \right. \\ & \quad - \left. \left(\frac{1 - (1/2)^{np - \beta p + 1}}{np - \beta p + 1} \right)^{1/p} \left(\frac{1}{2^{q\alpha + 1}(q\alpha + 1)} \right)^{1/q} + \left(\frac{(1/2)^{np - \beta p + 1}}{np - \beta p + 1} \right)^{1/p} \left(\frac{2^{q\alpha + 1} - 1}{2^{q\alpha + 1}(q\alpha + 1)} \right)^{1/q} \right) \\ & \quad - Cm \left(\frac{b}{m} - a \right)^2 \left[\left(\frac{1 - (1/2)^{np - \beta p + 1}}{np - \beta p + 1} \right)^{1/p} \left[\left(\frac{(1/2)^{q\alpha + 1}}{q\alpha + 1} \right)^{1/q} - \left(\frac{(1/2)^{2q\alpha + 1}}{2q\alpha + 1} \right)^{1/q} \right] \right. \\ & \quad + \left. \left(\frac{(1/2)^{np - \beta p + 1}}{np - \beta p + 1} \right)^{1/p} \left[\left(\frac{2^{2q\alpha + 1} - 1}{2^{2q\alpha + 1}(2q\alpha + 1)} \right)^{1/q} - \left(\frac{2^{q\alpha + 1} - 1}{2^{q\alpha + 1}(q\alpha + 1)} \right)^{1/q} \right] \right. \\ & \quad \left. + \frac{2 \left[(n - \beta + \alpha + 1)(1 - 2^{n - \beta + 2\alpha}) + 2^\alpha (n - \beta + 2\alpha + 1)(2^{n - \beta + \alpha} - 1) \right]}{2^{n - \beta + 2\alpha + 1}(n - \beta + \alpha + 1)(n - \beta + 2\alpha + 1)} \right], \end{aligned} \tag{40}$$

with $\beta > 0$ and $n = [\beta] + 1$.

Proof. Since $|\psi^{(n+1)}|$ is strongly (α, m) -convex function on $[a, b]$ and $z \in [0, 1]$, we have

$$\begin{aligned} |\psi^{(n+1)}(za + (1 - z)b)| &= \left| \psi^{(n+1)}\left(za + m(1 - z)\frac{b}{m}\right) \right| \\ &\leq z^\alpha |\psi^{(n+1)}(a)| + m(1 - z^\alpha) \left| \psi^{(n+1)}\left(\frac{b}{m}\right) \right| - Cmz^\alpha (1 - z^\alpha) \left(\frac{b}{m} - a \right)^2. \end{aligned} \tag{41}$$

By using Lemma 1 and inequality (41), we have

$$\begin{aligned}
 & \left| \frac{\psi^{(n)}(a) + \psi^{(n)}(b)}{2} - \frac{\Gamma(n-\beta+1)}{2(b-a)^{n-\beta}} \left[({}^C D_{a^+}^\beta \psi)(b) + (-1)^n ({}^C D_{b^-}^\beta \psi)(a) \right] \right| \\
 & \leq \frac{b-a}{2} \int_0^1 (1-z)^{n-\beta} - z^{n-\beta} \left| \psi^{(n+1)} \left(za + m(1-z)\frac{b}{m} \right) \right| dz \\
 & \leq \frac{b-a}{2} \int_0^1 (1-z)^{n-\beta} - z^{n-\beta} \left(\left| z^\alpha |\psi^{(n+1)}(a)| + m(1-z^\alpha) \left| \psi^{(n+1)}\left(\frac{b}{m}\right) \right| \right. \right. \\
 & \quad \left. \left. - Cmz^\alpha (1-z^\alpha) \left(\frac{b}{m} - a\right)^2 \right) dz \right. \tag{42} \\
 & \leq \frac{b-a}{2} \left[\int_0^{1/2} ((1-z)^{n-\beta} - z^{n-\beta}) \left(\left| z^\alpha |\psi^{(n+1)}(a)| + m(1-z^\alpha) \left| \psi^{(n+1)}\left(\frac{b}{m}\right) \right| \right. \right. \right. \\
 & \quad \left. \left. - Cmz^\alpha (1-z^\alpha) \left(\frac{b}{m} - a\right)^2 \right) dz + \int_{1/2}^1 (z^{n-\beta} - (1-z)^{n-\beta}) \right. \\
 & \quad \left. \left(\left| z^\alpha |\psi^{(n+1)}(a)| + m(1-z^\alpha) \left| \psi^{(n+1)}\left(\frac{b}{m}\right) \right| - Cmz^\alpha (1-z^\alpha) \left(\frac{b}{m} - a\right)^2 \right) dz \right].
 \end{aligned}$$

