Research Article

List Edge Colorings of Planar Graphs without Adjacent 7-Cycles

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Received 3 January 2021; Revised 29 March 2021; Accepted 3 April 2021; Published 15 April 2021

1. Introduction

Edge coloring and list edge coloring of graphs are very old fashioned problems in graph theory, and the research on such problems has a long history. Denote \( \mathbb{Z}^+ \) as the set of the integers. Now, we only consider the list edge coloring problem of finite simple undirected graphs. Before describing the concept of list edge coloring in detail, we have to revisit fundamental conception of normal edge coloring. A graph \( G \) is \( k \)-edge-colorable if all edges of \( G \) can be colored by \( k \) colors such that no two adjacent edges get the same colors. Denoted \( \chi' (G) \) as the edge chromatic number of a graph \( G \), which is the smallest \( k \in \mathbb{Z}^+ \) satisfying \( G \) is \( k \)-edge-colorable. For each edge \( e \) of graph \( G \), if we can assign a list \( L(e) \) of colors to it, then \( L \) is an edge assignment. Graph \( G \) is edge-L-colorable if \( G \) has a proper edge-coloring \( \phi \) such that \( \phi (e) \in L(e) \) for each edge \( e \) of \( G \), and \( \phi \) is an edge-L-coloring. Graph \( G \) is edge-\( k \)-choosable if for every \( L \) satisfying \( |L(e)| \geq k (k \in \mathbb{Z}^+) \) for each edge \( e \). Denote \( \chi'_l (G) \) as the list-edge chromatic number of \( G \) which is the smallest \( k \) in \( \mathbb{Z}^+ \) such that \( G \) is edge-\( k \)-choosable.

We will study the list edge colorings of planar graphs. Planar graph is a kind of graph broadly studied in graph coloring theory. The so-called plane graph is actually a special drawing method of planar graph, which can be embedded in the plane satisfying no two edges intersect geometrically except at a vertex to which they are both incident. Let us introduce some definitions and symbols needed. Given a plane graph \( G \), we use \( V(G) \), \( E(G) \), \( F(G) \), \( \Delta (G) \), and \( \delta (G) \) to indicate its vertex set, edge set, face set, maximum degree, and minimum degree, respectively. For a vertex \( v \in V(G) \), let \( E_G (v) \) or \( E (v) \) be the set of edges which are incident with \( v \). We use \( d_G (v) \) or \( d (v) \) to indicate the degree of \( v \) in \( G \), which is the number of edges in \( E (v) \). We use \( N_G (v) \) or \( N (v) \) to indicate the set of the vertices which are adjacent to \( v \) in \( G \). Denoted \( k \)-vertex, \( k^- \)-vertex, or \( k^+ \)-vertex as a vertex of degree \( k \), at most \( k \) or at least \( k \), respectively. A \( k \)-\( (k^+) \)-neighbor of \( v \) is a \( k \) (or \( k^+ \))-vertex which is adjacent to a vertex \( v \). A \( k \)-cycle is a cycle of length \( k \). Two cycles are adjacent, that is, the two cycles share at least a common edge. A \( 2 \)-alternating cycle is an even cycle in which the \( 2 \)-vertices appear alternately. For \( f \in F(G) \), we use \( d_c (f) \) to indicate the degree of a face, which is the number of edges incident with \( f \) where each cut edge is counted twice. Denote \( k^- \)-face as a face of degree \( k \), at least \( k \). For a \( k \)-face of \( G \), we called it \( (i_1, i_2, \ldots, i_k) \)-face if the vertices incident with it are of degrees \( i_1, i_2, \ldots, i_k \), respectively. We use \( f_k (v) \) to indicate the number of \( k \)-faces which are incident with \( v \), \( d_k (f) \) the number of \( k \)-faces which are incident with \( f \), and \( d_k (v) \) the number of \( k \)-vertices which are incident with \( v \).

List edge coloring was firstly put forward by Vizing [1] and later by Bollobas and Harris [2]. They posed Conjecture 1 which is later called the List Coloring Conjecture.

Conjecture 1. For every multigraph \( G \), \( \chi'_l (G) = \chi' (G) \).

So far, Conjecture 1 has only been proved to be true for a few graphs, including bipartite multigraphs [3], complete graphs of odd order [4], multicircuits [5], graphs embedded in a surface of nonnegative characteristic and \( \Delta (G) \geq 12 \) [6], and outer planar graphs [7]. For planar graphs, the readers can see [8–12].
Wang and Wu [13] proved that $G$ is edge-$k$-choosable ($k = \max(10, \Delta(G))$) for a planar graph $G$ without adjacent 7-cycles with three chords. Now, we will study the planar graphs without adjacent 7-cycles and obtain:

**Theorem 1.** Suppose that $G$ is a planar graph which contains no adjacent 7-cycles. Then, $G$ is edge-$k$-choosable, where $k = \max(10, \Delta(G))$.

