Research Article

Existence and Uniqueness of Weak Solutions for a New Class of Convex Optimization Problems Related to Image Analysis

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Received 24 December 2020; Revised 20 March 2021; Accepted 25 March 2021; Published 21 April 2021

Academic Editor: Jen-Chih Yao

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This paper proposes a new anisotropic diffusion model in image restoration that is understood from a variational optimization of an energy functional. Initially, a family of new diffusion functions based on cubic Hermite spline is provided for optimal image denoising. After that, the existence and uniqueness of weak solutions for the corresponding Euler–Lagrange equation are proven in an appropriate functional space, and a consistent and stable numerical model is also shown. We complement this work by illustrating some experiments on different actual brain Magnetic Resonance Imaging (MRI) scans, showing the proposed model's impressive results.

1. Introduction

During the last few decades, the digital image has been significantly utilized as a noninvasive medical diagnosis or treatment tool. For instance, a different method has been developed in the electroencephalogram (EEG) (see [1]). Since a digital image contains relevant information that should be carefully extracted from it, many engineers and mathematicians have established numerous techniques and theories [2–4] to collect quantitative or qualitative data. At that point, by analyzing these reliable data and understanding its contents, one can make the right decisions at the diagnosis time.

Along with the meaningful details, digital images carry false information as noise and other undesirable artifacts due to various reasons. This erroneous information can occur during image formation, transmission, or recording processes. Therefore, one needs good models to identify and eliminate these noises while preserving relevant information and structures.

Image denoising is a necessary preprocessing step for many applications that rely on image quality, such as image

segmentation and pattern recognition. Hence, how to restore a degraded image to its original form? Many approaches have been developed to process images and showed great interest in using the variational approach [5–15], which attempts to establish mathematical models strictly related to the physical world by defining a diffusion equation from a particular optimization problem, which generally has the following form:

$$\min_u \left\{ F(u) := \int_{\Omega} \left(\varphi(|\nabla u|) + \frac{\rho}{2} (u - u_0)^2 \right) d\Omega \right\}, \quad (1)$$

where $\Omega \subset \mathbb{R}^2$ is an open bounded domain, u_0 and u are the observed and the reconstructed images defined as functions of $\Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ that associate each pixel $x \in \Omega$ to the gray level $u(x)$ or $u_0(x)$, $\varphi: \mathbb{R}^+ \rightarrow \mathbb{R}$ is a nonnegative increasing function with $\varphi(0) = 0$, and $\rho \in \mathbb{R}_+$ is a weighting parameter that enables to adjust the influence of the data term in the regularizing term.

One common approach to solve the problem (1) is by seeking the steady-state solution of the following PDE, which corresponds to the Euler–Lagrange equation of the energy functional $F(u)$:

$$\frac{\partial u}{\partial t} = \nabla \cdot \left[\frac{\varphi'(|\nabla u|)}{|\nabla u|} \nabla u \right] - \rho(u - u_0). \quad (2)$$

Perona and Malik (Perona–Malik) [16] were the first to propose such a model in image processing, which corresponds to nonconvex energy functional. Unfortunately, this property leads to multiple solutions with staircase effects observed in practice [6, 17]. In [8], Charbonnier et al. suggested a strictly convex energy functional to circumvent the Perona–Malik model's ill-posedness. Besides, many variational approaches have also been developed during the last thirty years. The most famous one was the total variation denoising suggested by Rudin et al. in their seminal paper [5]. Several improved nonlinear diffusion models for image denoising derived from Perona–Malik or TV models have been proposed in the last twenty years; for more detailed information, we refer to [2, 10–15, 18] and the references therein.

Nevertheless, because of the above model's isotropy, the diffusion equation (2) is controlled by the function $s \rightarrow (\varphi'(s)/s)$ and produces the same amount of blurring in all its directions. This means that the process cannot successfully eliminate noises at the edges. Thus, it would be important to consider smoothing along edges by adopting anisotropic diffusion [19].

Furthermore, the type (1) model uses the gradient magnitude as a local descriptor operator for the image edge detector. However, digital images present some difficulties in their discrete structure; they are discrete in space and discrete in intensity value. Consequently, one may need to adapt to the digital image structure by considering differential operators that respond to vertical, horizontal, and diagonal edges and using a consistent approximation. Therefore, motivated by the above reasoning and inspired by Weickert's anisotropic idea [19], we conceived new convex optimization problems that lead to anisotropic diffusion equations with a novel matrix diffusion tensor.

Therefore, this paper provides a new variational PDE model based on directional edge detectors in Section 2. The construction of a new diffusion function using cubic Hermite spline and the existence and uniqueness of weak solutions are illustrated in Section 3. Next, Section 4 implements a consistent explicit symmetric finite difference approximation that involves the 3×3 neighborhoods at each point and shows the experiments on different real images. We terminate this work with a conclusion in Section 5.

2. Novel Anisotropic Diffusion Model for Image Restoration

Let us now consider the rectangular domain $\Omega := (0, a) \times (0, b) \subset \mathbb{R}^2$ with $\Gamma := \partial\Omega$ being its boundary. This approach's general idea is to search for a solution among the minima of

some functional energy. The proposed functional energy has a regularizing term that depends on the combination of a function φ and local structures of the image expressed by the four directional derivatives:

$$\begin{cases} \partial_{x_1} u := u_{x_1} := \nabla u \cdot e_1, \\ \partial_{x_2} u := u_{x_2} := \nabla u \cdot e_2, \\ \partial_{x_{12}} u := u_{x_{12}} := \nabla u \cdot \frac{e_1 + e_2}{|e_1 + e_2|}, \\ \partial_{x_{-12}} u := u_{x_{-12}} := \nabla u \cdot \frac{-e_1 + e_2}{|-e_1 + e_2|}, \end{cases} \quad (3)$$

where (e_1, e_2) is the canonical basis of $\mathbb{R}^2$. Hence, our problem consists of solving the following energy minimization problem:

$$\min_{u \in U} E(u), \quad (4)$$

such that

$$\begin{aligned} E(u) &= \int_{\Omega} \left(\left[\varphi(|u_{x_1}|) + \varphi(|u_{x_2}|) + \varphi(|u_{x_{12}}|) + \varphi(|u_{x_{-12}}|) \right] \right. \\ &\quad \left. + \frac{\rho}{2}(u - u_0)^2 \right) dx \\ &= \int_{\Omega} L(x, u, u_{x_1}, u_{x_2}) dx, \end{aligned} \quad (5)$$

under the following assumptions:

$$(\mathbf{H}_1) \begin{cases} \varphi: \mathbb{R}_+ \rightarrow \mathbb{R} \text{ is a } \mathcal{C}^2 \text{ - function } / \varphi(0) = \varphi'(0) = 0, \\ \varphi(s) > 0, \varphi'(s) > 0, \quad s > 0, \\ \varphi''(s) > 0, \quad s \geq 0, \\ \lim_{s \rightarrow +\infty} \frac{\varphi(s)}{s} = +\infty, \quad \rho \in \mathbb{R}_+^*. \end{cases} \quad (6)$$

The Euler–Lagrange equation of $E(u)$ (5) is obtained from

$$\frac{\partial L}{\partial u} - \partial_{x_1} \left(\frac{\partial L}{\partial u_{x_1}} \right) - \partial_{x_2} \left(\frac{\partial L}{\partial u_{x_2}} \right) = 0, \quad (7)$$

which leads to

$$\begin{aligned} \rho(u - u_0) - \left[\partial_{x_1} \left(\frac{\varphi'(|u_{x_1}|)}{|u_{x_1}|} u_{x_1} + \frac{\varphi'(|u_{x_{12}}|)}{\sqrt{2}|u_{x_{12}}|} u_{x_{12}} - \frac{\varphi'(|u_{x_{-12}}|)}{\sqrt{2}|u_{x_{-12}}|} u_{x_{-12}} \right) \right. \\ \left. + \partial_{x_2} \left(\frac{\varphi'(|u_{x_2}|)}{|u_{x_2}|} u_{x_2} + \frac{\varphi'(|u_{x_{12}}|)}{\sqrt{2}|u_{x_{12}}|} u_{x_{12}} + \frac{\varphi'(|u_{x_{-12}}|)}{\sqrt{2}|u_{x_{-12}}|} u_{x_{-12}} \right) \right] = 0. \end{aligned} \quad (8)$$

