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Research Article

A New Approach to Concavity Fuzzification

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In this paper, we introduce a more general approach to the fuzzification of fuzzy concavity. More specifically, the degree of (L, M)-fuzzy concavity is introduced and characterized as a generalization of L-concave structure and (L, M)-fuzzy concave structure. Based on that, the degree of (L, M)-fuzzy concavity preserving and (L, M)-fuzzy concave-to-concave of a function are defined. Some properties and relationships between the degree of (L, M)-fuzzy concavity preserving and (L, M)-fuzzy concave-to-concave functions are discussed.

1. Introduction

The concept of convexity is a fundamentally important geometrical property that plays a significant role in pure and applied mathematics. It can be sorted into concrete and abstract convexity. In this paper, we are mainly focusing on the abstract convexity. Fuzzification of the abstract convexity was initiated by Rosa [1–3] in 1994. He introduced fuzzy convex to convex and fuzzy convexity preserving functions and discussed some of their properties. To be worth mentioning, he developed the theory of fuzzy convexity by introducing its subspace, product, and quotient structures.

Many researchers have been involved in extending the notion of fuzzy convexity to the broader frame work of lattice-valued setting. In [4], Maruyama extended fuzzy convexity to L-fuzzy setting framework, where L is a completely distributive lattice. Jin and Li [5] proposed two functors between the categories of convex spaces and stratified L-convex spaces, where L is a continuous lattice. Both of the functors have been used to prove the embedding of convex spaces in stratified L-convex spaces as a reflective and coreflective subcategory, where L satisfies the multiplicative condition. Also, stratified L-convex spaces, convex-generated L-convex spaces, weakly induced L-convex spaces, and induced L-convex spaces are introduced and their relationships are discussed category-theoretically by Pang and Shi [6]. In 2016, Pang and Zhao [7] introduced the concept of L-concave spaces, concave L-neighborhood systems, and concave L-interior operators which are the dual concepts of L-convex spaces, convex L-neighborhood systems, and convex L-interior operators. The isomorphism between these categories and the category of L-convex spaces are discussed and studied when L is a completely distributive lattice with an order-reversing involution.

Shi and Xiu [8] initiated the concept of M-fuzzifying convex structure as a new approach to convexity fuzzification. Recently, Shi and Li [9] generalized the classical restricted hull operators to M-fuzzifying restricted hull operators and used it to characterize M-fuzzifying convex structures. In [10], Wu and Bai discussed some properties of hull operator and introduced M-fuzzifying JHC property and M-fuzzifying Peano property. Further, Xiu and his colleagues [11–14] verified the categorical relationship between M-fuzzifying convexity and other related spatial structures. Later, Shi and Xiu [15] introduced and characterized the notion of (L, M)-fuzzy convexity as an extension to L-convexity and M-fuzzifying convexity. In the new structure, each L-fuzzy subset can be regarded as an

L-convex set to some degree. M-fuzzifying convex spaces and (L, M)-fuzzy convex spaces have also been investigated in many studies [16–21].

In 1992, Šostak [22] introduced the concept of fuzzy category. In a fuzzy category, the potential objects and morphisms could be such to some degrees. Later, Kubiak and Šostak [23] fuzzified the category of M-valued L-fuzzy topological spaces. In [24], Zhong and Shi characterized the degree to which a function $\mathscr{T}\colon L^X \longrightarrow M$ is an (L, M)-fuzzy topology. Moreover, the degree to which an L-subset is an L-open set with respect to $\mathscr{T}$ is studied. Further, the degrees of continuity, openness, closeness, and quotient of a function $f\colon (X, \mathscr{T}_X) \longrightarrow (Y, \mathscr{T}_Y)$ with respect to the (L, M)-fuzzy topologies $\mathscr{T}_X$ and $\mathscr{T}_Y$, are given and their properties are characterized. Further, several kinds of continuity, compactness, and connectedness are generalized with their elementary properties to (L, M)-fuzzy topological spaces setting based on graded concepts (see [25–30]). In [31], Ghareeb et al. studied the (L, M)-fuzzy measurability in view of degree. Firstly, they generalized (L, M)-fuzzy σ-algebra by presenting the degree of an (L, M)-fuzzy σ-algebra with respect to a mapping $\sigma\colon L^X \longrightarrow M$. Moreover, they defined and discussed some special degrees such as the degree of (L, M)-fuzzy measurable mapping, (L, M)-fuzzy measurable-to-measurable mapping, isomorphic mapping, and quotient mapping with respect to mappings between two (L, M)-fuzzy measurable spaces in detail.

The aim of this paper is to present the degree of (L, M)-fuzzy concavity as an extension of (L, M)-fuzzy concave structure. Moreover, we present the degree of (L, M)-fuzzy concavity preserving functions and (L, M)-fuzzy concave-to-concave functions. Some properties and relationship between the degree of (L, M)-fuzzy concavity preserving and (L, M)-fuzzy concave-to-concave functions are characterized.

2. Preliminaries

This section begins with some introductory material on (L, M)-fuzzy convexity. In the sequel, X refers to a finite set, both L and M denote completely distributive lattices. The zero and the unit elements in L and M are symbolized by $\bot_L$, $\bot_M$ and $\top_L$, $\top_M$, respectively. By $\{\lambda_i\}_{i\in\Omega}^{\text{dir}} \subseteq L^X$ (resp. $\{\lambda_i\}_{i\in\Omega}^{\text{cdir}} \subseteq L^X$), we refer to the directed (resp. codirected) subfamily $\{\lambda_i\}_{i\in\Omega}$ of L^X. For a completely distributive lattice M, there exists a residual implication $\longmapsto\colon M \times M \longrightarrow M$ which is right adjoint for meet operation $\wedge$ and given by

$$r \longmapsto s = \vee\{t \in M | r\wedge t \leq s\}. \tag{1}$$

Moreover, the operation $\longleftrightarrow$ is defined by

$$r \longleftrightarrow s = (r \longmapsto s)\wedge(s \longmapsto r). \tag{2}$$

The following lemma lists some properties of implication operation.

Lemma 1 (see [32]). *For any r, s, $t \in M$, $\{r_i\}_{i\in\Omega}$, and $\{s_i\}_{i\in\Omega} \in M$, we have the following statements:*

(1) $\top_M \longmapsto r = r$

(2) $t \leq r \longmapsto s \Longleftrightarrow r\wedge t \leq s$

(3) $r \longmapsto s = \top_M \Longleftrightarrow r \leq s$

(4) $r \longmapsto \wedge_{i\in\Omega} r_i = \wedge_{i\in\Omega}(r \longmapsto r_i)$, hence $r \longmapsto s \leq r \longmapsto t$ whenever $s \leq t$

(5) $\vee_{i\in\Omega} r_i \longmapsto s = \vee_{i\in\Omega}(r_i \longmapsto s)$, hence $r \longmapsto t \geq s \longmapsto t$ whenever $r \leq s$

(6) $(r \longmapsto t)\wedge(t \longmapsto s) \leq r \longmapsto s$

An element $r \in M$ is said to be a prime element if $r \leq s\wedge t$ leads to $r \leq s$ or $r \leq t$. Also, $r \in M$ is said to be coprime if $r \geq s\vee t$ leads to $r \geq s$ or $r \geq t$. The collection of all nonunit prime and nonzero coprime elements in M are symbolized by $P(M)$ and $J(M)$, respectively.

The binary relation $\ll$ on M is defined as follows: for r, $s \in M$, $r \ll s$ if and only if for every subset $B \subseteq M$, the relation $s \leq \sup B$ always leads to the existence of $t \in B$ with $r \leq t$. The family $\{r \in M\colon r \ll s\}$ is called the greatest minimal family of s, symbolized by $\beta(s)$, and $\beta^*(s) = \beta(s) \cap J(M)$. Moreover, for $s \in M$, we define $\alpha(s) = \{r \in M\colon r' \ll s'\}$ and $\alpha^*(s) = \alpha(s) \cap P(M)$. In a completely distributive lattice L, there exist $\alpha(s)$ and $\beta(s)$ for each $s \in M$, $s = \wedge\alpha(s) = \vee\beta(s)$, and $\alpha(\top_M) = \beta(\bot_M) = \varnothing$ (see [33–35] for more details).

In [7], Pang and Zhao introduced the concept of L-concave spaces, which is a dual concept of L-convex spaces as follows.

Definition 1 (see [7]). An L-concave structure $\mathfrak{C}$ on a nonempty set X is a subset of L^X such that

(L1) $\top_{L^X}, \bot_{L^X} \in \mathfrak{C}$

(L2) $\wedge_{i\in\Omega}\lambda_i \in \mathfrak{C}$ for each $\{\lambda_i\}_{i\in\Omega}^{\text{cdir}} \subseteq L^X$

(L3) $\vee_{in\in\Omega}\lambda_i \in \mathfrak{C}$ for each $\{\lambda_i\}_{i\in\Omega} \subseteq L^X$

The pair $(X, \mathfrak{C})$ is called an L-concave space. A function $f\colon (X, \mathfrak{C}_X) \longrightarrow (Y, \mathfrak{C}_Y)$ is called L-concavity preserving if $\nu \in \mathfrak{C}_Y$ implies that $f_L^{\leftarrow}(\nu) \in \mathfrak{C}_X$, and f is called concave-to-concave function if $f_L^{\longrightarrow}(\lambda) \in \mathfrak{C}_Y$ for each $\lambda \in L^X$.

The following definition extends L-concavity to (L, M)-fuzzy setting.

Definition 2. A function $\mathscr{C}\colon L^X \longrightarrow M$ is called an (L, M)-fuzzy concavity on X if $\mathscr{C}$ satisfies the following statements:

(C1) $\mathscr{C}(\top_{L^X}) = \mathscr{C}(\bot_{L^X}) = \top_M$

(C2) $\mathscr{C}(\wedge_{i\in\Omega}\lambda_i) \geq \wedge_{i\in\Omega}\mathscr{C}(\lambda_i)$, for every $\varnothing \neq \{\lambda_i\}_{i\in\Omega}^{\text{cdir}} \subseteq L^X$

(C3) $\mathscr{C}(\vee_{i\in\Omega}\lambda_i) \geq \wedge_{i\in\Omega}\mathscr{C}(\lambda_i)$, for every $\varnothing \neq \{\lambda_i\}_{i\in\Omega} \subseteq L^X$

The pair $(X, \mathscr{C})$ is called an (L, M)-fuzzy concave structure. For all $\lambda \in L^X$, the value $\mathscr{C}(\lambda)$ represents the degree to which λ is concave L-subset. For any two (L, M)-fuzzy concavities $\mathscr{C}$ and $\mathscr{D}$ on X, we say $\mathscr{C}$ is coarser than $\mathscr{D}$ (i.e., $\mathscr{D}$ is finer than $\mathscr{C}$) if and only if $\mathscr{C}(\lambda) \leq \mathscr{D}(\lambda)$, for every $\lambda \in L^X$. For any two (L, M)-fuzzy concave structures $(X, \mathscr{C}_X)$ and $(Y, \mathscr{C}_Y)$, the function $f\colon (X, \mathscr{C}_X) \longrightarrow (Y, \mathscr{C}_Y)$ is said to be

(1) An (L,M)-fuzzy concavity preserving function if $\mathscr{C}_X(f_L^{\leftarrow}(\nu)) \geq \mathscr{C}_Y(\nu)$ for any $\nu \in L^Y$

(2) An (L,M)-fuzzy concave-to-concave function if $\mathscr{C}_Y(f_L^{\longrightarrow}(\lambda)) \geq \mathscr{C}_X(\lambda)$ for any $\lambda \in L^X$

Obviously, a $(2,M)$-fuzzy concave structure can be viewed as an M-fuzzifying concave structure, where $2=\{\top,\bot\}$. Moreover, an $(L,2)$-fuzzy concave structure is called an L-concave structure [7]. Further, when $L=M=[0,1]$, the (L,M)-fuzzy concave structure is called a $[0,1]$-fuzzy concave structure. A crisp convex structure can be regarded as a $(2,2)$-fuzzy convex structure.

