Hindawi
Journal of Mathematics
Volume 2021, Article ID 6692063, 7 pages
https://doi.org/10.1155/2021/6692063

Research Article

On Orthogonal Partial *b*-Metric Spaces with an Application

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Received 21 November 2020; Revised 21 December 2020; Accepted 23 December 2020; Published 12 January 2021

Academic Editor: Xiaolong Qin

In this paper, we initiate the concept of orthogonal partial *b*-metric spaces. We ensure the existence of a unique fixed point for some orthogonal contractive type mappings. Some useful examples are given, and an application is also provided in support of the obtained results.

1. Introduction

The notion of a metric space plays a vital role in metric fixed point theory. The Banach contraction principle [1] is a very famous result in the literature. Fixed point hypotheses are significant apparatuses for demonstrating the presence and uniqueness of solutions for different numerical models. Given a nonempty set $\widetilde{U}$ and a map Ψ from Ψ into itself, the problem of finding a point $\varkappa \in \widetilde{U}$ such that $\Psi(\varkappa) = \varkappa$ is considered as a fixed point problem, and the point $\varkappa \in \widetilde{U}$ is called a fixed point of Ψ.

A natural question is that, under what conditions on $\varkappa$ and $\widetilde{U}$ does a fixed point exist? Theorems which establish the existence (and uniqueness) of such points are called fixed point theorems. These results allow us to find the existence of solutions that satisfy certain conditions for operator equations.

There exist many generalizations of the concept of a metric space in the literature. In [2], Matthews introduced the notion of a partial metric space as part of the study of denotational semantics of dataflow networks. In this setting, the self-distance of any point may not be zero. A lot of fixed point theorems have been investigated in partial spaces (see [3–7] and references therein). Another important generalization of a metric space introduced by Czerwik [8] is a *b*-metric space, where the triangular inequality was replaced by $s[d(x,z) + d(z,y)]$ with $s \geq 1$. Since then, many articles dealing with fixed point theory and variation principle in the setting of *b*-metric spaces for single- and multivalued operators have been appeared (see [9–20]).

Recently, Eshaghi Gordji et al. [21] initiated the concept of orthogonal sets and gave an extension of the Banach contraction principle. Furthermore, they presented applications for their results to ensure the existence and uniqueness of solutions for first-order differential equations.

The purpose of this paper is to improve and generalize the concept of an orthogonal contraction in the sense of metric spaces due to Gordji et al. [22]. Namely, we introduce the concept of an orthogonal partial *b*-metric and establish some fixed point theorems for related contractions. We also enrich this paper with a nontrivial example involving an orthogonal partial *b*-metric, which is not a partial *b*-metric. We set up some hypotheses for the proposed construction, and additionally, we present a potential application for the arrangement of Volterra integral equation to guarantee the legitimacy of the outcomes.

2. Preliminaries

In 1993, Czerwik [8] presented the idea of *b*-metric spaces (in short, *b*-M.Ss).

Definition 1 (see [8]). Let $\widetilde{U}$ be a nonempty universal set and $\rho: \widetilde{U} \times \widetilde{U} \longrightarrow \mathbb{R}^+$ be a mapping so that for all $\varkappa_1, \varkappa_2, \varkappa_3 \in \widetilde{U}$ and $s \geq 1$:

$$\begin{aligned}(bm1)\rho(\varkappa_1, \varkappa_2) &= 0, \quad \text{if and only if } \varkappa_1 = \varkappa_2,\\ \rho(\varkappa_1, \varkappa_2) &= \rho(\varkappa_2, \varkappa_1),\\ (bm3)\rho(\varkappa_1, \varkappa_2) &\leq s[\rho(\varkappa_1, \varkappa_3) + \rho(\varkappa_3, \varkappa_2)].\end{aligned} \tag{1}$$

Then, ρ is said as a b-metric on $\widetilde{U}$ and $(\widetilde{U}, \rho)$ is said as a b-M.S.

The notion of a partial metric space (in short, ρ-M.S) has been initiated by Matthews [2].

Definition 2 (see [2]). Let $\widetilde{U} \neq \varnothing$ be a universal set and $\zeta^*: \widetilde{U} \times \widetilde{U} \longrightarrow \mathbb{R}^+$ be a mapping so that for all $\varkappa_1, \varkappa_2, \varkappa_3 \in \widetilde{U}$:

$$\begin{aligned}(pm1)\varkappa_1 = \varkappa_2, \quad \text{if and only if } \zeta^*(\varkappa_1, \varkappa_1) &= \zeta^*(\varkappa_1, \varkappa_2)\\ &= \zeta^*(\varkappa_2, \varkappa_2),\\ (pm2)\zeta^*(\varkappa_1, \varkappa_1) &\leq \zeta^*(\varkappa_1, \varkappa_2),\\ (pm3)\zeta^*(\varkappa_1, \varkappa_2) &= \zeta^*(\varkappa_2, \varkappa_1),\\ (pm4)\zeta^*(\varkappa_1, \varkappa_2) &\leq \zeta^*(\varkappa_1, \varkappa_3)\\ &+ \zeta^*(\varkappa_3, \varkappa_2)\\ &- \zeta^*(\varkappa_3, \varkappa_3).\end{aligned} \tag{2}$$

Then, ζ^* is said as a ρ-metric on $\widetilde{U}$ and $(\widetilde{U}, \zeta^*)$ is said as a ρ-M.S.

In 2014, Shukla [23] introduced the following concept of partial b-metric spaces (in short, ρ_b-M.Ss) and proved some fixed point results.