In the following, we compute integrals appearing on the right hand side of inequality (42) by using Holder inequality:

$$\begin{aligned}
 & \int_0^{1/2} ((1-z)^{n-\beta} - z^{n-\beta}) \left(\left| z^\alpha |\psi^{(n+1)}(a)| + m(1-z^\alpha) \left| \psi^{(n+1)}\left(\frac{b}{m}\right) \right| \right. \right. \\
 & \quad \left. \left. - Cmz^\alpha (1-z^\alpha) \left(\frac{b}{m} - a\right)^2 \right) dz \\
 & = \left| \psi^{(n+1)}(a) \right| \int_0^{1/2} (z^\alpha (1-z)^{n-\beta} - z^{n-\beta+\alpha}) dz \\
 & \quad + m \left| \psi^{(n+1)}\left(\frac{b}{m}\right) \right| \int_0^{1/2} ((1-z)^{n-\beta} (1-z^\alpha) - (1-z^\alpha) z^{n-\beta}) dz \\
 & \quad - Cm \left(\frac{b}{m} - a\right)^2 \left(\int_0^{1/2} (z^\alpha - z^{2\alpha}) (1-z)^{n-\beta} dz - \int_0^{1/2} z^{n-\beta+\alpha} (1-z^\alpha) dz \right) \\
 & = \left| \psi^{(n+1)}(a) \right| \left[\left(\frac{1 - (1/2)^{np-\beta p+1}}{np - \beta p + 1} \right)^{1/p} \left(\frac{1}{2^{q\alpha+1} (q\alpha + 1)} \right)^{1/q} - \frac{(1/2)^{n-\beta+\alpha+1}}{n-\beta+\alpha+1} \right] \\
 & \quad + m \left| \psi^{(n+1)}\left(\frac{b}{m}\right) \right| \left[\frac{2^{\alpha+1} (2^{n-\beta} - 1) (n-\beta + \alpha + 1) + (n-\beta + 1)}{2^{n-\beta+\alpha+1} (n-\beta + 1) (n-\beta + \alpha + 1)} \right. \\
 & \quad \left. - \left(\frac{1 - (1/2)^{np-\beta p+1}}{np - \beta p + 1} \right)^{1/p} \left(\frac{1}{2^{q\alpha+1} (q\alpha + 1)} \right)^{1/q} \right] \\
 & \quad - Cm \left(\frac{b}{m} - a\right)^2 \left[\frac{(n-\beta + \alpha + 1) - 2^\alpha (n-\beta + 2\alpha + 1)}{2^{n-\beta+2\alpha+1} (n-\beta + \alpha + 1) (n-\beta + 2\alpha + 1)} \right. \\
 & \quad \left. + \left(\frac{1 - (1/2)^{np-\beta p+1}}{np - \beta p + 1} \right)^{1/p} \left[\left(\frac{(1/2)^{q\alpha+1}}{q\alpha + 1} \right)^{1/q} - \left(\frac{(1/2)^{2q\alpha+1}}{2q\alpha + 1} \right)^{1/q} \right] \right],
 \end{aligned}$$