Example 1. If an (L,M)-fuzzy cotopology $\mathscr{O}\colon L^X \longrightarrow M$ (see [36, 37]) satisfies

$$\mathscr{O}\Big(\bigvee_{i\in\Omega}\lambda_i\Big) \geq \bigwedge_{i\in\Omega}\mathscr{O}(\lambda_i), \tag{3}$$

for every $\{\lambda_i\}_{i\in\Omega} \subseteq L^X$, then $\mathscr{O}$ is called saturated (L,M)-fuzzy cotopology. The pair $(X,\mathscr{O})$ is called an Alexandroff (L,M)-fuzzy cotopological space. It can be easily verified that Alexandroff (L,M)-fuzzy cotopological space is an (L,M)-fuzzy concave structure.

For each $\mathscr{C}\colon L^X \longrightarrow M$ and $r \in M$, we have the following two cut sets:

$$\begin{aligned}\mathscr{C}_{[r]} &= \left\{\lambda \in L^X\colon \mathscr{C}(\lambda) \geq r\right\},\\ \mathscr{C}^{[r]} &= \left\{\lambda \in L^X\colon r \notin \alpha(\mathscr{C}(\lambda))\right\}.\end{aligned} \tag{4}$$

Theorem 1. *Let $\mathscr{C}\colon L^X \longrightarrow M$ be a function. Then, the following statements are equivalent:*

(1) $(X,\mathscr{C})$ is an (L,M)-fuzzy concave structure

(2) For any $r \in M\{\bot_M\}$, $\mathscr{C}_{[r]}$ is an L-concavity on X

(3) For any $r \in \alpha(\bot_M)$, $\mathscr{C}^{[r]}$ is an L-concavity on X

In the following theorem, we assume the existence of an order-reversing involution "′" with the completely distributive lattice $(M,\vee,\wedge)$, i.e., $(M,\vee,\wedge,')$ is a completely distributive DeMorgan algebra.

Theorem 2 (see [20, 38, 39]) *.ie closure (resp. hull) operator $co_{\mathscr{C}}\colon L^X \longrightarrow M^{J(L^X)}$ of an (L,M)-fuzzy concave structure $(X,\mathscr{C})$ is defined by*

$$co_{\mathscr{C}}(\lambda)(x_r) = \bigwedge_{x_r \nleq \mu \geq \lambda}(\mathscr{C}(\mu'))', \quad \forall \lambda \in L^X, \forall x_r \in J\big(L^X\big). \tag{5}$$

Then, for every $\lambda,\mu \in L^X$ and $x_r \in J(L^X)$, $co_{\mathscr{C}}$ achieves the following statements:

(H1) $co_{\mathscr{C}}(\bot_{L^X})(x_r) = \bot_M$

(H2) If $x_r \leq \lambda$, then $co_{\mathscr{C}}(\lambda)(x_r) = \top_M$

(H3) If $\lambda \leq \mu$, then $co_{\mathscr{C}}(\lambda) \leq co_{\mathscr{C}}(\mu)$

(H4) $co_{\mathscr{C}}(\lambda)(x_r) = \wedge_{x_r \leq \mu \geq \lambda}\vee_{y_s \leq \mu} co_{\mathscr{C}}(\mu)(y_s)$

Conversely, let $co\colon L^X \longrightarrow M^{J(L^X)}$ be an operator achieving (H1)–(H4) and the function $\mathscr{C}_{co}\colon L^X \longrightarrow M$ is given by

$$\mathscr{C}_{co}(\lambda) = \bigwedge_{x_r \nleq \lambda'}(co(\lambda')(x_r))', \quad \forall \lambda \in L^X. \tag{6}$$

Then, $\mathscr{C}_{co}$ is an (L,M)-fuzzy concave structure.

3. The Degree of (L,M)-Fuzzy Concavity

In this section, we present and characterize the degree of (L,M)-fuzzy concavity. Moreover, M-fuzzifying concavity degree is presented and its characterizations are introduced.

Definition 3. Let $\mathscr{C}\colon L^X \longrightarrow M$ be a function. Then, $\mathbf{Coc}(\mathscr{C})$ defined by

$$\begin{aligned}\mathbf{Coc}(\mathscr{C}) = {} & \mathscr{C}(\top_{L^X}) \wedge \mathscr{C}(\bot_{L^X})\\ & \wedge\left\{\bigwedge_{\{\lambda_i\}_{i\in\Omega}^{\text{cdir}} \subseteq L^X}\left(\left(\bigwedge_{i\in\Omega}\mathscr{C}(\lambda_i)\right) \longmapsto \mathscr{C}\left(\bigwedge_{i\in\Omega}\lambda_i\right)\right)\right\}\\ & \wedge\left\{\bigwedge_{\{\lambda_i\}_{i\in\Omega} \subseteq L^X}\left(\left(\bigwedge_{i\in\Omega}\mathscr{C}(\lambda_i)\right) \longmapsto \mathscr{C}\left(\bigvee_{i\in\Omega}\lambda_i\right)\right)\right\},\end{aligned} \tag{7}$$

is called an (L,M)-fizzy concavity degree (i.e., the degree to which $\mathscr{C}$ is an (L,M)-fuzzy concavity on X).

Remark 1

(1) If $\mathbf{Coc}(\mathscr{C}) = \top_M$, then $\mathscr{C}(\top_{L^X}) = \mathscr{C}(\bot_{L^X}) = \top_M$, $\wedge_{i\in\Omega}\mathscr{C}(\lambda_i) \leq \mathscr{C}(\wedge_{i\in\Omega}\lambda_i)$ for each $\{\lambda_i\}_{i\in\Omega}^{\text{cdir}} \subseteq L^X$ and $\wedge_{i\in\Omega}\mathscr{C}(\lambda_i) \leq \mathscr{C}(\vee_{i\in\Omega}\lambda_i)$ for each $\{\lambda_i\}_{i\in\Omega} \subseteq L^X$. It is exactly the definition of (L,M)-fuzzy concave structure on X (see Definition 2). Moreover, $\mathscr{C}$ is an (L,M)-fuzzy concavity if and only if $\mathbf{Coc}(\mathscr{C}) = \top_M$.

(2) If $L=\{\bot,\top\}$ in Definition 3, then $\mathbf{Coc}(\mathscr{C})$ is called an M-fuzzifying concavity degree of $\mathscr{C}$.

Example 2. If $\mathscr{C}\colon L^X \longrightarrow [0,1]$ is a function such that $\mathscr{C}(\lambda) = (1/2)$ for any $\lambda \in L^X$. By Definition 3, we get $\mathbf{Coc}(\mathscr{C}) = (1/2) \neq 1$.

Lemma 2. *Let $\mathscr{C}\colon L^X \longrightarrow M$ be a function. For any $r \in M$, $r \leq \mathbf{Coc}(\mathscr{C})$ if and only if $r \leq \mathscr{C}(\top_{L^X})$, $r \leq \mathscr{C}(\bot_{L^X})$, $\wedge_{i\in\Omega}\mathscr{C}(\lambda_i) \wedge r \leq \mathscr{C}(\wedge_{i\in\Omega}\lambda_i)$ for each $\{\lambda_i\}_{i\in\Omega}^{cdir} \subseteq L^X$ and $\wedge_{i\in\Omega}\mathscr{C}(\lambda_i) \wedge r \leq \mathscr{C}(\vee_{i\in\Omega}\lambda_i)$ for each $\{\lambda_i\}_{i\in\omega} \subseteq L^X$.*

The following theorem can be proved using the previous lemma.

Theorem 3. *For the function $\mathscr{C}\colon L^X \longrightarrow M$, we have*

$$\mathbf{Coc}(\mathscr{C}) = \vee\left\{r \in M \;\middle|\; \begin{array}{l} r \leq \mathscr{C}(\top_{L^X}), r \leq \mathscr{C}(\bot_{L^X}),\\ \left(\bigwedge_{i\in\Omega}\mathscr{C}(\lambda_i)\right) \wedge r \leq \mathscr{C}\left(\bigwedge_{i\in\Omega}\lambda_i\right), \quad \forall\{\lambda_i\}_{i\in\Omega}^{\text{cdir}} \subseteq L^X,\\ \left(\bigwedge_{i\in\Omega}\mathscr{C}(\lambda_i)\right) \wedge r \leq \mathscr{C}\left(\bigvee_{i\in\Omega}\lambda_i\right), \quad \forall\{\lambda_i\}_{i\in\omega} \subseteq L^X. \end{array}\right\}. \tag{8}$$

Theorem 4. *Let* $\mathscr{C}\colon L^X \longrightarrow M$ *be a function. Then,*

$$\mathbf{Coc}(\mathscr{C}) = \vee\left\{r \in M \mid \forall s \le r, \mathscr{C}_{[s]} \text{ is an } L-\text{concavity}\right\}. \tag{9}$$

Proof. Let $r \le \mathscr{C}(\top_{L^X})$, $r \le \mathscr{C}(\bot_{L^X})$, and $(\wedge_{i\in\Omega}\mathscr{C}(\lambda_i))\wedge r \le \mathscr{C}(\wedge_{i\in\Omega}\lambda_i)$ for every $\{\lambda_i\}_{i\in\Omega}^{\text{cdir}} \subseteq L^X$ and $(\wedge_{i\in\Omega}\mathscr{C}(\lambda_i))\wedge r \le \mathscr{C}(\vee_{i\in\Omega}\lambda_i)$ for every $\{\lambda_i\}_{i\in\Omega} \subseteq L^X$. For $s \le r$, $\{\lambda_i\}_{i\in\Omega} \subseteq \mathscr{C}_{[s]}$, and $\{\lambda_i\}_{i\in\Omega}^{\text{cdir}} \subseteq \mathscr{C}_{[s]}$, we have $\mathscr{C}(\top_{L^X}) \ge s$, $\mathscr{C}(\bot_{L^X}) \ge s$, $\mathscr{C}(\wedge_{i\in\Omega}\lambda_i) \ge (\wedge_{i\in\Omega}\mathscr{C}(\lambda_i))\wedge r \ge s$, and $\mathscr{C}(\vee_{i\in\Omega}\lambda_i) \ge (\wedge_{i\in\Omega}\mathscr{C}(\lambda_i))\wedge r \ge s$. Therefore, $\top_{L^X}, \bot_{L^X} \in \mathscr{C}_{[s]}$, $\wedge_{i\in\Omega}\lambda_i \in \mathscr{C}_{[s]}$, and $\vee_{i\in\Omega}\lambda_i \in \mathscr{C}_{[s]}$ and hence $\mathbf{Coc}(\mathscr{C}) \le R$, where R denotes the right hand side of the equality.