Definition 3 (see [23]). Let $\widetilde{U}$ be a nonempty universal set and $\sigma: \widetilde{U} \times \widetilde{U} \longrightarrow \mathbb{R}^+$ be a mapping so that for all $\varkappa_1$, $\varkappa_2$, $\varkappa_3 \in \widetilde{U}$ and $s \geq 1$:

$$\begin{aligned}(\sigma1)\varkappa_1 = \varkappa_2, \quad \text{if and only if } \sigma(\varkappa_1, \varkappa_1) &= \sigma(\varkappa_1, \varkappa_2) = \sigma(\varkappa_2, \varkappa_2),\\ (\sigma2)\sigma(\varkappa_1, \varkappa_1) &\leq \sigma(\varkappa_1, \varkappa_2),\\ (\sigma3)\sigma(\varkappa_1, \varkappa_2) &= \sigma(\varkappa_2, \varkappa_1),\\ (\sigma4)\sigma(\varkappa_1, \varkappa_2) &\leq s[\sigma(\varkappa_1, \varkappa_3) + \sigma(\varkappa_3, \varkappa_2)]\\ &- \sigma(\varkappa_3, \varkappa_3).\end{aligned} \tag{3}$$

Then, σ is said as a ρ_b-metric on $\widetilde{U}$ and $(\widetilde{U}, \sigma)$ is said as a ρ_b-M.S with coefficient $s \geq 1$.

Remark 1 (see [23]). If $\varkappa_1$ and $\varkappa_2$ are in a ρ_b-M.S $(\widetilde{U}, \sigma)$ so that $\sigma(\varkappa_1, \varkappa_2) = 0$, then $\varkappa_1 = \varkappa_2$. However, the converse may not be true.

Example 1 (see [23]). Let $\widetilde{U} = \mathbb{R}^+$, $\varkappa > 1$ be a constant, and $\sigma: \widetilde{U} \times \widetilde{U} \longrightarrow \mathbb{R}^+$ be defined by

$$\sigma(\varkappa_1, \varkappa_2) = \max\{\varkappa_1, \varkappa_2\}^2 + |\varkappa_1 - \varkappa_2|^2, \quad \text{for all } \varkappa_1, \varkappa_2 \in \widetilde{U}. \tag{4}$$

Then, $(\widetilde{U}, \sigma)$ is a ρ_b-M.S with coefficient $s = 2^{\varkappa} > 1$. However, it is neither a b-M.S nor a ρ -M.S.

In 2017, the authors of [22] introduced the notion of orthogonal sets and gave a real generalization of Banach fixed point theorem.

Definition 4 (see [22]). Let $\widetilde{U} \neq \varnothing$ and $\perp$ be a binary relation defined on $\widetilde{U} \times \widetilde{U}$. Then, $(\widetilde{U}, \perp)$ is called an orthogonal set (OS) if there is $\varkappa_1 \in \widetilde{U}$ so that

$$\begin{aligned}&(\forall \varkappa_2 \in \widetilde{U}, \varkappa_1 \perp \varkappa_2)\\ \text{or } &(\forall \varkappa_2 \in \widetilde{U}, \varkappa_2 \perp \varkappa_1).\end{aligned} \tag{5}$$

The element $\varkappa_1$ is said to be an O-element.

Definition 5 (see [22]). Let $(\widetilde{U}, \perp)$ be an OS. A sequence $\{\varkappa_\eta\}$ is said to be an orthogonal sequence $(\mathrm{O}-\mathrm{seq})$ if

$$\begin{aligned}&(\forall \eta \in \mathbb{N}, \varkappa_\eta \perp \varkappa_{\eta+1})\\ \text{or } &(\forall \eta \in \mathbb{N}, \varkappa_{\eta+1} \perp \varkappa_\eta).\end{aligned} \tag{6}$$

Definition 6 (see [21]). A mapping $\Psi: \widetilde{U} \longrightarrow \widetilde{U}$ is orthogonal continuous (OC) in $\varkappa \in \widetilde{U}$ if for each O-sequence $\{\varkappa_\eta\}_{\eta\in\mathbb{N}}$ in $\widetilde{U}$ such that $\varkappa_\eta \longrightarrow \varkappa$, then $\Psi(\varkappa_\eta) \longrightarrow \Psi(\varkappa)$. Also, Ψ is said to be OC on $\widetilde{U}$ if Ψ is OC at each $\varkappa \in \widetilde{U}$.

Definition 7 (see [22]). Let $(\widetilde{U}, \perp, \sigma_\perp)$ be an orthogonal M.S. Then, $\widetilde{U}$ is called O-complete if every Cauchy O-seq is convergent.

Definition 8 (see [21]). Let $(\widetilde{U}, \perp, \sigma\perp)$ be an orthogonal M.S and $0 < z < 1$. A mapping $\Psi: \widetilde{U} \longrightarrow \widetilde{U}$ is called an orthogonal contraction $(\mathrm{O}-\mathrm{contraction})$ with the Lipschitz constant z if

$$\sigma_\perp(\Psi\varkappa_1, \Psi\varkappa_2) \leq z\sigma_\perp(\varkappa_1, \varkappa_2), \tag{7}$$

for all $\varkappa_1$, $\varkappa_2 \in \widetilde{U}$ with $\varkappa_1 \perp \varkappa_2$.

Definition 9 (see [22]). Let $(\widetilde{U}, \perp)$ be an OS. A mapping $\Psi: \widetilde{U} \longrightarrow \widetilde{U}$ is said to be O-preserving (OP) if $\Psi\varkappa_1 \perp \Psi\varkappa_2$, whenever $\varkappa_1 \perp \varkappa_2$.

3. Main Results

Throughout the paper, O-comp designs orthogonal complete. First, we introduce the concept of an orthogonal partial b-metric space (in short, orthogonal ρ_b-M.S).