This is equivalent to

$$\begin{aligned} \rho(u - u_0) - \left[\partial_{x_1} \left(\frac{\varphi'(|u_{x_1}|)}{|u_{x_1}|} u_{x_1} \right) + \partial_{x_2} \left(\frac{\varphi'(|u_{x_2}|)}{|u_{x_2}|} u_{x_2} \right) \right. \\ \left. + \partial_{x_{12}} \left(\frac{\varphi'(|u_{x_{12}}|)}{|u_{x_{12}}|} u_{x_{12}} \right) + \partial_{x_{-12}} \left(\frac{\varphi'(|u_{x_{-12}}|)}{|u_{x_{-12}}|} u_{x_{-12}} \right) \right] = 0. \end{aligned} \quad (9)$$

Let $g: \mathbb{R}_+ \rightarrow \mathbb{R}$ be a positive function such that $g(s) = (\varphi'(s)/s)$. We may reformulate equation (9) and present it as follows.

Proposition 1. *The Euler-Lagrange equation (9) is equivalent to*

$$\rho(u - u_0) - \nabla \cdot [\mathbf{D}_{\nabla u} \nabla u] = 0, \quad (10)$$

where $\mathbf{D}_{\nabla u}$ is a real symmetric positive definite matrix of $\mathbb{R}^{2 \times 2}$ defined as follows:

$$\mathbf{D}_{\nabla u} = \begin{pmatrix} g(|u_{x_1}|) + \frac{g(|u_{x_{12}}|) + g(|u_{x_{-12}}|)}{2} & \frac{g(|u_{x_{12}}|) - g(|u_{x_{-12}}|)}{2} \\ \frac{g(|u_{x_{12}}|) - g(|u_{x_{-12}}|)}{2} & g(|u_{x_2}|) + \frac{g(|u_{x_{12}}|) + g(|u_{x_{-12}}|)}{2} \end{pmatrix}. \quad (11)$$

Proof. First, it is clear that

$$\partial_{x_1} (g(|u_{x_1}|) u_{x_1}) + \partial_{x_2} (g(|u_{x_2}|) u_{x_2}) = \nabla \cdot \left[\begin{pmatrix} g(|u_{x_1}|) & 0 \\ 0 & g(|u_{x_2}|) \end{pmatrix} \begin{pmatrix} u_{x_1} \\ u_{x_2} \end{pmatrix} \right]. \quad (12)$$

On the other hand, we have

$$\begin{aligned}
& \partial_{x_{12}} \left(g(|u_{x_{12}}|) u_{x_{12}} \right) + \partial_{x_{-12}} \left(g(|u_{x_{-12}}|) u_{x_{-12}} \right) \\
&= \nabla \left(g(|u_{x_{12}}|) \nabla u \cdot \frac{e_1 + e_2}{|e_1 + e_2|} \right) \cdot \frac{e_1 + e_2}{|e_1 + e_2|} \\
&\quad + \nabla \left(g(|u_{x_{-12}}|) \nabla u \cdot \frac{-e_1 + e_2}{|-e_1 + e_2|} \right) \cdot \frac{-e_1 + e_2}{|-e_1 + e_2|} \\
&= \frac{1}{2} \left[\partial_{x_1} \left(g(|u_{x_{12}}|) (u_{x_1} + u_{x_2}) \right) + \partial_{x_2} \left(g(|u_{x_{12}}|) (u_{x_1} + u_{x_2}) \right) \right. \\
&\quad \left. - \partial_{x_1} \left(g(|u_{x_{-12}}|) (-u_{x_1} + u_{x_2}) \right) + \partial_{x_2} \left(g(|u_{x_{-12}}|) (-u_{x_1} + u_{x_2}) \right) \right] \\
&= \frac{1}{2} \left[\partial_{x_1} \left((g(|u_{x_{12}}|) + g(|u_{x_{-12}}|)) u_{x_1} \right) + \partial_{x_2} \left((g(|u_{x_{12}}|) + g(|u_{x_{-12}}|)) u_{x_2} \right) \right. \\
&\quad \left. + \partial_{x_2} \left((g(|u_{x_{12}}|) - g(|u_{x_{-12}}|)) u_{x_1} \right) + \partial_{x_1} \left((g(|u_{x_{12}}|) - g(|u_{x_{-12}}|)) u_{x_2} \right) \right] \\
&= \frac{1}{2} \nabla \cdot \left[\left((g(|u_{x_{12}}|) + g(|u_{x_{-12}}|)) \begin{pmatrix} u_{x_1} \\ u_{x_2} \end{pmatrix} + (g(|u_{x_{12}}|) - g(|u_{x_{-12}}|)) \begin{pmatrix} u_{x_2} \\ u_{x_1} \end{pmatrix} \right) \right] \\
&= \frac{1}{2} \nabla \cdot \left[\begin{pmatrix} (g(|u_{x_{12}}|) + g(|u_{x_{-12}}|)) & (g(|u_{x_{12}}|) - g(|u_{x_{-12}}|)) \\ (g(|u_{x_{12}}|) - g(|u_{x_{-12}}|)) & (g(|u_{x_{12}}|) + g(|u_{x_{-12}}|)) \end{pmatrix} \begin{pmatrix} u_{x_1} \\ u_{x_2} \end{pmatrix} \right].
\end{aligned} \tag{13}$$

Besides, the matrix $\mathbf{D}_{\nabla u}$ has two eigenvalues $\lambda_{\pm}$ as follows:

$$\lambda_{\pm} = \frac{1}{2} \left(g(|u_{x_1}|) + g(|u_{x_2}|) + g(|u_{x_{12}}|) + g(|u_{x_{-12}}|) \pm \sqrt{(g(|u_{x_1}|) - g(|u_{x_2}|))^2 + (g(|u_{x_{12}}|) - g(|u_{x_{-12}}|))^2} \right). \tag{14}$$

Since $g > 0$ and

$$\lambda_- = \frac{2(g(|u_{x_1}|)g(|u_{x_2}|) + g(|u_{x_{12}}|)g(|u_{x_{-12}}|)) + (g(|u_{x_1}|) + g(|u_{x_2}|))(g(|u_{x_{12}}|) + g(|u_{x_{-12}}|))}{2\lambda_+}, \tag{15}$$

then we deduce that the matrix $\mathbf{D}_{\nabla u}$ is symmetric positive definite, which completes the proof. $\square$

By applying the steepest descent method, the Euler-Lagrange equation (10) can be solved considering the steady state of the following evolution equation:

$$\begin{cases} \frac{\partial u}{\partial t} = \nabla \cdot [\mathbf{D}_{\nabla u} \nabla u] - \rho(u - u_0), & \text{in } \Omega \times (0, T], \\ \langle \mathbf{D}_{\nabla u} \nabla, \mathbf{n} \rangle = 0, & \text{on } \partial\Omega \times (0, T], \\ u(x; 0) = u_0(x), & \text{in } \Omega. \end{cases} \tag{16}$$

In [20], the existence and uniqueness of weak solutions for problem (16) were proven if the weighting parameter $\rho = 0$.

