Research Article

Numerical Solution for Third-Order Two-Point Boundary Value Problems with the Barycentric Rational Interpolation Collocation Method

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The numerical solution for a kind of third-order boundary value problems is discussed. With the barycentric rational interpolation collocation method, the matrix form of the third-order two-point boundary value problem is obtained, and the convergence and error analysis are obtained. In addition, some numerical examples are reported to confirm the theoretical analysis.

1. Introduction

Differential equations can give full play to their mathematical advantages in various disciplines. Combining the theory of differential equations with practical problems can build models of practical problems. Many engineering and physical problems can be transformed into the initial boundary value problems of differential equations. In these problems, only a few simple cases can be solved analytically, and most engineering problems need to be solved by numerical methods. Compared with polynomial interpolation, rational function interpolation has higher interpolation accuracy and can effectively overcome the instability of interpolation [1–4]. Barycentric rational interpolation not only has high interpolation accuracy on special distributed nodes but also has high interpolation accuracy for equidistant nodes [5–7]. This method has been used to solve certain problems such as Volterra integral equations [2, 8, 9], delay Volterra integrodifferential equations [10, 11], plane elastic problems [12], nonlinear problems [13], heat conduction equation [14], and so on [15–17].

The third-order differential equation has a wide range of applications and important theoretical values in many

scientific fields, such as applied mathematics and physics. Therefore, the third-order boundary value problem has been widely concerned by many scholars [18–20]. In this paper, we consider the numerical solution of the third-order two-point boundary value problem,

$$u'''(x) + pu''(x) + qu'(x) + ru(x) = f(x), \quad a < x < b, \quad (1)$$

$$\begin{aligned} u(a) = A, u'(a) = B, u'(b) = C, \quad (\text{or}) \\ u(a) = A, u'(a) = B, u'''(a) = C, \end{aligned} \quad (2)$$

by the barycentric rational interpolation collocation method.

Barycentric rational interpolation collocation method means using barycentric interpolation polynomials to find the differential matrix of a function at each discrete point; thus, the solution of the differential equation can be obtained by matrix operation. The barycentric rational interpolation has excellent numerical stability and high approximation accuracy, and the barycentric rational interpolation formula has a compact calculation formula of all order derivatives. Therefore, the barycentric rational interpolation collocation

method is an effective method for solving boundary value problems of differential equations.

2. Formula of the Barycentric Interpolation Collocation Method

Discretize the interval $[a, b]$ into n uniform parts with $h = ((b - a)/n)$, and suppose $u_1, u_2, \dots, u_n$ is the function value of an unknown function u at discrete nodes $x_1, x_2, \dots, x_n$.

For any $0 \leq d \leq n$, $P_i(x), i = 0, 1, \dots, n - d$, is the interpolation function at the point $x_i, x_{i+1}, \dots, x_{i+d}$; then, we have $P_i(x_k) = f(x_k), k = i, i + 1, \dots, i + d$, and

$$r(x) = \frac{\sum_{i=0}^{n-d} \lambda_i(x) P_i(x)}{\sum_{i=0}^{n-d} \lambda_i(x)}, \quad (3)$$

where

$$\lambda_i(x) = \frac{(-1)^i}{(x - x_i) \cdots (x - x_{i+d})}. \quad (4)$$

By changing the polynomial $P_i(x)$ into the Lagrange interpolation form as

$$P_i(x) = \sum_{k=i}^{i+d} \prod_{j=i, j \neq k}^{i+d} \frac{x - x_j}{x_k - x_j} f_k \quad (5)$$

and combining (4) and (5) together, we get

$$\sum_{i=0}^{n-d} \lambda_i(x) P_i(x) = \sum_{i=0}^{n-d} (-1)^i \sum_{k=i}^{i+d} \frac{1}{x - x_k} \prod_{j=i, j \neq k}^{i+d} \frac{1}{x_k - x_j} f_k = \sum_{k=0}^n \frac{w_k}{x - x_k} f_k, \quad (6)$$

where $w_k = \sum_{i \in J_k} (-1)^i \prod_{j=i, j \neq k}^{i+d} (1 / (x_k - x_j))$, $J_k = \{i \in I; k - d \leq i \leq k\}$.