Definition 10. A map $\sigma_\perp: \widetilde{U} \times \widetilde{U} \longrightarrow \mathbb{R}^+$ is called an orthogonal ρ_b-M.S on the OS $(\widetilde{U}, \perp)$ if the following axioms are satisfied:

$$\begin{aligned}
&(\sigma_\perp 1)\varkappa_1 = \varkappa_2, \quad \text{if and only if } \sigma_\perp(\varkappa_1,\varkappa_1) = \sigma_\perp(\varkappa_1,\varkappa_2) \\
&\qquad\qquad = \sigma_\perp(\varkappa_2,\varkappa_2), \\
&(\sigma_\perp 2)\sigma_\perp(\varkappa_1,\varkappa_1) \le \sigma_\perp(\varkappa_1,\varkappa_2), \\
&(\sigma_\perp 3)\sigma_\perp(\varkappa_1,\varkappa_2) = \sigma_\perp(\varkappa_2,\varkappa_1), \\
&(\sigma_\perp 4)\sigma_\perp(\varkappa_1,\varkappa_2) \le s[\sigma_\perp(\varkappa_1,\varkappa_3) \\
&\qquad + \sigma_\perp(\varkappa_3,\varkappa_2)] \\
&\qquad - \sigma_\perp(\varkappa_3,\varkappa_3),
\end{aligned} \tag{8}$$

for all $\varkappa_1, \varkappa_2, \varkappa_3 \in \widetilde{U}$ with $\varkappa_1 \perp \varkappa_3 \perp \varkappa_2$.

The pair $(\widetilde{U}, \sigma_\perp)$ is said as an orthogonal ρ_b-M.S with a coefficient $s \ge 1$.

Remark 2. Every orthogonal partial b-metric space is a partial b-metric, but the converse is not true in general. Example 2 describes an orthogonal partial b-metric, which is not a partial b-metric. In [22], Eshaghi Gordji and Habibi considered orthogonal metric spaces to prove some fixed point theorems, while in Theorem 3.2 and Theorem 3.3 of the paper [21], the authors considered generalized orthogonal metric spaces to prove some related fixed point theorems. In this paper, we consider orthogonal partial b metric spaces to prove some fixed point results. Note that an orthogonal partial b-metric is more generalized than an orthogonal metric, an orthogonal b-metric, and an orthogonal partial metric. That is, our results are generalizations and extensions of the results given in [21, 22].

Example 2. Let $\widetilde{U} = \{-1,-2,-3,1,2,3\}$ and let the binary relation be defined by $\varkappa_1 \perp \varkappa_2$ iff $\varkappa_1 = \varkappa_2$ or $\varkappa_1, \varkappa_2 > 0$. It is easy to prove that $\sigma_\perp(\varkappa_1,\varkappa_2) = \max\{|\varkappa_1|, |\varkappa_2|\}$ is an orthogonal ρ_b-metric on $\widetilde{U}$ (with a coefficient $s \ge 1$). However, $\sigma_\perp$ is not a ρ_b-metric on $\widetilde{U}$ (with a coefficient $s \ge 1$). Indeed, for $\varkappa_1 = -3$ and $\varkappa_2 = 3$, we have $\sigma_\perp(\varkappa_1,\varkappa_1) = \sigma_\perp(\varkappa_1,\varkappa_2) = \sigma_\perp(\varkappa_2,\varkappa_2) = 3$.

As related topological notions for this new setting, we state the following definitions.

Definition 11. Let $(\widetilde{U}, \sigma_\perp)$ be an orthogonal ρ_b-M.S with $s \ge 1$. Then, an O-seq $\{\varkappa_\eta\}$ is called

(i) Convergent iff there exists $\varkappa \in \widetilde{U}$ such that $\sigma_\perp(\varkappa_\eta, \varkappa) \longrightarrow 0$ as $\eta \longrightarrow \infty$

(ii) Cauchy iff $\sigma_\perp(\varkappa_\eta, \varkappa_m) \longrightarrow 0$ as $\eta, m \longrightarrow \infty$

Definition 12. Let $(\widetilde{U}, \sigma_\perp)$ be an orthogonal ρ_b-M.S. Then, $\Psi: \widetilde{U} \longrightarrow \widetilde{U}$ is called O-continuous (OC) at $\varkappa \in \widetilde{U}$ if for each O-seq $\{\varkappa_\eta\}$ in $\widetilde{U}$ with $\sigma_\perp(\varkappa_\eta, \varkappa) \longrightarrow 0$, we have $\sigma_\perp(\Psi\varkappa_\eta, \Psi\varkappa) \longrightarrow 0$. Also, Ψ is said to be OC on $\widetilde{U}$ if Ψ is OC at each $\varkappa \in \widetilde{U}$.

Definition 13. Let $(\widetilde{U}, \sigma_\perp)$ be an orthogonal ρ_b-M.S with $s \ge 1$. Then, $\widetilde{U}$ is called O-complete if every Cauchy O-seq is convergent in $\widetilde{U}$.

Our first essential main result is as follows.

Theorem 1. *Let $(\widetilde{U}, \sigma_\perp)$ be an O-comp ρ_b-M.S with $s \ge 1$ and $\Psi: \widetilde{U} \longrightarrow \widetilde{U}$ be an OP and OC mapping so that*

$$\sigma_\perp(\Psi\varkappa_1, \Psi\varkappa_2) \le z\sigma_\perp(\varkappa_1,\varkappa_2), \quad \text{for all } \varkappa_1, \varkappa_2 \in \widetilde{U} \text{ with } \varkappa_1 \perp \varkappa_2, \tag{9}$$

where $z \in [0, 1/s)$. Then, Ψ has a unique fixed point $\varkappa^* \in \widetilde{U}$ and $\sigma_\perp(\varkappa^*, \varkappa^*) = 0$.