$$\begin{aligned}
 & \int_{1/2}^1 (z^{n-\beta} - (1-z)^{n-\beta}) \left(z^\alpha |\psi^{(n+1)}(a)| + m(1-z^\alpha) \left| \psi^{(n+1)}\left(\frac{b}{m}\right) \right| - Cmz^\alpha(1-z^\alpha) \left(\frac{b}{m} - a\right)^2 \right) dz \\
 &= |\psi^{(n+1)}(a)| \int_{1/2}^1 (z^{n-\beta+\alpha} - z^\alpha(1-z)^{n-\beta}) dz \\
 & \left| \psi^{(n+1)}\left(\frac{b}{m}\right) \right| \int_{1/2}^1 ((1-z^\alpha)z^{n-\beta} - (1-z^\alpha)(1-z)^{n-\beta}) dz \\
 & \quad - Cm \left(\frac{b}{m} - a\right)^2 \left(\int_{1/2}^1 (1-z^\alpha)z^{n-\beta+\alpha} dz - \int_{1/2}^1 (z^\alpha - z^{2\alpha})(1-z)^{n-\beta} dz \right) \\
 &= |\psi^{(n+1)}(a)| \left(\frac{2^{n-\beta+\alpha+1} - 1}{2^{n-\beta+\alpha+1}(n-\beta+\alpha+1)} - \left(\frac{(1/2)^{np-\beta p+1}}{np-\beta p+1}\right)^{1/p} \left(\frac{2^{q\alpha+1} - 1}{2^{q\alpha+1}(q\alpha+1)}\right)^{1/q} \right) \\
 & \quad + m \left| \psi^{(n+1)}\left(\frac{b}{m}\right) \right| \left(\frac{\left(\frac{(1/2)^{np-\beta p+1}}{np-\beta p+1}\right)^{1/p} \left(\frac{2^{q\alpha+1} - 1}{2^{q\alpha+1}(q\alpha+1)}\right)^{1/q}}{+ \frac{2^{\alpha+1}(2^{n-\beta} - 1)(n-\beta+\alpha+1) - (n-\beta+1)(2^{n-\beta+\alpha+1} - 1)}{2^{n-\beta+\alpha+1}(n-\beta+1)(n-\beta+\alpha+1)}} \right) \\
 & \quad - Cm \left(\frac{b}{m} - a\right)^2 \left[\frac{2^\alpha(n-\beta+2\alpha+1)(2^{n-\beta+\alpha+1} - 1) - (n-\beta+\alpha+1)(2^{n-\beta+2\alpha+1} - 1)}{2^{n-\beta+2\alpha+1}(n-\beta+\alpha+1)(n-\beta+2\alpha+1)} \right. \\
 & \quad \left. + \left(\frac{(1/2)^{np-\beta p+1}}{np-\beta p+1}\right)^{1/p} \left[\left(\frac{2^{2q\alpha+1} - 1}{2^{2q\alpha+1}(2q\alpha+1)}\right)^{1/q} - \left(\frac{2^{q\alpha+1} - 1}{2^{q\alpha+1}(q\alpha+1)}\right)^{1/q} \right] \right].
 \end{aligned} \tag{44}$$

By putting the values of (43) and (44) in (42), we get (40).

By using Lemma 2, we give the following error bounds of Caputo fractional derivative inequality (26). \square

Theorem 5. Let $\psi: [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) with $\psi \in C^{n+1}[a, b]$, $(\alpha, m) \in [0, 1]^2$, and $0 \leq a < mb, m \neq 0$. If $|\psi^{(n+1)}|^q$ is strongly (α, m) -convex function on $[a, b]$ for $q \geq 1$, then the following inequality for Caputo fractional derivatives holds:

$$\begin{aligned}
 & \left| \frac{2^{n-\beta-1}\Gamma(n-\beta+1)}{(mb-a)^{n-\beta}} \left[\left({}^C D_{(a+bm/2)^+}^\beta \psi\right)(mb) + m^{n-\beta+1}(-1)^n \left({}^C D_{(a+mb/2m)^-}^\beta \psi\right)\left(\frac{a}{m}\right) \right] \right. \\
 & \quad \left. - \frac{1}{2} \left[\psi^{(n)}\left(\frac{a+mb}{2}\right) + m\psi^{(n)}\left(\frac{a+mb}{2m}\right) \right] \right| \leq \frac{mb-a}{4(n-\beta+1)^{1/p}} \left(\frac{1}{n-\beta+\alpha+1}\right)^{1/q} \\
 & \quad \times \left\{ \left[\frac{|\psi^{(n+1)}(a)|^q}{2^\alpha} + \frac{m|\psi^{(n+1)}(b)|^q (2^\alpha(n-\beta+\alpha+1) - (n-\beta+1))}{2^\alpha(n-\beta+1)} - \frac{Cm(b-a)^2 (2^\alpha(n-\beta+2\alpha+1) - (n-\beta+\alpha+1))}{4(n-\beta+2\alpha+1)} \right]^{1/q} \right. \\
 & \quad \left. + \left[\frac{m|\psi^{(n+1)}(a/m^2)|^q (2^\alpha(n-\beta+\alpha+1) - (n-\beta+1))}{2^\alpha(n-\beta+1)} \right. \right. \\
 & \quad \left. \left. + \frac{|\psi^{(n+1)}(b)|^q}{2^\alpha} - \frac{Cm(b-(a/m^2))^2 (2^\alpha(n-\beta+2\alpha+1) - (n-\beta+\alpha+1))}{4(n-\beta+2\alpha+1)} \right]^{1/q} \right\},
 \end{aligned} \tag{45}$$

with $\beta > 0$ and $n = [\beta] + 1$.

Proof. By taking $k = 1$ in Lemma 2 and using power mean inequality, we have

$$\begin{aligned} & \left| \frac{2^{n-\beta-1} \Gamma(n-\beta+1)}{(mb-a)^{n-\beta}} \left[\left({}^C D_{(a+bm/2)^+}^\beta \psi \right) (mb) + m^{n-\beta+1} (-1)^n \left({}^C D_{(a+bm/2m)^-}^\beta \psi \right) \left(\frac{a}{m} \right) \right] \right. \\ & \quad \left. - \frac{1}{2} \left[\psi^{(n)} \left(\frac{a+mb}{2} \right) + m \psi^{(n)} \left(\frac{a+mb}{2m} \right) \right] \right| \\ & \leq \frac{mb-a}{4} \left[\int_0^1 z^{n-\beta} \left(\left| \psi^{(n+1)} \left(\frac{z}{2} a + m \left(\frac{2-z}{2} \right) b \right) \right| + \left| \psi^{(n+1)} \left(\frac{2-z}{2m} a + \frac{z}{2} b \right) \right| \right) dz \right] \\ & \leq \frac{mb-a}{4} \left(\int_0^1 z^{n-\beta} dz \right)^{1-(1/q)} \left[\left(\int_0^1 z^{n-\beta} \left| \psi^{(n+1)} \left(a \frac{z}{2} + m \left(\frac{2-z}{2} \right) b \right) \right|^q dz \right)^{1/q} \right. \\ & \quad \left. + \left(\int_0^1 z^{n-\beta} \left| \psi^{(n+1)} \left(a \left(\frac{2-z}{2m} \right) + b \frac{z}{2} \right) \right|^q dz \right)^{1/q} \right]. \end{aligned} \tag{46}$$