Then, we get

$$r(x) = \frac{\sum_{j=0}^n (w_j / (x - x_j)) f_j}{\sum_{j=0}^n (w_j / (x - x_j))}, \quad (7)$$

where its basis function is

$$L_j(x) = \frac{(w_j / (x - x_j))}{\sum_{k=0}^n (w_k / (x - x_k))}. \quad (8)$$

For the equidistant point, the weight function is

$$w_j = (-1)^{n-j} C_n^j. \quad (9)$$

For the Chebyshev point of the second kind,

$$x_j = \cos \frac{j\pi}{n}, \quad j = 0, 1, \dots, n, \quad (10)$$

the weight function is

$$w_j = (-1)^j \delta_j, \delta_j = \begin{cases} \frac{1}{2}, & j = 0, n, \\ 1, & \text{otherwise.} \end{cases} \quad (11)$$

By formula (8), the m -th order derivative of $u(x)$ at the nodes $x_1, x_2, \dots, x_n$ can be expressed as

$$\begin{aligned} u^{(m)}(x_j) &:= u_i^{(m)} = \frac{d^m u(x_j)}{dx^m} = \sum_{k=1}^n L_k^{(m)}(x_j) u_k \\ &= \sum_{k=1}^n D_{ij}^{(m)} u_k, \quad m = 1, 2, \dots, \end{aligned} \quad (12)$$

and then (12) can be written in the matrix form as

$$u^{(m)} = D^{(m)} u, \quad (13)$$

where $u^{(m)} = [u_1^{(m)}, u_2^{(m)}, \dots, u_n^{(m)}]^T$ and $u = [u_1, u_2, \dots, u_n]^T$.

By using the barycentric interpolation function as

$$u_n(x) = \sum_{j=0}^n L_j(x) u_j, \quad (14)$$

equation (1) can be written in the numerical form as

$$\sum_{j=0}^n u_j L_j''(x) + p \sum_{j=0}^n u_j L_j''(x) + q \sum_{j=0}^n u_j L_j'(x) + r \sum_{j=0}^n u_j L_j(x) = f(x). \quad (15)$$

By using the notation of the differential matrix, (15) can also be denoted as

$$\begin{aligned} \sum_{j=0}^n D_{ij}^{(3)} u_j + p \sum_{j=0}^n D_{ij}^{(2)} u_j + q \sum_{j=0}^n D_{ij}^{(1)} u_j + r \\ \sum_{j=0}^n \delta_{ij} u_j = f(x_i), \quad i = 1, 2, \dots, n, \end{aligned} \quad (16)$$

or the simple matrix form

$$[D^{(3)} + pD^{(2)} + qD^{(1)} + rI]\mathbf{u} = \mathbf{f}. \quad (17)$$

Boundary conditions (2) can be divided into

$$\begin{aligned} u_1 = A, u'(x_1) = \sum_{j=0}^n D_{1j}^{(1)} u_j = B, \\ u'(x_n) = \sum_{j=0}^n D_{nj}^{(1)} u_j = C, \text{ (or)} \\ u_1 = A, u'(x_1) = \sum_{j=0}^n D_{1j}^{(1)} u_j = B, \\ u''(x_1) = \sum_{j=0}^n D_{1j}^{(2)} u_j = C, \end{aligned} \quad (18)$$

where $D^{(k)} = [D_{ij}^{(k)}]_{(n+1) \times (n+1)}$,

$$\begin{aligned} D_{ij}^{(1)} &= \begin{cases} \frac{w_i}{x_i - x_j}, & i \neq j, \\ -\sum_{k \neq i} D_{ik}^{(1)}, & i = j, \end{cases} \\ D_{ij}^{(2)} &= \begin{cases} 2 \left(D_{ii}^{(1)} D_{ij}^{(1)} - \frac{D_{ij}^{(1)}}{x_i - x_j} \right), & i \neq j, \\ -\sum_{k \neq i} D_{ik}^{(2)}, & i = j, \end{cases} \\ D_{ij}^{(3)} &= \begin{cases} 3 \left(D_{ii}^{(2)} D_{ij}^{(1)} - \frac{D_{ij}^{(2)}}{x_i - x_j} \right), & i \neq j, \\ -\sum_{k \neq i} D_{ik}^{(3)}, & i = j. \end{cases} \end{aligned} \quad (19)$$

$$\mathbf{u} = [u_0, u_1, \dots, u_n]^T,$$

$$\mathbf{f} = [f(x_0), f(x_1), \dots, f(x_n)]^T.$$

3. Convergence and Error Analysis

In this section, we will consider the error problem of equidistant interpolation nodes:

$$x_i = a + \frac{(b-a)}{n} i, \quad i = 0, 1, \dots, n. \quad (20)$$

Let $u(x)$ be the solution of (1); for any $0 \leq d \leq n$, suppose $P(x_i), i = 0, 1, \dots, n-d$, to be the barycentric interpolation function at the point $x_i, x_{i+1}, \dots, x_{i+d}$; then, we have $P_i(x_k) = f(x_k), k = i, i+1, \dots, i+d$, and

$$r(x) = \frac{\sum_{i=0}^{n-d} \lambda_i(x) P_i(x)}{\sum_{i=0}^{n-d} \lambda_i(x)}, \quad (21)$$

where

$$\lambda_i(x) = \frac{(-1)^i}{(x - x_i) \cdots (x - x_{i+d})}. \quad (22)$$

Then, the error function is defined as

$$\begin{aligned} e(x) &:= u(x) - P(x) = (x - x_i) \cdots \\ &(x - x_{i+d}) [x_i, x_{i+1}, \dots, x_{i+d}, x] f, \end{aligned} \quad (23)$$

and

$$e(x) = \frac{\sum_{i=0}^{n-d} \lambda_i(x) (u(x) - P_i(x))}{\sum_{i=0}^{n-d} \lambda_i(x)} = \frac{A(x)}{B(x)} = O(h^{d+1}), \quad (24)$$

where $A(x) := \sum_{i=0}^{n-d} (-1)^i [x_i, \dots, x_{i+d}, x] f$ and $B(x) := \sum_{i=0}^{n-d} \lambda_i(x)$.

Taking the numerical form,

$$\begin{aligned} \sum_{j=0}^n u_j L_j''(x) + p \sum_{j=0}^n u_j L_j''(x) + q \sum_{j=0}^n u_j L_j'(x) + r \\ \sum_{j=0}^n u_j L_j(x) = f(x), \end{aligned} \quad (25)$$

and combining (24) and (1), we get

$$e'''(x) + pe''(x) + qe'(x) + re(x) = R_f(x), \quad (26)$$

where $R_f(x) = f(x) - f(x_k), k = 0, 1, 2, \dots, n$.

Lemma 1. For $e(x)$ defined in (23), we have

$$\begin{cases} |e(x)| \leq Ch^{d+1}, & u \in C^{d+2}[a, b], \\ |e'(x)| \leq Ch^d, & u \in C^{d+2}[a, b], \\ |e''(x)| \leq Ch^{d-1}, & u \in C^{d+3}[a, b], \quad d \geq 1, \\ |e'''(x)| \leq Ch^{d-2}, & u \in C^{d+4}[a, b], \quad d \geq 2. \end{cases} \quad (27)$$

Let $u(x)$ be the solution of (1) and $u_n(x)$ be the numerical solution; then, we have

$$\begin{aligned} u_n''(x_k) + pu_n''(x_k) + qu_n'(x_k) + ru_n(x_k) \\ = f(x_k), \quad k = 0, 1, 2, \dots, n, \end{aligned} \quad (28)$$

and

$$\lim_{n \rightarrow \infty} u_n(x) = u(x). \quad (29)$$

The results can be obtained in [1].

Based on Lemma 1, we can get the following theorem.

Theorem 1. Let $f(x) \in C[a, b]$, $Tu(x) = u'''(x) + pu''(x) + qu'(x) + ru(x)$, and

$u_n(x): Tu_n(x) = f(x), u_n^*(x): Tu_n^*(x) = f^*(x)$; then, we have

$$|u_n(x) - u_n^*(x)| \leq Ch^{d-2}. \quad (30)$$

Proof. Let $L = D^{(3)} + pD^{(2)} + qD^{(1)} + rI$

$$= \begin{bmatrix} D_{00}^{(3)} + pD_{00}^{(2)} + qD_{00}^{(1)} + r & D_{01}^{(3)} + pD_{01}^{(2)} + qD_{01}^{(1)} & \cdots & D_{0n}^{(3)} + pD_{0n}^{(2)} + qD_{0n}^{(1)} \\ D_{10}^{(3)} + pD_{10}^{(2)} + qD_{10}^{(1)} & D_{11}^{(3)} + pD_{11}^{(2)} + qD_{11}^{(1)} + r & \cdots & D_{1n}^{(3)} + pD_{1n}^{(2)} + qD_{1n}^{(1)} \\ \cdots & \cdots & \cdots & \cdots \\ D_{n0}^{(3)} + pD_{n0}^{(2)} + qD_{n0}^{(1)} & D_{n1}^{(3)} + pD_{n1}^{(2)} + qD_{n1}^{(1)} & \cdots & D_{nn}^{(3)} + pD_{nn}^{(2)} + qD_{nn}^{(1)} + r \end{bmatrix}. \quad (31)$$