3. Mathematical Investigation of the Proposed Model

3.1. Description of the Proposed Model. Our model's main objective is to allow strong directional smoothing within the

areas where $|u_{x_1}|$, $|u_{x_2}|$, $|u_{x_{12}}|$, or $|u_{x_{-12}}|$ is small and prevent blurring boundaries, contours, or corners that separate neighboring areas, where one or a combination of these differential operators has substantial value. Some sufficient assumptions have to be made to ensure that our nonlinear diffusion meets acceptable behavior along these lines. Therefore, we can assume, as exhibited by Perona–Malik [16], some minimal hypotheses on g :

$$(\mathbf{H}_2): \begin{cases} g: \mathbb{R}_+ \longrightarrow \mathbb{R} \text{ is } \mathcal{C}^1 - \text{function such that } g(s) > 0, & g'(s) \leq 0, \\ \lim_{s \rightarrow 0} g(s) > 0, & \lim_{s \rightarrow \infty} g(s) = 0. \end{cases} \quad (17)$$

Besides, due to the decreasing positivity of the function g , it is evident that our model's behavior (16) encourages smoothing along edges in the $\mathbf{e}_1$, $\mathbf{e}_2$, $(\mathbf{e}_1 + \mathbf{e}_2)$, or $(-\mathbf{e}_1 + \mathbf{e}_2)$ directions and ceases across them.

Moreover, we may provide another investigation analysis of our model by examining the eigenvalues-eigenvectors

of the matrix $\mathbf{D}_{\nabla u}$. The matrix $\mathbf{D}_{\nabla u}$ has two eigenvalues $\lambda_{\pm}$ with $\theta_{\pm}$ being the corresponding eigenvectors:

$$\theta_{\pm} = \frac{\eta_{\pm}}{|\eta_{\pm}|}, \quad (18)$$

where

$$\eta_{\pm} = \begin{pmatrix} g(|u_{x_1}|) - g(|u_{x_2}|) \pm \sqrt{(g(|u_{x_1}|) - g(|u_{x_2}|))^2 + (g(|u_{x_{12}}|) - g(|u_{x_{-12}}|))^2} \\ g(|u_{x_{12}}|) - g(|u_{x_{-12}}|) \end{pmatrix}, \quad (19)$$

provided that $|u_{x_{12}}| \neq |u_{x_{-12}}|$. We can then expand the first equation of (16) into

$$\frac{\partial u}{\partial t} = \nabla \cdot [\lambda_+ \theta_+ \theta_+^T \nabla u] + \nabla \cdot [\lambda_- \theta_- \theta_-^T \nabla u] - \rho(u - u_0). \quad (20)$$

Accordingly, the diffusion caused by (20) is measured by the $\lambda_{\pm}$ values and oriented toward $\theta_{\pm}$. Specifically, it is clear from the expression of $\lambda_{\pm}$ that $\lambda_+ \geq \lambda_- > 0$, which means that the diffusion toward θ_+ is privileged over θ_- . Furthermore, we can deduce the following results:

(i) In flat areas, we have

$$\left\{ \begin{array}{l} |u_{x_1}| = |u_{x_2}|, \\ |u_{x_{12}}| = |u_{x_{-12}}| \end{array} \right\} \iff \left\{ |u_{x_1}| = |u_{x_2}| = |u_{x_{12}}| = |u_{x_{-12}}| = 0 \right\}, \quad (21)$$

which means that $\lambda_+ = \lambda_- = 2g(0) > 0$. Then, the diffusion is isotropic and linear, and the smoothing effect is the same in all directions.

(ii) At straight edges: $(\lambda_+ > \lambda_- = 0)$.

For instance, we can assume that $u_{x_1} = u_{x_2}$ and $|u_{x_1}| \gg 0$. Then, we obtain $|u_{x_{12}}| = 0$ and $|u_{x_{-12}}| \gg 0$, which implies that

$$\begin{cases} \lambda_+ = g(0), & \theta_+ = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right)^T, \\ \lambda_- = 0, & \theta_- = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right)^T. \end{cases} \quad (22)$$

Consequently, the diffusion process is anisotropic and oriented along the $(-\mathbf{e}_1 + \mathbf{e}_2)$ direction.

(iii) Corners: $(\lambda_+ \geq \lambda_- > 0)$.

According to the characteristics of the function g , diffusion is anisotropic and oriented along θ_+ and θ_- directions.

In fact, the difference $(\lambda_+ - \lambda_-)^2 = (g(|u_{x_1}|) - g(|u_{x_2}|))^2 + (g(|u_{x_{12}}|) - g(|u_{x_{-12}}|))^2$ gives insights into our diffusion model's anisotropic property. In other words, it indicates the isotropic diffusion for the zero value and the anisotropic diffusion for larger values.

3.2. New Adaptive Diffusion Function Using Hermite Spline. For effective image denoising and better control of the diffusion via the process (16), we will approximate numerically the unknown diffusion function g that meets the requirements $(\mathbf{H}_1 + \mathbf{H}_2)$ by using Hermite spline [21]. Therefore, we can use cubic Hermite spline to interpolate

numeric data specified at 0, k , and $h \geq 1$ ($0 < k < h$) to obtain a function that meets the requirements $(\mathbf{H}_1 + \mathbf{H}_2)$. To this end, we propose a function g as follows:

$$g(s) = \begin{cases} p_0 P_{1,0k}(s) + v_0 P_{2,0k}(s) + p_k P_{1,k0}(s) + v_k P_{2,k0}(s), & s \in [0, k[, \\ p_k P_{1,kh}(s) + v_k P_{2,kh}(s) + p_h P_{1,hk}(s) + v_h P_{2,hk}(s), & s \in [k, h[, \\ p_h g_{h,1}(s) + v_h g_{h,2}(s), & s \in [h, \infty[, \end{cases} \quad (23)$$

where p and v are the coefficients used to define the position and the velocity vector at a specific point, k and h are two threshold parameters, and $\{P_{i,cd}\}$ is a family of the basis functions composed of polynomials of degree 3 used on the interval $[c, d[$ such that

$$\begin{cases} P_{1,cd}(s) = \frac{(s-d)^2(2s+d-3c)}{(d-c)^3}, \\ P_{2,cd}(s) = \frac{(s-d)^2(s-c)}{(d-c)^2}. \end{cases} \quad (24)$$

And, we may consider

$$\begin{cases} g_{h,1}(s) = \frac{h}{\log(h)+2} \frac{2s(\log(s)+1) - h \log(h)}{s^2}, \\ g_{h,2}(s) = \frac{h^2}{\log(h)+2} \frac{s(\log(s)+1) - h(\log(h)+1)}{s^2}. \end{cases} \quad (25)$$

Hence, we can reformulate the expression of g as follows:

$$g(s) = \begin{cases} \sum_{i=0}^3 A_{0k,i} s^i, & s \in [0, k[, \\ \sum_{i=0}^3 A_{kh,i} s^i, & s \in [k, h[, \\ A_{h,2} \frac{\log(s)+1}{s} + A_{h,1} \frac{1}{s^2}, & s \in [h, \infty[, \end{cases} \quad (26)$$

By considering the continuity of φ at k and h , and using $\varphi(0) = 0$, it follows then

$$\begin{aligned} \varphi(s) &= \begin{cases} \sum_{i=0}^3 \frac{A_{0k,i}}{i+2} s^{i+2}, & s \in [0, k[, \\ \sum_{i=0}^3 \left[\frac{A_{kh,i}}{i+2} s^{i+2} + \left(\frac{A_{0k,i} - A_{kh,i}}{i+2} \right) k^{i+2} \right], & s \in [k, h[, \\ A_{h,2} s \log(s) + A_{h,1} \log(s) + A_{h,0}, & s \in [h, \infty[, \end{cases} \\ & (27) \end{aligned}$$

where

$$\begin{aligned} A_{h,0} &= \sum_{i=0}^3 \left[\frac{A_{kh,i}}{i+2} h^{i+2} + \left(\frac{A_{0k,i} - A_{kh,i}}{i+2} \right) k^{i+2} \right] \\ &\quad - A_{h,2} h \log(h) - A_{h,1} \log(h). \end{aligned} \quad (28)$$

The values of the coefficients $A_{0k,\cdot}$ and $A_{kh,\cdot}$ are determined experimentally under the conditions of $(\mathbf{H}_1 + \mathbf{H}_2)$ on the intervals $[0, k[$ and $[k, h[$. Besides, we may introduce some sufficient conditions on the coefficients $A_{h,\cdot}$ to ensure that the functions φ and g satisfy $(\mathbf{H}_1 + \mathbf{H}_2)$ on $[h, \infty[$:

$$\begin{cases} A_{h,2} > 0, \\ A_{h,1} < h A_{h,2}, \\ A_{h,1} \geq -\frac{h \log(h)}{2} A_{h,2}. \end{cases} \quad (29)$$

Now, we establish a useful growth condition for the function φ .