From Theorem 1, we can obtain the following corollary.

**Corollary 1.** Suppose that $G$ is a planar graph which contains no adjacent 7-cycles, and $\chi^*(G) = \Delta(G)$ for $\Delta(G) \geq 10$.

### 2. The Proofs of Theorem

**Proof.** Let $G = (V(G), E(G), F(G))$ be a minimal graph satisfying the number of $|E(G)|$ as little as possible; then, the graph $G$ has the following properties.

**Lemma 1** (see [14]). Let $G$ be a planar graph, by the minimality hypothesis of graph $G$, and we have

1. $\delta(G) \geq 2$ and $G$ is a connected graph
2. $G$ does not contain edge $v_1v_2$ satisfying $\min\{d(v_1), d(v_2)\} \leq (k/2)$ and $d(v_1) + d(v_2) \leq k + 1$
3. $G$ does not contain a 2-alternating cycle

Denote $G_2$ as the induced subgraph of $G$ by all 2-vertices of $G$, where $E(G_2) = \{v_1v_2 : d(v_1) = 2\}$. By Lemma 1(2) and (3), $G_2$ contains no odd cycle and even cycle. Therefore, $G_2$ must be a forest. Thereby, there must be a matching $M$ in $G_2$ and all 2-vertices in $M$ are saturated. If $v_1v_2 \in M$ and $d_{G_2}(v_1) = 2$, then $v_2$ is named the 2-master of $v_1$ and $v_1$ is the dependent of $v_2$. Obviously, each 2-vertex has a 2-master and each k-vertex may be the 2-master of no more than one 2-vertex.

**Lemma 2** (see [15]). Given $X = \{x \in V(G) \mid d_G(x) \leq 3\}$ and $Y = \cup x \in N(x)$. Then, there is a bipartite subgraph $M'$ of $G$ with parts $X$ and $Y$ satisfying $d_M'(x) = 1$, for any $x \in X$, and $d_M'(y) \leq 2$, for any $y \in Y$.

Note that, in Lemma 2, we mark $y$ as the 3-master of $x$ if $xy \in M'$ and $x \in X$.

**Lemma 3** Suppose that $G$ is a planar graph which contains no adjacent 7-cycles and $d(v) = 9$. Then,

1. If $f_3(v) = 6$ and one of its edges is not incident with any 3-face (as in Figure 1, (1–10), (1–11), (1–13)), then $f_7'(v) \geq 1$
2. If $f_3(v) = 7$, then $f_7'(v) = 2$

**Proof.** Let $N_G(v) = \{v_1, v_2, \ldots, v_k\}$, where all the neighbors $v_i$ ($1 \leq i \leq 9$) of $v$ are in an anticlockwise order. We use $f_1$ to indicate the face which is incident with $v$, $v_i$, and $v_{i+1}$ ($1 \leq i \leq 9$).

(1) Now, we prove (1–10) in Figure 1. Suppose that $f_1, f_2, \ldots, f_5$ and $f_6$ are 3-faces. If $f_6$ is a 4-face or 5-face or 6-face, then there will be adjacent 7-cycles in $G$. It must be $d(f_6) \geq 7$ and $f_7'(v) \geq 1$. The proof process of (1–11) and (1–13) is similar, so we will not repeat it.

(2) Its proof process is similar to (1), so we will not repeat it.

Similarly, we can get the following two lemmas.

**Lemma 4.** Suppose that $G$ is a planar graph which contains no adjacent 7-cycles and $d(v) = 10$. Then,

1. If $f_3(v) = 5$ and four of its edges are not on any 3-face (as shown in Figure 2 (1–2)), then $f_7'(v) \geq 1$
2. If $f_3(v) = 6$ (as shown in figure (2–16) – (2–27)), then $f_7'(v) \geq 1$
3. If $f_3(v) = 7$ (as shown in figure (2–28) – (2–33)), then $f_7'(v) \geq 2$
4. If $f_3(v) = 8$ (as shown in figure (2–34), (2–35)), then $f_7'(v) = 2$

**Lemma 5.** Suppose that $G$ is a planar graph which contains no adjacent 7-cycles and $d(v) = 11$. Then,

1. If $f_3(v) = 8$ and one of its edges is not on any 3-face (as shown in Figure 3 (3–41), (3–43)), then $f_7'(v) \geq 1$
2. If $f_3(v) = 9$ (as in figure (3–47)), then $f_7'(v) = 2$