Now, by using the strongly (α, m) -convexity of $|\psi^{(n+1)}|^q$, we have the last expression

$$\begin{aligned} & \leq \frac{mb-a}{4(n-\beta+1)^{(1/p)}} \left[\left(|\psi^{(n+1)}(a)|^q \int_0^1 \frac{z^{n-\beta+\alpha}}{2^\alpha} dz + m |\psi^{(n+1)}(b)|^q \int_0^1 \frac{z^{n-\beta} (2^\alpha - z^\alpha)}{2^\alpha} dz \right. \right. \\ & \quad \left. - \frac{Cm(b-a)^2}{4} \int_0^1 (2^\alpha z^{n-\beta+\alpha} - z^{n-\beta+2\alpha}) dz \right)^{1/q} + \left(m \left| \psi^{(n+1)} \left(\frac{a}{m^2} \right) \right|^q \int_0^1 \frac{z^{n-\beta} (2^\alpha - z^\alpha)}{2^\alpha} dz \right. \\ & \quad \left. + |\psi^{(n+1)}(b)|^q \int_0^1 \frac{z^{n-\beta+\alpha}}{2^\alpha} dz - \frac{Cm(b-(a/m^2))^2}{4} \int_0^1 (2^\alpha z^{n-\beta+\alpha} - z^{n-\beta+2\alpha}) dz \right)^{(1/q)} \Big] \\ & = \frac{mb-a}{4(n-\beta+1)^{(1/p)}} \left[\left(\frac{|\psi^{(n+1)}(a)|^q}{2^\alpha(n-\beta+\alpha+1)} + \frac{m |\psi^{(n+1)}(b)|^q (2^\alpha(n-\beta+\alpha+1) - (n-\beta+1))}{2^\alpha(n-\beta+1)(n-\beta+\alpha+1)} \right. \right. \\ & \quad \left. - \frac{Cm(b-a)^2 (2^\alpha(n-\beta+2\alpha+1) - (n-\beta+\alpha+1))^{1/q}}{4(n-\beta+\alpha+1)(n-\beta+2\alpha+1)} \right)^{1/q} \\ & \quad + \left(\frac{m |\psi^{(n+1)}(a/m^2)|^q (2^\alpha(n-\beta+\alpha+1) - (n-\beta+1))}{2^\alpha(n-\beta+1)(n-\beta+\alpha+1)} \right. \\ & \quad \left. + \frac{|\psi^{(n+1)}(b)|^q}{2^\alpha(n-\beta+\alpha+1)} - \frac{Cm(b-(a/m^2))^2 (2^\alpha(n-\beta+2\alpha+1) - (n-\beta+\alpha+1))^{1/q}}{4(n-\beta+\alpha+1)(n-\beta+2\alpha+1)} \right)^{1/q} \Big], \end{aligned} \tag{47}$$

and inequality (45) is obtained. □

Remark 3. If $C = 0, m = 1,$ and $\alpha = 1$ in (45), then we get the fractional Hadamard inequality for convex function given in [7], Theorem 3.2.

Theorem 6. Let $\psi: [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) with $\psi \in C^{n+1}[a, b], (\alpha, m) \in [0, 1]^2,$ and $0 \leq a < mb, m \neq 0.$ If $|\psi^{(n+1)}|^q$ is a strongly (α, m) -convex function on $[a, b]$ for $q > 1,$ then the following inequality for Caputo fractional derivatives holds:

$$\begin{aligned} & \left| \frac{2^{n-\beta-1} \Gamma(n-\beta+1)}{(mb-a)^{n-\beta}} \left[\left({}^C D_{(a+bm/2)^+}^\beta \psi \right)(mb) + m^{n-\beta+1} (-1)^n \left({}^C D_{(a+mb/2m)^-}^\beta \psi \right) \left(\frac{a}{m} \right) \right] \right. \\ & \quad \left. - \frac{1}{2} \left[\psi^{(n)} \left(\frac{a+mb}{2} \right) + m \psi^{(n)} \left(\frac{a+mb}{2m} \right) \right] \right| \leq \frac{bm-a}{4(np-\beta p+1)^{1/p} (2^\alpha(\alpha+1))^{1/q}} \\ & \quad \times \left[\left(\left| \psi^{(n+1)}(a) \right| + m^{1/q} \left| \psi^{(n+1)}(b) \right| \right) (2^\alpha(\alpha+1)-1)^{1/q} \right]^q \\ & \quad - \frac{Cm(b-a)^2 (2^\alpha(2\alpha+1) - (\alpha+1))^{1/q}}{2^\alpha(2\alpha+1)} \\ & \quad + \left(\left(m^{1/q} \left| \psi^{(n+1)} \left(\frac{a}{m^2} \right) \right| \right) (2^\alpha(\alpha+1)-1)^{1/q} + \left| \psi^{(n+1)}(b) \right| \right)^q \\ & \quad \left. - \frac{Cm(b-(a/m^2))^2 (2^\alpha(2\alpha+1) - (\alpha+1))^{1/q}}{2^\alpha(2\alpha+1)} \right], \end{aligned} \tag{48}$$