Proposition 2. For all $s \in \mathbb{R}_+$, we can prove that

$$\varphi(s) \geq As + B, \quad (30)$$

where $A > 0$ and $B \in \mathbb{R}$.

Proof. We know that $g'(s) \leq 0$. Then,

$$\begin{aligned} s &\in [0, k[: \\ \varphi'(s) &= sg(s) > g(k)s \implies \varphi(s) \geq g(k)(s^2/2) + C \\ &\geq g(k)s - (g(k)/2) + C \text{ with } C \in \mathbb{R}. \end{aligned}$$

$s \in [k, h[$: $\varphi'(s) = sg(s) > kg(h) \implies \varphi(s) \geq kg(h)s + B$ with $B \in \mathbb{R}$.

$s \in [h, \infty[$: we have $\lim_{s \rightarrow +\infty} (\varphi(s)/s) = +\infty$. Then, we may find an $l = h$ for an $A > 0$ such that $\varphi(s) > As$, for all $s \geq l$.

3.3. Existence and Uniqueness. In this section, we investigate the existence and uniqueness of weak solutions of the Euler–Lagrange equation associated with the energy functional $E(u)$ defined in (5). For $\rho > 0$ and $u_0 \in L^2(\Omega)$, we consider

$$\begin{cases} \rho(u - u_0) - \operatorname{div}(\mathbf{D}_{\nabla u} \nabla u) = 0, & \text{in } \Omega, \\ \langle \mathbf{D}_{\nabla u} \nabla u, \mathbf{n} \rangle = 0, & \text{on } \partial\Omega. \end{cases} \quad (31)$$

First, we introduce a new functional space $L\log L^h(\Omega)$:

$$L\log L^h(\Omega) = \left\{ u: \Omega \longrightarrow \mathbb{R} \mid \int_{\Omega \cap \{|u| \geq h\}} |u| \log(|u|) dx < \infty \right\}. \quad (32)$$

Next, we define a weak solution for problem (31).

Definition 1 A function $u \in L^2(\Omega) \cap W^{1,1}(\Omega)$ with $\partial_{x_i} u \in L\log L^h(\Omega)$ for $i = 1, 2$ is called a weak solution for problem (31) if, for any $v \in C^1(\overline{\Omega})$ we have

$$\int_{\Omega} \rho(u - u_0) v dx + \int_{\Omega} \mathbf{D}_{\nabla u} \nabla u \cdot \nabla v dx = 0. \quad (33)$$

And when v is a constant function, we obtain

$$U = \left\{ u \in L^2(\Omega) \cap W^{1,1}(\Omega) \mid \partial_{x_i} u \in L\log L^h(\Omega) \text{ with } i = 1, 2, \int_{\Omega} u dx = \int_{\Omega} u_0 dx \right\}, \quad (39)$$

and the functional E as defined in (5). It is obvious that $u_{0,\Omega} \in U$ knowing that $u_{0,\Omega} = (1/|\Omega|) \int_{\Omega} u_0 dx$.

Since

$$0 \leq \inf_{u \in U} E(u) \leq E(u_{0,\Omega}) = \frac{\rho}{2} \int_{\Omega} (u_{0,\Omega} - u_0)^2 dx, \quad (40)$$

then we can construct a minimizing sequence $\{u_m\}_{m=1}^{\infty}$ in U such that $E(u_m) < E(u_{0,\Omega}) + 1$ and

$$\lim_{m \rightarrow \infty} E(u_m) = \inf_{u \in U} E(u). \quad (41)$$

Besides,

$$\begin{aligned} \int_{\Omega} u_m^2 dx &= \int_{\Omega} (u_m - u_0 + u_0 - u_{0,\Omega} + u_{0,\Omega})^2 dx \\ &\leq 4 \int_{\Omega} (u_m - u_0)^2 dx + 4 \int_{\Omega} (u_0 - u_{0,\Omega})^2 dx + 2 \int_{\Omega} u_{0,\Omega}^2 dx \\ &\leq \frac{8}{\rho} (E(u_m) + E(u_{0,\Omega})) + 2u_{0,\Omega}^2 |\Omega| \\ &\leq \frac{8}{\rho} (2E(u_{0,\Omega}) + 1) + 2u_{0,\Omega}^2 |\Omega|. \end{aligned} \quad (42)$$

$$\int_{\Omega} u dx = \int_{\Omega} u_0 dx. \quad (34)$$

Before stating our main theorem, we need first to introduce a useful lemma.

Lemma 1 (uniform integrability and weak convergence [22]). Assume that Ω is bounded and let $\{u_m\}_{m=1}^{\infty}$ be a sequence of functions in $L^1(\Omega)$ satisfying

$$\sup_m \|u_m\|_{L^1(\Omega)} < \infty. \quad (35)$$

Suppose also

$$\lim_{l \rightarrow \infty} \sup_m \int_{\Omega \cap \{|u_m| \geq l\}} |u_m| dx = 0. \quad (36)$$

Then, there exist a subsequence $\{u_{m_j}\}_{j=1}^{\infty}$ and $\tilde{u} \in L^1(\Omega)$ such that

$$u_{m_j} \rightharpoonup \tilde{u} \text{ weakly in } L^1(\Omega). \quad (37)$$

Now, we state our main theorem.

Theorem 1. There exists a unique weak solution for problem (31).

Proof. We consider the variational problem

$$\min\{E(u) \mid u \in U\}, \quad (38)$$

where

It follows then

$$\sup_m \|u_m\|_{L^2(\Omega)} < \infty. \quad (43)$$

On the other hand, from Proposition 2, we may find $\alpha > 0$ and $\beta \in \mathbb{R}$ such that

$$\begin{aligned} E(u_{0,\Omega}) + 1 &> E(u_m) \geq \int_{\Omega} \left[\varphi(|\partial_{x_1} u_m|) + \varphi(|\partial_{x_2} u_m|) \right] dx \\ &\geq \alpha \|\nabla u_m\|_{L^1(\Omega)} + \beta. \end{aligned} \quad (44)$$

Then, we get

$$\sup_m \|\nabla u_m\|_{L^1(\Omega)} < \infty. \quad (45)$$

Moreover, we have

$$\lim_{s \rightarrow +\infty} \frac{\varphi(s)}{s \log(s)} = A_{h,2} > 0. \quad (46)$$

Then, for $i = 1, 2$ there exist $l \geq h \geq 1$ and positive constants C and C_1 (independent of l) such that

$$\begin{aligned} & \int_{\Omega \cap \{|\partial_{x_i} u_m| \geq l\}} \varphi(|\partial_{x_i} u_m|) dx \\ & \leq C \int_{\Omega \cap \{|\partial_{x_i} u_m| \geq h\}} |\partial_{x_i} u_m| \log(|\partial_{x_i} u_m|) dx \leq C_1. \end{aligned} \quad (47)$$