Proof. By the definition of orthogonality, there is $\varkappa_1 \in \widetilde{U}$ such that for all $\varkappa_2 \in \widetilde{U}$, $\varkappa_1 \perp \varkappa_2$ or for all $\varkappa_2 \in \widetilde{U}$, $\varkappa_2 \perp \varkappa_1$. It follows that $\varkappa_1 \perp \Psi\varkappa_1$ or $\Psi\varkappa_1 \perp \varkappa_1$. Let $\varkappa_1 = \Psi\varkappa_0$, $\varkappa_2 = \Psi\varkappa_1$, $\varkappa_3 = \Psi\varkappa_2, \varkappa_4 = \Psi\varkappa_3, \ldots, \varkappa_{\eta+1} = \Psi\varkappa_\eta$, for all $\eta \in \mathbb{N}$. Since Ψ is OP, $\{\varkappa_\eta\}$ is an O-seq. Then, by (9), we obtain

$$\sigma_\perp(\varkappa_\eta, \varkappa_{\eta+1}) = \sigma_\perp(\Psi\varkappa_{\eta-1}, \Psi\varkappa_\eta) \le z^\eta \sigma_\perp(\varkappa_0, \varkappa_1), \tag{10}$$

for all $\eta \in \mathbb{N}$. For all $m \ge 1$ and $p \ge 1$, it follows that

$$\begin{aligned}
\sigma_\perp(\varkappa_{m+p}, \varkappa_m) &\le s[\sigma_\perp(\varkappa_{m+p}, \varkappa_{m+p-1}) + \sigma_\perp(\varkappa_{m+p-1}, \varkappa_m)] - \sigma_\perp(\varkappa_{m+p-1}, \varkappa_{m+p-1}) \\
&\le s\sigma_\perp(\varkappa_{m+p}, \varkappa_{m+p-1}) + s^2[\sigma_\perp(\varkappa_{m+p-1}, \varkappa_{m+p-2}) + \sigma_\perp(\varkappa_{m+p-2}, \varkappa_m)] - s\sigma_\perp(\varkappa_{m+p-2}, \varkappa_{m+p-2}) \\
&\le s\sigma_\perp(\varkappa_{m+p}, \varkappa_{m+p-1}) + s^2\sigma_\perp(\varkappa_{m+p-1}, \varkappa_{m+p-2}) + s^2\sigma_\perp(\varkappa_{m+p-2}, \varkappa_m) \\
&\le s\sigma_\perp(\varkappa_{m+p}, \varkappa_{m+p-1}) + s^2\sigma_\perp(\varkappa_{m+p-1}, \varkappa_{m+p-2}) + s^3\sigma_\perp(\varkappa_{m+p-2}, \varkappa_{m+p-3}) + \cdots + s^{p-1}\sigma_\perp(\varkappa_{m+2}, \varkappa_{m+1}) + s^{p-1}\sigma_\perp(\varkappa_{m+1}, \varkappa_m) \\
&\le sz^{m+p-1}\sigma_\perp(\varkappa_1, \varkappa_0) + s^2 z^{m+p-2}\sigma_\perp(\varkappa_1, \varkappa_0) + s^3 z^{m+p-3}\sigma_\perp(\varkappa_1, \varkappa_0) + \cdots + s^{p-1}z^{m+1}\sigma_\perp(\varkappa_1, \varkappa_0) + s^{p-1}z^m\sigma_\perp(\varkappa_1, \varkappa_0) \\
&\le [sz^{m+p-1} + s^2 z^{m+p-2} + s^3 z^{m+p-3} + \cdots + s^{p-1}z^{m+1} + s^{p-1}z^m]\sigma_\perp(\varkappa_1, \varkappa_0) \\
&\le \frac{s^p z^m}{1 - sz}\sigma_\perp(\varkappa_1, \varkappa_0).
\end{aligned} \tag{11}$$

Taking limit as $m \longrightarrow \infty$, we have

$$\lim_{\eta\longrightarrow\infty} \sigma_\perp\left(\varkappa_{m+p}, \varkappa_m\right) = 0. \tag{12}$$

Therefore, $\{\varkappa_\eta\}$ is a Cauchy O-seq. Since $\widetilde{U}$ is O-comp, there is $\varkappa_1^* \in \widetilde{U}$ so that $\varkappa_\eta \longrightarrow \varkappa^*$ as $\eta \longrightarrow \infty$. Since Ψ is OC, we obtain

$$\Psi\varkappa^* = \Psi\left(\lim_{\eta\longrightarrow\infty} \varkappa_\eta\right) = \lim_{\eta\longrightarrow\infty} \Psi\varkappa_\eta = \lim_{\eta\longrightarrow\infty} \varkappa_{\eta+1} = \varkappa^*. \tag{13}$$

Hence, $\varkappa^*$ is a fixed point of Ψ. Next, we demonstrate its uniqueness. Let $\varkappa_2^* \in \widetilde{U}$ be a fixed point of Ψ. So, we obtain $\Psi^\eta\varkappa^* = \varkappa^*$ and $\Psi^\eta\varkappa_2^* = \varkappa_2^*$ for all $\eta \in \mathbb{N}$. By the definition of orthogonality, there is $\varkappa_1 \in \widetilde{U}$ so that

$$\begin{gathered}\left[\varkappa_1 \perp \varkappa^* \text{ and } \varkappa_1 \perp \varkappa_2^*\right] \\ \text{or } \left[\varkappa^* \perp \varkappa_1 \text{ and } \varkappa_2^* \perp \varkappa_1\right].\end{gathered} \tag{14}$$