Conversely, suppose that $\mathscr{C}_{[s]}$ is an L-concavity on X for any $s \le r$. Let $s = r$, then $\top_{L^X}, \bot_{L^X} \in \mathscr{C}_{[s]}$, which means $\mathscr{C}(\top_{L^X}) \ge r$ and $\mathscr{C}(\bot_{L^X}) \ge r$. For any $\{\lambda_i\}_{i\in\Omega}^{\text{cdir}} \subseteq L^X$, let $s = (\wedge_{i\in\Omega}\mathscr{C}(\lambda_i))\wedge r$, then $s \le r$ and $\{\lambda_i\}_{i\in\Omega}^{\text{cdir}} \subseteq \mathscr{C}_{[s]}$. Thus, $\wedge_{i\in\Omega}\lambda_i \in \mathscr{C}_{[s]}$, i.e., $\mathscr{C}(\wedge_{i\in\Omega}\lambda_i) \ge (\wedge_{i\in\Omega}\mathscr{C}(\lambda_i))\wedge r$. For any $\{\lambda_i\}_{i\in\Omega} \subseteq L^X$, let $s = (\wedge_{i\in\Omega}\mathscr{C}(\lambda_i))\wedge r$, then $s \le r$ and $\{\lambda_i\}_{i\in\Omega} \subseteq \mathscr{C}_{[s]}$. Thus, $\vee_{i\in\Omega}\lambda_i \in \mathscr{C}_{[s]}$, i.e., $\mathscr{C}(\vee_{i\in\Omega}\lambda_i) \ge (\wedge_{i\in\Omega}\mathscr{C}(\lambda_i))\wedge r$. Hence, $\mathbf{Coc}(\mathscr{C}) \ge R$.

Theorem 5. *Let* $\mathscr{C}\colon L^X \longrightarrow M$ *be a function. Then,*

$$\mathbf{Coc}(\mathscr{C}) = \vee\left\{r \in M \mid \forall s \notin \alpha(r), \mathscr{C}^{[s]} \text{ is an } L-\text{concavity}\right\}. \tag{10}$$

Proof. Let $r \le \mathscr{C}(\top_{L^X})$, $r \le \mathscr{C}(\bot_{L^X})$, and $(\wedge_{i\in\Omega}\mathscr{C}(\lambda_i))\wedge r \le \mathscr{C}(\wedge_{i\in\Omega}\lambda_i)$ for each $\{\lambda_i\}_{i\in\Omega}^{\text{cdir}} \subseteq L^X$ and $(\wedge_{i\in\Omega}\mathscr{C}(\lambda_i))\wedge r \le \mathscr{C}(\vee_{i\in\Omega}\lambda_i)$ for each $\{\lambda_i\}_{i\in\Omega} \subseteq L^X$. For any $s \notin \alpha(r)$, $\{\lambda_i\}_{i\in\Omega} \subseteq \mathscr{C}^{[s]}$, and $\{\lambda_i\}_{i\in\Omega}^{\text{cdir}} \subseteq \mathscr{C}^{[s]}$, we have $s \notin \alpha(r) \cup (\cup_{i\in\Omega}\alpha(\mathscr{C}(\lambda_i)))$. Then,

$$\begin{aligned}
\alpha(r) \cup \left(\bigcup_{i\in\Omega}\alpha(\mathscr{C}(\lambda_i))\right) &= \alpha\left(r \wedge \left(\bigwedge_{i\in\Omega}\mathscr{C}(\lambda_i)\right)\right) \supseteq \alpha\left(\mathscr{C}\left(\bigwedge_{i\in\Omega}\lambda_i\right)\right),\\
\alpha(r) \cup \left(\bigcup_{i\in\Omega}\alpha(\mathscr{C}(\lambda_i))\right) &= \alpha\left(r \wedge \left(\bigwedge_{i\in\Omega}\mathscr{C}(\lambda_i)\right)\right) \supseteq \alpha\left(\mathscr{C}\left(\bigvee_{i\in\Omega}\lambda_i\right)\right).
\end{aligned} \tag{11}$$

Thus, $s \notin \alpha(\mathscr{C}(\wedge_{i\in\Omega}\lambda_i))$ and $s \notin \alpha(\mathscr{C}(\vee_{i\in\Omega}\lambda_i))$. This shows $\wedge_{i\in\Omega}\lambda_i \in \mathscr{C}^{[s]}$ and $\vee_{i\in\Omega}\lambda_i \in \mathscr{C}^{[s]}$. Since $\alpha(r) \supseteq \alpha(\mathscr{C}(\top_{L^X}))$ and $s \notin \alpha(\mathscr{C}(\bot_{L^X}))$, we know $s \notin \alpha(\mathscr{C}(\top_{L^X}))$ and $s \notin \alpha(\mathscr{C}(\bot_{L^X}))$, which proves $\top_{L^X}, \bot_{L^X} \in \mathscr{C}^{[s]}$. Hence, $\mathbf{Coc}(\mathscr{C}) \le R$, where R denotes the right hand side of the equality.

Conversely, suppose that for any $s \notin \alpha(r)$, $\mathscr{C}^{[s]}$ is an L-concavity and $\top_{L^X}, \bot_{L^X} \in \mathscr{C}^{[s]}$, i.e., $s \notin \alpha(\mathscr{C}(\top_{L^X}))$ and $s \notin \alpha(\mathscr{C}(\bot_{L^X}))$. Then, $\alpha(\mathscr{C}(\top_{L^X})) \subseteq \alpha(r)$ and $\alpha(\mathscr{C}(\bot_{L^X})) \subseteq \alpha(r)$, i.e., $r \le \mathscr{C}(\top_{L^X})$ and $r \le \mathscr{C}(\bot_{L^X})$. For $\{\lambda_i\}_{i\in\Omega}^{\text{cdir}} \subseteq L^X$, take any $s \notin \alpha((\wedge_{i\in\Omega}\mathscr{C}(\lambda_i))\wedge r)$. By $\alpha((\wedge_{i\in\Omega}\mathscr{C}(\lambda_i))\wedge r) = (\cup_{i\in\Omega}\alpha(\mathscr{C}(\lambda_i))) \cup \alpha(r)$, we have $s \notin \alpha(r)$, $s \notin \alpha(\mathscr{C}(\lambda_i))$, for all $i \in \Omega$, which means $\{\lambda_i\}_{i\in\Omega}^{\text{cdir}} \subseteq \mathscr{C}^{[s]}$. Since $\mathscr{C}^{[s]}$ is an L-concavity, we have $\wedge_{i\in\Omega}\lambda_i \in \mathscr{C}^{[s]}$, i.e., $s \notin \alpha(\mathscr{C}(\wedge_{i\in\Omega}\lambda_i))$. Then, $\alpha(\mathscr{C}(\wedge_{i\in\Omega}\lambda_i)) \subseteq \alpha((\wedge_{i\in\Omega}\mathscr{C}(\lambda_i))\wedge r)$, i.e., $(\wedge_{i\in\Omega}\mathscr{C}(\lambda_i))\wedge r \le \mathscr{C}(\wedge_{i\in\Omega}\lambda_i)$. Similarly, we can show that $(\wedge_{i\in\Omega}\mathscr{C}(\lambda_i))\wedge r \le \mathscr{C}(\vee_{i\in\Omega}\lambda_i)$ for each $\{\lambda_i\}_{i\in\Omega} \subseteq L^X$. Hence, $\mathbf{Coc}(\mathscr{C}) \ge R$.

The following corollary gives similar characterization of M-fuzzifying concavity.

Corollary 1. *Let* $\mathscr{C}\colon 2^X \longrightarrow M$ *be a function. Then,*

(1) $\mathbf{Coc}(\mathscr{C}) = \vee\{r \in M,\ r \le \mathscr{C}(X), r \le \mathscr{C}(\varnothing),\ (\wedge_{i\in\Omega}\mathscr{C}(A_i))\wedge r \le \mathscr{C}(\cap_{i\in\Omega}A_i),\ \forall\{A_i\}_{i\in\Omega}^{cdir} \subseteq 2^X,\ (\wedge_{i\in\Omega}\mathscr{C}(A_i))\wedge r \le \mathscr{C}(\cup_{i\in\Omega}A_i),\ \forall\{A_i\}_{i\in\omega} \subseteq 2^X\}$

(2) $\mathbf{Coc}(\mathscr{C}) = \vee\left\{r \in M \mid \forall tsn \le qrh,_{\mathscr{C}_{[s]}} x \textit{ is a concavity}\right\}$

(3) $\mathbf{Coc}(\mathscr{C}) = \vee\left\{r \in M \mid \forall s \notin \alpha(r), \mathscr{C}^{[s]} \textit{ is a concavity}\right\}$

An (L,M)*-fuzzy concavity degree* $\mathbf{Coc}(\mathscr{C})$ *can also be treated as a function* $\mathbf{Coc}\colon M^{L^X} \longrightarrow M$ *given by* $\mathscr{C} \longrightarrow \mathbf{Coc}(\mathscr{C})$*. This function has the following property.*

Theorem 6. *Let* $\left\{\mathscr{C}_j \mid \mathscr{C}_j t\colon nL^X q \longrightarrow hM\right\}_{j\in J}$ *be a family of functions. Then,*

$$\bigwedge_{j\in J}\mathbf{Coc}(\mathscr{C}_j) \le \mathbf{Coc}\left(\bigwedge_{j\in J}\mathscr{C}_j\right). \tag{12}$$