Add column 2, column 3, . . . , column n to column 1, and we have

$$L = \begin{bmatrix} r & D_{01}^{(3)} + pD_{01}^{(2)} + qD_{01}^{(1)} & \cdots & D_{0n}^{(3)} + pD_{0n}^{(2)} + qD_{0n}^{(1)} \\ r & D_{11}^{(3)} + pD_{11}^{(2)} + qD_{11}^{(1)} + r & \cdots & D_{1n}^{(3)} + pD_{1n}^{(2)} + qD_{1n}^{(1)} \\ \cdots & \cdots & \cdots & \cdots \\ r & D_{n1}^{(3)} + pD_{n1}^{(2)} + qD_{n1}^{(1)} & \cdots & D_{nn}^{(3)} + pD_{nn}^{(2)} + qD_{nn}^{(1)} + r \end{bmatrix}. \quad (32)$$

Then, we have $|L| \neq 0$ with $r \neq 0$, $u_n(x) = \sum_{j=0}^n L_j(x)f_j$, and $u_n^*(x) = \sum_{j=0}^n L_j(x)f_j^*$, where $U_n = (f(x_0), f(x_1), \dots, f(x_n))^T$ and $U_n^* = (f^*(x_0), f^*(x_1), \dots, f^*(x_n))^T$.

By

$$U_n - U_n^* = L^{-1}(LU_n - F_n^*), \quad (33)$$

which means

$$u_n(x) - u_n^*(x) = \sum M_j(x)Te(x), \quad (34)$$

where $M_j(x)$ is the element of matrix L^{-1} , we have

$$|u_n(x) - u_n^*(x)| \leq \left| \sum M_j(x) \right| |Te(x)| \leq Ch^{d-2}. \quad (35)$$

The proof is completed. $\square$

4. Numerical Example

As an example, we consider the two-point boundary value problem:

$$y''' + y = f(x), \quad -1 < x < 1, \quad (36)$$

$$y(-1) = 0, y(1) = 0, y'(-1) = 0. \quad (37)$$

For this problem, we can find a function $f(x)$ such that the analysis solution is

$$y = (1 - x^2)(1 + x)e^{\lambda x}, \quad (38)$$

where λ is a freely selected parameter.

Substituting (38) into (36), we get

$$f(x) = [-6 - 6\lambda(1 + 3x) + 3\lambda^2(1 - 2x - 3x^2) + \lambda^3(1 + x - x^2 - x^3) + (1 - x^2)(1 + x)]e^{\lambda x}. \quad (39)$$

For different values of d and different number of nodes, we can calculate the corresponding relative error and convergence rate; some of the data are shown in Tables 1 and 2.

In Table 1, the convergence rate of equidistant nodes with different d is $O(h^{d-2})$; in Table 2, the convergence rate of the Chebyshev point of the second kind with different d is $O(h^{d+2})$, $d \geq 2$.

For different values of λ and different number of nodes, we can calculate the corresponding relative error; some of the data are shown in Tables 3 and 4.

From Tables 3 and 4, we can find that, for different values of λ , the convergence rate can reach $O(h^{d+2})$ ($d \geq 2$) both for equidistant and nonequidistant nodes.

TABLE 1: Errors and convergence rate of the equidistant nodes with different d .

n	$(\lambda = 2)$							
	$d = 2$		$d = 3$		$d = 4$		$d = 5$	
	Error	h^α	Error	h^α	Error	h^α	Error	h^α
10	7.1976e+00		4.6412e+00		2.4217e+00		1.1522e+00	
20	3.5555e+00	1.0175	1.2336e+00	1.9117	3.6419e-01	2.7333	9.9594e-02	3.5322
40	1.4588e+00	1.2853	2.6378e-01	2.2254	4.1431e-02	3.1359	6.0309e-03	4.0456
80	5.5050e-01	1.4060	5.1045e-02	2.3695	4.1384e-03	3.3236	3.1073e-04	4.2786
160	2.0058e-01	1.4566	9.4229e-03	2.4375	3.8810e-04	3.4146	1.4799e-05	4.3921
320	7.1879e-02	1.4806	1.7004e-03	2.4703	3.5301e-05	3.4586	6.7836e-07	4.4473
640	2.5569e-02	1.4912	3.0357e-04	2.4858	3.1616e-06	3.4810	3.1718e-08	4.4187
1280	9.0650e-03	1.4960	5.3912e-05	2.4933	2.8666e-07	3.4632	1.2685e-08	1.3222