Since Ψ is OP, one writes

$$\left[\Psi^\eta\varkappa_1 \perp \Psi^\eta\varkappa^* \text{ and } \Psi^\eta\varkappa_1^* \perp \Psi^\eta\varkappa_2^*\right], \tag{15}$$

or

$$\left[\Psi^\eta\varkappa^* \perp \Psi^\eta\varkappa_1 \text{ and } \Psi^\eta\varkappa_2^* \perp \Psi^\eta\varkappa_1\right], \tag{16}$$

for all $\eta \in \mathbb{N}$. Therefore, by the triangle inequality, we obtain

$$\begin{aligned}\sigma_\perp\left(\varkappa_1^*, \varkappa_2^*\right) &= \sigma_\perp\left(\Psi^\eta\varkappa_1^*, \Psi^\eta\varkappa_2^*\right) \\ &= s\left[\sigma_\perp\left(\Psi^\eta\varkappa_1^*, \Psi^\eta\varkappa_1\right) + \sigma_\perp\left(\Psi^\eta\varkappa_1, \Psi^\eta\varkappa_2^*\right)\right] \\ &\quad - \sigma_\perp\left(\Psi^\eta\varkappa_1, \Psi^\eta\varkappa_1\right) \\ &\leq sz^\eta\sigma_\perp\left(\varkappa_1^*, \varkappa_1\right) + sz^\eta\sigma_\perp\left(\varkappa_1, \varkappa_2^*\right).\end{aligned} \tag{17}$$

Taking limit as $\eta \longrightarrow \infty$, we obtain

$$\sigma_\perp\left(\varkappa_1^*, \varkappa_2^*\right) = 0, \tag{18}$$

and so $\varkappa_1^* = \varkappa_2^*$. □

Example 3. Consider $\widetilde{U} = \mathbb{R}$. Given $\sigma_\perp: \widetilde{U} \times \widetilde{U} \longrightarrow \mathbb{R}^+$ as

$$\sigma_\perp\left(\varkappa_1, \varkappa_2\right) = \left\{\begin{array}{ll} \left|\varkappa_1 - \varkappa_2\right|^2, & \text{if } \varkappa_1, \varkappa_2 \geq 0, \\ 0, & \text{otherwise.}\end{array}\right\} \tag{19}$$

Define $\perp$ on $\widetilde{U}$ by $\varkappa_1 \perp \varkappa_2$ iff $\varkappa_1 = \varkappa_2$ or $\varkappa_1, \varkappa_2 \geq 0$. Then, $(\widetilde{U}, \sigma_\perp)$ is an O-comp ρ_b-M.S with coefficient $s = 4$. Define the mapping $\Psi: \widetilde{U} \longrightarrow \widetilde{U}$ by

$$\Psi(\varkappa) = \left\{\begin{array}{ll} \dfrac{\varkappa}{4}, & \varkappa \geq 0, \\ 0, & \text{otherwise.}\end{array}\right\} \tag{20}$$

We have the following cases:

(a) If $\varkappa_1 = \varkappa_2$, then $\Psi(\varkappa_1) = \Psi(\varkappa_2)$. If $\varkappa_1, \varkappa_2 \geq 0$, then $\Psi(\varkappa_1), \Psi(\varkappa_2) \geq 0$. Thus, Ψ is OP.

(b) If any O-seq $\{\varkappa_\eta\}$ in $\widetilde{U}$ with $\lim_{n\longrightarrow\infty}\sigma_\perp(\varkappa_\eta, \varkappa) = \sigma_\perp(\varkappa, \varkappa)$, $\varkappa_\eta \perp \varkappa$ for some $\eta \geq k$ and $\lim_{n\longrightarrow\infty}\sigma_\perp(\varkappa_\eta, \Psi\varkappa) = \sigma_\perp(\varkappa, \varkappa)$, then we obtain $\varkappa \perp \Psi\varkappa$.

(c) If $\varkappa_0 \geq 0$ any real number, then $\Psi(\varkappa_0) \geq 0$. Thus, $\varkappa_0$ and $\Psi(\varkappa_0) \geq 0$, that is, $\varkappa_0 \perp \Psi(\varkappa_0)$.

(d) Let $\varkappa_1, \varkappa_2 \in \widetilde{U}$ with $\varkappa_1 \perp \varkappa_2$.

If $\varkappa_1 = \varkappa_2$, then the following result holds:

$$\sigma_\perp\left(\Psi\varkappa_1, \Psi\varkappa_2\right) = 0 = \frac{1}{16}\sigma_\perp\left(\varkappa_1, \varkappa_2\right). \tag{21}$$

If $\varkappa_1, \varkappa_2 \geq 0$, then the following result holds:

$$\begin{aligned}\sigma_\perp\left(\Psi\varkappa_1, \Psi\varkappa_2\right) &= \left|\frac{\varkappa_1}{4} - \frac{\varkappa_2}{4}\right|^2 \\ &= \frac{1}{16}\left|\varkappa_1 - \varkappa_2\right|^2 \\ &= \frac{1}{16}\sigma_\perp\left(\varkappa_1, \varkappa_2\right).\end{aligned} \tag{22}$$

By Theorem 1, Ψ has a fixed point.

Our second result as a generalization of Theorem 1 is as follows.

Theorem 2. *Let $(\widetilde{U}, \sigma_\perp)$ be an O-comp ρ_b-M.S with $s \geq 1$ and $\Psi: \widetilde{U} \longrightarrow \widetilde{U}$ be an OP and OC mapping so that*

$$\sigma_\perp\left(\Psi\varkappa_1, \Psi\varkappa_2\right) \leq z \max\left\{\sigma_\perp\left(\varkappa_1, \varkappa_2\right), \sigma_\perp\left(\varkappa_1, \Psi\varkappa_1\right), \sigma_\perp\left(\varkappa_2, \Psi\varkappa_2\right)\right\}, \tag{23}$$

for all $\varkappa_1, \varkappa_2 \in \widetilde{U}$, where $z \in [0, 1/s)$. Then, Ψ has a unique fixed point $\varkappa^* \in \widetilde{U}$ and $\sigma_\perp(\varkappa^*, \varkappa^*) = 0$.