On the other hand, we know that

$$\lim_{s \rightarrow +\infty} \frac{\varphi(s)}{s} = +\infty. \quad (48)$$

Then, given $\epsilon > 0$, let $\eta_\epsilon = C_1/\epsilon$ and choose $l \geq h \geq 1$ such that for all $s \geq l$, we have $\varphi(s) > \eta_\epsilon s$. Hence, for $i = 1, 2$, we obtain

$$\begin{aligned} \int_{\Omega \cap \{|\partial_{x_i} u_m| \geq l\}} |\partial_{x_i} u_m| dx & \leq \frac{1}{\eta_\epsilon} \int_{\Omega \cap \{|\partial_{x_i} u_m| \geq l\}} \varphi(|\partial_{x_i} u_m|) dx \\ & \leq \frac{C_1}{\eta_\epsilon} = \epsilon, \end{aligned} \quad (49)$$

and this is true for all m and arbitrary $\epsilon > 0$. It follows then that, for $i = 1, 2$, we have

$$\lim_{l \rightarrow \infty} \sup_m \int_{\Omega \cap \{|\partial_{x_i} u_m| \geq l\}} |\partial_{x_i} u_m| dx = 0. \quad (50)$$

From (43), (45)–(50) and by Lemma 1 and the weak compactness of $L^2(\Omega)$, we can find a subsequence $\{u_{m_j}\}_{j=1}^\infty$ of $\{u_m\}_{m=1}^\infty$ and a function $\tilde{u} \in L^2(\Omega) \cap W^{1,1}(\Omega)$ such that

$$\begin{aligned} u_{m_j} & \rightharpoonup \tilde{u} \text{ weakly in } L^2(\Omega), \\ \nabla u_{m_j} & \rightharpoonup \nabla \tilde{u} \text{ weakly in } L^1(\Omega). \end{aligned} \quad (51)$$

Therefore, we have

$$\int_{\Omega} \tilde{u} dx = \lim_{j \rightarrow \infty} \int_{\Omega} u_{m_j} dx = \int_{\Omega} u_0 dx. \quad (52)$$

Additionally, knowing that the function $f(s) = s \log(s)$ for $s \geq 1$ is increasing and convex, the function $f(|s|)$ is also convex for all $s \geq 1$. Therefore, we obtain for $i = 1, 2$

$$f(|\partial_{x_i} \tilde{u}|) \leq f(|\partial_{x_i} u_{m_j}|) + f'(|\partial_{x_i} \tilde{u}|) (\partial_{x_i} \tilde{u} - \partial_{x_i} u_{m_j}). \quad (53)$$

Integrating the above inequality over $\Omega_M \cap \{h \leq |\partial_{x_i} u_{m_j}|\}$ with $\Omega_M = \Omega \cap \{h \leq |\partial_{x_i} \tilde{u}| \leq M\}$ with $M > 0$, we have

$$\begin{aligned} & \int_{\Omega_M} f(|\partial_{x_i} \tilde{u}|) dx \leq \int_{\Omega \cap \{h \leq |\partial_{x_i} u_{m_j}|\}} f(|\partial_{x_i} u_{m_j}|) dx \\ & \quad + \int_{\Omega_M \cap \{h \leq |\partial_{x_i} u_{m_j}|\}} f'(|\partial_{x_i} \tilde{u}|) (\partial_{x_i} \tilde{u} - \partial_{x_i} u_{m_j}) dx \\ & \leq \int_{\Omega \cap \{h \leq |\partial_{x_i} u_{m_j}|\}} f(|\partial_{x_i} u_{m_j}|) dx \\ & \quad + \int_{\Omega \cap \{h \leq |\partial_{x_i} u_{m_j}|\}} f'(|\partial_{x_i} \tilde{u}|) \chi_{\{h \leq |\partial_{x_i} \tilde{u}| \leq M\}} (\partial_{x_i} \tilde{u} - \partial_{x_i} u_{m_j}) dx. \end{aligned} \quad (54)$$

Since $f'(|\partial_{x_i} \tilde{u}|) \chi_{\{h \leq |\partial_{x_i} \tilde{u}| \leq M\}} \in L^\infty(\Omega)$ and by letting $j \rightarrow \infty$, we get

$$\int_{\Omega_M} f(|\partial_{x_i} \tilde{u}|) dx \leq \liminf_{j \rightarrow \infty} \int_{\Omega \cap \{h \leq |\partial_{x_i} \tilde{u}| \}} f(|\partial_{x_i} u_{m_j}|) dx < \infty. \quad (55)$$

Then, by letting $M \rightarrow \infty$, we deduce

$$\int_{\Omega \cap \{h \leq |\partial_{x_i} \tilde{u}| \}} f(|\partial_{x_i} \tilde{u}|) dx < \infty. \quad (56)$$

It follows then $\partial_{x_i} \tilde{u} \in L \log L^h(\Omega)$. Besides, we have

$$\int_{\Omega} (\tilde{u} - u_0)^2 dx \leq \liminf_{j \rightarrow \infty} \int_{\Omega} (u_{m_j} - u_0)^2 dx. \quad (57)$$

And by following the same reasoning as set out above, we know that the function $\varphi(s)$ for $s \geq 0$ is increasing and convex. Then, we can easily deduce that

$$\int_{\Omega} \varphi(|\partial_{x_i} \tilde{u}|) dx \leq \liminf_{j \rightarrow \infty} \int_{\Omega} \varphi(|\partial_{x_i} u_{m_j}|) dx. \quad (58)$$

Therefore, by combining (57) with (58), we conclude

$$E(\tilde{u}) \leq \liminf_{j \rightarrow \infty} E(u_{m_j}) = \inf_{u \in U} E(u), \quad (59)$$

which signifies that $\tilde{u} \in U$ is a minimizer of the energy functional $E(u)$, i.e.,

$$E(\tilde{u}) = \min_{u \in U} E(u). \quad (60)$$

Furthermore, for all $v \in C^1(\overline{\Omega})$ and for all $t \in \mathbb{R}$, we have $\tilde{u} + t(v - v_\Omega) \in U$ with $v_\Omega = (1/|\Omega|) \int_{\Omega} v dx$. Then, $r(0) \leq r(t)$, where

$$r(t) = E(\tilde{u} + t(v - v_\Omega)). \quad (61)$$

Hence, we have $r'(0) = 0$, which means

$$\int_{\Omega} \rho(\tilde{u} - u_0)(v - v_\Omega) dx + \int_{\Omega} \mathbf{D}_{\tilde{u}} \nabla \tilde{u} \cdot \nabla v dx = 0. \quad (62)$$

Because of (52), we have

$$\int_{\Omega} \rho(\tilde{u} - u_0) v dx + \int_{\Omega} \mathbf{D}_{\tilde{u}} \nabla \tilde{u} \cdot \nabla v dx = 0. \quad (63)$$

We conclude then that $\tilde{u}$ is a weak solution for problem (31).

Now, assuming that there is another minimizer $\hat{u}$ of E and using the fact that E is strictly convex (5) and (6), we have

$$\min_{u \in U} E(u) = \frac{1}{2} E(\tilde{u}) + \frac{1}{2} E(\hat{u}) > E\left(\frac{\tilde{u} + \hat{u}}{2}\right) \geq \min_{u \in U} E(u), \quad (64)$$

a contradiction. Thus, there is only one minimizer, which completes the proof.