Now, we complete the proof by Euler’s formula. In [6], the authors proved $\chi^*(G) = \Delta(G)$ for every planar graph with $\Delta(G) \geq 12$, so we only assume that $\Delta(G) \leq 11$ in our proof. Suppose that $G$ is already embedded in the plane. We can obtain

$$
\sum_{x \in V(G)} (d(x) - 4) + \sum_{x \in F(G)} (d(x) - 4) = -8 < 0, \tag{1}
$$

by practical Euler’s formula $|V(G)| - |E(G)| + |F(G)| = 2$.

Firstly, denote $ch$ as the original charge. For each $x \in V(F), let ch(x) = d(x) - 4$. Therefore, $\sum_{x \in V(F)} ch(x) < 0$. Secondly, we formulate some rules to redistribute the original charge and each $x \in V \cup F$ will get a new charge $ch'(x)$. Note that the rules we formulated only move between the vertices and faces of the graph and have no effect on the total charge. Thirdly, we will show that $ch'(x) > 0 \ (x \in V \cup F)$. If we can do so, then we will obtain an apparent contradiction $0 \leq \sum_{x \in V \cup F} ch(x) \leq \sum_{x \in V \cup F} ch(x) < 0$. The proof of Theorem 1 is completed.

The discharging rules are formulated as in R1–R5. We use $c(x \to y)$ to indicate the charge from $x$ to $y$.

### R1. Let $d(v) = 2$, $v_1$ be a 3-master vertex and $v_2$ be 2-master vertex of $v$. Firstly, $c(v_1 \to v) = 1$. Secondly, if $v$ is on a face $f$ with $d(f) \geq 5$, then $c(f \to v) = (1/2)$ and $c(v_2 \to v) = (1/2)$, otherwise $c(v_1 \to v) = 1$.

### R2. Let $d(v) = 3$ and $v_1$ and $v_2$ be two 3-masters of $v$, then $c(v_1 \to v) = 1$ and $c(v_2 \to v) = 1$. 

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R3. Every 5-vertex receives \( \frac{2}{15} \) from each of its 7+ neighbors.

R4. Let \( f \) be a 3-face \((v_1v_2v_3)\) with \( d(v_1) \leq d(v_2) \leq d(v_3)\).

R4.1 If \( d(v_3) = 3 \) or 4, then \( c(v_2 \rightarrow f) = \frac{1}{2} \) and \( c(v_3 \rightarrow f) = \frac{1}{2} \).

R4.2 If \( d(v_1) \geq 5 \), then \( c(v_i \rightarrow f) = \frac{1}{3} \) for \( i = 1, 2, 3 \).

Figure 1: Vertex \( v \) is a 9-vertex, the various cases of \( f_3(v) \geq 5 \).

Figure 2: Vertex \( v \) is a 10-vertex, the various cases of \( f_3(v) \geq 5 \).
R5. Let $d(f) \geq 5$ and $t = d_5(f)$. Each of 2-vertices on $f$ receives $(1/2)$ from $f$ and other vertices remaining on $f$ receives $(d(f) - 4 - t \cdot (1/2))/(d(f) - t)$ from $f$.

Now, let us start to test and verify $\chi'$ is greater than or equal to 0 for all vertices and faces. It is easy to verify faces, so let us verify the new charge of every face firstly. Obviously, $d(f) \geq 3$. If $d(f) = 3$, then $\chi'(f) = -1 + \min[2 \times (1/2), 3 \times (1/3)] = 0$ by R4; if $d(f) = 4$, then $\chi'(f) = \chi(f) = 0$; otherwise $d(f) \geq 5$, by R5, $\chi'(f) \geq 1 - t \times (1/2) - ((d(f) - 4 - t \times (1/2))/(d(f) - t)) \times (d(f) - t) = 0$.

Let us verify the new charge for every vertex. If $d(v) = i$ for $2 \leq i \leq 4$, then $\chi'(v) = -2 + 1 + \min[2 \times (1/2), 1] = 0$ by R1, $\chi'(v) = -1 + 1 = 0$ by R2, and $\chi'(v) = 0$, respectively. If $d(v) = 5$, then $v$ must be adjacent to 7*-vertices by Lemma 1(2). So, by R3 and R4, $\chi'(v) \geq 1 + 5 \times (2/15) - 5 \times (1/3) = 0$. If $d(v) = 6$, then $v$ must be adjacent to 6*-vertices by Lemma 1(2). So, by R4, $\chi'(v) \geq 2 - 6 \times (1/3) = 0$. If $d(v) = 7$, then $f_3(v) \leq 5$, and by Lemma 1(2), $v$ must be adjacent to 5*-vertices. So, by R3 and R4, $\chi'(v) \geq 7 - 4 - 5 \times (1/3) - 7 \times (2/15) = (8/15) > 0$.