TABLE 2: Errors and convergence rate of the Chebyshev point with different d .

n	$(\lambda = 2)$							
	$d = 2$		$d = 3$		$d = 4$		$d = 5$	
	Error	h^α	Error	h^α	Error	h^α	Error	h^α
10	2.4329e+00		1.3897e+00		2.5216e-01		1.1033e-01	
20	8.8604e-01	1.4572	1.3680e-02	6.6665	4.5076e-03	5.8058	1.5195e-03	6.1821
40	1.8242e-01	2.2801	2.8639e-03	2.2561	1.4419e-04	4.9664	7.2659e-06	7.7082
80	3.2188e-02	2.5026	1.7517e-04	4.0311	3.0543e-06	5.5610	1.0125e-07	6.1652
160	5.2215e-03	2.6240	8.5892e-06	4.3501	5.9098e-08	5.6916	1.0071e-06	—
320	8.0770e-04	2.6926	2.9760e-07	4.8511	1.9760e-06	—	2.2948e-05	—
640	1.2077e-04	2.7416	2.8632e-06	—	6.856e-05	—	1.0211e-03	—
1280	1.7925e-05	2.7522	7.4861e-05	—	1.2910e-02	—	9.6669e-02	—

TABLE 3: Errors and convergence rate of the equidistant nodes with different λ .

n	$(d = 4)$							
	$\lambda = -5$		$\lambda = 1$		$\lambda = 5$		$\lambda = 20$	
	Error	h^α	Error	h^α	Error	h^α	Error	h^α
10	3.6809e+01		2.2939e-01		2.1508e+02		3.0238e+08	
20	1.3259e+01	1.4731	2.6894e-02	3.0925	6.1838e+01	1.7983	8.0305e+08	—
40	2.2319e+00	2.5707	2.7208e-03	3.3052	9.5448e+00	2.6957	6.7632e+08	2.4778e-01
80	2.6875e-01	3.0539	2.5675e-04	3.4056	1.1054e+00	3.1101	1.6623e+08	2.0246
160	2.7599e-02	3.2835	2.3416e-05	3.4548	1.1147e-01	3.3098	2.4028e+07	2.7904
320	2.6249e-03	3.3943	2.1009e-06	3.4784	1.0509e-02	3.4070	2.7001e+06	3.1536

TABLE 4: Errors and convergence rate of the Chebyshev point with different λ .

n	$(d = 4)$							
	$\lambda = -5$		$\lambda = 1$		$\lambda = 5$		$\lambda = 20$	
	Error	h^α	Error	h^α	Error	h^α	Error	h^α
10	1.0975e+01		1.8992e-02		4.4245e+01		1.3431e_01	
20	1.5012e-01	6.1919	2.6112e-04	6.1845	1.3603e+00	5.0235	2.5740e+08	2.3835
40	5.5289e-03	4.7630	8.0762e-06	5.0149	4.7712e-02	4.8334	1.4800e+07	4.1204
80	1.1729e-04	5.5589	1.6777e-07	5.5891	1.0350e-03	5.5267	3.5715e+05	5.3729
160	2.2528e-06	5.7022	1.8130e-08	3.2100	2.3528e-05	5.4591	8.3139e+03	5.4249
320	1.3043e-05	—	2.5236e-06	—	2.4682e-06	3.2529	2.1982e+02	5.2411

5. Conclusion

In this paper, the barycentric rational collocation method for solving third-order two-point boundary value equations is presented, and the error function of the convergence rate $O(h^{d-2})$ is also obtained. For the constant coefficient and variable coefficient of two-point boundary value equations, numerical results show that the convergence rate can reach $O(h^{d-2})$ for the equidistant nodes and Chebyshev point of the second kind with $d \geq 2$, so the barycentric rational collocation method is an effective method. Compared with other methods, the advantage of this method is that the matrix equation can be easily obtained, the program is simple, and high computational accuracy can be obtained by using a few points.

Data Availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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