4. Numerical Implementation and Experimental Results

4.1. Consistency and Stability of Finite Difference Approximation. For a consistent and stable discretization scheme, one can use the following accurate finite difference scheme

approximation: at time $t_n = n\delta_t$, $n \geq 0$, and mesh points $x_i = i\delta$, $y_j = j\delta$ ($0 \leq i \leq N+1$ and $0 \leq j \leq M+1$), we denote by $u_{i,j}^n$ the finite difference approximation of $u(x_i, y_j; t_n)$ and exhibit the time-space derivatives discretization as follows:

$$\begin{aligned}
 u_{x_1}(x_i, y_j; t_n) &= \frac{u(x_{i+(1/2)}, y_j; t_n) - u(x_{i-(1/2)}, y_j; t_n)}{\delta} + \mathcal{O}(\delta^2), \\
 u_{x_2}(x_i, y_j; t_n) &= \frac{u(x_i, y_{j+(1/2)}; t_n) - u(x_i, y_{j-(1/2)}; t_n)}{\delta} + \mathcal{O}(\delta^2), \\
 u_{x_{12}}(x_i, y_j; t_n) &= \frac{u(x_{i+(1/2)}, y_{j+(1/2)}; t_n) - u(x_{i-(1/2)}, y_{j-(1/2)}; t_n)}{\sqrt{2}\delta} + \mathcal{O}(\delta^2), \\
 u_{x_{-12}}(x_i, y_j; t_n) &= \frac{u(x_{i-(1/2)}, y_{j+(1/2)}; t_n) - u(x_{i+(1/2)}, y_{j-(1/2)}; t_n)}{\sqrt{2}\delta} + \mathcal{O}(\delta^2), \\
 u_t(x_i, y_j; t_n) &= \frac{u(x_i, y_j; t_{n+1}) - u(x_i, y_j; t_n)}{\delta_t} + \mathcal{O}(\delta_t).
 \end{aligned} \tag{65}$$

By assuming $\delta = 1$ and denoting

$$\left\{ \begin{aligned} g_{N_{i,j}}^n &= g(|\Delta_N u_{i,j}^n|), \\ g_{E_{i,j}}^n &= g(|\Delta_E u_{i,j}^n|), \\ g_{S_{i,j}}^n &= g(|\Delta_S u_{i,j}^n|), \\ g_{W_{i,j}}^n &= g(|\Delta_W u_{i,j}^n|), \\ g_{NE_{i,j}}^n &= g\left(\left|\frac{\Delta_{NE} u_{i,j}^n}{\sqrt{2}}\right|\right), \\ g_{SE_{i,j}}^n &= g\left(\left|\frac{\Delta_{SE} u_{i,j}^n}{\sqrt{2}}\right|\right), \\ g_{SW_{i,j}}^n &= g\left(\left|\frac{\Delta_{SW} u_{i,j}^n}{\sqrt{2}}\right|\right), \\ g_{NW_{i,j}}^n &= g\left(\left|\frac{\Delta_{NW} u_{i,j}^n}{\sqrt{2}}\right|\right), \end{aligned} \right. \text{ with } \left\{ \begin{aligned} \Delta_N u_{i,j}^n &= u_{i,j+1}^n - u_{i,j}^n, \\ \Delta_E u_{i,j}^n &= u_{i+1,j}^n - u_{i,j}^n, \\ \Delta_S u_{i,j}^n &= u_{i,j-1}^n - u_{i,j}^n, \\ \Delta_W u_{i,j}^n &= u_{i-1,j}^n - u_{i,j}^n, \\ \Delta_{NE} u_{i,j}^n &= u_{i+1,j+1}^n - u_{i,j}^n, \\ \Delta_{SE} u_{i,j}^n &= u_{i+1,j-1}^n - u_{i,j}^n, \\ \Delta_{SW} u_{i,j}^n &= u_{i-1,j-1}^n - u_{i,j}^n, \\ \Delta_{NW} u_{i,j}^n &= u_{i-1,j+1}^n - u_{i,j}^n, \end{aligned} \right. \tag{66}$$

then we may approximate the solution for problem (16) by the above scheme to obtain the following discrete diffusion filter:

$$u_{i,j}^{n+1} = u_{i,j}^n + \delta_t \left[g_N \Delta_N u + g_E \Delta_E u + g_S \Delta_S u + g_W \Delta_W u + \frac{g_{NE} \Delta_{NE} u + g_{SE} \Delta_{SE} u + g_{SW} \Delta_{SW} u + g_{NW} \Delta_{NW} u}{2} \right]_{i,j}^n - \rho \delta_t (u_{i,j}^n - u_{i,j}^0), \quad (67)$$

where $u_{i,j}^0$ is the degraded image, $1 \leq i \leq N$, $1 \leq j \leq M$, and $n \geq 0$. Furthermore, we use the discrete Neumann boundary condition:

$$\begin{cases} u_{0,j}^n = u_{1,j}^n, u_{N+1,j}^n = u_{N,j}^n, & \text{for } 1 \leq j \leq M, \\ u_{i,0}^n = u_{i,1}^n, u_{i,M+1}^n = u_{i,M}^n, & \text{for } 1 \leq i \leq N, \\ u_{0,0}^n = u_{1,1}^n, u_{N+1,0}^n = u_{N,1}^n, \\ u_{0,M+1}^n = u_{1,M}^n, u_{N+1,M+1}^n = u_{N,M}^n. \end{cases} \quad (68)$$

Then, using the filter (67) on every initial image u^0 yields a unique sequence $(u^n)_{n \in \mathbb{N}}$ [19]. Besides, due to the continuity of the function g , for every finite n , u^n depends continuously on u^0 . Thus, under a specific condition, equation (67) satisfies the following maximum-minimum

principle, which describes a stability property for the discrete scheme.

Theorem 2 (discrete extremum principle). For an iteration step satisfying

$$0 < \delta_t < \frac{1}{\rho + 6 \max_{s \in \mathbb{R}_+} g(s)}, \quad (69)$$

scheme (67) satisfies

$$\min_{i,j} u_{i,j}^0 \leq u_{i,j}^n \leq \max_{i,j} u_{i,j}^0, \quad (70)$$

for all $1 \leq i \leq N$, $1 \leq j \leq M$, and $n \in \mathbb{N}$.

Proof. For all $1 \leq i \leq N$, $1 \leq j \leq M$, and $n \in \mathbb{N}$, we have

$$\begin{aligned} u_{i,j}^{n+1} &= \delta_t \left(g_{N_{i,j}}^n u_{i,j+1}^n + g_{E_{i,j}}^n u_{i+1,j}^n + g_{S_{i,j}}^n u_{i,j-1}^n + g_{W_{i,j}}^n u_{i-1,j}^n \right) \\ &\quad + \frac{\delta_t}{2} \left(g_{NE_{i,j}}^n u_{i+1,j+1}^n + g_{SE_{i,j}}^n u_{i+1,j-1}^n + g_{SW_{i,j}}^n u_{i-1,j-1}^n + g_{NW_{i,j}}^n u_{i-1,j+1}^n \right) \\ &\quad + \left[1 - \delta_t \left(g_{N_{i,j}}^n + g_{E_{i,j}}^n + g_{S_{i,j}}^n + g_{W_{i,j}}^n + \frac{g_{NE_{i,j}}^n + g_{SE_{i,j}}^n + g_{SW_{i,j}}^n + g_{NW_{i,j}}^n}{2} + \rho \right) \right] u_{i,j}^n + \rho \delta_t u_{i,j}^0. \end{aligned} \quad (71)$$

Since $0 < \delta_t < (1/(\rho + 6 \max_{s \in \mathbb{R}_+} g(s)))$, $\rho \in \mathbb{R}_+^*$, and $g > 0$, the following inequality holds:

$$1 - \delta_t \left(g_{N_{i,j}}^n + g_{E_{i,j}}^n + g_{S_{i,j}}^n + g_{W_{i,j}}^n + \frac{g_{NE_{i,j}}^n + g_{SE_{i,j}}^n + g_{SW_{i,j}}^n + g_{NW_{i,j}}^n}{2} + \rho \right) > 0. \quad (72)$$