Proof. By the definition of orthogonality, there is $\varkappa_1 \in \widetilde{U}$ so that for all $\varkappa_2 \in \widetilde{U}$, $\varkappa_1 \perp \varkappa_2$ or for all $\varkappa_2 \in \widetilde{U}$, $\varkappa_2 \perp \varkappa_1$. It follows that $\varkappa_1 \perp \Psi\varkappa_1$ or $\Psi\varkappa_1 \perp \varkappa_1$. Let $\varkappa_1 = \Psi\varkappa_0, \varkappa_2 = \Psi\varkappa_1, \ldots, \varkappa_{\eta+1} = \Psi\varkappa_\eta$ for all $\eta \in \mathbb{N}$. Since Ψ is OP, $\{\varkappa_\eta\}$ is an O-seq. Then, by (23), we have

$$\begin{aligned}\sigma_\perp\left(\varkappa_{\eta+1}, \varkappa_\eta\right) &= \sigma_\perp\left(\Psi\varkappa_\eta, \Psi\varkappa_{\eta-1}\right) \\ &\leq z \max\left\{\sigma_\perp\left(\varkappa_\eta, \varkappa_{\eta-1}\right), \sigma_\perp\left(\varkappa_\eta, \Psi\varkappa_\eta\right), \sigma_\perp\left(\varkappa_{\eta-1}, \Psi\varkappa_{\eta-1}\right)\right\} \\ &= z \max\left\{\sigma_\perp\left(\varkappa_\eta, \varkappa_{\eta-1}\right), \sigma_\perp\left(\varkappa_\eta, \varkappa_{\eta+1}\right), \sigma_\perp\left(\varkappa_{\eta-1}, \varkappa_\eta\right)\right\} \\ &= z \max\left\{\sigma_\perp\left(\varkappa_\eta, \varkappa_{\eta-1}\right), \sigma_\perp\left(\varkappa_\eta, \varkappa_{\eta+1}\right)\right\}.\end{aligned} \tag{24}$$

If for some η, $\max\{\sigma_\perp(\varkappa_\eta, \varkappa_{\eta-1}), \sigma_\perp(\varkappa_\eta, \varkappa_{\eta+1})\} = \sigma_\perp(\varkappa_{\eta+1}, \varkappa_\eta)$, we obtain that $\sigma_\perp(\varkappa_{\eta+1}, \varkappa_\eta) \leq z\, \sigma_\perp(\varkappa_{\eta+1}, \varkappa_\eta) < \sigma_\perp(\varkappa_{\eta+1}, \varkappa_\eta)$, which is a contradiction. Thus,

$$\max\left\{\sigma_\perp\left(\varkappa_\eta, \varkappa_{\eta-1}\right), \sigma_\perp\left(\varkappa_\eta, \varkappa_{\eta+1}\right)\right\} = \sigma_\perp\left(\varkappa_\eta, \varkappa_{\eta-1}\right), \quad \text{for all } \eta \geq 1. \tag{25}$$

Again, we have

$$\sigma_\perp\left(\varkappa_{\eta+1}, \varkappa_\eta\right) \leq z\sigma_\perp\left(\varkappa_\eta, \varkappa_{\eta-1}\right). \tag{26}$$

Repeating this cycle, we obtain

$$\sigma_\perp(\varkappa_{\eta+1},\varkappa_\eta)\le z^\eta\sigma_\perp(\varkappa_1,\varkappa_0), \tag{27}$$

for all $\eta\ge 0$. For m, $\eta\in\mathbb{N}$ with $m>\eta$, we obtain

$$\begin{aligned}\sigma_\perp(\varkappa_\eta,\varkappa_m)&\le s\left[\sigma_\perp(\varkappa_\eta,\varkappa_{\eta+1})+\sigma_\perp(\varkappa_{\eta+1},\varkappa_m)\right]-\sigma_\perp(\varkappa_{\eta+1},\varkappa_{\eta+1})\\&\le s\sigma_\perp(\varkappa_\eta,\varkappa_{\eta+1})+s^2\left[\sigma_\perp(\varkappa_{\eta+1},\varkappa_{\eta+2})+\sigma_\perp(\varkappa_{\eta+2},\varkappa_m)\right]-s\sigma_\perp(\varkappa_{\eta+2},\varkappa_{\eta+2})\\&\le s\sigma_\perp(\varkappa_\eta,\varkappa_{\eta+1})+s^2\sigma_\perp(\varkappa_{\eta+1},\varkappa_{\eta+2})+s^3\sigma_\perp(\varkappa_{\eta+2},\varkappa_{\eta+3})+\cdots+s^{m-\eta}\sigma_\perp(\varkappa_{m-1},\varkappa_m).\end{aligned} \tag{28}$$

Using (27) in the above inequality,

$$\begin{aligned}\sigma_\perp(\varkappa_\eta,\varkappa_m)&\le sz^\eta\sigma_\perp(\varkappa_1,\varkappa_0)+s^2z^{\eta+1}\sigma_\perp(\varkappa_1,\varkappa_0)+s^3z^{\eta+2}\sigma_\perp(\varkappa_1,\varkappa_0)+\cdots+s^{m-\eta}z^{m-1}\sigma_\perp(\varkappa_1,\varkappa_0)\\&\le sz^\eta\left[1+sz+(sz)^2+\cdots\right]\sigma_\perp(\varkappa_1,\varkappa_0)\\&=\frac{sz^\eta}{1-sz}\sigma_\perp(\varkappa_1,\varkappa_0).\end{aligned} \tag{29}$$

As $z\in[0,1/s)$ and $s'\ge 1$, it follows from the above inequality that

$$\lim_{\eta,m\longrightarrow\infty}\sigma_\perp(\varkappa_\eta,\varkappa_m)=0. \tag{30}$$