Proof. From Definition 3, we have

$$\begin{aligned}
\mathbf{Coc}\left(\bigwedge_{j\in J}\mathscr{C}_j\right) &= \left(\bigwedge_{j\in J}\mathscr{C}_j(\top_{L^X})\right)\wedge\left(\bigwedge_{j\in J}\mathscr{C}_j(\bot_{L^X})\right)\wedge\left\{\bigwedge_{\{\lambda_i\}_{i\in\Omega}^{\text{cdir}} \subseteq L^X}\left\{\left(\bigwedge_{i\in\Omega}\left(\bigwedge_{j\in J}\mathscr{C}_j(\lambda_i)\right)\right) \longmapsto \left(\bigwedge_{j\in J}\mathscr{C}_j\left(\bigwedge_{i\in\Omega}\lambda_i\right)\right)\right\}\right\}\\
&\quad \wedge\left\{\bigwedge_{\{\lambda_i\}_{i\in\Omega} \subseteq L^X}\left\{\left(\bigwedge_{i\in\Omega}\left(\bigwedge_{j\in J}\mathscr{C}_j(\lambda_i)\right)\right) \longmapsto \left(\bigwedge_{j\in J}\mathscr{C}_j\left(\bigwedge_{i\in\Omega}\lambda_i\right)\right)\right\}\right\}\\
&= \left(\bigwedge_{j\in J}\mathscr{C}_j(\top_{L^X})\right)\wedge\left(\bigwedge_{j\in J}\mathscr{C}_j(\bot_{L^X})\right)\wedge\left\{\bigwedge_{\{\lambda_i\}_{i\in\Omega}^{\text{cdir}} \subseteq L^X}\bigwedge_{j\in J}\left\{\left(\bigwedge_{i\in\Omega}\bigwedge_{j\in J}\mathscr{C}_j(\lambda_i)\right) \longmapsto \mathscr{C}_j\left(\bigwedge_{i\in\Omega}\lambda_i\right)\right\}\right\}\\
&\quad \wedge\left\{\bigwedge_{\{\lambda_i\}_{i\in\Omega} \subseteq L^X}\bigwedge_{j\in J}\left\{\left(\bigwedge_{i\in\Omega}\bigwedge_{j\in J}\mathscr{C}_j(\lambda_i)\right) \longmapsto \mathscr{C}_j\left(\bigvee_{i\in\Omega}\lambda_i\right)\right\}\right\}
\end{aligned}$$

$$\geq \left(\bigwedge_{j\in J}\mathscr{C}_j\left(\top_{L^X}\right)\right)\wedge\left(\bigwedge_{j\in J}\mathscr{C}_j\left(\bot_{L^X}\right)\right)\wedge\left\{\bigwedge_{j\in J}\bigwedge_{\{\lambda_i\}_{i\in\Omega}^{\text{cdir}}\subseteq L^X}\left\{\left(\bigwedge_{i\in\Omega}\mathscr{C}_j\left(\lambda_j\right)\right)\longmapsto\mathscr{C}_j\left(\bigwedge_{i\in\Omega}\lambda_i\right)\right\}\right\}$$
$$\wedge\left\{\bigwedge_{j\in J}\bigwedge_{\{\lambda_i\}_{i\in\Omega}\subseteq L^X}\left\{\left(\bigwedge_{i\in\Omega}\mathscr{C}_j\left(\lambda_j\right)\right)\longmapsto\mathscr{C}_j\left(\bigvee_{i\in\Omega}\lambda_i\right)\right\}\right\}$$
$$= \bigwedge_{j\in J}\mathbf{Coc}\left(\mathscr{C}_j\right). \tag{13}$$

4. The Degree of Concavity Preserving and Concave-to-Concave Functions

This section presents the degree of (L, M)-fuzzy concavity preserving and (L, M)-fuzzy concave-to-concave functions and discuss their properties.

Definition 4. Given a function $\mathscr{C}\colon L^X \longrightarrow M$, (L, M)-fuzzy concavity degree $\mathbf{Coc}(\mathscr{C})$ of $\mathscr{C}$, for any $\lambda \in L^X$, we define $\mathbf{P}_{\mathbf{Coc}}(\lambda)$ by $\mathbf{P}_{\mathbf{Coc}}(\lambda) = \mathbf{Coc}(\mathscr{C})\wedge\mathscr{C}(\lambda)$ is called the degree to which λ is an L-concave set with respect to $\mathscr{C}$ (or the L-concave set degree of λ with respect to $\mathscr{C}$).

Remark 2. If $\mathbf{Coc}(\mathscr{C}) = \top_M$, which means that $\mathscr{C}$ is an (L, M)-fuzzy concavity, then $\mathbf{P}_{\mathbf{Coc}}(\lambda) = \mathscr{C}(\lambda)$, which can be regarded as a generalization of $\mathscr{C}(\lambda)$.

Proposition 1. *Given a function $\mathscr{C}\colon L^X \longrightarrow M$ and (L, M)-fuzzy concavity degree $\mathbf{Coc}(\mathscr{C})$ of $\mathscr{C}$, for each $\lambda \in L^X$, $\mathbf{P}_{\mathbf{Coc}}(\lambda)$ denotes the L-concave set degree of λ with respect to $\mathscr{C}$. Then,*

(1) $\wedge_{i\in\Omega}\mathbf{P}_{\mathbf{Coc}}(\lambda_i) \leq \mathbf{P}_{\mathbf{Coc}}(\wedge_{i\in\Omega}\lambda_i)$, $\forall\{\lambda_i\}_{i\in\Omega}^{cdir} \subseteq L^X$

(2) $\wedge_{i\in\Omega}\mathbf{P}_{\mathbf{Coc}}(\lambda_i) \leq \mathbf{P}_{\mathbf{Coc}}(\vee_{i\in\Omega}\lambda_i)$, $\forall\{\lambda_i\}_{i\in\Omega} \subseteq L^X$

That is, if $\mathbf{P}_{\mathbf{Coc}}(\lambda)$ is regarded as a function $\mathbf{P}_{\mathbf{Coc}}\colon L^X \longrightarrow M$ defined by $\lambda \longmapsto \mathbf{P}_{\mathbf{Coc}}(\lambda)$, then $\mathbf{P}_{\mathbf{Coc}}$ is an (L, M)-fuzzy concavity on X.

Proof

(1) Based on Definition 4, it suffices to prove that $\mathbf{Coc}(\mathscr{C})\wedge(\wedge_{i\in\Omega}\mathscr{C}(\lambda_i)) \leq \mathbf{Coc}(\mathscr{C})$ and $\mathbf{Coc}(\mathscr{C})\wedge(\wedge_{i\in\Omega}\mathscr{C}(\lambda_i)) \leq \mathscr{C}(\wedge_{i\in\Omega}\lambda_i)$. By Definition 3, we have

$$\mathbf{Coc}(\mathscr{C})\wedge\left(\bigwedge_{i\in\Omega}\mathscr{C}(\lambda_i)\right) \leq \left\{\left(\bigwedge_{i\in\Omega}\mathscr{C}(\lambda_i)\right)\longmapsto\mathscr{C}\left(\bigwedge_{i\in\Omega}\lambda_i\right)\right\}$$
$$\wedge\left(\bigwedge_{i\in\Omega}\mathscr{C}(\lambda_i)\right) \leq \mathscr{C}\left(\bigwedge_{i\in\Omega}\lambda_i\right). \tag{14}$$

(2) It is similar to (3).

The following theorem characterizes L-concave set degree.

Theorem 7. *Let $\mathscr{C}\colon L^X \longrightarrow M$ be a function and $\mathbf{Coc}(\mathscr{C})$ be (L, M)-fuzzy concavity degree of $\mathscr{C}$. For each $\lambda \in L^X$, $\mathbf{P}_{\mathbf{Coc}}(\lambda)$ denotes the L-concave set degree of λ with respect to $\mathscr{C}$. Then,*

(1) $\mathbf{P}_{\mathbf{Coc}}(\lambda) = \vee\{r \in M,\ r \leq \mathscr{C}(\lambda),\ r \leq \mathscr{C}(\top_{L^X}),\ r \leq \mathscr{C}(\bot_{L^X}),\ (\wedge_{i\in\Omega}\mathscr{C}(\lambda_i))\wedge r \leq \mathscr{C}(\wedge_{i\in\Omega}\lambda_i)\forall\{\lambda_i\}_{i\in\Omega}^{cdir} \subseteq L^X,\ (\wedge_{i\in\Omega}\mathscr{C}(\lambda_i))\wedge r \leq \mathscr{C}(\vee_{i\in\Omega}\lambda_i)\forall\{\lambda_i\}_{i\in\omega} \subseteq L^X\}$

(2) $\mathbf{P}_{\mathbf{Coc}}(\lambda) = \vee\{r \in M|\forall s \leq r, \mathscr{C}_{[s]}$ is an $L-$concavity and $\lambda \in \mathscr{C}_{[s]}\}$

(3) $\mathbf{P}_{\mathbf{Coc}}(\lambda)=\vee\{r\in M|\forall s\notin\alpha(r), \mathscr{C}^{[s]}$ is an $L-$concavity and $\lambda\in\mathscr{C}_{[s]}\}$

Proof. The proofs are similar to those of Theorems 3–5.

Definition 5. Given two functions $\mathscr{C}_X\colon L^X \longrightarrow M$ and $\mathscr{C}_Y\colon L^Y \longrightarrow M$, let $f\colon (X, \mathscr{C}_X) \longrightarrow (Y, \mathscr{C}_Y)$ be a function. Then,

(1) The concavity preserving degree $\mathfrak{p}$ of f with respect to $\mathscr{C}_X$ and $\mathscr{C}_Y$ is defined by

$$\mathfrak{p}(f) = \bigwedge_{\nu\in L^Y}\left\{\mathbf{P}_{\mathbf{Coc}_Y}(\nu)\longmapsto\mathbf{P}_{\mathbf{Coc}_X}\left(f_L^{\leftarrow}(\nu)\right)\right\}. \tag{15}$$

(2) The concave-to-concave degree $\mathfrak{c}$ of f with respect to $\mathscr{C}_X$ and $\mathscr{C}_Y$ is defined by

$$\mathfrak{c}(f) = \bigwedge_{\lambda\in L^X}\left\{\mathbf{P}_{\mathbf{Coc}_X}(\lambda)\longmapsto\mathbf{P}_{\mathbf{Coc}_Y}\left(f_L^{\longrightarrow}(\lambda)\right)\right\}, \tag{16}$$

where $\mathbf{Coc}_X = \mathbf{Coc}(\mathscr{C}_X)$ and $\mathbf{Coc}_Y = \mathbf{Coc}(\mathscr{C}_Y)$.

Theorem 8. *Let $\mathscr{C}_X\colon L^X \longrightarrow M$, $\mathscr{C}_Y\colon L^Y \longrightarrow M$ be two functions and let $\mathbf{Coc}(\mathscr{C}_X)$, $\mathbf{Coc}(\mathscr{C}_Y)$ refer to the (L, M)-fuzzy concavity degrees of $\mathscr{C}_X$ and $\mathscr{C}_Y$, respectively. Then,*

(1) $\mathfrak{p}(f) = \wedge\{r \in M|\mathbf{Coc}(\mathscr{C}_Y)\wedge\mathscr{C}_Y(\nu)\wedge r \leq \mathbf{Coc}(\mathscr{C}_X)\wedge\mathscr{C}_X(f_L^{\leftarrow}(\nu)),\ \forall\nu \in L^Y\}$

(2) $\mathfrak{p}(f) = \wedge\left\{r \in M\middle|\begin{array}{l}\forall s \leq \mathbf{Coc}(\mathscr{C}_Y)\wedge r,\ \forall\nu \in (\mathscr{C}_Y)_{[s]},\\ s \leq \mathbf{Coc}(\mathscr{C}_X),\ f_L^{\leftarrow}(\nu) \in (\mathscr{C}_X)_{[s]}\end{array}\right\}$

(3) $\mathfrak{p}(f)=\wedge\left\{r\in M\middle|\begin{array}{l}\forall s\notin\alpha(\mathbf{Coc}(\mathscr{C}_Y)\wedge r), \forall\nu\in(\mathscr{C}_Y)^{[s]}\\ s\notin\alpha(\mathbf{Coc}(\mathscr{C}_X)), f_L^{\leftarrow}(\nu)\in(\mathscr{C}_X)^{[s]}\end{array}\right\}$

Proof

(1) Since for each $r \in M$ and $\nu \in L^Y$, we have

$$r \leq \mathfrak{p}(f) \Longleftrightarrow r \wedge \mathbf{P}_{\mathbf{Coc}_Y}(\nu) \leq \mathbf{P}_{\mathbf{Coc}_X}\left(f_L^{\leftarrow}(\nu)\right) \Longleftrightarrow \mathbf{Coc}\left(\mathscr{C}_Y\right) \wedge \mathscr{C}_Y(\nu) \wedge r \leq \mathbf{Coc}\left(\mathscr{C}_X\right) \wedge \mathscr{C}_X\left(f_L^{\leftarrow}(\nu)\right). \tag{17}$$

The proof of (1) is clear.