It follows then

$$\begin{aligned} u_{i,j}^{n+1} &< \delta_t \left(g_{N_{i,j}}^n + g_{E_{i,j}}^n + g_{S_{i,j}}^n + g_{W_{i,j}}^n \right) \max_{i,j} u_{i,j}^n \\ &\quad + \frac{\delta_t}{2} \left(g_{NE_{i,j}}^n + g_{SE_{i,j}}^n + g_{SW_{i,j}}^n + g_{NW_{i,j}}^n \right) \max_{i,j} u_{i,j}^n \\ &\quad + \left[1 - \delta_t \left(g_{N_{i,j}}^n + g_{E_{i,j}}^n + g_{S_{i,j}}^n + g_{W_{i,j}}^n + \frac{g_{NE_{i,j}}^n + g_{SE_{i,j}}^n + g_{SW_{i,j}}^n + g_{NW_{i,j}}^n}{2} + \rho \right) \right] \max_{i,j} u_{i,j}^n + \rho \delta_t \max_{i,j} u_{i,j}^0 \\ &= (1 - \rho \delta_t) \max_{i,j} u_{i,j}^n + \rho \delta_t \max_{i,j} u_{i,j}^0. \end{aligned} \quad (73)$$

Similarly, we have

$$\begin{aligned}
u_{i,j}^{n+1} &> \delta_t \left(g_{N_{i,j}}^n + g_{E_{i,j}}^n + g_{S_{i,j}}^n + g_{W_{i,j}}^n \right) \min_{i,j} u_{i,j}^n \\
&+ \frac{\delta_t}{2} \left(g_{NE_{i,j}}^n + g_{SE_{i,j}}^n + g_{SW_{i,j}}^n + g_{NW_{i,j}}^n \right) \min_{i,j} u_{i,j}^n \\
&+ \left[1 - \delta_t \left(g_{N_{i,j}}^n + g_{E_{i,j}}^n + g_{S_{i,j}}^n + g_{W_{i,j}}^n + \frac{g_{NE_{i,j}}^n + g_{SE_{i,j}}^n + g_{SW_{i,j}}^n + g_{NW_{i,j}}^n}{2} + \rho \right) \right] \min_{i,j} u_{i,j}^n + \rho \delta_t \min_{i,j} u_{i,j}^0 \\
&= (1 - \rho \delta_t) \min_{i,j} u_{i,j}^n + \rho \delta_t \min_{i,j} u_{i,j}^0.
\end{aligned} \tag{74}$$

Consequently, the result (70) is true for $n = 1$ because

$$\begin{aligned}
\min_{i,j} u_{i,j}^0 &= (1 - \rho \delta_t) \min_{i,j} u_{i,j}^0 + \rho \delta_t \min_{i,j} u_{i,j}^0 < u_{i,j}^1 \\
&< (1 - \rho \delta_t) \max_{i,j} u_{i,j}^0 + \rho \delta_t \max_{i,j} u_{i,j}^0 = \max_{i,j} u_{i,j}^0.
\end{aligned} \tag{75}$$

Therefore, by induction, it is easy to prove the result (70), which completes the proof.

4.2. Experimental Procedures and Results

4.2.1. Experimental Procedures. This section is devoted to comparing our model (16), as an image denoising algorithm, with the ones proposed by Wang and Zhou (Wang–Zhou) [10] and Maiseli [15]. All the experiments are conducted under Windows 10 and GNU Octave version 6.2.0, running on a laptop with an Intel® Core™ i7-10510U (8 MB cache, 4 Core, up to 4.9 GHz), 16 GB memory (LPDDR3, Dual-channel, 2133 MHz), and 512 GB storage (PCIe, SSD, 3 × 4). The experiments were done on three actual MRI scans (Figure 1) affected by different σ^2 -values of zero-mean white Gaussian noise and restored using our filter (67) with the boundary-initial conditions (68), provided that the diffusion function (23) satisfies the assumptions $(\mathbf{H}_1 + \mathbf{H}_2)$.

We also used the discrete diffusion filter as described in [15] for Wang–Zhou and Maiseli models with the following diffusion functions:

Wang–Zhou [10] diffusion function:

$$g(s) = \frac{1}{s+1} + \frac{\log(s+1)}{s}, \quad s \geq 0. \tag{76}$$

Maiseli [15] diffusion function is a combination between the Perona–Malik diffusion function [16] and the Charbonnier diffusion function [8]:

$$g(s) = \begin{cases} \frac{1}{1 + (s/k_1)^2}, & s < k_1, \\ \frac{1}{\sqrt{1 + (s/k_2)^2}}, & s \geq k_1, \end{cases} \tag{77}$$

where k_1 and k_2 are positive constants.

Besides, in order to evaluate the quality of the restored images from different image denoising methods, we used two image quality metrics:

(i) Peak signal-to-noise ratio (PSNR) [26]:

$$\text{PSNR} = 10 \log_{10} \left(\frac{255^2 MN}{\|u - I\|_2^2} \right), \tag{78}$$

where I is the original or the uncorrupted image and u is the distorted or the restored image. PSNR is one of the oldest image quality metrics evaluating an image's signal strength relative to noise, and it is always positive. We evaluate the PSNR metric by using the Octave built-in function “psnr.” However, due to its limitations and its failure in some circumstances as an adequate measure of visual quality [27], we used another metric that significantly correlates with the human visual perception.

(ii) Structural similarity index (SSIM) [28]:

$$\text{SSIM} = \frac{(2\mu_u \mu_I + c_1)(2\sigma_{uI} + c_2)}{(\mu_u^2 + \mu_I^2 + c_1)(\sigma_u^2 + \sigma_I^2 + c_2)}, \tag{79}$$

where c_1 and c_2 are tuning parameters. μ_u , σ_u^2 , and σ_{uI} stand for the mean, variance, and covariance, respectively. It is a method for measuring the similarity between a degraded image and a perfect one, and it is bounded between zero and one. For a good similarity between the original and the restored images, we need higher values of SSIM-index. In all experiments, we estimate the SSIM-index using “ssim.m” with automatic downsampling, downloaded from the website: <https://ece.uwaterloo.ca/~z70wang/research/ssim/>.

In all experiments and to prioritize SSIM-index over PSNR, we used a combined metric

$$\text{IQM} = \text{PSNR} + 100 \times \text{SSIM}, \tag{80}$$

to quantify the quality of the restored images. The iterations are stopped when the IQM value reaches the maximum.