Therefore, $\{\varkappa_\eta\}$ is a Cauchy O-seq. Since $\widetilde{U}$ is O-comp, there is $\varkappa^*\in\widetilde{U}$ so that $\varkappa_\eta\longrightarrow\varkappa^*$ as $\eta\longrightarrow\infty$. As Ψ is OC, one writes

$$\Psi\varkappa^*=\Psi\left(\lim_{\eta\longrightarrow\infty}\varkappa_\eta\right)=\lim_{\eta\longrightarrow\infty}\Psi\varkappa_\eta=\lim_{\eta\longrightarrow\infty}\varkappa_{\eta+1}=\varkappa^*. \tag{31}$$

Therefore, $\varkappa^*$ is a fixed point. To show that it is unique, consider $\varkappa_2^*\,(\neq\varkappa_1^*)\in\widetilde{U}$ as a fixed point of Ψ. So, we obtain $\Psi^\eta\varkappa_1^*=\varkappa_1^*$ and $\Psi^\eta\varkappa_2^*=\varkappa_2^*$ for all $\eta\in\mathbb{N}'$. By the definition of orthogonality, there exists $\varkappa_1\in\widetilde{U}$ so that

$$\begin{gathered}[\varkappa_1\perp\varkappa_1^*\text{ and }\varkappa_1\perp\varkappa_2^*]\\\text{or }[\varkappa_1^*\perp\varkappa_1\text{ and }\varkappa_2^*\perp\varkappa_1].\end{gathered} \tag{32}$$

As Ψ is OP, we obtain

$$[\Psi^\eta\varkappa_1\perp\Psi^\eta\varkappa_1^*\text{ and }\Psi^\eta\varkappa_1\perp\Psi^\eta\varkappa_2^*], \tag{33}$$

or

$$[\Psi^\eta\varkappa_1^*\perp\Psi^\eta\varkappa_1\text{ and }\Psi^\eta\varkappa_2^*\perp\Psi^\eta\varkappa_1], \tag{34}$$

for all $\eta\in\mathbb{N}$. Then, we obtain

$$\begin{aligned}\sigma_\perp(\varkappa_1^*,\varkappa_2^*)=\sigma_\perp(\Psi^\eta\varkappa_1^*,\Psi^\eta\varkappa_2^*)&\le z\max\{\sigma_\perp(\varkappa_1^*,\varkappa_2^*),\sigma_\perp(\varkappa_1^*,\Psi\varkappa_1^*),\sigma_\perp(\varkappa_2^*,\Psi\varkappa_2^*)\}\\&=z\max\{\sigma_\perp(\varkappa_1^*,\varkappa_2^*),\sigma_\perp(\varkappa_1^*,\varkappa_1^*),\sigma_\perp(\varkappa_2^*,\varkappa_2^*)\}\\&=z\sigma_\perp(\varkappa_1^*,\varkappa_2^*)\\&<\sigma_\perp(\varkappa_1^*,\varkappa_2^*).\end{aligned} \tag{35}$$

It is a contradiction. So, we need to have $\sigma_\perp(\varkappa_1^*,\varkappa_2^*)=0$, that is, $\varkappa_1^*=\varkappa_2^*$. Thus, if fixed point of Ψ exists, then it is unique. □

The following corollary is the analog of Kannan fixed point theorem [24] in orthogonal partial b-metric spaces.

Corollary 1. *Let $(\widetilde{U},\sigma_\perp)$ be an O-comp ρ_b-M.S with $s\ge 1$ and $\Psi\colon\widetilde{U}\longrightarrow\widetilde{U}$ be an OP and OC mapping so that*

$$\sigma_\perp(\Psi\varkappa_1,\Psi\varkappa_2)\le z\{\sigma_\perp(\varkappa_1,\Psi\varkappa_1)+\sigma_\perp(\varkappa_2,\Psi\varkappa_2\}, \tag{36}$$

for all $\varkappa_1$, $\varkappa_2\in\widetilde{U}$, where $z\in[0,1/s)$. Then, Ψ has a unique fixed point $\varkappa^*\in\widetilde{U}$ and $\sigma_\perp(\varkappa^*,\varkappa^*)=0$.

The following corollary is the analog of Bianchini fixed point theorem [25] in orthogonal partial b-metric spaces.

Corollary 2. *Let $(\widetilde{U},\sigma_\perp)$ be an O-comp ρ_b-M.S with $s\ge 1$ and $\Psi\colon\widetilde{U}\longrightarrow\widetilde{U}$ be an OP and OC mapping so that*

$$\sigma_\perp(\Psi\varkappa_1,\Psi\varkappa_2)\le z\max\{\sigma_\perp(\varkappa_1,\Psi\varkappa_1),\sigma_\perp(\varkappa_2,\Psi\varkappa_2)\}, \tag{37}$$

for all $\varkappa_1, \varkappa_2 \in \widetilde{U}$, *where* $0 \le z < 1$. *Then,* Ψ *has a unique fixed point* $\varkappa^* \in \widetilde{U}$ *and* $\sigma_\perp(\varkappa^*, \varkappa^*) = 0$.

4. Application

In this section, we consider the Volterra integral type equation:

$$\varkappa_1(e) = h(e) + k\int_0^1 u(e, s) f(s, \varkappa_1(s))\mathrm{d}s, \quad e \in I = [0,1], k \ge 0. \tag{38}$$

Take the space $\widetilde{U} = C(I)$ of continuous functions defined on I endowed with a metric given by

$$p(\varkappa_1, \varkappa_2) = \sup_{e\in I}|\varkappa_1(e) - \varkappa_2(e)|, \tag{39}$$

for all $\varkappa_1, \varkappa_2 \in C(I)$.

Let γ denote the class of function γ: $\mathbb{R}^+ \longrightarrow \mathbb{R}^+$ so that $(\gamma(\mu))^q \le \gamma(\mu^q)$ for each $q \ge 1$ and $\mu \ge 0$.