(2) Suppose that $\mathbf{Coc}(\mathscr{C}_Y) \wedge \mathscr{C}_Y(\nu) \wedge r \leq \mathbf{Coc}(\mathscr{C}_X) \wedge \mathscr{C}_X(f_L^{\leftarrow}(\nu))$ for each $\nu \in L^Y$. For any $s \leq \mathbf{Coc}(\mathscr{C}_Y) \wedge r$ and $\nu \in (\mathscr{C}_Y)_{[s]}$, i.e., $s \leq \mathscr{C}_Y(\nu)$, we have $s \leq \mathbf{Coc}(\mathscr{C}_X) \wedge \mathscr{C}_X(f_L^{\leftarrow}(\nu))$. Thus, $s \leq \mathbf{Coc}(\mathscr{C}_X)$ and $s \leq \mathscr{C}_X(f_L^{\leftarrow}(\nu))$, i.e., $f_L^{\leftarrow}(\nu) \in (\mathscr{C}_X)_{[s]}$. By (1), hence $\mathfrak{p}(f) \leq R$, where R refers to the right hand side of equality.

Conversely, let $s \leq \mathbf{Coc}(\mathscr{C}_Y) \wedge \mathscr{C}_Y(\nu) \wedge r$. Then, $s \leq \mathbf{Coc}(\mathscr{C}_Y) \wedge r$ and $s \leq \mathscr{C}_Y(\nu)$, i.e., $\nu \in (\mathscr{C}_Y)_{[s]}$. Thus, $\mathbf{Coc}(\mathscr{C}_X) \geq s$ and $f_L^{\leftarrow}(\nu) \in (\mathscr{C}_X)_{[s]}$, i.e., $\mathscr{C}_X(f_L^{\leftarrow}(\nu)) \geq s$. This implies $\mathbf{Coc}(\mathscr{C}_X) \wedge \mathscr{C}_X(f_L^{\leftarrow}(\nu)) \geq s$. Since s is arbitrary, we have $\mathbf{Coc}(\mathscr{C}_Y) \wedge \mathscr{C}_Y(\nu) \wedge r \leq \mathbf{Coc}(\mathscr{C}_X) \wedge \mathscr{C}_X(f_L^{\leftarrow}(\nu))$. From this result together with (1), we have $\mathfrak{p}(f) \geq R$.

(3) Suppose that $\mathbf{Coc}(\mathscr{C}_Y) \wedge \mathscr{C}_Y(\nu) \wedge r \leq \mathbf{Coc}(\mathscr{C}_X) \wedge \mathscr{C}_X(f_L^{\leftarrow}(\nu))$ for each $\nu \in L^Y$. For any $s \notin \alpha(\mathbf{Coc}(\mathscr{C}_Y) \wedge r)$ and $\nu \in (\mathscr{C}_Y)^{[s]}$, i.e., $s \notin \alpha(\mathscr{C}_Y(\nu))$, we have $s \notin \alpha(\mathscr{C}_Y(\nu)) \cup \alpha(\mathbf{Coc}(\mathscr{C}_Y) \wedge r)$. Since α is a $\wedge$-$\cup$ map, it follows that

$$\begin{aligned}\alpha\left(\mathscr{C}_Y(\nu)\right) \cup \alpha\left(\mathbf{Coc}\left(\mathscr{C}_Y\right) \wedge r\right) &= \alpha\left(\mathbf{Coc}\left(\mathscr{C}_Y\right) \wedge \mathscr{C}_Y(\nu) \wedge r\right) \supseteq \alpha\left(\mathbf{Coc}\left(\mathscr{C}_X\right) \wedge \mathscr{C}_X\left(f_L^{\leftarrow}(\nu)\right)\right)\\ &= \alpha\left(\mathbf{Coc}\left(\mathscr{C}_X\right)\right) \cup \alpha\left(\mathscr{C}_X\left(f_L^{\leftarrow}(\nu)\right)\right).\end{aligned} \tag{18}$$

This implies $s \notin \alpha(\mathbf{Coc}(\mathscr{C}_X))$ and $s \notin \alpha(\mathscr{C}_X(f_L^{\leftarrow}(\nu)))$, i.e., $f_L^{\leftarrow}(\nu) \in (\mathscr{C}_X)^{[s]}$. By **(1)**, we have $\mathfrak{p}(f) \leq R$, where R refers to the right hand side of the equality.

Conversely, take any $s \notin \alpha(\mathbf{Coc}(\mathscr{C}_Y) \wedge \mathscr{C}_Y(\nu) \wedge r)$. By

$$\alpha\left(\mathbf{Coc}\left(\mathscr{C}_Y\right) \wedge \mathscr{C}_Y(\nu) \wedge r\right) = \alpha\left(\mathbf{Coc}\left(\mathscr{C}_Y\right) \wedge r\right) \cup \alpha\left(\mathscr{C}_Y(\nu)\right), \tag{19}$$

we have $s \notin \alpha(\mathbf{Coc}(\mathscr{C}_Y) \wedge r)$ and $s \notin \alpha(\mathscr{C}_Y(\nu))$, i.e., $\nu \in (\mathscr{C}_Y)^{[s]}$. Thus, $s \notin \alpha(\mathbf{Coc}(\mathscr{C}_X))$ and $f_L^{\leftarrow}(\nu) \in (\mathscr{C}_X)^{[s]}$, i.e., $s \notin \alpha(\mathscr{C}_X(f_L^{\leftarrow}(\nu)))$. This implies

$$\begin{aligned} s &\notin \alpha\left(\mathbf{Coc}\left(\mathscr{C}_X\right)\right) \cup \alpha\left(\mathscr{C}_X\left(f_L^{\leftarrow}(\nu)\right)\right)\\ &= \alpha\left(\mathbf{Coc}\left(\mathscr{C}_X\right) \wedge \mathscr{C}_X\left(f_L^{\leftarrow}(\nu)\right)\right).\end{aligned} \tag{20}$$

Since s is arbitrary, we have $\mathbf{Coc}(\mathscr{C}_Y) \wedge \mathscr{C}_Y(\nu) \wedge r \leq \mathbf{Coc}(\mathscr{C}_X) \wedge \mathscr{C}_X(f_L^{\leftarrow}(\nu))$. From this result together with (1), we get $\mathfrak{p}(f) \geq R$.

Theorem 9. *Given two functions $\mathscr{C}_X\colon L^X \longrightarrow M$ and $\mathscr{C}_Y\colon L^Y \longrightarrow M$, if $\mathbf{Coc}(\mathscr{C}_X)$ and $\mathbf{Coc}(\mathscr{C}_Y)$ denote the (L,M)-fuzzy concavity degrees of $\mathscr{C}_X$ and $\mathscr{C}_Y$, respectively, then*

(1) $\mathfrak{c}(f) = \wedge\left\{r \in M \,\middle|\, \begin{array}{c}\mathbf{Coc}(\mathscr{C}_X) \wedge \mathscr{C}_X(\lambda) \wedge r \leq \mathbf{Coc}(\mathscr{C}_Y) \wedge \mathscr{C}_Y(f_L^{\rightarrow}(\lambda)),\\ \forall \lambda \in L^X\end{array}\right\}$

(2) $\mathfrak{c}(f) = \wedge\left\{r \in M \,\middle|\, \begin{array}{c}\forall s \leq \mathbf{Coc}(\mathscr{C}_X) \wedge r, \forall \lambda \in (\mathscr{C}_X)_{[s]}\\ s \leq \mathbf{Coc}(\mathscr{C}_Y), f_L^{\rightarrow}(\lambda) \in (\mathscr{C}_Y)_{[s]}\end{array}\right\}$

(3) $\mathfrak{c}(f) = \wedge\left\{r \in M \,\middle|\, \begin{array}{c}\forall s \notin \alpha(\mathbf{Coc}(\mathscr{C}_X) \wedge r), \forall \lambda \in (\mathscr{C}_X)^{[s]},\\ s \notin \alpha(\mathbf{Coc}(\mathscr{C}_Y)), f_L^{\rightarrow}(\lambda) \in (\mathscr{C}_Y)^{[s]}\end{array}\right\}$

Proof. Similar to the proof of Theorem 8.

Proposition 2. *If M is a complete distributive DeMorgan algebra and $f\colon (X, \mathscr{C}_X) \longrightarrow (Y, \mathscr{C}_Y)$ is a function between two (L,M)-fuzzy concavity structures $(X, \mathscr{C}_X)$ and $(Y, \mathscr{C}_Y)$, then*

$$\mathfrak{p}(f) = \bigwedge_{x_r \in J(L^X)} \bigwedge_{\nu \in L^Y} \left\{\mathrm{co}_{\mathscr{C}_Y}(\nu')\left(f(x)_r\right)' \longmapsto \mathrm{co}_{\mathscr{C}_X}\left(f_L^{\leftarrow}(\nu')\right)(x_r)'\right\}, \tag{21}$$

where $\mathrm{co}_{\mathscr{C}_X}$ and $\mathrm{co}_{\mathscr{C}_Y}$ are the corresponding hull operators.

Proof. Straightforward.