Suppose that $v$ is a 8-vertex. Then, $f_3(v) \leq 6$ and $v$ must be adjacent to 4*-vertices by Lemma 1(2). If $f_3(v) \leq 5$, then by R3-

R4 $\chi'(v) \geq 8 - 4 - 5 \times (1/2) - 8 \times (2/15) = (13/30) > 0$. Otherwise, $f_3(v) = 6$. Since all the neighbors of 5-vertex must be 6*-vertices by Lemma 1(2), so every 3-face which is incident with 8-vertex has no more than a 5-vertex; that is, $d_5(v) \leq 6$. So, by R3-R4, $\chi'(v) \geq 8 - 4 - 6 \times (1/2) - 6 \times (2/15) = (1/5) > 0$.

Suppose that $v$ be a 9-vertex. Then, $f_3(v) \leq 7$ and $v$ must be adjacent to 3*-vertices by Lemma 1(2), and it may be the 3-master of no more than two 3-vertices. In the following, we will test it from three cases. Firstly, it is not the 3-master of any 3-vertex, and then, $\chi'(v) \geq 9 - 4 - 7 \times (1/2) - 9 \times (2/15) = (3/10) > 0$ by R3 and R4.

Secondly, $v$ is the 3-master of only one 3-vertex. If $f_3(v) \leq 5$, then $\chi'(v) \geq 9 - 4 - 1 - 5 \times (1/2) - 8 \times (2/15) = (13/30) > 0$ by R3, R4, and R5. Suppose $f_3(v) = 6$ (as shown in Figure 1, (1–10)–(1–15)). Since all the neighbors of 5-vertex should be 6*-vertices by Lemma 1(2), so every 3-face incident with 9-vertex has no more than one 5-vertex, that is, $d_5(v) \leq 6$. So, $\chi'(v) \geq 9 - 4 - 1 - 6 \times (1/2) - 6 \times (2/15) = (1/5) > 0$ by R3–R5. Suppose $f_3(v) = 7$ (as shown in figure, (1–16)–(1–17)). If $v$ has a 5-neighbor $u$, then the 3-face which is incident with $u$ and $v$ must be a (5, 6*, 9)-face and by R4 this 3-face receives (1/3) from $v$. 

Figure 3: Vertex $v$ is a 11-vertex, the various cases of $f_3(v) \geq 6$. 

(3-1)  (3-2)  (3-3)  (3-4)  (3-5)  (3-6)  (3-7)  (3-8)  (3-9)  (3-10)  (3-11)  (3-12)  (3-13)  (3-14)  (3-15)  (3-16)  (3-17)  (3-18)  (3-19)  (3-20)  (3-21)  (3-22)  (3-23)  (3-24)  (3-25)  (3-26)  (3-27)  (3-28)  (3-29)  (3-30)  (3-31)  (3-32)  (3-33)  (3-34)  (3-35)  (3-36)  (3-37)  (3-38)  (3-39)  (3-40)  (3-41)  (3-42)  (3-43)  (3-44)  (3-45)  (3-46)  (3-47)  (3-48)  (3-49)  (3-50)  (3-51)  (3-52)
\[\text{So, } ch'(v) \geq 9 - 4 - 1 - \max(7 \times (1/2), 6 \times (1/2)+ (1/3)) + (2/15), 5 \times (1/2) + 2 \times (1/3) + 2 \times (2/15), 4 \times (1/2)+ 3 \times (1/3) + 3 \times (2/15), 3 \times (1/2) + 4 \times (1/3) + 4 \times (2/15), 2 \times (1/2) + 5 \times (1/3) + 5 \times (2/15), (1/2) + 6 \times (1/3) + 6 \times (2/15)) = (1/2) > 0 \text{ by R3, R4, and R5.}\]