We consider the following assumptions:

(i) f: $I \times \mathbb{R} \longrightarrow \mathbb{R}$ is nondecreasing with respect to its second variable and continuous so that there is $0 \le L \le 1$:

$$|f(e, v_1) - f(e, v_2)| \le L\gamma(v_1 - v_2), \tag{40}$$

for all $v_1, v_2 \in \mathbb{R}$ with $v_1 \ge v_2$.

(ii) h: $I \longrightarrow \mathbb{R}$ is continuous on I.

(iii) u: $I \times I \longrightarrow \mathbb{R}$ is continuous with respect to its first variable and measurable with respect to its second variable such that for each $e \in I$,

$$\int_0^1 u(e, s)\mathrm{d}s \le K. \tag{41}$$

(iv) $L^q k^q K^q \le (1/2^{4q-4})$.

We consider on $\widetilde{U}$ the following: $\varkappa_1, \varkappa_2 \in C(I)$ and $\varkappa_1 \perp \varkappa_2 \Longleftrightarrow \varkappa_1 \le \varkappa_2$.

Now, for $q \ge 1$, we define

$$\begin{aligned}\sigma_\perp(\varkappa_1, \varkappa_2) &= (p(\varkappa_1, \varkappa_2))^q = (\sup_{e\in I}|\varkappa_1(e) - \varkappa_2(e)|)^q \\ &= \sup_{e\in I}|\varkappa_1(e) - \varkappa_2(e)|^q,\end{aligned} \tag{42}$$

for $\varkappa_1, \varkappa_2 \in C(I)$.

We conclude that $(\widetilde{U}, t\rho_b)$ is a O-comp ρ_b M.S with $s = 2^{q-1}$.

Now, we formulate the main result of this section.

Theorem 3. *Under the assumptions (i)-(iv), equation (38) has a unique solution in* $C(I)$.

Proof. We consider the operator Ψ: $\widetilde{U} \longrightarrow \widetilde{U}$ defined by

$$\Psi\varkappa(e) = h(e) + k\int_0^1 u(e, s) f(s, \varkappa(s))\mathrm{d}s, \tag{43}$$

for $e \in I$ and $k \ge 0$.

By virtue of our assumptions, Ψ is well defined (if $\varkappa \in \widetilde{U}$, then $\Psi(\varkappa) \in \widetilde{U}$).

For $\varkappa_1 \le \varkappa_2$ and $e \in I$, we have

$$\begin{aligned}\Psi\varkappa_1(e) - \Psi\varkappa_2(e) &= h(e) + k\int_0^1 u(e, s) f(s, \varkappa_1(s))\mathrm{d}s - h(e) \\ &\quad - k\int_0^1 u(e, s) f(s, \varkappa_2(s))\mathrm{d}s \\ &= k\int_0^1 u(e, s)[f(s, \varkappa_1(s)) \\ &\quad - f(s, \varkappa_2(s))]\mathrm{d}s \le 0.\end{aligned} \tag{44}$$

Therefore, Ψ has the monotone nondecreasing property. Also, for $\varkappa_1 \perp \varkappa_2$, we have

$$\begin{aligned}|\Psi\varkappa_1(e) - \Psi\varkappa_2(e)| &= \left|h(e) + k\int_0^1 u(e, s) f(s, \varkappa_1(s))\mathrm{d}s \right. \\ &\quad \left. - h(e) - k\int_0^1 u(e, s) f(s, \varkappa_2(s))\mathrm{d}s\right| \\ &\le k\int_0^1 u(e, s)|f(s, \varkappa_1(s)) - f(s, \varkappa_2(s))|\mathrm{d}s \\ &\le k\int_0^1 u(e, s)L\gamma|\varkappa_1 - \varkappa_2|.\end{aligned} \tag{45}$$

Since $\varkappa_1 \perp \varkappa_2$, we have

$$\gamma(\varkappa_2(s) - \varkappa_1(s)) \le \gamma\left(\sup_{e\in I}|\varkappa_1(s) - \varkappa_2(s)|\right) = \gamma(p(\varkappa_1, \varkappa_2)), \tag{46}$$

hence

$$\begin{aligned}|\Psi\varkappa_1(e) - \Psi\varkappa_2(e)| &\le k\int_0^1 u(e, s)L\gamma(p(\varkappa_1, \varkappa_2))\mathrm{d}s \\ &\le kKL\gamma(p(\varkappa_1, \varkappa_2)).\end{aligned} \tag{47}$$

Then, we obtain

$$\begin{aligned}\sigma_\perp(\Psi(\varkappa_1), \Psi(\varkappa_2)) &= \sup_{e\in I}|\Psi\varkappa_1(e) - \Psi\varkappa_2(e)|^q \\ &\le \{kKL\gamma p(\varkappa_1, \varkappa_2)\}^q \\ &= k^q K^q L^q \gamma(p(\varkappa_1, \varkappa_2))^q \\ &= k^q K^q L^q \gamma\sigma_\perp(\varkappa_1, \varkappa_2) \\ &\le \frac{1}{2^{4q-4}}\sigma_\perp(\varkappa_1, \varkappa_2).\end{aligned} \tag{48}$$

This proves that the operator Ψ satisfies the contractive condition (9) appearing in Theorem 1. So, (38) has a solution and the proof is complete. □

5. Conclusion

The study of fixed points of mappings satisfying orthogonal sets has been focused vigorously on different research activities in the recent decade. As a consequence, many mathematicians obtained more results in this direction. In this paper, the concept of generalized orthogonal contractive conditions in partial b-metric spaces was introduced. Based on this notion, fixed point results have been discussed. Some illustrative examples are furnished, which demonstrate the validity of the hypotheses and degree of utility of the proposed results. It would be interesting to consider more generalized orthogonal contractions in this setting.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest.

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