with $\beta > 0$ and $n = [\beta] + 1.$

Proof. By taking $k = 1$ in Lemma 2 and with the help of modulus property, we get

$$\begin{aligned} & \left| \frac{2^{n-\beta-1} \Gamma(n-\beta+1)}{(mb-a)^{n-\beta}} \left[\left({}^C D_{(a+bm/2)^+}^\beta \psi \right)(mb) + m^{n-\beta+1} (-1)^n \left({}^C D_{(a+mb/2)^-}^\beta \psi \right) \left(\frac{a}{m} \right) \right] \right. \\ & \quad \left. - \frac{1}{2} \left[\psi^{(n)} \left(\frac{a+mb}{2} \right) + m \psi^{(n)} \left(\frac{a+mb}{2m} \right) \right] \right| \\ & \leq \frac{mb-a}{4} \left[\int_0^1 z^{n-\beta} \left| \psi^{(n+1)} \left(\frac{z}{2} a + m \frac{(2-z)}{2} b \right) \right| dz + \int_0^1 z^{n-\beta} \left| \psi^{(n+1)} \left(\frac{2-z}{2m} a + \frac{z}{2} b \right) \right| dz \right]. \end{aligned} \tag{49}$$

Now, applying Holder's inequality, we get

$$\begin{aligned} & \left| \frac{2^{n-\beta-1} \Gamma(n-\beta+1)}{(mb-a)^{n-\beta}} \left[\left({}^C D_{(a+bm/2)^+}^\beta \psi \right)(mb) + m^{n-\beta+1} (-1)^n \left({}^C D_{(a+mb/2)^-}^\beta \psi \right) \left(\frac{a}{m} \right) \right] \right. \\ & \quad \left. - \frac{1}{2} \left[\psi^{(n)} \left(\frac{a+mb}{2} \right) + m \psi^{(n)} \left(\frac{a+mb}{2m} \right) \right] \right| \\ & \leq \frac{mb-a}{4} \left(\frac{1}{(np-\beta p+1)} \right)^{(1/p)} \left[\left(\int_0^1 \left| \psi^{(n+1)} \left(\frac{z}{2} a + m \frac{(2-z)}{2} b \right) \right| dz \right)^{(1/q)} \right. \\ & \quad \left. + \left(\int_0^1 \left| \psi^{(n+1)} \left(\frac{2-z}{2m} a + \frac{z}{2} b \right) \right| dz \right)^{(1/q)} \right]. \end{aligned} \tag{50}$$