Proposition 3. *Given five functions $\mathscr{C}_X\colon L^X \longrightarrow M$, $\mathscr{C}_Y\colon L^Y \longrightarrow M$, $\mathscr{C}_Z\colon L^Z \longrightarrow M$, $f\colon (X, \mathscr{C}_X) \longrightarrow (Y, \mathscr{C}_Y)$, and $g\colon (Y, \mathscr{C}_Y) \longrightarrow (Z, \mathscr{C}_Z)$, then*

(1) $\mathfrak{p}(f) \wedge \mathfrak{p}(g) \leq \mathfrak{p}(g^{\circ} f)$

(2) $\mathfrak{c}(f) \wedge \mathfrak{c}(g) \leq \mathfrak{c}(g^{\circ} f)$

Proof

(1) Since $(g^{\circ} f)_L^{\leftarrow}(\xi) = f_L^{\leftarrow}(g_L^{\leftarrow}(\xi))$ for each $\xi \in L^Z$, by Definition 5, we have

$$\begin{aligned}\mathfrak{p}(f) \wedge \mathfrak{p}(g) &= \left\{\bigwedge_{\nu \in L^Y}\left(\mathbf{P}_{\mathbf{Coc}_Y}(\nu) \longmapsto \mathbf{P}_{\mathbf{Coc}_X}\left(f_L^{\leftarrow}(\nu)\right)\right)\right\} \wedge \left\{\bigwedge_{\gamma \in L^Z}\left(\mathbf{P}_{\mathbf{Coc}_Z}(\gamma) \longmapsto \mathbf{P}_{\mathbf{Coc}_Y}\left(g_L^{\leftarrow}(\gamma)\right)\right)\right\}\\ &\leq \left\{\bigwedge_{\xi \in L^Z}\left(\mathbf{P}_{\mathbf{Coc}_Y}\left(g_L^{\leftarrow}(\xi)\right) \longmapsto \mathbf{P}_{\mathbf{Coc}_X}\left(f_L^{\leftarrow}\left(g_L^{\leftarrow}(\xi)\right)\right)\right)\right\} \wedge \left\{\bigwedge_{\gamma \in L^Z}\left(\mathbf{P}_{\mathbf{Coc}_Z}(\gamma) \longmapsto \mathbf{P}_{\mathbf{Coc}_Y}\left(g_L^{\leftarrow}(\gamma)\right)\right)\right\}\\ &= \left\{\bigwedge_{\xi \in L^Z}\left(\mathbf{P}_{\mathbf{Coc}_Y}\left(g_L^{\leftarrow}(\xi)\right) \longmapsto \mathbf{P}_{\mathbf{Coc}_X}\left(\left(g^{\circ} f\right)_L^{\leftarrow}(\xi)\right)\right)\right\} \wedge \left\{\bigwedge_{\gamma \in L^Z}\left(\mathbf{P}_{\mathbf{Coc}_Z}(\gamma) \longmapsto \mathbf{P}_{\mathbf{Coc}_Y}\left(g_L^{\leftarrow}(\gamma)\right)\right)\right\}\end{aligned}$$

$$
\begin{aligned}
&= \bigwedge_{\gamma\in L^Z}\left\{\left(\mathbf{P}_{\mathbf{Coc}_Y}\left(g_L^{\leftarrow}(\gamma)\right)\longmapsto \mathbf{P}_{\mathbf{Coc}_X}\left(\left(g^\circ f\right)_L^{\leftarrow}(\gamma)\right)\right)\wedge\left(\mathbf{P}_{\mathbf{Coc}_Z}(\gamma)\longmapsto \mathbf{P}_{\mathbf{Coc}_Y}\left(g_L^{\leftarrow}(\gamma)\right)\right)\right\}\\
&\leq \bigwedge_{\gamma\in L^Z}\left\{\mathbf{P}_{\mathbf{Coc}_Z}(\gamma)\longmapsto \mathbf{P}_{\mathbf{Coc}_X}\left(\left(g^\circ f\right)_L^{\leftarrow}(\gamma)\right)\right\}\\
&= \mathfrak{p}\left(g^\circ f\right).
\end{aligned}
\tag{22}
$$

(2) similar to the proof of (1).

Proposition 4. *Given three functions $\mathscr{C}_X\colon L^X \longrightarrow M$, $\mathscr{C}_Y\colon L^Y \longrightarrow M$, and $\mathscr{C}_Z\colon L^Z \longrightarrow M$, let $f\colon (X,\mathscr{C}_X) \longrightarrow (Y,\mathscr{C}_Y)$ and $g\colon (Y,\mathscr{C}_Y) \longrightarrow (Z,\mathscr{C}_Z)$. Then,*

(1) If f is surjective, then $\mathfrak{c}(g^\circ f)\wedge\mathfrak{p}(f)\leq\mathfrak{c}(g)$.

(2) If g is injective, then $\mathfrak{c}(g^\circ f)\wedge\mathfrak{p}(g)\leq\mathfrak{c}(f)$.

Proof

(1) Since f is surjective, we know $f_L^{\longrightarrow}(f_L^{\leftarrow}(\nu)) = \nu$ for all $\nu \in L^Y$. Then,

$$
\left(g^\circ f\right)_L^{\longrightarrow}\left(f_L^{\leftarrow}(\nu)\right) = g_L^{\longrightarrow}\left(f_L^{\longrightarrow}\left(f_L^{\leftarrow}(\nu)\right)\right) = g_L^{\longrightarrow}(\nu).
\tag{23}
$$

Hence,

$$
\begin{aligned}
\mathfrak{c}\left(g^\circ f\right)\wedge\mathfrak{p}(f) &= \left\{\bigwedge_{\lambda\in L^X}\left(\mathbf{P}_{\mathbf{Coc}_X}(\lambda)\longmapsto \mathbf{P}_{\mathbf{Coc}_Z}\left(\left(g^\circ f\right)_L^{\longrightarrow}(\lambda)\right)\right)\right\}\wedge\left\{\bigwedge_{\nu\in L^Y}\left(\mathbf{P}_{\mathbf{Coc}_Y}(\nu)\longmapsto \mathbf{P}_{\mathbf{Coc}_X}\left(f_L^{\leftarrow}(\nu)\right)\right)\right\}\\
&\leq \left\{\bigwedge_{\gamma\in L^Y}\left(\mathbf{P}_{\mathbf{Coc}_X}\left(f_L^{\leftarrow}(\gamma)\right)\longmapsto \mathbf{P}_{\mathbf{Coc}_Z}\left(\left(g^\circ f\right)_L^{\longrightarrow}\left(f_L^{\leftarrow}(\gamma)\right)\right)\right)\right\}\wedge\left\{\bigwedge_{\nu\in L^Y}\left(\mathbf{P}_{\mathbf{Coc}_Y}(\nu)\longmapsto \mathbf{P}_{\mathbf{Coc}_X}\left(f_L^{\leftarrow}(\nu)\right)\right)\right\}\\
&= \left\{\bigwedge_{\gamma\in L^Y}\left(\mathbf{P}_{\mathbf{Coc}_X}\left(f_L^{\leftarrow}(\gamma)\right)\longmapsto \mathbf{P}_{\mathbf{Coc}_Z}\left(g_L^{\longrightarrow}(\gamma)\right)\right)\right\}\wedge\left\{\bigwedge_{\nu\in L^Y}\left(\mathbf{P}_{\mathbf{Coc}_X}(\nu)\longmapsto \mathbf{P}_{\mathbf{Coc}_X}\left(f_L^{\leftarrow}(\nu)\right)\right)\right\}\\
&= \bigwedge_{\nu\in L^Y}\left\{\left(\mathbf{P}_{\mathbf{Coc}_X}\left(f_L^{\leftarrow}(\nu)\right)\longmapsto \mathbf{P}_{\mathbf{Coc}_Z}\left(g_L^{\longrightarrow}(\nu)\right)\right)\wedge\left(\mathbf{P}_{\mathbf{Coc}_Y}(\nu)\longmapsto \mathbf{P}_{\mathbf{Coc}_X}\left(f_L^{\leftarrow}(\nu)\right)\right)\right\}\\
&\leq \bigwedge_{\nu\in L^Y}\left\{\mathbf{P}_{\mathbf{Coc}_Y}(\nu)\longmapsto \mathbf{P}_{\mathbf{Coc}_Z}\left(g_L^{\longrightarrow}(\nu)\right)\right\} = \mathfrak{c}(g).
\end{aligned}
\tag{24}
$$

(2) Since g is injective, we have $g_L^{\leftarrow}(g_L^{\longrightarrow}(\nu)) = \nu$ for each $\nu \in L^Y$. Then, $g_L^{\leftarrow}((g^\circ f)_L^{\longrightarrow}(\mu)) = f_L^{\longrightarrow}(\mu)$ for all $\mu \in L^X$. Hence,

$$
\begin{aligned}
\mathfrak{c}\left(g^\circ f\right)\wedge\mathfrak{p}(g) &= \left\{\bigwedge_{\lambda\in L^X}\left(\mathbf{P}_{\mathbf{Coc}_X}(\lambda)\longmapsto \mathbf{P}_{\mathbf{Coc}_Z}\left(\left(g^\circ f\right)_L^{\longrightarrow}(\lambda)\right)\right)\right\}\wedge\left\{\bigwedge_{\xi\in L^Z}\left(\mathbf{P}_{\mathbf{Coc}_Z}(\xi)\longmapsto \mathbf{P}_{\mathbf{Coc}_Y}\left(g_L^{\leftarrow}(\xi)\right)\right)\right\}\\
&\leq \left\{\bigwedge_{\lambda\in L^X}\left(\mathbf{P}_{\mathbf{Coc}_X}(\lambda)\longmapsto \mathbf{P}_{\mathbf{Coc}_Z}\left(\left(g^\circ f\right)_L^{\longrightarrow}(\lambda)\right)\right)\right\}\wedge\left\{\bigwedge_{\gamma\in L^X}\left(\mathbf{P}_{\mathbf{Coc}_Z}\left(\left(g^\circ f\right)_L^{\longrightarrow}(\gamma)\right)\longmapsto \mathbf{P}_{\mathbf{Coc}_Y}\left(g_L^{\leftarrow}\left(\left(g^\circ f\right)_L^{\longrightarrow}(\gamma)\right)\right)\right)\right\}\\
&= \left\{\bigwedge_{\lambda\in L^X}\left(\mathbf{P}_{\mathbf{Coc}_X}(\lambda)\longmapsto \mathbf{P}_{\mathbf{Coc}_Z}\left(g_L^{\longrightarrow}\left(f_L^{\longrightarrow}(\lambda)\right)\right)\right)\right\}\wedge\left\{\bigwedge_{\gamma\in L^X}\left(\mathbf{P}_{\mathbf{Coc}_Z}\left(g_L^{\longrightarrow}\left(f_L^{\longrightarrow}(\gamma)\right)\right)\longmapsto \mathbf{P}_{\mathbf{Coc}_Y}\left(f_L^{\longrightarrow}(\gamma)\right)\right)\right\}\\
&= \bigwedge_{\lambda\in L^X}\left\{\left(\mathbf{P}_{\mathbf{Coc}_X}(\lambda)\longmapsto \mathbf{P}_{\mathbf{Coc}_Z}\left(g_L^{\longrightarrow}\left(f_L^{\longrightarrow}(\lambda)\right)\right)\right)\wedge\left(\mathbf{P}_{\mathbf{Coc}_Z}\left(g_L^{\longrightarrow}\left(f_L^{\longrightarrow}(\lambda)\right)\right)\longmapsto \mathbf{P}_{\mathbf{Coc}_Y}\left(f_L^{\longrightarrow}(\lambda)\right)\right)\right\}\\
&\leq \bigwedge_{\lambda\in L^X}\left\{\mathbf{P}_{\mathbf{Coc}_X}(\lambda)\longmapsto \mathbf{P}_{\mathbf{Coc}_Y}\left(f_L^{\longrightarrow}(\lambda)\right)\right\}\\
&= \mathscr{c}(f).
\end{aligned}
\tag{25}
$$

Definition 6. For any two functions $\mathscr{C}_X\colon L^X \longrightarrow M$ and $\mathscr{C}_Y\colon L^Y \longrightarrow M$, if the function $f\colon (X,\mathscr{C}_X) \longrightarrow (Y,\mathscr{C}_Y)$ is bijective, then the isomorphism degree $\mathscr{iso}(f)$ of the function f is given by $\mathscr{iso}(f) = \mathfrak{p}(f)\wedge\mathfrak{p}(f^{-1})$.