FIGURE 1: Brain MRI scans (512×512). (a) Patient 30: sagittal T1 of a 30-years-old female patient [23]. (b) Patient 50: coronal T2 of a 50-year-old female patient [24]. (c) Patient 55: axial T2 of a 55-year-old patient [25].

```

input:  $I$ .
output:  $u$ , iter, IQM.
(1) Initialize:  $I$ ;
(2)  $u^0 \leftarrow \text{imnoise}(I, \text{"gaussian"}, 0, \sigma^2)$ ;
(3)  $\text{IQM}_0 \leftarrow \text{PSNR}(I, u^0) + 100 \times \text{SSIM}(I, u^0)$ ;
(4)  $u \leftarrow u^0$ ;
(5) iter  $\leftarrow 0$ ;
(6)  $\text{IQM} \leftarrow \text{IQM}_0$ ;
(7)  $n \leftarrow 0$ ;
(8) While true do
(9)  $\Delta_N u^n \leftarrow [\text{zeros}(1, M); (u^n(1:N-1, :) - u^n(2:N, :))]$ ;
(10)  $\Delta_S u^n \leftarrow [(-\Delta_N u^n(2:N, :)); \text{zeros}(1, M)]$ ;
(11)  $\Delta_E u^n \leftarrow [(u^n(:, 2:M) - u^n(:, 1:M-1)) \text{zeros}(N, 1)]$ ;
(12)  $\Delta_W u^n \leftarrow [\text{zeros}(N, 1) (-\Delta_E u^n(:, 1:M-1))]$ ;
(13)  $\Delta_{NE} u^n \leftarrow [\text{zeros}(1, M); (u^n(1:N-1, 2:M) - u^n(2:N, 1:M-1)) \text{zeros}(N-1, 1)]$ ;
(14)  $\Delta_{SW} u^n \leftarrow [\text{zeros}(N-1, 1) (-\Delta_{NE} u^n(2:N, 1:M-1)); \text{zeros}(1, M)]$ ;
(15)  $\Delta_{SE} u^n \leftarrow [(u^n(2:N, 2:M) - u^n(1:N-1, 1:M-1)) \text{zeros}(N-1, 1); \text{zeros}(1, M)]$ ;
(16)  $\Delta_{NW} u^n \leftarrow [\text{zeros}(1, M); \text{zeros}(N-1, 1) (-\Delta_{SE} u^n(1:N-1, 1:M-1))]$ ;
(17)  $g_\alpha^n \leftarrow g_\alpha^n(|\Delta_\alpha u^n|)$ ,  $\alpha \in \Lambda_1 = \{N, E, S, W\}$ ;
(18)  $g_\beta^n \leftarrow g_\beta^n(|\Delta_\beta u^n|/\sqrt{2})$ ,  $\beta \in \Lambda_2 = \{NE, SE, SW, NW\}$ ;
(19)  $u^{n+1} \leftarrow u^n + \delta_t [\sum_{\alpha \in \Lambda_1} g_\alpha^n \Delta_\alpha u^n + \sum_{\beta \in \Lambda_2} (g_\beta^n \Delta_\beta u^n / 2) - \rho(u^n - u^0)]$ ;
(20)  $\text{IQM}_{n+1} \leftarrow \text{PSNR}(I, u^{n+1}) + 100 \times \text{SSIM}(I, u^{n+1})$ ;
(21) if ( $\text{IQM}_{n+1} > \text{IQM}$ ) then
(22)    $\text{IQM} \leftarrow \text{IQM}_{n+1}$ ;
(23)   iter  $\leftarrow n + 1$ ;
(24)    $u \leftarrow u^{n+1}$ ;
(25) else
(26)   Break;
(27) end
(28)  $n \leftarrow n + 1$ ;
(29) end

```

ALGORITHM 1: Proposed image denoising algorithm.

The image denoising process using the proposed model (67) can be implemented as shown in Algorithm 1. First, we consider I as one of the three original brain MRI scans

(Figure 1) and generate in it a Gaussian white noise with zero-mean and σ^2 -value, using the Octave built-in function “imnoise.” Next, the initial value u^0 is set to be the noisy

TABLE 1: PSNR values of the images in Figure 1 are affected by different σ^2 values of zero-mean white Gaussian noise and restored from different methods with their corresponding iteration number.

	σ^2	Noisy	Wang and Zhou [10]		Maiseli [15]		Proposed	
		PSNR	PSNR	Iter	PSNR	Iter	PSNR	Iter
Patient 30	0.005	23.4404	32.8436	36	33.4928	19	33.4628	7
	0.010	20.6935	30.9182	52	31.5063	27	31.4409	9
	0.015	19.0923	29.6302	64	30.1488	33	29.9034	11
Patient 50	0.005	23.6551	31.0313	29	31.3403	15	31.2913	5
	0.010	20.7956	29.0496	42	29.2157	22	29.0662	7
	0.015	19.1404	27.8208	52	27.8922	28	27.7835	8
Patient 55	0.005	24.0326	31.3462	30	31.4897	17	30.6296	6
	0.010	21.2339	29.1926	43	29.1900	24	28.4105	8
	0.015	19.5726	27.8086	53	27.7439	29	26.8229	10

TABLE 2: SSIM-index values of the images in Figure 1 are affected by different σ^2 values of zero-mean white Gaussian noise and restored from different methods with their corresponding iteration number.

	σ^2	Noisy	Wang and Zhou [10]		Maiseli [15]		Proposed	
		SSIM	SSIM	Iter	SSIM	Iter	SSIM	Iter
Patient 30	0.005	0.7022	0.9089	36	0.9230	19	0.9420	7
	0.010	0.5889	0.8649	52	0.8899	27	0.9163	9
	0.015	0.5180	0.8282	64	0.8619	33	0.8960	11
Patient 50	0.005	0.7832	0.9139	29	0.9157	15	0.9182	5
	0.010	0.6767	0.8650	42	0.8702	22	0.8736	7
	0.015	0.6099	0.8285	52	0.8367	28	0.8397	8
Patient 55	0.005	0.7174	0.9065	30	0.9155	17	0.9313	6
	0.010	0.6041	0.8559	43	0.8706	24	0.8913	8
	0.015	0.5358	0.8138	53	0.8327	29	0.8587	10

FIGURE 2: Continued.

FIGURE 2: Visual comparison on different actual MRI scans corrupted by zero-mean white Gaussian noise with $\sigma^2 = 0.015$ and restored from various methods. (a) Original: patient 30. (b) Original: patient 50. (c) Original: patient 55. (d) Noisy: patient 30. (e) Noisy: patient 50. (f) Noisy: patient 55. (g) Wang-Zhou: patient 30. (h) Wang-Zhou: patient 50. (i) Wang-Zhou: patient 55. (j) Maiseli: patient 30. (k) Maiseli: patient 50. (l) Maiseli: patient 55. (m) Proposed: patient 30. (n) Proposed: patient 50. (o) Proposed: patient 55.

image in the while-loop. After that, a while-loop is executed, and in each iteration, our filter is used to build a new image with a new IQM value. Finally, when the stopping criterion is confirmed, we obtain the restored image with the corresponding iteration.

4.2.2. Experimental Results. Several tests have been performed on the images in Figure 1 to obtain the best possible parameters:

The time step:

$$\delta_t = 0.10917. \quad (81)$$

The weighting parameter:

$$\rho = 0.00577. \quad (82)$$

The threshold parameters for Maiseli diffusion function (77):

$$\begin{aligned} k_1 &= 2.82261, \\ k_2 &= 8.39061. \end{aligned} \quad (83)$$

The different parameters for our diffusion function (23):

$$\begin{cases} p_0 = 1.46493, & v_0 = -0.00003, & k = 16.76863, \\ p_k = 0.60493, & v_k = -0.00735, & h = 195.5899, \\ p_h = 0.15094, & v_h = -0.00002. \end{cases} \quad (84)$$

Tables 1 and 2 show quantitative results on real images, corrupted with various white Gaussian noises, filtered by different filters. From Table 1, it is clear that the PSNR value of the restored images via Maiseli method has, in most cases, higher values with slight differences over the other methods. However, as shown in Table 2, when it comes to the SSIM-index, the proposed method reveals impressive results against the Wang-Zhou and Maiseli methods. Additionally, our method converges more rapidly to the solution than the others with the less iteration number.

From a visual comparison, Figure 2 shows that the denoising process using our diffusion function removes noises more efficiently and preserves essential image features. Besides, it can be seen from different images that all the restored images via Wang-Zhou and Maiseli methods are more affected by blocky artifacts than the proposed one, despite the higher values in PSNR. This can be attributed to the fact that the Wang-Zhou and Maiseli methods have lower SSIM-indices.

5. Conclusion

This paper presented a new anisotropic diffusion model for image restoration, eliminating the corruption caused by white Gaussian noise. The existence and uniqueness of weak solutions of functions in Orlicz-Sobolev space that minimize the energy functional E have been proven, provided that the functions φ and g satisfy some specific conditions $(H_1 + H_2)$. Besides, we have established a consistent and stable numerical model for the denoising process and used the cubic Hermite spline to approximate the best possible diffusion function that revealed its efficiency regarding the

optimal image denoising via (67). We have also proved that our method provides better results compared to the Wang-Zhou and Maiseli methods.

The next stage of the research attempts to prove the existence and uniqueness of weak solutions for the evolution problem (16) and use different numerical methods, such as finite element and mixed finite element methods.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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