Using strong (α, m) -convexity of $|\psi^{(n+1)}|^q$, we get

$$\begin{aligned}
 & \left| \frac{2^{n-\beta-1}\Gamma(n-\beta+1)}{(mb-a)^{n-\beta}} \left[\left({}^C D_{(a+bm/2)^+}^\beta \psi \right) (mb) + m^{n-\beta+1} (-1)^n \left({}^C D_{(a+mb/2m)^-}^\beta \psi \right) \left(\frac{a}{m} \right) \right] \right. \\
 & \quad \left. - \frac{1}{2} \left[\psi^{(n)} \left(\frac{a+mb}{2} \right) + m\psi^{(n)} \left(\frac{a+mb}{2m} \right) \right] \right| \\
 & \leq \frac{mb-a}{4} \left(\frac{1}{np-\beta p+1} \right)^{1/p} \left[\left(|\psi^{(n+1)}(a)|^q \int_0^1 \left(\frac{z}{2} \right)^\alpha dz + m |\psi^{(n+1)}(b)|^q \int_0^1 \left(1 - \left(\frac{z}{2} \right)^\alpha \right) dz \right. \right. \\
 & \quad \left. - \frac{Cm(b-a)^2}{2^{2\alpha}} \int_0^1 (2^\alpha z^\alpha - z^{2\alpha}) dz \right)^{1/q} + \left(m \left| \psi^{(n+1)} \left(\frac{a}{m^2} \right) \right|^q \int_0^1 \left(1 - \left(\frac{z}{2} \right)^\alpha \right) dz \right. \\
 & \quad \left. + |\psi^{(n+1)}(b)|^q \int_0^1 \left(\frac{z}{2} \right)^\alpha dz - \frac{Cm(b-(a/m^2))^2}{2^{2\alpha}} \int_0^1 (2^\alpha z^\alpha - z^{2\alpha}) dz \right)^{1/q} \Big] \\
 & = \frac{mb-a}{4} \left(\frac{1}{np-\beta p+1} \right)^{1/p} \left[\left(\frac{|\psi^{(n+1)}(a)|^q + m |\psi^{(n+1)}(b)|^q (2^\alpha(\alpha+1) - 1)}{2^\alpha(\alpha+1)} \right. \right. \\
 & \quad \left. - \frac{Cm(b-a)^2 (2^\alpha(2\alpha+1) - (\alpha+1))}{2^{2\alpha}(\alpha+1)(2\alpha+1)} \right)^{1/q} \\
 & \quad + \left(\frac{m \left| \psi^{(n+1)} \left(\frac{a}{m^2} \right) \right|^q (2^\alpha(\alpha+1) - 1) + |\psi^{(n+1)}(b)|^q}{2^\alpha(\alpha+1)} \right. \\
 & \quad \left. - \frac{Cm(b-(a/m^2))^2 (2^\alpha(2\alpha+1) - (\alpha+1))}{2^{2\alpha}(\alpha+1)(2\alpha+1)} \right)^{1/q} \Big] \\
 & \leq \frac{bm-a}{4} \left(\frac{1}{np-\beta p+1} \right)^{1/p} \left[\left(\frac{\left(|\psi^{(n+1)}(a)| + m^{1/q} |\psi^{(n+1)}(b)| (2^\alpha(\alpha+1) - 1)^{1/q} \right)^q}{2^\alpha(\alpha+1)} \right. \right. \\
 & \quad \left. - \frac{Cm(b-a)^2 (2^\alpha(2\alpha+1) - (\alpha+1))}{2^{2\alpha}(\alpha+1)(2\alpha+1)} \right)^{1/q} \\
 & \quad + \left(\frac{\left(m^{1/q} |\psi^{(n+1)} \left(\frac{a}{m^2} \right) \right| (2^\alpha(\alpha+1) - 1)^{1/q} + |\psi^{(n+1)}(b)| \right)^q}{2^\alpha(\alpha+1)} \right. \\
 & \quad \left. - \frac{Cm(b-(a/m^2))^2 (2^\alpha(2\alpha+1) - (\alpha+1))}{2^{2\alpha}(\alpha+1)(2\alpha+1)} \right)^{1/q} \Big].
 \end{aligned} \tag{51}$$

Here, we have used the fact that $(a_1 + b_1)^q \geq (a_1)^q + (b_1)^q$, where $q > 1$, $a_1, b_1 \geq 0$. This completes the proof. \square

Remark 4. If $C = 0$, $m = 1$, and $\alpha = 1$ in (48), then we get the fractional Hadamard inequality for convex function given in [7], Theorem 3.3.

Data Availability

No data were used in this paper.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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