Theorem 10. *For any two functions* $\mathscr{C}_X\colon L^X \longrightarrow M$ *and* $\mathscr{C}_Y\colon L^X \longrightarrow M$, *if the function* $f\colon (X,\mathscr{C}_X) \longrightarrow (Y,\mathscr{C}_Y)$ *is bijective, then* $\mathfrak{p}(f^{-1}) = \mathfrak{c}(f)$ *and* $\mathscr{iso}(f) = \mathfrak{p}(f)\wedge\mathfrak{p}(f^{-1}) = \mathfrak{p}(f)\wedge\mathfrak{c}(f)$.

Proof. From the bijectivity of the function f, we have $(f^{-1})_L^{\leftarrow}(\lambda) = f_L^{\longrightarrow}(\lambda)$ for any $\lambda \in L^X$. Thus,

$$\begin{aligned}\mathfrak{p}\left(f^{-1}\right) &= \bigwedge_{\lambda\in L^X}\left\{\mathbf{P}_{\mathbf{Coc}_X}(\lambda) \longmapsto \mathbf{P}_{\mathbf{Coc}_Y}\left(\left(f^{-1}\right)_L^{\leftarrow}(\lambda)\right)\right\}\\ &= \bigwedge_{\lambda\in L^X}\left\{\mathbf{P}_{\mathbf{Coc}_X}(\lambda) \longmapsto \mathbf{P}_{\mathbf{Coc}_Y}\left(f_L^{\longrightarrow}(\lambda)\right)\right\} = \mathfrak{c}(f).\end{aligned} \tag{26}$$

Therefore, $\mathscr{iso}(f) = \mathfrak{p}(f)\wedge\mathfrak{p}(f^{-1}) = \mathfrak{p}(f)\wedge\mathfrak{c}(f)$. This completes the proof.

Proposition 5. *Given a function* $\mathscr{C}_X\colon L^X \longrightarrow M$, *if* $id\colon (X,\mathscr{C}_X) \longrightarrow (X,\mathscr{C}_X)$ *is the identity function, then*

$$\mathfrak{iso}(\mathrm{id}) = \mathfrak{p}(\mathrm{id}) = \mathfrak{c}(\mathrm{id}) = \top_M. \tag{27}$$

Proof. Straightforward.

Proposition 6. *Given three functions* $\mathscr{C}_X\colon L^X \longrightarrow M$, $\mathscr{C}_Y\colon L^Y \longrightarrow M$, *and* $\mathscr{C}_Z\colon L^Z \longrightarrow M$, *let* $f\colon (X,\mathscr{C}_X) \longrightarrow (Y,\mathscr{C}_Y)$ *and* $g\colon (Y,\mathscr{C}_Y) \longrightarrow (Z,\mathscr{C}_Z)$ *be two bijective functions, then* $\mathscr{iso}(f)\wedge\mathscr{iso}(g) \le \mathscr{iso}(g^\circ f)$.

Proof. Straightforward.

Lemma 3. *For any two functions* $\mathscr{C}_X\colon L^X \longrightarrow M$ *and* $\mathscr{C}_Y\colon L^Y \longrightarrow M$, *if the function* $f\colon (X,\mathscr{C}_X) \longrightarrow (Y,\mathscr{C}_Y)$ *is bijective, then*

(1) $\mathfrak{p}(f) = \wedge_{\lambda\in L^X}\left\{\mathbf{P}_{\mathbf{Coc}_Y}\left(f_L^{\longrightarrow}(\lambda)\right) \longmapsto \mathbf{P}_{\mathbf{Coc}_X}(\lambda)\right\}$

(2) $\mathfrak{c}(f) = \wedge_{\nu\in L^Y}\left\{\mathbf{P}_{\mathbf{Coc}_X}\left(f_L^{\leftarrow}(\nu)\right) \longmapsto \mathbf{P}_{\mathbf{Coc}_Y}(\nu)\right\}$

Proof

(1) From the bijectivity of the function f, we get $f_L^{\leftarrow}(f_L^{\longrightarrow}(\lambda)) = \lambda$ for any $\lambda \in L^X$ and $f_L^{\longrightarrow}(f_L^{\leftarrow}(\nu)) = \nu$ for any $\nu \in L^Y$. Thus,

$$\begin{aligned}\bigwedge_{\lambda\in L^X}\left\{\mathbf{P}_{\mathbf{Coc}_Y}\left(f_L^{\longrightarrow}(\lambda)\right) \longmapsto \mathbf{P}_{\mathbf{Coc}_X}(\lambda)\right\} &= \bigwedge_{\lambda\in L^X}\left\{\mathbf{P}_{\mathbf{Coc}_Y}\left(f_L^{\longrightarrow}(\lambda)\right) \longmapsto \mathbf{P}_{\mathbf{Coc}_X}\left(f_L^{\leftarrow}\left(f_L^{\longrightarrow}(\lambda)\right)\right)\right\}\\ &\ge \bigwedge_{\lambda\in L^X}\left\{\mathbf{P}_{\mathbf{Coc}_Y}(\nu) \longmapsto \mathbf{P}_{\mathbf{Coc}_X}\left(f_L^{\leftarrow}(\nu)\right)\right\} = \mathfrak{p}(f)\\ &= \bigwedge_{\nu\in L^X}\left\{\mathbf{P}_{\mathbf{Coc}_Y}\left(f_L^{\longrightarrow}\left(f_L^{\leftarrow}(\nu)\right)\right) \longmapsto \mathbf{P}_{\mathbf{Coc}_X}\left(f_L^{\leftarrow}(\nu)\right)\right\}\\ &\ge \bigwedge_{\lambda\in L^X}\left\{\mathbf{P}_{\mathbf{Coc}_Y}\left(f_L^{\longrightarrow}(\lambda)\right) \longmapsto \mathbf{P}_{\mathbf{Coc}_X}(\lambda)\right\}.\end{aligned} \tag{28}$$

Therefore,

$$\mathfrak{p}(f) = \bigwedge_{\lambda\in L^X}\left\{\mathbf{P}_{\mathbf{Coc}_Y}\left(f_L^{\longrightarrow}(\lambda)\right) \longmapsto \mathbf{P}_{\mathbf{Coc}_X}(\lambda)\right\}. \tag{29}$$

(2) similar to (1).

Theorem 11. *For any two functions* $\mathscr{C}_X\colon L^X \longrightarrow M$ *and* $\mathscr{C}_Y\colon L^Y \longrightarrow M$, *if the function* $f\colon (X,\mathscr{C}_X) \longrightarrow (Y,\mathscr{C}_Y)$ *is bijective, then*

(1) $\mathscr{iso}(f) = \wedge_{\lambda\in L^X}\left\{\mathbf{P}_{\mathbf{Coc}_X}(\lambda) \longleftrightarrow \mathbf{P}_{\mathbf{Coc}_Y}\left(f_L^{\longrightarrow}(\lambda)\right)\right\}$

(2) $\mathscr{iso}(f) = \wedge_{\nu\in L^Y}\left\{\mathbf{P}_{\mathbf{Coc}_Y}(\nu) \longleftrightarrow \mathbf{P}_{\mathbf{Coc}_X}\left(f_L^{\leftarrow}(\nu)\right)\right\}$

5. The Quotient Degree of Functions between (L,M)-Fuzzy Concave Structures

In this section, we endow the quotient functions with some degree and discuss the relationship with the degree of (L,M)-fuzzy concavity preserving functions and the degree of (L,M)-fuzzy concave-to-concave functions.

Definition 7. For any two functions $\mathscr{C}_X\colon L^X \longrightarrow M$ and $\mathscr{C}_Y\colon L^Y \longrightarrow M$, if the function $f\colon X \longrightarrow Y$ is surjective, then the quotient degree of the function f with respect to $\mathscr{C}_X$ and $\mathscr{C}_Y$, denoted by $\mathfrak{q}(f)$, is given by

$$\mathfrak{q}(f) = \bigwedge_{\nu\in L^Y}\left\{\mathbf{P}_{\mathbf{Coc}_Y}(\nu) \longleftrightarrow \mathbf{P}_{\mathbf{Coc}_X}\left(f_L^{\leftarrow}(\nu)\right)\right\}. \tag{30}$$

The proof of the following theorem is similar to the proof of Theorem 8.

Theorem 12. *For any two functions* $\mathscr{C}_X\colon L^X \longrightarrow M$ *and* $\mathscr{C}_Y\colon L^Y \longrightarrow M$, *if* $\mathbf{Coc}_X$ *and* $\mathbf{Cov}_Y$ *refer to the* (L,M)*-fuzzy concavity degree of* $\mathscr{C}_X$ *and* $\mathscr{C}_Y$, *respectively, and* $f\colon (X,\mathscr{C}_X) \longrightarrow (Y,\mathscr{C}_Y)$ *is a surjective function, then*

(1) $\mathfrak{q}(f) = \wedge\left\{r \in M \,\middle|\, \begin{array}{c}\mathbf{Coc}_Y\wedge\mathscr{C}_Y(\nu)\wedge r \le \mathbf{Coc}_X\wedge\mathscr{C}_X\left(f_L^{\leftarrow}(\nu)\right),\\ \mathbf{Coc}_X\wedge\mathscr{C}_X\left(f_L^{\leftarrow}(\nu)\right)\wedge r \le \mathbf{Coc}_Y\wedge\mathscr{C}_Y(\nu),\ \forall \nu\in L^Y\end{array}\right\}$

(2) $\mathfrak{q}(f) = \wedge\left\{r \in M \,\middle|\, \begin{array}{c}\forall s \le \mathbf{Coc}_Y\wedge r,\ \forall \nu \in (\mathscr{C}_Y)_{[s]}\, s \le \mathbf{Coc}_X,\ f_L^{\leftarrow}(\nu) \in (\mathscr{C}_X)_{[s]},\\ \forall t \le \mathbf{Coc}_X\wedge r,\ \forall f_L^{\leftarrow}(\nu) \in (\mathscr{C}_X)_{[t]}\, t \le \mathbf{Coc}_Y,\ \nu \in (\mathscr{C}_Y)_{[t]}\end{array}\right\}$

(3) $\mathfrak{q}(f) = \wedge\left\{r \in M \,\middle|\, \begin{array}{c}\forall s \notin \alpha(\mathbf{Coc}_Y\wedge r),\ \forall \nu \in (\mathscr{C}_Y)^{[s]}\, s \notin \alpha(\mathbf{Coc}_X),\ f_L^{\leftarrow}(\nu) \in (\mathscr{C}_X)^{[s]},\\ \forall t \notin \alpha(\mathbf{Coc}_X\wedge r),\ \forall f_L^{\leftarrow}(\nu) \in (\mathscr{C}_X)^{[t]}\, t \notin \alpha(\mathbf{Coc}_Y),\ \nu \in (\mathscr{C}_Y)^{[t]}\end{array}\right\}$

Theorem 13. *For any two functions* $\mathscr{C}_X\colon L^X \longrightarrow M$ *and* $\mathscr{C}_Y\colon L^Y \longrightarrow M$, *if* $f\colon (X,\mathscr{C}_X) \longrightarrow (Y,\mathscr{C}_Y)$ *is a surjective function, then* $\mathfrak{q}(f) \le \mathfrak{p}(f)$.