Thirdly, \(v\) is the 3-master of two 3-vertices. If \(f_3(v) \leq 4\), then \(ch'(v) \geq 9 - 4 - 2 - 4 \times (1/2) - 7 \times (2/15) = (1/15) > 0\) by R3, R4, and R5. Suppose \(f_3(v) = 5\) (as shown in Figure 1, (1-1)-(1-9)). Then, \(ch'(v) \geq 9 - 4 - 2 - \max(5 \times (1/2)+ 3 \times (2/15), 4 \times (1/2) + 1 \times (1/3) + 4 \times (2/15), 3 \times (1/2) + 2 \times (1/3) + 5 \times (2/15), 2 \times (1/2) + 3 \times (1/3) + 6 \times (2/15), (1/2) + 4 \times (1/3) + 7 \times (2/15), 5 \times (1/3) + 7 \times (2/15)) = (1/10) > 0 \) by R3, R4, and R5. Suppose \(f_3(v) = 6\) (as shown in Figure 1: (1-10)-(1-15)). In Figure 1, (1-12), (1-14), (1-15), \(ch'(v) \geq 9 - 4 - 2 - \max(6 \times (1/2), 5 \times (1/2) + (1/3) + (2/15), 4 \times (1/2) + 2 \times (1/3) + 2 \times (2/15), 3 \times (1/2) + 3 \times (1/3) + 3 \times (2/15), 2 \times (1/2) + 4 \times (1/3) + 4 \times (2/15)) = (29/120) > 0 \) by R3, R4, and R5. Now, we consider Figure 1: (1-10), (1-11), and (1-13). By Lemma 3, \(f_{ch}(v) \geq 1\). By R5, the 7'-face gives no less than \(3/8\) to \(v\). So, \(ch'(v) \geq 9 - 4 - 3/8 - 2 - \max(6 \times (1/2), 5 \times (1/2) + 1 \times (1/3) + (2/15), 4 \times (1/2) + 2 \times (1/3) + 2 \times (2/15), 3 \times (1/2) + 3 \times (1/3) + 3 \times (2/15), 2 \times (1/2) + 4 \times (1/3) + 4 \times (2/15)) = (29/120) > 0 \) by R3, R4, and R5.

Suppose \(f_3(v) = 7\) (as shown in Figure 1, (1-16), (1-17), then, by Lemma 3, \(f_3'(v) = 2\). By R5, each 7'-face gives no less than \(3/8\) to \(v\). So, \(ch'(v) \geq 9 - 4 - 2 \times (3/8) = 2 - \max(7 \times (1/2), 6 \times (1/2) + (1/3) + (2/15), 5 \times (1/2) + 2 \times (1/3) + 2 \times (2/15), 4 \times (1/2) + 3 \times (1/3) + 3 \times (2/15), 3 \times (1/2) + 4 \times (1/3) + 4 \times (2/15), 2 \times (1/2) + 5 \times (1/3) + 5 \times (2/15)) = (1/4) > 0 \) by R3, R4, and R5.

Suppose that \(v\) is a 10-vertex. Obviously, \(f_3(v) \leq 8\) and it could be the 3-master of no more than two 3-vertices and the 2-master of no more than one 2-vertex. If \(f_3(v) \leq 4\), then by R3–R5, \(ch'(v) \geq 10 - 4 - 1 - 2 - 4 \times (1/2) - 7 \times (2/15) = (1/15) > 0\). Let us discuss it in four ways.

Firstly, \(f_3(v) = 6\) (as shown in Figure 3, (3-1)-(3-25)). Obviously, by R3–R5, \(ch'(v) \geq 11 - 4 - 1 - 2 - 6 \times (1/2) - 3 \times (2/15) = (3/5) > 0\).

Secondly, \(f_3(v) = 7\) (as shown in Figure 3, (3-26)-(3-40)). From Figure 3, (3-26)-(3-40), by R3–R5, \(ch'(v) \geq 11 - 4 - 1 - 2 - 7 \times (1/2) - 2 \times (2/15) = (7/30) > 0\).

Thirdly, \(f_3(v) = 8\) (as shown in Figure 3, (3-41)-(3-46)). By Lemma 5, \(f_{ch}(v) \geq 1\) in Figure 3: (3-41) and (3-43). So, \(ch'(v) \geq 11 - 4 + (3/8) - 1 - 2 - 8 \times (1/2) - (2/15) = (129/120) > 0\) by R3, R4, and R5. In Figure 3, (3-42) and (3-44)-(3-46), \(ch'(v) \geq 11 - 4 - 1 - 2 - 8 \times (1/2) = 0\) by R3–R5.

Fourthly, \(f_3(v) = 9\) (as shown in the figure, (3-47)), then, by Lemma 5, \(f_{ch}(v) = 2\). Hence, \(ch'(v) \geq 11 - 4 + 2 \times (3/8) - 1 - 2 - 9 \times (1/2) = (1/4) > 0\) by R3–R5.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (NSFC) under Grant 12001332.

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