Proof. Straightforward

Theorem 14. *For any two functions* $\mathscr{C}_X\colon L^X \longrightarrow M$ *and* $\mathscr{C}_Y\colon L^Y \longrightarrow M$, *if the function* $f\colon (X,\mathscr{C}_X) \longrightarrow (Y,\mathscr{C}_Y)$ *is surjective, then* $\mathfrak{p}(f)\wedge\mathfrak{c}(f) \le \mathfrak{q}(f)$.

Proof. From the surjectivity of f, $f_L^{\longrightarrow}(f_L^{\leftarrow}(\gamma) = \gamma$ for each $\gamma \in L^Y$. Then,

$$\begin{aligned}
\mathfrak{p}(f)\wedge\mathfrak{c}(f) &= \bigwedge_{\nu\in L^Y}\left\{\mathbf{P}_{\mathbf{Coc}_Y}(\nu)\longmapsto\mathbf{P}_{\mathbf{Coc}_X}(f_L^{\leftarrow}(\nu))\right\}\wedge\bigwedge_{\lambda\in L^X}\left\{\mathbf{P}_{\mathbf{Coc}_X}(\lambda)\longmapsto\mathbf{P}_{\mathbf{Coc}_Y}(f_L^{\longrightarrow}(\lambda))\right\}\\
&\leq \bigwedge_{\nu\in L^Y}\left\{\mathbf{P}_{\mathbf{Coc}_Y}(\nu)\longmapsto\mathbf{P}_{\mathbf{Coc}_X}(f_L^{\leftarrow}(\nu))\right\}\wedge\bigwedge_{\gamma\in L^Y}\left\{\mathbf{P}_{\mathbf{Coc}_X}(f_L^{\leftarrow}(\gamma))\longmapsto\mathbf{P}_{\mathbf{Coc}_Y}(f_L^{\longrightarrow}(f_L^{\leftarrow}(\gamma)))\right\}\\
&= \bigwedge_{\nu\in L^Y}\left\{\mathbf{P}_{\mathbf{Coc}_Y}(\nu)\longmapsto\mathbf{P}_{\mathbf{Coc}_X}(f_L^{\leftarrow}(\nu))\right\}\wedge\bigwedge_{\gamma\in L^Y}\left\{\mathbf{P}_{\mathbf{Coc}_X}(f_L^{\leftarrow}(\gamma))\longmapsto\mathbf{P}_{\mathbf{Coc}_Y}(\gamma)\right\}\\
&= \bigwedge_{\nu\in L^Y}\left\{\left(\mathbf{P}_{\mathbf{Coc}_Y}(\nu)\longmapsto\mathbf{P}_{\mathbf{Coc}_X}(f_L^{\leftarrow}(\nu))\right)\wedge\left(\mathbf{P}_{\mathbf{Coc}_X}(f_L^{\leftarrow}(\nu))\longmapsto\mathbf{P}_{\mathbf{Coc}_Y}(\nu)\right)\right\}\\
&= \bigwedge_{\nu\in L^Y}\left\{\mathbf{P}_{\mathbf{Coc}_Y}(\nu)\longleftrightarrow\mathbf{P}_{\mathbf{Coc}_X}(f_L^{\leftarrow}(\nu))\right\} = \mathfrak{q}(f).
\end{aligned}\tag{31}$$

Theorem 15. *For any three functions $\mathscr{C}_X\colon L^X \longrightarrow M$, $\mathscr{C}_Y\colon L^Y \longrightarrow M$, and $\mathscr{C}_Z\colon L^Z \longrightarrow M$, if the functions $f\colon (X,\mathscr{C}_X) \longrightarrow (Y,\mathscr{C}_Y)$ and $g\colon (Y,\mathscr{C}_Y) \longrightarrow (Z,\mathscr{C}_Z)$ are surjective, then*

(1) $\mathfrak{q}(f)\wedge\mathfrak{q}(g)\leq\mathfrak{q}(g^\circ f)$

(2) $\mathfrak{q}(g^\circ f)\wedge\mathfrak{p}(f)\wedge\mathfrak{p}(g)\leq\mathfrak{q}(g)$

Proof

(1)

$$\begin{aligned}
\mathfrak{q}(f)\wedge\mathfrak{q}(g) &= \bigwedge_{\nu\in L^Y}\left\{\mathbf{P}_{\mathbf{Coc}_Y}(\nu)\longleftrightarrow\mathbf{P}_{\mathbf{Coc}_X}(f_L^{\leftarrow}(\nu))\right\}\wedge\bigwedge_{\gamma\in L^Z}\left\{\mathbf{P}_{\mathbf{Coc}_Z}(\gamma)\longleftrightarrow\mathbf{P}_{\mathbf{Coc}_Y}(g_L^{\leftarrow}(\gamma))\right\}\\
&\leq \bigwedge_{\xi\in L^Z}\left\{\mathbf{P}_{\mathbf{Coc}_Y}(g_L^{\leftarrow}(\xi))\longleftrightarrow\mathbf{P}_{\mathbf{Coc}_X}(f_L^{\leftarrow}(g_L^{\leftarrow}(\xi)))\right\}\wedge\bigwedge_{\gamma\in L^Z}\left\{\mathbf{P}_{\mathbf{Coc}_Z}(\gamma)\longleftrightarrow\mathbf{P}_{\mathbf{Coc}_Y}(g_L^{\leftarrow}(\gamma))\right\}\\
&= \bigwedge_{\xi\in L^Z}\left\{\mathbf{P}_{\mathbf{Coc}_Y}(g_L^{\leftarrow}(\xi))\longleftrightarrow\mathbf{P}_{\mathbf{Coc}_X}\left((g^\circ f)_L^{\leftarrow}(\xi)\right)\right\}\wedge\bigwedge_{\gamma\in L^Z}\left\{\mathbf{P}_{\mathbf{Coc}_Z}(\gamma)\longleftrightarrow\mathbf{P}_{\mathbf{Coc}_Y}(g_L^{\leftarrow}(\gamma)\right\}\\
&= \bigwedge_{\gamma\in L^Z}\left\{\left(\mathbf{P}_{\mathbf{Coc}_Y}(g_L^{\leftarrow}(\gamma))\longleftrightarrow\mathbf{P}_{\mathbf{Coc}_X}\left((g^\circ f)_L^{\leftarrow}(\gamma)\right)\right)\wedge\left(\mathbf{P}_{\mathbf{Coc}_Z}(\gamma)\longleftrightarrow\mathbf{P}_{\mathbf{Coc}_Y}(g_L^{\leftarrow}(\gamma))\right)\right\}\\
&\leq \bigwedge_{\gamma\in L^Z}\left\{\mathbf{P}_{\mathbf{Coc}_X}(\gamma)\longleftrightarrow\mathbf{P}_{\mathbf{Coc}_X}\left((g^\circ f)_L^{\leftarrow}(\gamma)\right)\right\}\\
&= \mathfrak{q}(g\circ f).
\end{aligned}\tag{32}$$

(2) Since

$$\begin{aligned}
\mathfrak{q}(g^\circ f) &= \bigwedge_{\gamma\in L^Z}\left\{\mathbf{P}_{\mathbf{Coc}_Z}(\gamma)\longleftrightarrow\mathbf{P}_{\mathbf{Coc}_X}\left((g^\circ f)_L^{\leftarrow}(\gamma)\right)\right\}\leq\bigwedge_{\gamma\in L^Z}\left\{\mathbf{P}_{\mathbf{Coc}_X}\left((g^\circ f)_L^{\leftarrow}(\gamma)\right)\longmapsto\mathbf{P}_{\mathbf{Coc}_Z}(\gamma)\right\},\\
\mathfrak{p}(f) &= \bigwedge_{\nu\in L^Y}\left\{\mathbf{P}_{\mathbf{Coc}_Y}(\nu)\longmapsto\mathbf{P}_{\mathbf{Coc}_X}(f_L^{\leftarrow}(\nu))\right\}\leq\bigwedge_{\xi\in L^Z}\left\{\mathbf{P}_{\mathbf{Coc}_Y}(g_L^{\leftarrow}(\xi))\longmapsto\mathbf{P}_{\mathbf{Coc}_X}(f_L^{\leftarrow}(g_L^{\leftarrow}(\xi)))\right\}\\
&= \bigwedge_{\xi\in L^Z}\left\{\mathbf{P}_{\mathbf{Coc}_Y}(g_L^{\leftarrow}(\xi))\longmapsto\mathbf{P}_{\mathbf{Coc}_X}\left((g^\circ f)_L^{\leftarrow}(\xi)\right)\right\},
\end{aligned}\tag{33}$$

then

$$\mathfrak{q}(g\circ f)\wedge\mathfrak{p}(f)\leq\bigwedge_{\xi\in L^Z}\left\{\mathbf{P}_{\mathbf{Coc}_Y}\left(g_L^{\leftarrow}(\xi)\right)\longmapsto\mathbf{P}_{\mathbf{Coc}_Z}(\xi)\right\}. \tag{34}$$

Therefore,

$$\begin{aligned}\mathfrak{q}(g^{\circ}f)\wedge\mathfrak{p}(f)\wedge\mathfrak{p}(g)&\leq\bigwedge_{\xi\in L^Z}\left\{\mathbf{P}_{\mathbf{Coc}_Y}\left(g_L^{\leftarrow}(\xi)\right)\longmapsto\mathbf{P}_{\mathbf{Coc}_Z}(\xi)\right\}\wedge\bigwedge_{\gamma\in L^Z}\left\{\mathbf{P}_{\mathbf{Coc}_Z}(\gamma)\longmapsto\mathbf{P}_{\mathbf{Coc}_Y}\left(g_L^{\leftarrow}(\gamma)\right)\right\}\\&=\bigwedge_{\gamma\in L^Z}\left\{\left(\mathbf{P}_{\mathbf{Coc}_Y}\left(g_L^{\leftarrow}(\gamma)\right)\longmapsto\mathbf{P}_{\mathbf{Coc}_Z}(\gamma)\right)\wedge\left(\mathbf{P}_{\mathbf{Coc}_Z}(\gamma)\longmapsto\mathbf{P}_{\mathbf{Coc}_Y}\left(g_L^{\leftarrow}(\gamma)\right)\right)\right\}\\&=\bigwedge_{\gamma\in L^Z}\left\{\mathbf{P}_{\mathbf{Coc}_Z}(\gamma)\longleftrightarrow\mathbf{P}_{\mathbf{Coc}_Y}\left(g_L^{\leftarrow}(\gamma)\right)\right\}=\mathfrak{q}(g).\end{aligned} \tag{35}$$

This completes the proof.

6. Conclusion

In this paper, we presented the degree to which a function $\mathscr{C}\colon L^X\longrightarrow M$ is an (L,M)-fuzzy concavity on a nonempty set X. Moreover, the degree to which an L-subset is an L-concave set with respect to $\mathscr{C}$ was considered. Also, we defined the concavity preserving, concave-to-concave degree, and quotient degree for functions between (L,M)-fuzzy concave structures. Their characterizations were given and the relationships among them were discussed. We think our results will be useful to consider many properties of concave structures under degree of (L,M)-fuzzy concavity. We think that studying the topological properties of such degree is the most abstract and generalization of these properties, which leads to the results of previous studies as soon as the degree equals to $\top_M$. Thus, the fuzzification theory would be applied in a better and more generalized way.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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